Solutions Manual

to accompany

Introduction to Operations Research

Seventh Edition

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INTRODUCTION TO OPERATIONS RESEARCH
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Chapter 2

2.1-1

(a) San Francisco Police Department has a total police force of 1900, with 850 officers on patrol. The total budget of SFPD in 1986 was $176 million with patrol coverage cost of $79 million. This brings out the importance of the problem.

Like most police departments, SFPD was also operated with manually designed schedules. It was impossible to know if the manual schedules were optimal in serving residents' needs. It was difficult to evaluate alternative policies for scheduling and deploying officers. There was also the problem of poor response time and low productivity, pressure of increasing demands for service with decreasing budgets. The scheduling system was facing the problem of providing the highest possible correlation between the number of officers needed and the number actually on duty during each hour. All these problems led the Task Force to search for a new system and thus undertake this study.

(b) After reviewing the manual system, the Task Force decided to search for a new system. The criteria it specified included the following six directives :

-- the system must use the CAD (computer aided dispatching) system, which provides a large and rich data base on resident calls for service. The CAD system was used to dispatch patrol officers to call for service and to maintain operating statistics such as call types, waiting times, travel time and total time consumed in servicing calls. The directive was to use this data on calls for service and consumed times to establish work load by day of week and hour of day

-- it must generate optimal and realistic integer schedules that meet management policy guidelines using a micro-computer

-- it must allow easy adjustment of optimal schedules to accommodate human considerations without sacrificing productivity

-- it must create schedules in less than 30 minutes and make changes in less than 60 seconds

-- it must be able to perform both tactical scheduling and strategic policy testing in one integrated system

-- the user interface must be flexible and easy, allowing the users (captains) to decide the sequence of functions to be executed instead of forcing them to follow a restrictive sequence.

(c) The computer aided dispatching (CAD) system also is used to gather hourly data on historical patrol activity. For each of the 168 hours of the week, data are gathered on call types, consumed times by call type, and percent of each call type requiring two or more officers. Other operating statistics maintained include waiting times and travel times. When needed for forecasting purposes, the call and consumed time data are downloaded from the mainframe to a microcomputer. The other needed data are input by the captains and stored in a file for reuse. (See Figures 1 and 2 of the article for further information.)

(d) A list of the various tangible and intangible benefits are as follows :

-- total savings of $11 million per year
-- finding 4/10 plan (4 day 10 hour shift) is superior to 5/8 plan (5 day 8 hour shift) which leads to a saving of 5.8 million
-- added 176,000 productive hours to the patrol staff every year --> a saving of $5.2 million per year
-- response times dropped an average of 20 percent which led to potential crime reduction
-- an approximate 50 percent reduction in shortages and surpluses of police officers
-- resulted in a 32 percent increase in traffic citation --> a $3 million increase in citation revenues
-- a cost benefit analysis of the scheduling aspects shows a one-time cost of $50,000 and benefits of $5.2 million per year
-- a 36 percent decrease in days lost due to sick leave
-- a 21 percent increase in self-initiated office activities
-- an improved morale of the officers revealed by a survey by an approval rating of 96 percent
-- it allowed management to adjust schedules easily and quickly to meet seasonal changes in pattern of crime.

2.1.2

(a) Taking all the statistics of AIDS cases into account it was inferred that just one-third of all cases nation-wide involved some aspect of Injection Drug Use(IDU). But in contrast to this national picture, over 60% of 500 cases reported in New Haven , Connecticut was traced to drug use. Though it was realized previously , by 1987 it was clear that the dominant mode of HIV transmission in New Haven was the practice of needle sharing for drug injection.

This was the background of the study and in 1987 a street outreach program was implemented which included a survey of drug addicts with partial intent to determine why IDUs continued to share needles given the threat of HIV infection and AIDS. It was claimed by the survey respondents that IDUs shared needles since they were scared and feared arrest for possessing a syringe without prescription which was forbidden by law in Connecticut. Respondents also pointed out difficulties involved in entering drug treatment program. The officials recognized that logical intervention was needle exchange whereby IDUs exchanged their used needles for clean ones. This would remove infectious drug injection equipment from circulation and also ease access to clean needles. Further, contacts made as a result of needle exchange might lead some active IDUs to consider counseling or enter drug treatment. After a lot of lobbying finally the bill for the first legal needle exchange program became effective on July 1, 1990.

(b) The design for the needle exchange program was achieved over the summer of 1980. The relevant committee decided that IDUs would be treated with respect and so no identification information was asked of program clients. The program began operating on November 13, 1990.

The needle exchange operate on an outreach basis. A van donated by Yale university visits neighborhoods with high concentration of IDUs. Outreach staff members try to educate the clients over there by different means like distributing literature documenting risks of HIV infection, dispensing condoms , clean packets, etc.

The primary goal of needle exchange is to reduce incidence of new HIV infection among IDUs. While studies showed consistent self-reported reductions in risky behavior among IDUs participating in needle exchange programs the studies were not convincing. So the mechanics of needle exchange require that the behavior of needles must change What was required was to reduce the time needles spend circulating in the population. As needles circulate for shorter period of time, needles share fewer people which lower the number of infected needles in the pool of circulating needles which in effect lowers chances of an IDU becoming infected being injected with a previously infected needle. To use this theory required invention of new data collection system which is as follows.

A syringe tracking and testing is a system developed to interview the needles returned to the program. All clients participating in the needle exchange are given unique code names and every needle distributed receives a code. Everytime a client exchanges needles, an outreach worker records the date and location of exchange. He also records the code name of the client receiving the needles alongside the codes of the needles. The client then places the returned needles in a canister to which the worker puts a label with the date and location of exchange and code name of client.

All returned needles are brought to a laboratory at Yale University where a technician collates the information on the canister labels with the tracking numbers on the returned needles. For non-program or street needles returned to needle exchange , the

location, date, and client code are recorded. A sample of the returned needles are tested for HIV.

(c) The initial results from this system were both shocking and decisive. At the start of the program, the IDUs presented the needles in their possession for clear needles. These street needles are representative of risk faced by an IDU prior to operation of needle exchange which showed a prevalence level of 67.5 percent which tested HIV positive. As of middle of March 1991, 50.3 percent of the program needles tested positive. Since March 1991, additional program needles have been tested of which 40.5 percent tested positive. This gave further support to the protection offered by needle exchange program.

Though these results are encouraging, they do not link operations of needle exchange to changes in the rate of new HIV infections . To achieve this required a development of a mathematical model describing HIV transmission among IDUs via needle sharing. The syringe tracking and testing system in concert with limited observation obtained from surveying program clients provided data required to estimate parameters for this model. Though the model developed was conservative, the results were interesting. It estimated that in absence of behavioral changes on part of IDUs in the program, rate of new HIV infections among needle exchange clients would drop. It estimates a 33 percent reduction in new HIV infections.

(d) To understand the impact of this study requires both a local and national perspective. In the local aspect, it is possible to construct a conservative estimate of the actual number of infections averted. As many clients who joined the needle exchange apparently dropped out, the conservative impact of the program can be estimated by multiplying the cumulative number of person years spent in the program over all clients by the incidence reduction of 2 HIV infections per 100 client years. This assumes all those who apparently dropped out of the program are truly recidivists, an assumption that may be patently false. Calculations have shown that between $1 million and $2 million dollar in public health care expenditure have been avoided over the first two years of the program.

This only hints at the true impact of this work. Needle exchange has been returned to the menu of legitimate AIDS intervention in major American cities in large part due to the evaluation of New Haven. In some calculations made as to how much public health care costs could be avoided only the annual reduction in HIV incidence among needle exchange program clients is considered, as opposed to changes in lifetime probability of acquiring HIV infection. While decrease in lifetime risk will be less than decrease in annual incidence the effect of placing clients in drug treatment via needle exchange has been ignored. If this point is considered impact of needle exchange on probability of their acquiring HIV could be substantial.

2.2-1

(a) The Dutch Government has been facing problems regarding its water management the past it was too much water but now it is the scarcity of fresh water and pollution due to increased industrialization and a growing population with high standard of living. Some features of the Dutch landscape exaggerate the problem.

Netherlands , one of the densely populated countries of the world and the seventh largest wealthiest nation derives a huge amount of wealth from crops grown in irrigated land. Since agriculture is the largest user of fresh water in Netherlands water shortages can cause large economic losses. The Rhine river is Netherlands major source of surface water for agriculture, irrigation and other purposes. Along with other rivers and canals it is a major artery for the inland shipping fleet of Western Europe. Low water levels in rivers and canals can cause shipping delays and economic losses too because only partially laden ships can navigate the inland waterways. Besides this mines and industries along the Rhine discharge into the water different type of pollutants which also contribute to the rivers increasing level of salinity which in turn damage crops and threaten environment and personal health. Power plants on the banks can degrade quality of water by discharging excess heat into streams, which may in turn endanger the ecological balance in the neighborhood. Besides salinity, the most important water quality problem is eutrophication heavy growth of algae in relatively stagnant water of storage reservoirs and lakes which cause the water to smell and taste foul.

Though in mid 1970s supply met the demand of fresh surface water except in dry years, it was predicted not to be true for late 1980s. But ground water sources were already facing scarcity. Rapid increases in ground water extraction recently have resulted in drop of its level in many areas. This in turn can cause agricultural and environmental damage in areas where water level was higher. Facing such water management problem the Netherlands Government agency responsible for water control and public works, Rijkswaterstaat commissioned an analysis on which to base a new national water management policy, which resulted in PAWN , the Policy Analysis for the Water Management of the Netherlands in April, 1977.

(b) The purpose of the five mathematical models are as follows:

The Water Distribution Model is the heart of the analytic method. The infrastructure of the surface water system consists of rivers and canals that transport water, lakes and reservoirs that store water, weirs, locks and lock bypasses , sluices and pumping stations that are used to control the transport of water throughout the country. The model simulates the major components of this system in detail and contains aggregated representations of the other components. The model provides information on the water management system, including flows, level of water, extractions, discharges,

depth of shipping and concentration of pollutants. It provides information on different costs, including investment and operating for technical and managerial tactics and irrigation, as well as shortage and salinity losses for agriculture, low water shipping loss and shipping delay losses. This information is provided for each 10-day period and is given in a summary of totals on averages for the entire year.

The Industry Response Simulation Model : When water becomes more expensive or less available, firms respond by modifying their production process to consume less water, usually with an accompanying increase in costs, part or all which may be passed on to their customers. To find out these responses and their costs, PAWN developed and used this model. The model simulates the behavior of industrial firms in response to a change in ground water extractions or an increase in the price of drinking water. In determining behavior, the model assumes that each firm will choose the least costly alternative available.

Although the model was developed to investigate the effect of ground water charges on industry it is also used to examine effect of imposing quotas that restrict ground water extractions.

Electric Power Reallocation and Cost Model : The Water Distribution Model provides an excess temperature table that shows rise in temperature at one node resulting from a reference heat discharge by a power plant at another node. To obtain the excess temperatures created by heat of power plant discharges other than the reference, this model scales the correct entries in the table by the ratio of the new to reference discharge.

The model calculates the optimal generating schedule for two basic conditions: one in which the thermal standards are relaxed and the other in which it is imposed. The difference is the cost attributable to the thermal standards, the thermal penalty cost. The model repeats this process for each 10-day period in the year and calculates the total thermal penalty annually as well as some other statistics.

The Nutrient Model : Eutrophication, a heavy growth of algae called an algae bloom occurs in the still water of lakes and reservoirs. This model estimates the amounts of nutrients , phosphates, nitrogen and silicon available to algae, given the nutrient flows entering the lake. The model calculates the composition of a column of water 1 square meter in area and as deep as the lake under investigation , in contact with the air and with the bottom sedimentations. The important nutrient processes include the inflow and outflow of nutrient bearing water, the flux of nutrients from the bottom, and the flux of nutrient to and from algae.

The Algae Bloom Model : PAWN used this model to analyze the effect on algae blooms of circumstances, including introducing control tactics. It predicts weekly size and species composition of the algae bloom, given amounts of nutrients and solar energy available to the algae.

(c) PAWN compares policies in terms of their impacts. In choosing impact measures, primary criterion was that they are sufficient to span quite a number of objectives. It includes both national and general water management as well as specific objectives mentioned by different interest groups. The objectives also had to reflect both equity and efficiency.

Impacts on water management system include investment and operating cost of technical and managerial tactics, as well as flood risk in the Ijssal lake.

Direct impact on users include change in profit, expenditure, revenue for each user group like agriculture, shipping, electric power generation, industries, drinking water supply companies.

Environmental impacts include violation of water quality standards, damage to nature areas caused by construction of new facilities and total amount of ground water extracted.

Impact on entire nation include net monetary benefit to nation after deducting transfer payments, total economic effects-- both Government revenues and charges-- in production , employment and imports-- that occurs in both industries directly involved in construction of major new facilities and interrelated industries and effects on public health.

PAWN also pays attention to distributional effects that show uneven distribution of monetary benefit and costs among producers, consumers, Government and uneven distribution of other impacts among different groups and locations.

(d) The several tangible benefits are:

-- the building of Brielse Mier pipeline which will yield $38 million investment savings and $15 million annual net benefit in decreased salinity damage to agriculture.

-- rejection of plan to build the second dike to separate Markermeer from an adjacent saline lake-- saving more than $95 million in investment costs, 0.2% of Dutch domestic product.

-- implementation of new flushing policy for Markermeer is expected to yield net benefits between $1.2 million and $5.4 million per year.

-- adoption of a more stringent thermal standard of an increase of 3 degree Celsius for canals by Dutch since PAWN showed it was practical and not costly. This led to a decrease in locally harmful ecological effects of power plant heat discharges.

The intangible benefits are:

-- drastic changes in Dutch approach to eutrophication, their most serious water quality problem

-- implementation of all recommendations by PAWN would lead to an expected profit between $53 million and $128 million per year

-- to deal with ground water extraction problem, priority had to be given to industry and drinking water companies. If practical methods to replenish the ground

water cannot be devised, regulatory measures will show growth of ground water sprinkling.

-- comprehensive methodology developed by PAWN has been adopted by the Government, other departments, laboratories and used in several major studies

-- PAWN provides method to educate decision makers and train analysts in analyses of complex natural resource and environmental questions.

-- the general approach and some of the techniques have potentially wide applicability.

2.2-2

(a) The author's example of a model in natural sciences is Newton's Law of Universal Gravitation. Tough he says it is one of the most important models in Physics this does not account for all details. For example, it is only approximate if the particles are objects with non-spherical shapes and model ignores relativity.

The model in OR identified by him is the Economic Order Quantity (EOQ) model. Like Newton's Law in Physics, this model too is simple and highlights important features of the real world. It identifies some critical relationships and also shows that a single model can be used for all types of orders. This model too ignores details of the real world which might be considered important. But just like Newton's law, EOQ model is one of the most important one in MS/OR.

(b) The MS/OR profession is often compared to natural sciences. Basic precepts in natural sciences can be used to guide research in MS/OR. The author believes a greater understanding of these precepts can provide needed focus form the profession and help resolve some recent debates.

To be useful, an MS/OR model must possess some qualities as models in natural science. The most important of them are :

-- understandibility

-- verifiability

-- reproducibility

The extent to which a model can be understood depends on tools available for evaluation. But models have inherent values which can be interpreted on inspection. In 1960s many MS/OR professionals tried to model the behavior of automobile traffic. The most successful models examined traffic from macro point of view and found similarities between traffic flow and fluid flow. Less successful models examined behavior of individual drivers with complicated queuing expressions. The latter models though more accurate have less value. They are too difficult to interpret.

A model should be verifiable and based on observable phenomena. It must capture the essence of a problem faced by MS/OR practitioners. It must include important parameters, decision variables, aims and objectives and their relationships.

As a criterion for publication, the phenomena underlying the model must be reproducible. To have a broader appeal the model must be sufficiently general.

2.3-1

(a) The role of evaluating a model is to extract information from it. It entails two, often simultaneous activities -- identifying alternatives and calculating objectives.

The most known technique for identifying alternatives is optimization. The process yields a single solution which maximizes or minimizes a single objective function. The most prevalent technique used for identifying multiple alternatives is sensitivity analysis. The process can show how the optimum changes when model parameters change or can provide near-optimal alternative solutions.

The author views that optimization should not be the sole goal, not just because models are abstractions of real world but because does not provide adequate information for making decisions. Its objective is to find only one solution. But the decision maker probably would prefer information on several alternatives. Though sensitivity analysis increases effectiveness of optimization , it is deficient. It only yields alternative solution near optimum. The decision maker rather needs unique solutions which offer distinct alternatives.

So the author opines that research should be devoted to identify multiple alternatives. One may begin in the solution process itself. Each solution is a feasible alternative, which the decision maker may choose over the optimum. New algorithms may be designed to identify distinct alternatives.

The second step of evaluation should involve calculating quantifiable objective for each alternative.

Thus summarizing, the author views that although optimization has dominated research in MS/OR it is but one technique for addressing one part of MS/OR process. It is deficient since does not provide adequate information for making important decisions/ Complex decisions rather require information on many alternatives and also an understanding of basic trade-offs and principles. Optimization alone cannot provide this information.

(b) The key to MS/OR is not only possessing knowledge. Though different practitioners take different approaches -- three key steps being

-- modeling

-- evaluating

-- deciding , which are all complementary.

In MS/OR systematized knowledge is reflected in better decisions. The key to good decisions is knowledge and judgment. Modeling and evaluation form a systematized way for acquiring knowledge; judgment is acquired through experience.

The problems which do not require judgment are the ones which can be formulated with well-defined objective functions and solved automatically with algorithms which are pretty efficient; an example being the shortest path algorithm. On the other hand, there are problems which are easy to formulate but difficult to solve, example a carpet store owner would not argue with the objective of the cutting stock problem but may not be happy with solutions provided by available software. He would benefit from models that offer help in cutting the carpet. Combining knowledge from modeling with judgment of store owner would give best result.

Generally, important questions facing management are not well-defined as shortest path or cutting stock problem. Neither there are related well-defined problems which can be optimized, example the facilities layout problem.

Thus the roles are all complementary. Most depend on both judgment of decision maker and knowledge gained from modeling and evaluating.

2.4-1

The credibility of analyses and therefore the probability that policies based upon them will be implemented depends on the perceived validity of the models.

The process of model validation though is a burden helps to learn lessons which may not lead to just improvements in the model but also to changes in the scientific theory and public policy. This happened in PAWN with the Nutrient model and eutrophication. When PAWN was started, the Dutch eutrophication control strategy was to decrease phosphate discharges into surface water from point sources mostly sewerage treatment plants.

To find out how effective this strategy is the Algae Bloom model was applied to some major Dutch lakes. It was revealed that in most cases this required enormous percentage decrease in phosphate concentrations.

Next question was what was to be done to achieve a particular percentage decrease in phosphate concentration. The Dutch strategy was based on the fact that large amount of phosphates and other nutrients accumulated in bottom of the lakes was bound permanently to the bottom and hence unavailable to support algae blooms. This was contradicted both in the Nutrient Model calibration process and validation process.

Studies taken convinced that nutrients particularly phosphate can be liberated from bottom sediments both in normal steady mode and explosive mode. This conclusion was widely accepted in the scientific community.

But the conclusion implied that use of a phosphate reduction program as the only way to limit algae bloom would have hardly any immediate success. But analysis with the Algae Bloom model suggests other tactics which could be effective and combination of tactics should be tailored to individual lakes.

2.4-2 The author feels that observation and experimentation are not emphasized in the MS/OR literature or in the training of its workers as much as experience would lead one to believe. As examples he has given some experiences with the US AirForce in early '50s which strengthens his belief.

He opines that observing actual operations as part of analysis process provides a required base for understanding what is going on in a problem situation. They can help to point out difficulties being encountered, suggest hypothesis and theories that may account for problems and offer evidence regarding the validity of the models built as part of problem solving process.

If a problem is in regard to a non-existing system or an operating system fulfills an important function that must continue, so that controlled experiments with are not possible-- one can build a theory about relevant phenomena and analyze the theory but numerical results obtained in this way clearly can be viewed with suspicion. Alternatively if a similar system exists, one can extrapolate from results with it to make estimates about the prospective system. Infact, administrative emergencies or an executive desire to try something new may cause the behavior of a system already in existence to change. The analyst may then be able to collect data useful for analyzing how the system would operate under changed circumstances or for identifying problems that might crop up under different operating regimes.

From his personal experiences he gives evidence to give substance to these remarks of his.

If data was used from one system to predict performance of another he believes that the parameter values form observing another similar system can be useful, and incorporating such estimates in a crude study can be better than not doing a study at all. Parameter values from one context to another cannot be expected to support detailed findings, but even crude findings are enough to provide indispensable information on which to base policy.

He has also analyzed the results of a continent wide Air Defense exercise. He says here that analysis must be carefully planned, and planning must begin early. Early work serves to put attention on the structure of the work and issues to be faced as well as other responsibilities.

Thus, in nutshell, the author views that skills involved in observation and experimentation are enumerous and should be part of the tool kit of many MS/OR analysts. He views that discriminating observation and carefully planned experimentation and analysis are central to MS/OR.

Observing actual operations and collection of data allow us to discern problems, develop hypotheses and validate models needing skill.

Similarly, accurate and complete data are required to estimate validity. Program evaluation brings together many of the issues of observation and experimentation.

Thus issues of scientific and professional craft related to observation and experimentation should occur important places in experience, literature and training of MS/OR workers.

2.4-3

(a) The author views that analysts do not believe that a model can be completely validated. He further opines that policy models can at best be invalidated. Thus the objective of validation or invalidation attempts is to increase the degree of confidence that the events obtained from the model will take place under conditions assumed. After trying all invalidation procedures, one will have a good understanding of strengths and weaknesses of the model and will be able to meet criticisms of omissions. Knowing the limitations of the model will enable one to express proper confidence on its results.

(b) Model Validity deals with correspondence of the model to the real world and related to pointing out all stated and implied assumptions, identification and inclusion of all decision variables and hypothesized relations among variables. Different assumptions are made and the analyst compares each assumption and hypothesis to the internal and external problem environments viewed by the decision maker and comments on the extent of divergence.

Data validity deals with raw and structured data, where structured data is manipulated raw data. Raw data validity is concerned with measurement problems and determining if the data is accurate, impartial and representative. Structured data validity needs review of each step of the manipulation and is a part of model verification.

Logical/mathematical validity deals with translating the model form into a numerical, computer process that produces solutions. There is no standard method to determine this. Approaches include comparing model outcomes with expected or historical results and a close scrutiny of the model form and its numerical representation on a flow chart.

Predictive validity is analyzing errors between actual and predicted outcomes for a model's components and relationships. Here one looks for errors and their magnitudes, why they exist and if how they can be corrected.

Operational validity attempts to assess the importance of errors found under technical validity. It must find out if the use of the model is appropriate for the observed and expected errors. It also deals with the fact whether the model can produce unacceptable answers for proper ranges of parameter values.

Dynamic validity is concerned with determining how the model will be maintained during its life cycle so it will continue to be an accepted representation of the real system. The two areas of interest thus are update and review.

(c) Sensitivity analysis plays an important role in testing the operational validity of a model. In this, values of model parameters are varied over some range of interest to determine if and how the recommended solution changes. If the solution is sensitive to certain parameter changes, the decision maker may want the model analysts to explore further or justify in detail values of these parameters. Sensitivity analysis also involves the relationship between small changes in parameter values and magnitude of related changes in outputs.

(d) Validating a model tests the agreement between behavior of the model and the real world system being modeled. Models of a non-existing system are the difficult to validate. Three concepts apply here : face validity or expert opinion, variable -parameter validity and sensitivity analysis and hypothesis validity. Though these concepts are applicable to all models, models of real systems can be subjected to further tests. Validity is measured by how well the real-system compares with model-generated data.The model is replicatively valid if it matches data acquired from the real system. It is predictively valid when it matches data before getting the data from the real system. A model is structurally valid if it reproduces the observed real system behavior as well as reflects the way in which real system works to produce this.

The author views that there is no validation methodology appropriate for all models. He says that a decision-aiding model can never be completely validated as there are never real data about the alternatives not implemented. Thus, analysts must be careful in devising, implementing, interpreting and reporting validation tests for their models.

(e) Basic validation steps have been cited in page 616 of the article.

2.5-1

(a) In late 1970s oil companies began to experience downward pressure on profitability due to rapid and continuing changes in external environment. Partially in response to these pressures Texaco's Computer Information Systems department developed an improved on-line interactive gasoline blending system called OMEGA. It was first installed in 1983 and is now used in all seven Texaco US refineries and in two foreign plants.

(b) Simple interactive user interface makes OMEGA easy to use. All input data can be entered by hand. OMEGA can interface with refinery data acquisition system. The user can access stock qualities, stock availabilities, blend specification and requirements, starting values and limits, optimization options, automatic stock selection, automatic blend specification and several other options.

Several features aid the user in performing planning functions. By choosing appropriate options user can obtain optimization options. User also has other options.

Each refinery uses different set of features depending on its availability of blending stocks. These vary depending on the configuration of the refinery and particular crudes being refined. Availability and easy use of OMEGA features has provided engineers and blenders with powerful and easy tool.

(c) OMEGA is constantly being updated and extended. It had to be modified to take into account EPA's regulation for a lead phase down for regular-leaded gasoline so that now OMEGA could be more accurate for these lower lead levels.

OMEGA is continuously modified to reflect changes in refinery operations. Differences in refineries required changes to the system.

When Texaco began installing OMEGA in their foreign refineries, additional changes had to be made to handle different requirements of different countries.

Improvements to OMEGA are needed to enable it to answer the new and unanticipated what-if questions often asked by refinery engineers.

(d) Each refinery uses OMEGA in varying degrees and for various purposes depending on their needs, complexity and configuration. Below the typical usage of the system is pointed out.

On a monthly basis, refineries use OMEGA to develop a gasoline blending plan for the month. The refinery planning model's projected blending stock volumes are input to OMEGA. The blending planner calculates 3 to 8 blends in a single OMEGA run. The refinery planning model's blend compositions are input into OMEGA as initial values. Once a reasonable blend is developed, the marketing department is contacted to discuss resulting grade splits. After marketing department does their job a finalized blending plan is developed for the month. The scheduler determines when each of the grades will be blended. All these work are done by using OMEGA.

(e) OMEGA contributes to overall profitability. To measure actual benefit, a method tried was comparing blend composition that blenders used with and without OMEGA. Here OMEGA achieved as much as 30 percent increase in profit. Average increase in profit is approximately 5 percent of gross gasoline revenue. If OMEGA is used to calculate blending recipes fewer blends fail to meet their quality specification. OMEGA's more reliable gasoline grade-split estimates provide significant aid to those developing marketing strategies and refinery production targets. OMEGA is used for what-if case studies performed for example for economic analysis of refinery improvement projects and analysis of how proposed Government regulations would affect Texaco. OMEGA's features have enabled Texaco with capacity to do things not possible with previous blending system, for example, to deal with mix stocks, consider new grades of gasoline, more control on inventory, etc. OMEGA's features make it easy and quick to explore new avenues of profitability for a refinery.

2.5-2

(a) Yellow Freight System, Inc. was founded in 1926 as a regional motor carrier serving the Mid-West. Today it is one of the largest motor carriers in the country. From a mixed operations in 1970s, Yellow now predominantly serves the less-than-truckload (LTL) portion of the freight market. The '80s were a difficult decade for the motor carrier industry. Deregulation made the way for tremendous opportunity for growth but also presented management with new and difficult challenges to manage these larger operations more efficiently than before. After 1980, motor carriers were forced to compete on price, which led to a lot of pressure to cut costs. The result was decrease in transportation rates. Between 1980 to 1990, transportation rates translated to a drop in real terms of 29%. In addition to real rate decreases, the shipping community in response to intense international competition, started to increase their expectation in service. For many shippers, Yellow Freight is a full partner in their total quality management programs. Another important component of the logistics system is timely delivery of freight. Service reliability is also critical. This heightened emphasis on service was a problem for some long-standing operating practices used by national LTL carriers. The effect of these pressures can be seen in the tremendous attrition the industry suffered. Out of top 20 revenue producing LTL carriers in 1979, only 6 are there today. In this period, Yellow Freight grew from 248 to 630 terminals. This growth has had the effect of creating an extremely large and complex operation. The large network also needs a greater degree of coordination.

In 1986, Yellow initiated a project to improve its ability to manage a complex system. Yellow was interested in using modern network method to simulate and optimize a large network. The project had a main goal-- improved service and service reliability through better management control of the network. This goal was supplemented by broader management objectives. There was also an expectation that improved planning would lead to higher productivity level and lower costs. Consequently, a project team was formed.

(b) The development effort at Yellow started with an existing model as a base and then were modified. The result of this effort was SYSNET. SYSNET is more than 80,000 lines of FORTRAN code for performing sophisticated optimizations using modern network tools. They developed an innovative, interactive optimization technology that puts human beings in the loop, placing sophisticated, up-to-date optimization methods in their hands. These methods were required in the development of a system that would handle the entire network without resorting to heuristic methods to decrease the size of the problem. As a result , user is able to analyze impacts of changes in the whole network in a simple but interactive fashion. Projects can be completed earlier new with greater precision. Decisions on shipment consolidations are now optimized taking into account the system effect of each decision.

Yellow uses SYSNET for two sets of applications:

-- main use is tactical load planning, which involves monthly planning and revision of set of instructions that govern handling and consolidation of shipments through the network.

-- the second set of applications involve longer range planning of the network itself. These problems cover the location and sizing of new facilities, and long range decisions that govern the flow of freight between terminals.

At Yellow SYSNET is more than just a piece of code. It embodies an entire planning methodology adopted by all levels of the company. From strategic planning studies communicated to high-level management to network routing instructions sent right to the field, SYSNET has become a comprehensive planning process that has allowed management to maintain control of a large complex operation. In addition, Yellow uses SYSNET as the central tool in the design and evaluation of projects of over $10 million in annual savings.

(c) The interactive aspects of the code proved important in two respects:

-- the user was needed to guide the search for changes in the network. For example, user may know that freight levels are in the rise in the Midwest or a particular breakbulk is facing problems with capacity. In other cases, user may know that current solution is a local minimum and a major change in the network is needed to achieve an overall improvement. A human being can easily point out these spatial patterns and test for promising configurations.

-- the second use proved critical to the adoption of the system and was the user's capability to accept and reject suggestions made by the computer. SYSNET displays suggested changes and allows the user to evaluate each one in terms of difficult to quantify factors. Also local factors, such as work rules or special operating practices that are not incorporated into the model can be accounted for by a knowledgeable user.

(d) For strategic planning, the outputs from SYSNET are a set of reports used to prepare management summaries on different options. SYSNET is also used on a operational basis to perform load planning. In this role, SYSNET is used to maintain a file that determines the actual routing of shipments through the service network. This file, which contains the load planning, is accessed directly by systems that are used by every terminal manager in the field. SYSNET'S control of load planning and its capability to communicate these instructions to the field is the most important accomplishment of the project.

(e) SYSNET's effect can be seen in four areas :

-- quality of planning practices and management culture

-- cost savings resulting directly from improvement in load planning

-- in analyzing projects
-- improved service to customers from more reliable transportation

Qualitative changes includes the following :

-- management had more control over network operations. SYSNET now allows managers to have direct control. The new load pattern closely controls the loading of directs and management can quickly change the load pattern in response to changing needs

-- it could set realistic performance standards. SYSNET allowed Yellow to set direct loading standards based on anticipated freight levels, creating more realistic performance expectations.

-- planners can better understand the total system now. Yellow can now evaluate new projects and ideas based on their impact on the entire system

-- SYSNET allows managers to analyze projects formally before making decisions

-- with SYSNET managers can analyze new options quickly in response to changing situations

-- Analysts can now try new ideas on computers which ultimately leads to new ideas in the field

-- because of SYSNET, Yellow is more open now to use of new information technologies

-- the new system has reduced claims. SYSNET has had a substantial impact on management culture at Yellow

Performance improvement due to better load planning include :

A study was undertaken to estimate savings that could be attributed to SYSNET. Total cost savings for the system were estimated at over $7.3 million annually. Savings in breakbulk handling costs also increased.

Besides this, reducing shipments handled in the long run may bring down investments in fixed facilities.

SYSNET brought down the cost of routing trailers in part by identifying directs with lower transportation costs-- savings due to better routing of trailers were estimated to be $1 million annually.

Ongoing projects include :

Operations planning uses SYSNET to scrutinize various projects with a wide range from relocating breakbulks to realigning satellites with breakbulks. Using SYSNET, operational planning now completes over 200 projects per year, mostly on an informal, exploratory basis. SYSNET'S speed in evaluating different ideas is critical to this process.

In 1990, Yellow used SYSNET to identify over $10 million in annual savings from different projects. SYSNET improved the speed with which such analyses could be completed and expanded the scope of each project thus allowing Yellow to study system impacts with more precision than before. SYSNET thus has played a main role in identification, design and evaluation of these projects.

Improved service includes :

Savings from SYSNET are substantial compared to the cost of its development and implementation. Following the implementation of SYSNET management can be better focused on improved service.

Yellow continues to use SYSNET for a number of planning projects and to continuously monitor and improve the load planning system which is now used directly within linehaul operations group responsible for day-to-day management of flows through the system. In addition, Yellow is using SYSNET as a foundation to expand the use of optimization methods for the other aspects of its operations.

SYSNET is now very popular within the company for its capability to carry out accurate, comprehensive network planning projects.

2.6-1

(a) Implementing this major change in operations needed involvement and support of all levels of the company. Process started with acceptance of system with operation planning department. Operation planning was responsible for guiding the project and managing with close cooperation from the information services department and all aspects of the implementation. The systems acceptance was ma lot due to use of interactive optimization which gave users the support needed to optimize such a large network while simultaneously keeping them in close control of the entire process. Users could also analyze suggested changes to the network based on changes in flows and costs, which could be compared against actual field totals.

The next step was to validate the cost model. They were able to compare both total system costs and different subcategories against actual cost summaries for these categories. The individual cost categories within SYSNET consistently match corporate statistics within a few percent and total costs often match with 1 or 2 percent.

The validation of the cost model, both in totality and individual components, played a vital role in gaining upper management's acceptance. The interactive reports and features that convinced operations planning also played a strong role in winning support of top management. They ran sessions for upper management to demonstrate how SYSNET made suggestions and generated supporting reports to back-up the numbers. They also demonstrated how standard operating practices could be detrimental and why coordinating the entire network was important. By taking all these efforts, they gained the needed confidence of upper management required to support a field implementation.

(b) With the support of upper management, they were able to develop an implementation strategy. The controlled direct program changed operating philosophy so drastically that a single corporate-wide transition was viewed as not safe. In implementing SYSNET, Yellow made a systematic change in the way it loaded directs. SYSNET encourages a greater proportion of directs to be loaded onto breakbulks. It was not possible to change this operation methods so easily over the whole network. It was also difficult to do it in a piecemeal fashion. To deal with this problem, they developed a phased implementation strategy that started with smallest breakbulks in the system and went up to larger ones. Careful planning made sure that no breakbulk would be over capacity during the intermediate stages of the process. The entire implementation was so planned as to ensure that no breakbulk would find itself over capacity during the transition period.

(c) To communicate the new concept to terminal managers in the field involved three steps :

-- designing new support tools so that SYSNET routing instructions were easy to follow

-- training terminal managers and dock personnel to use these new system and most important

-- convincing terminal managers that the new approach was a good idea.

They developed two new support tools to assist field operations :

-- first was a set of reports that managers or dock supervisors could access from their local computer terminals which would give them immediate access to SYSNET load pattern.

-- second, was a revised shipment movement bill. This provides a very high level of control over the routing of individual shipments.

The Operations Planning department handled training by organizing series of visits to all 25 breakbulks. During each visit, the staff members explained the principles behind the controlled direct program, new reports and use of new routing directions. Follow-up was done by phone calls.

The most important task was to convince terminal managers of the logic behind the new operations strategy. Terminal managers needed to understand that they had to follow the load planning since it was designed to coordinate different parts of the system. They used examples to illustrate the effect their decisions could have on other terminals. Generally, people in the field accepted the principle that their decisions should be coordinated with those in the rest of the system

(d) Following the implementation of SYSNET, they developed a target that represented the expected number of directs that they should be loading based on the SYSNET plan. Yellow then measured terminal manager's performance based on how

close they were to this target. After some period, it deemed compliance with the plan so good that it now measures terminal managers performance on other activities and Yellow continues to monitor compliance with the load plan informally. It then contacts managers that appear to be not in compliance to determine the reasons. In short, SYSNET has changed load planning from a decentralized process that depended on local management incentives to a centralized process that relies on monitoring and enforcement

2.6-2

(a) The information processing industry has experienced several decades of sustained profitable growth. Recently, competition has intensified leading to quick advances in computer technology. This in turn leads to proliferation of both-end products and services. These trends are especially relevant for after-sales service. Maintaining a service parts logistic system to support products installed in the field is essential to competing in this industry.

Growth in both sales and scope of products offered has dramatically increased the number of spare parts that must be maintained. For IBM, the number of installed machines and the annual usage of spare parts have both increased. This growth has increased the dollar value of service inventories, which are used to maintain the very high levels of service expected by IBM's customers. IBM has developed an extensive multiple-echelon logistic structure to provide ready service for the large population of installed machines, which are distributed through the United States.

IBM developed a large and sophisticated inventory management system to provide customers with prompt and reliable service. A fast changing business environment and pressures to decrease investment in inventory led IBM to look for improvements in its control system.

In response to these new needs, IBM initiated the development of a new planning and control system for management of service parts. The result of this was the creation and implementation of a system called Optimizer.

(b) The complicating factors faced by the OR team are as follows :

-- there are more than 15 million part-location combinations
-- there are more than 50000 product-location combinations
-- frequent updating (weekly) of system control parameters was a requirement in response to changes in the service environment and installed base
-- success of the system is important to IBM's daily operations and so can have a major impact on its future sales and revenues
-- employees could be expected to protest against any change since the existing control system was working and sophisticated and overall parts logistics problem was complex.

(c) The system developed in this phase had minimum interface to provide data inputs and multi-echelon algorithm without any improvements. Most of the big changes from the original design was in this phase.

They discovered that the echelon structure was in reality more complex than the one used in the analytic model. Consequently, they had to develop extensions to the demand pass-up methodology and incorporate them into the model.

The test was conducted in early 1986 and led to the finding that the value of the total inventory generated by the new system was smaller than expected. It was discovered that the problem was due to differences in criticality of parts. The algorithm made extensive use of inexpensive, non-functional parts to meet product-service objective. Another problem found out at this stage was the churn (instability) in the recommended stock levels every week. Although stock levels are expected to change periodically in response to changing failure rates and to changes in the installed base, it is desirable to keep the stock levels quasi-static in order to avoid logistic and supply problems. They developed control procedures and changed the model to take care of this problem.

(d) In this phase, they completed all functions required for implementation and developed a measurement system to monitor the field implementation test. After being done with the system coding for this phase, they conducted an extensive user acceptance test. Every program module was tested individually and jointly. Finally, a field implementation test went live on 7 machine types in early 1987. The working of the system fulfilled expectations. Scope of the field test was slowly expanded. Results were monitored on a weekly and then monthly basis by the measurement system.

(e) In this phase, they completed the development and installation of all the functions currently in place in Optimizer. The system was able to provide the specified service performance for all parts and locations. Improvements were made. User acceptance testing and integration of final system went smoothly. The project staging helped to sustain support for the project by demonstrating concrete progress throughout the implementation process. It also helped to eradicate problems in formulation and algorithm and programming bugs early. So very few problems occurred when the system went live in a national basis. The final Optimizer system for national implementation consisted of four major modules :

-- a forecasting system module
-- a data delivery system module
-- a decision system that solves multi-echelon stock control problem
-- the PIMS interface system

(f) The implementation of Optimizer yielded a variety of benefits :

-- a decrease in investment on inventory
-- improved services
-- enhanced flexibility in responding to changing service requirements
-- provision of a planning capability
-- improved understanding of the impact of parts operations
-- increased responsive of the control system
-- increased efficiency of NSD human resources
-- identifying the role of functional parts in providing product service is an example of benefits derived from implementation of Optimizer
-- ability to run Optimizer on a weekly basis has increased responsiveness of entire parts inventory system
-- for machines controlled by Optimizer, inventory analysts no longer have to specify parts stocking lists for each echelon in order to make sure that service objectives are attained. They can now focus on other critical management issues.

Optimizer thus has proved to be an extremely valuable planning and operating control tool.

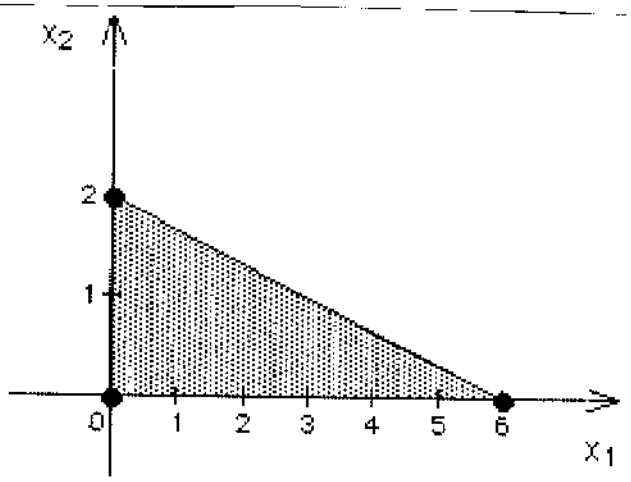
2.7-1

The 13 detailed phases of an OR study according to this reference are : Initiation(Embryonic), Feasibility, Formulation, Data, Design, Verification(Software Development), Validation, Training and Education, Installation, Implementation, Maintenance and Update, Evaluation and Review, Documentation and Dissemination.

Following are the 6 broader phases given in the chapter and in parentheses are given the detailed phases of the article that fall partially or primarily within the broader phase :

Define the problem of interest and gather relevant data (initiation, feasibility, data),
formulate a mathematical model to represent the problem (formulation),
develop a computer-based procedure for deriving solutions for the problem from the model(design, verification that is software development),
test the model and refine it as needed(validation),
prepare for ongoing application of the model as prescribed by management(training and education, installation),
implementation(implementation, maintenance and update, evaluation and review, documentation and dissemination).

3.1-1 a)



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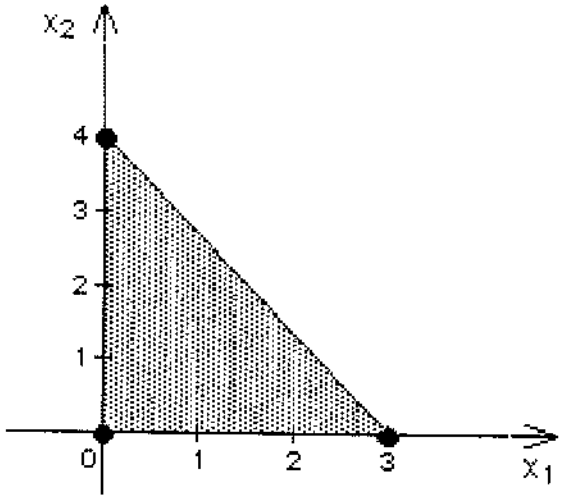

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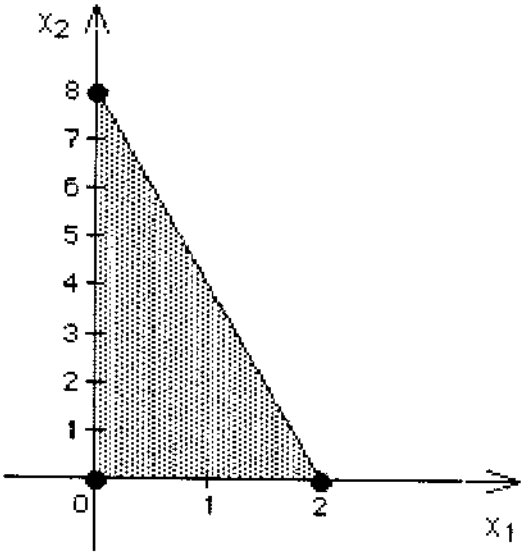

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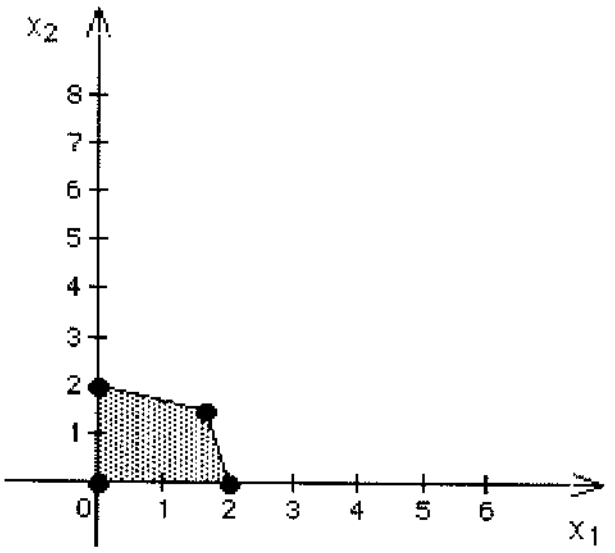
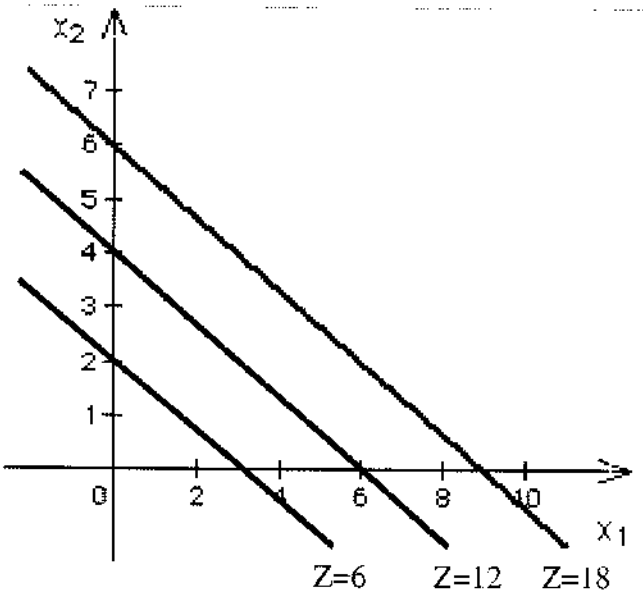


3.1-2

a)



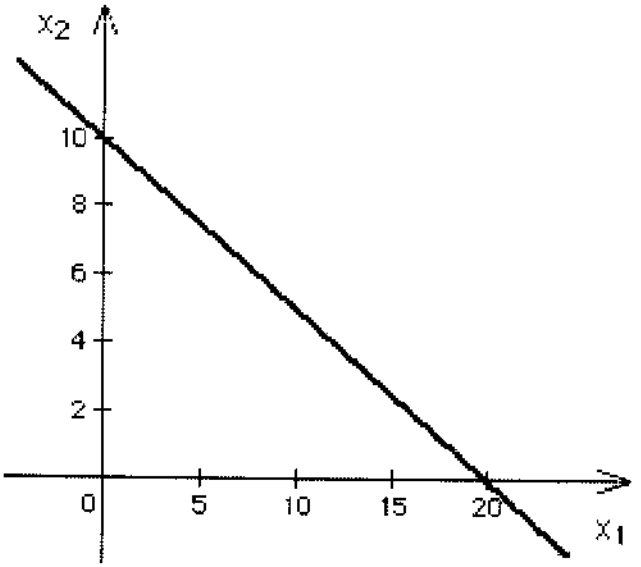
b)

	slope-intercept form	slope	x_2 intercept
Z=6	$x_2 = -\frac{2}{3}x_1 + 2$	$-\frac{2}{3}$	2
Z=12	$x_2 = -\frac{2}{3}x_1 + 4$	$-\frac{2}{3}$	4
Z=18	$x_2 = -\frac{2}{3}x_1 + 6$	$-\frac{2}{3}$	6

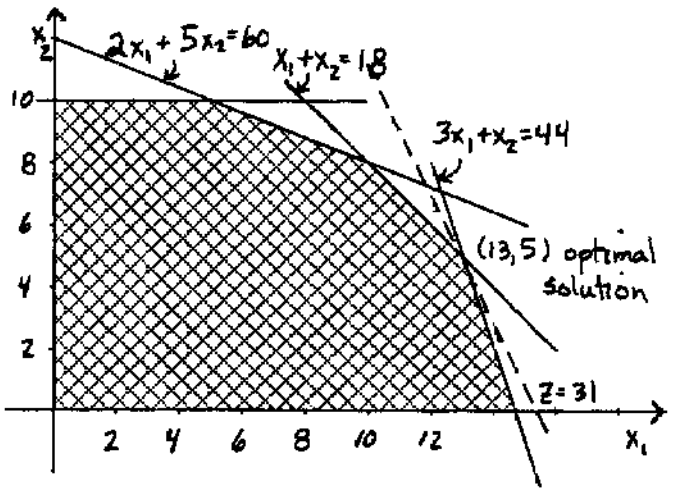
3.1-3 a) $x_2 = -\frac{1}{2}x_1 + 10$

b) slope $= -\frac{1}{2}$ x_2 intercept = 10

c)

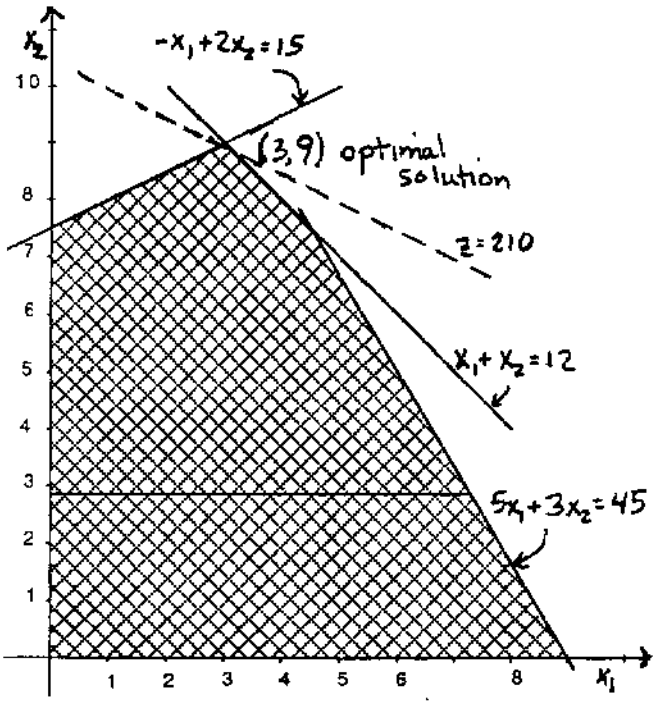


3.1-4



$(x_1, x_2) = (13,5)$ is optimal with $Z = 31$

3.1-5



$(x_1, x_2) = (3,9)$ is optimal with $Z = 210$

3.1-6 a) As in the Wyndor Glass Co. problem, we want to find the optimal levels of two activities that compete for limited resources. We want to find the optimal mix of the two activities.

Let W be the number of wood-framed windows to produce.
Let A be the number of aluminum-framed windows to produce.

The following table gives the data for the problem:

Resource	Resource Usage per Unit of Activity wood-framed	aluminum-framed	Amount of Resource Available
glass	6	8	48
aluminum	0	1	4
wood	1	0	6
Unit Profit	\$60	\$30	

b)

Maximize $P = 60W + 30A$,

subject to $6W + 8A \le 48$

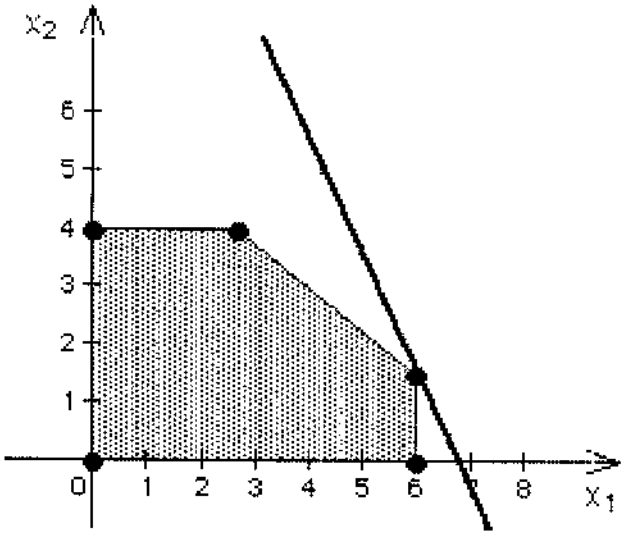
$W \le 6$

$A \le 4$

and $W \ge 0, \ A \ge 0.$

c)

Optimal Solution: $(W, A) = (x_1, x_2) = \left(6, 1\frac{1}{2}\right)$ and $P = 405$.



(CONT.)

CONT'D.

d)

Resource	Resources Used Per Unit Produced: wood-framed	aluminum-framed	Totals		Resource Available
glass	6	8	48	≤	48
aluminum	0	1	1.5	≤	4
wood	1	0	6	≤	6
Unit Profit	$40	$30	**$ 285**		
Solution	**6**	**1.5**			

Resource	Resources Used Per Unit Produced: wood-framed	aluminum-framed	Totals		Resource Available
glass	6	8	48	≤	48
aluminum	0	1	4	≤	4
wood	1	0	2.66667	≤	6
Unit Profit	$20	$30	**$ 173**		
Solution	**3**	**4**			

e)

Resource	Resources Used Per Unit Produced: wood-framed	aluminum-framed	Totals		Resource Available
glass	6	8	48	≤	48
aluminum	0	1	2.25	≤	4
wood	1	0	5	≤	5
Unit Profit	$60	$30	**$ 368**		
Solution	**5**	**2.25**			

31-7

a) Let x_1 = number of 27" TV sets to be produced per month
Let x_2 = number of 20" TV sets to be produced per month
Maximize $P = 120x_1 + 80x_2$,

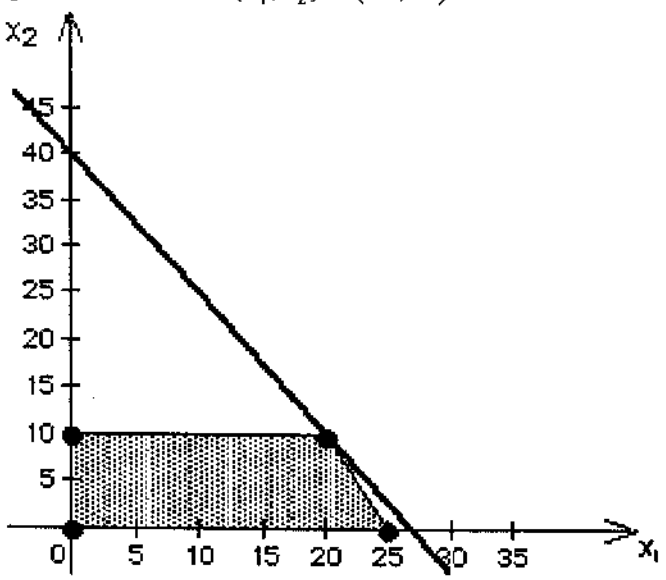
subject to $20x_1 + 10x_2 \le 500$

$x_1 \le 40$

$x_2 \le 10$

and $x_1 \ge 0,\ x_2 \ge 0.$

b) Optimal Solution: $(x_1, x_2) = (20, 10)$ and $P = 3200$.



3.1-8

a) Let x_1 = number of units of product 1 to produce
Let x_2 = number of units of product 2 to produce
Maximize $P = x_1 + 2x_2$,

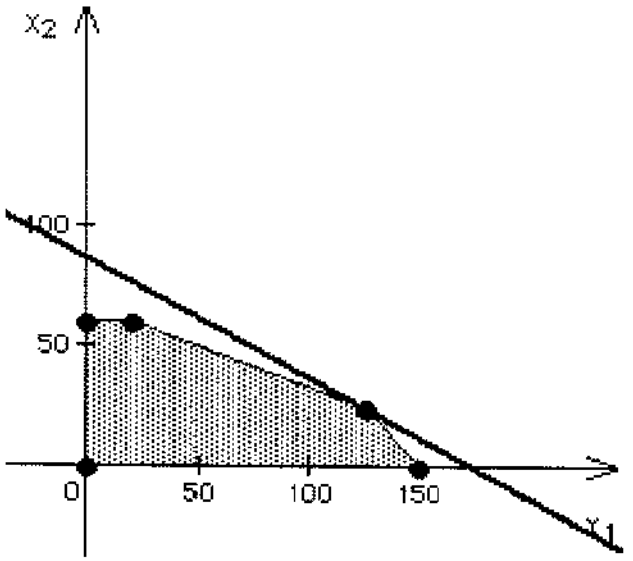
subject to $x_1 + 3x_2 \le 200$

$2x_1 + 2x_2 \le 300$

$x_2 \le 60$

and $x_1 \ge 0, \ x_2 \ge 0.$

b) Optimal Solution: $(x_1, x_2) = (125, 25)$ and $P = 175$.



3.1-9

a)

Let X_1 = # units on special risk insurance
X_2 = # " " mortgages

Maximize $Z = 5X_1 + 2X_2$

subject to $3X_1 + 2X_2 \le 2400$

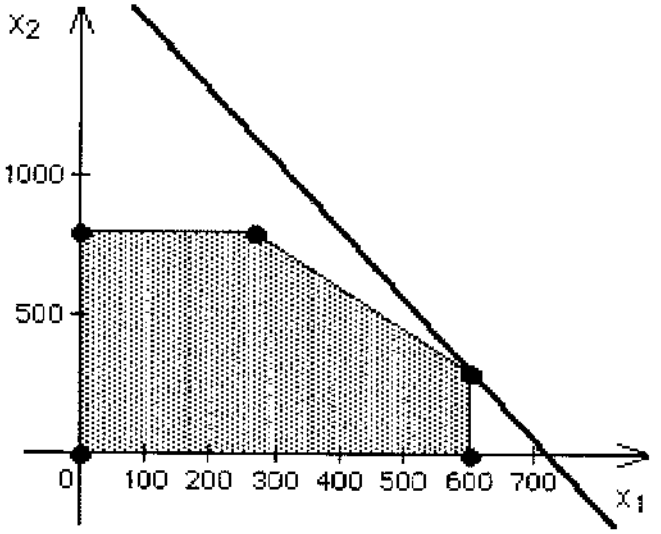
$X_2 \le 800$

$2X_1 \le 1200$

$X_1 \ge 0, X_2 \ge 0$

3.1-9 Optimal Solution: $(S,M) = (x_1, x_2) = (600,300)$ and $P = 3600$.

b)



c- we have :

$$3x_1 + 2x_2 = 2400$$
$$2x_1 = 1200$$

so $x_1 = 600$ and $x_2 = \frac{1}{2}(2400 - 3x_1) = 300$

and $Z = 5x_1 + 2x_2 = 5 \cdot 600 + 2 \cdot 300 = 3600$

3.1-10

a) Maximize $P = 0.2H + 0.1B$,

subject to $0.1B \le 200$

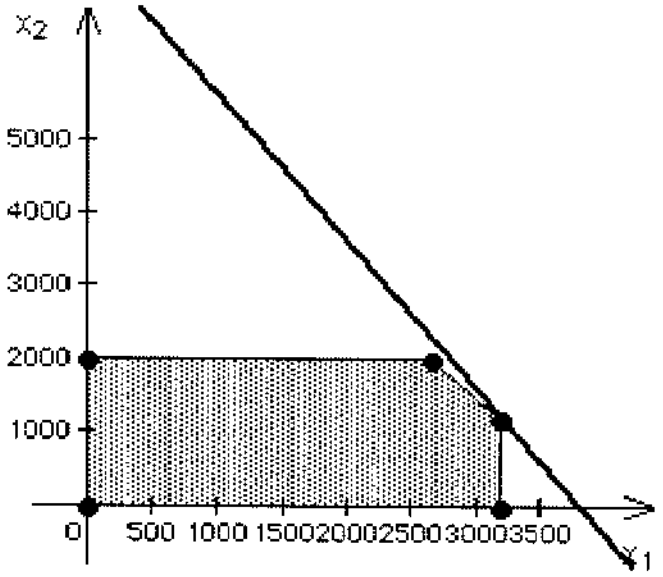
$0.25H \le 800$

$3H + 2B \le 12{,}000$

and $H \ge 0,\ B \ge 0.$

3.1-10

b) Optimal Solution: $(H, B) = (x_1, x_2) = (3200, 1200)$ and $P = 760$.



3.1-11

a)

Let x_1 = number of units of product 1 produced
x_2 = number of units of product 2 produced
x_3 = number of units of product 3 produced.

Maximize $Z = 50x_1 + 20x_2 + 25x_3$

subject to

$$9x_1 + 3x_2 + 5x_3 \le 500$$
$$5x_1 + 4x_2 \le 350$$
$$3x_1 + 2x_3 \le 150$$
$$x_3 \le 20$$
$$x_1 \ge 0 \quad x_2 \ge 0 \quad x_3 \ge 0$$

3.1-11

b)

Solve Automatically by the Simplex Method:

Optimal Solution

Value of the
Objective Function: Z = 2904.7619

Variable	Value
X_1	26.1905
X_2	54.7619
X_3	20

Constraint	Slack or Surplus	Shadow Price
1	0	4.7619
2	0	1.42857
3	31.4286	0
4	0	1.19048

Sensitivity Analysis

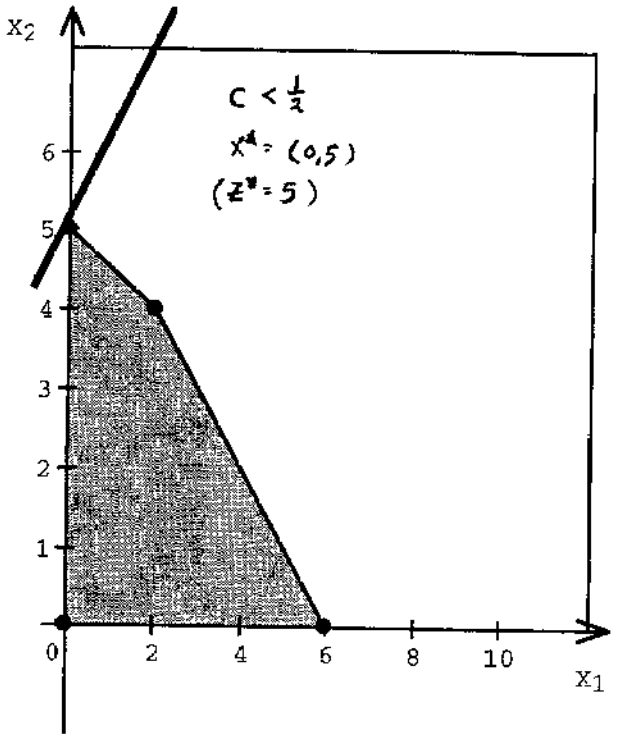
Objective Function Coefficient

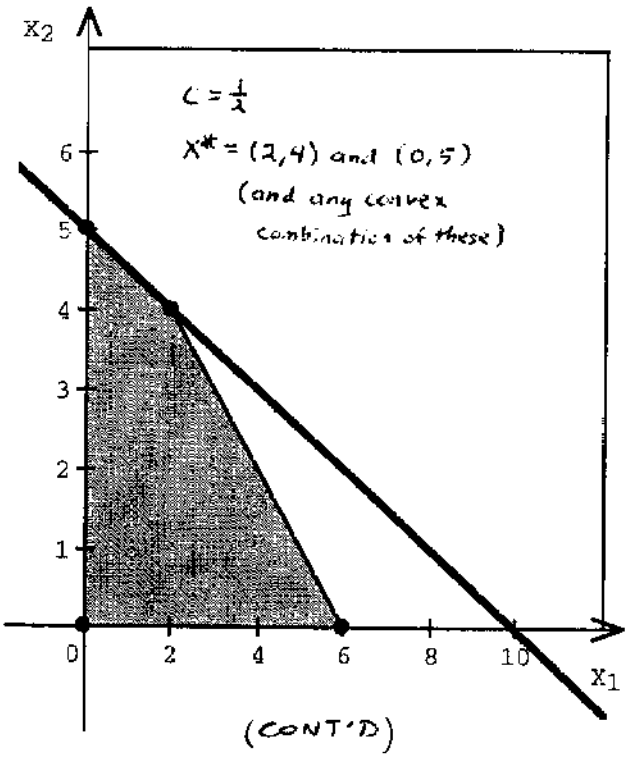
Current Value	Allowable Range To Stay Optimal Minimum	Maximum
50	25	51.25
20	19	40
25	23.8095	$+\infty$

Right Hand Sides

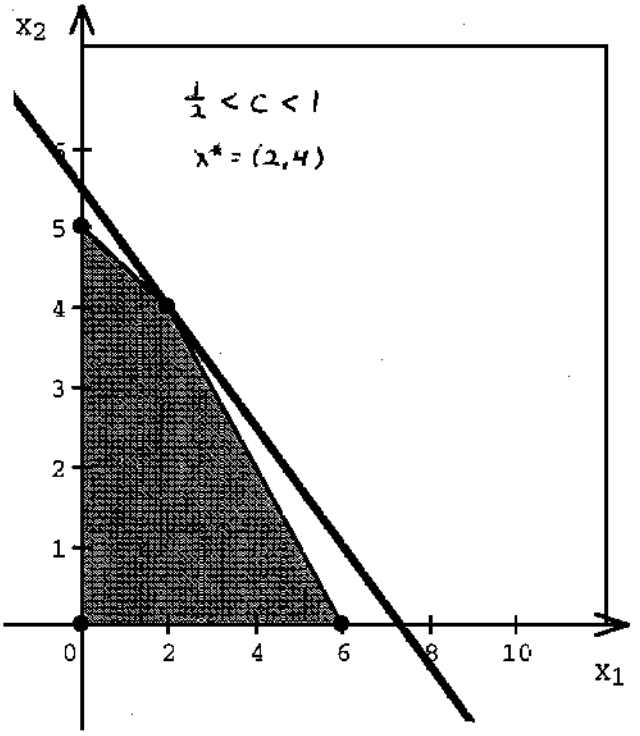
Current Value	Allowable Range To Stay Feasible Minimum	Maximum
500	362.5	555
350	276.667	533.333
150	118.571	$+\infty$
20	0	47.5

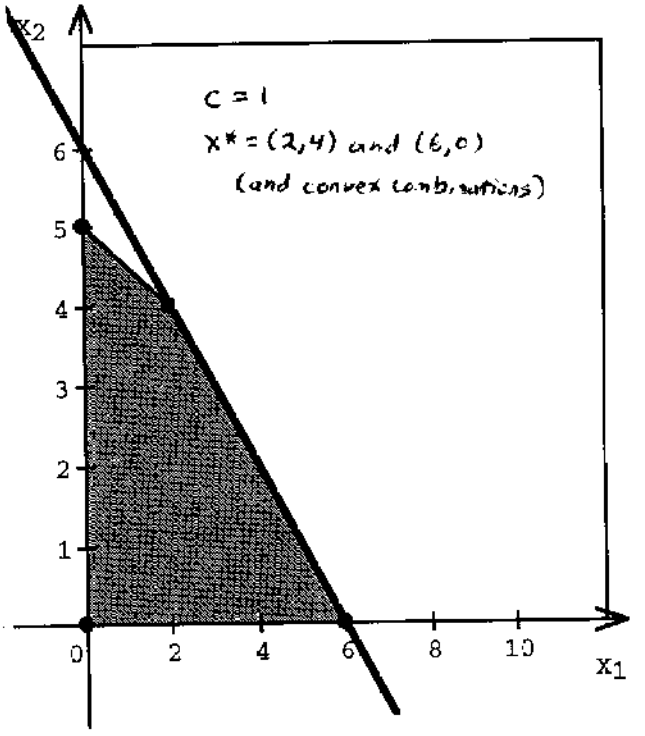
3.1-12

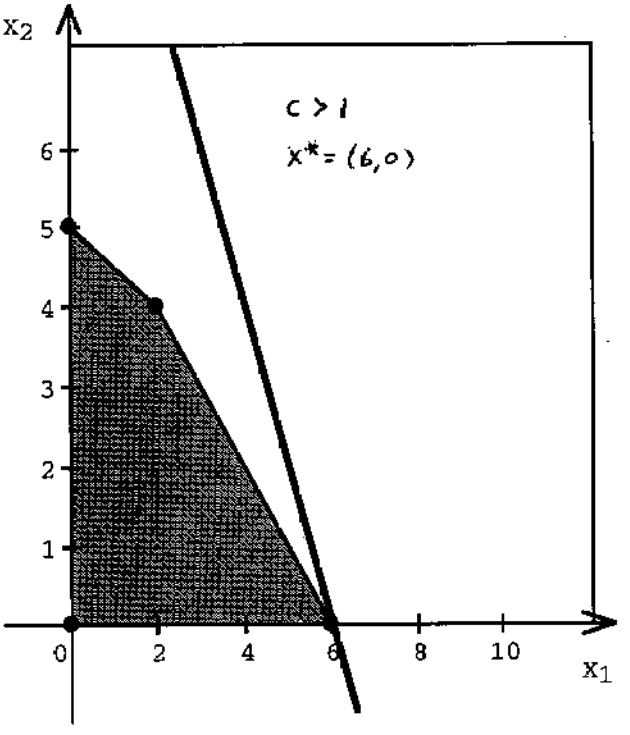




3.1-12 (CONTD.)


x_2
$\frac{1}{2} < c < 1$
$x^* = (2,4)$
0
2
4
6
8
10
x_1


x_2
$c = 1$
$x^* = (2,4)$ and $(6,0)$
(and convex combinations)
0
2
4
6
8
10
x_1


x_2
$c > 1$
$x^* = (6,0)$
0
2
4
6
8
10
x_1

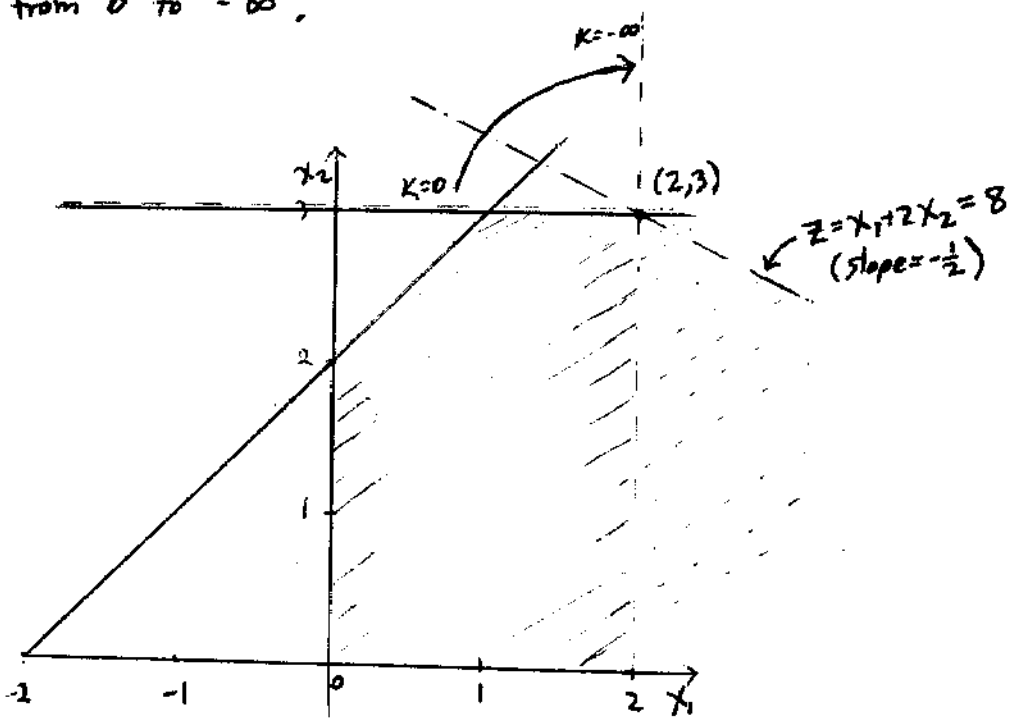
3.1-13

First note that (2,3) always satisfies the 3 constraints. (i.e. (2,3) is always feasible)

In fact since $kX_1 + X_2 = 2k+3$ at (2,3) for any k, the third constraint is always binding.

We only need to check if (2,3) is optimal.

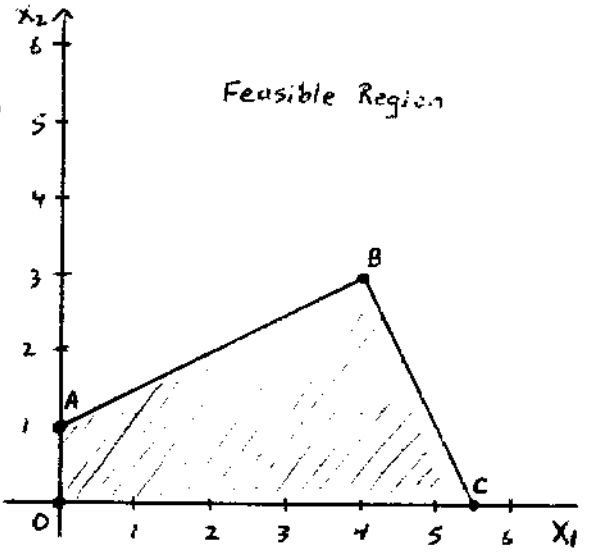
Since the line $kX_1 + X_2 = 2k+3$ always passes through the point (2,3), changing k simply rotates the line. Rewriting: $X_2 = -kX_1 + (2k+3)$, we see that the slope of the line is $-k$, and, therefore, the slope ranges from 0 to $-\infty$.



As we can see, (2,3) is optimal as long as the slope of the 3rd constraint line is less than $-\frac{1}{2}$ (the slope of the objective line). If $k < \frac{1}{2}$, then we can increase the objective by traveling along the 3rd constraint to point $(2+\frac{3}{k}, 0)$ which has an objective value of $2+\frac{3}{k} > 8$ if $k < \frac{1}{2}$.

Therefore, (2,3) is optimal for $k \geq \frac{1}{2}$.

3.1-14



Case 1: $c_2 = 0$ (Objective line is vertical)

If $c_1 > 0$, objective increases as x_1 increases,
so $x^* = (\frac{11}{2}, 0)$ (pt. C)

If $c_1 < 0$, opposite is true,
so $x^* = (0,0)$ and $(0,1)$ (line $\overline{OA}$)

(Note if $c_1 = 0$, every feasible point gives "optimal" of $0x_1 + 0x_2 = 0$)

Case 2: $c_2 > 0$ (Slope of obj. is $-\frac{c_1}{c_2}$)

If $-\frac{c_1}{c_2} > \frac{1}{2}$ (or equivalently, $\frac{c_1}{c_2} < -\frac{1}{2}$),

$x^* = (0,1)$ (pt. A)

If $-\frac{c_1}{c_2} < -2$,

$x^* = (\frac{11}{2}, 0)$ (pt. C)

If $\frac{1}{2} > -\frac{c_1}{c_2} > -2$,

$x^* = (4,3)$ (pt. B)

(Note, if $-\frac{c_1}{c_2} = \frac{1}{2}$ or -2, x^* is $\overline{AB}$ or $\overline{BC}$, respectively)

Case 3: $c_2 < 0$ (Slope is still $-\frac{c_1}{c_2}$, but the objective increases as the line is shifted down)

If $-\frac{c_1}{c_2} > 0$ (i.e. $c_1 > 0$), $x^* = (\frac{11}{2}, 0)$ (pt. C)

If $-\frac{c_1}{c_2} < 0$ ($c_1 < 0$), $x^* = (0,0)$ (pt. O)

(If $c_1 = 0$, x^* is $\overline{OC}$)

3.2-1

Maximize $P = 3A + 2B$,

subject to $2A + B \le 2$

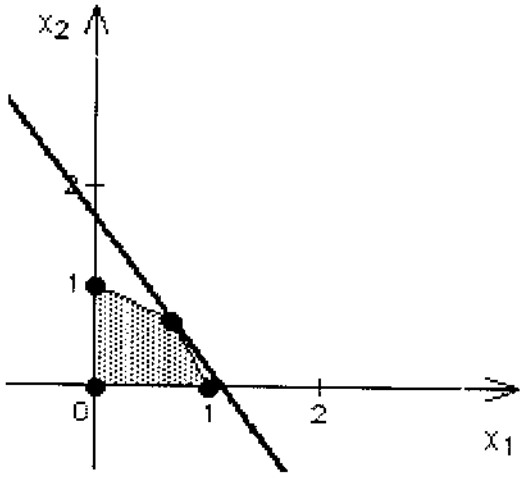
$A + 2B \le 2$

$3A + 3B \le 4$

and $A \ge 0,\ B \ge 0.$

3.2-1

b) Optimal Solution: $(A, B) = (x_1, x_2) = \left(\frac{2}{3}, \frac{2}{3}\right)$ and $P = 3.33$.



c- We have to solve:

$$2A + B = 2 \quad (A)$$
$$A + 2B = 2$$

Substracting the 2nd equation from the 1st equation:

$$A - B = 0 \; ; \text{ so } A = B$$

so in (A) we have:

$$2 = 2A + B = 3A \Rightarrow A = 2/3 \Rightarrow B = 2/3$$

3.2-2

a) True (eg. $\max z = -x_1 + 4x_2$)

b) True (eg. $\max z = -x_1 + 3x_2$)

c) False (eg. $\max z = -x_1 - x_2$)

3.2-3

a) As in the Wyndor Glass Co. problem, we want to find the optimal levels of two activities that compete for limited resources. We want to find the optimal mix of the two activities.
Let x_1 be the fraction purchased of the partnership in the first friends venture.
Let x_2 be the fraction purchased of the partnership in the second friends venture.
The following table gives the data for the problem:

Resource	Resource Usage per Unit of Activity 1	2	Amount of Resource Available
Fraction of partnership in first friends venture	1	0	1
Fraction of partnership in second friends venture	0	1	1
Money	\$5000	\$4000	\$6000
Summer Work Hours	400	500	600
Unit Profit	\$4500	\$4500	

b)

Maximize $P = 4500x_1 + 4500x_2$,

subject to $x_1 \le 1$

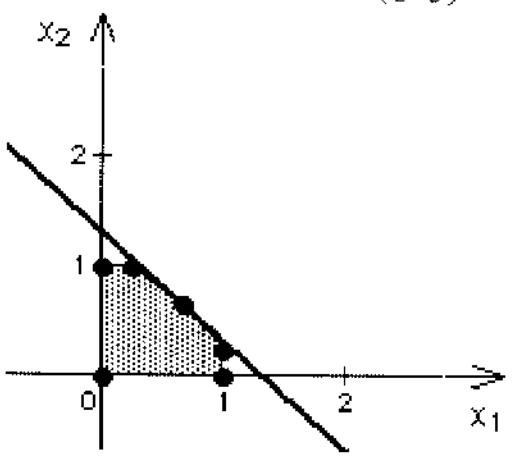
$x_2 \le 1$

$5000x_1 + 4000x_2 \le 6000$

$400x_1 + 500x_2 \le 600$

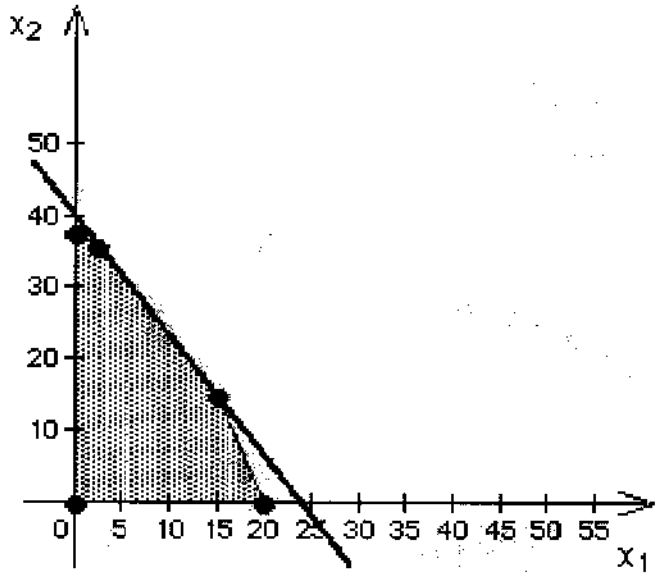
and $x_1 \ge 0, \ x_2 \ge 0.$

c) Optimal Solution: $(x_1, x_2) = \left(\frac{2}{3}, \frac{2}{3}\right)$ and $P = 6000$.

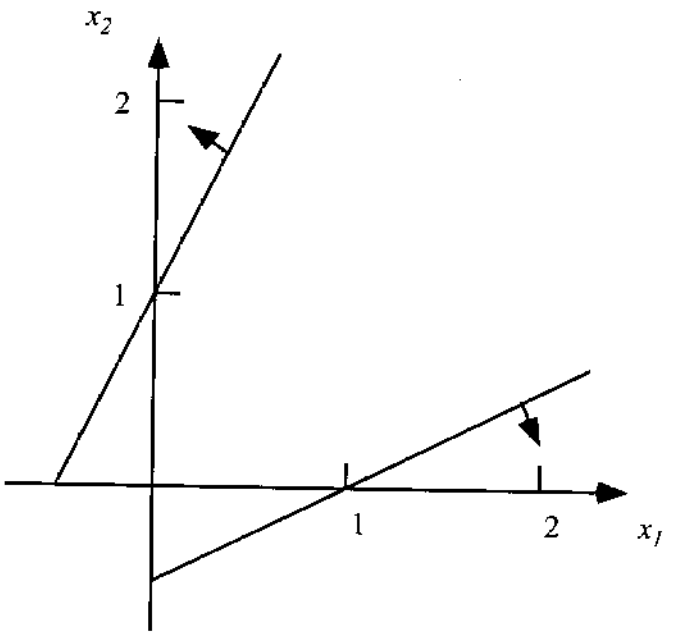


3.2-4

Optimal Solution: $(x_1, x_2) = (15,15)$, (2.5, 35.833) and all points on the connecting line. $Z = 12{,}000$.

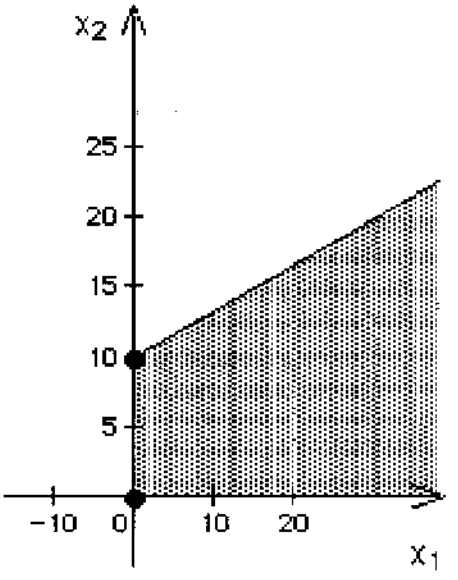


3.2-5

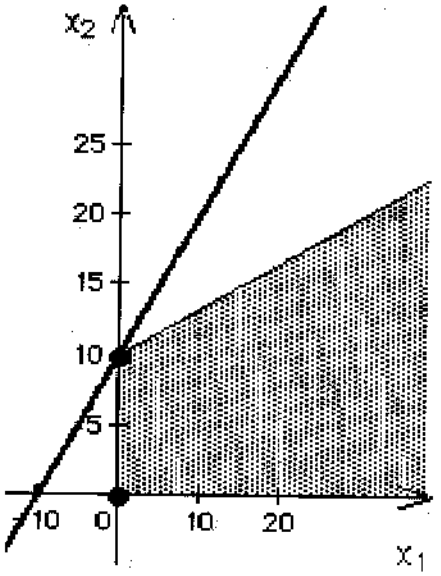


3.2-6

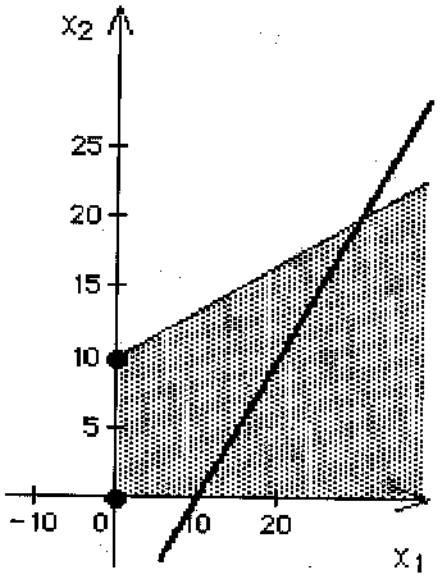
a)



b) Yes. Optimal Solution: $(x_1, x_2) = (0,10)$ and $Z = 10$.



c) No. The objective function value is maximized by sliding the objective function line to the right. This can be done forever, so there is no optimal solution.



d) No, solutions exist that will make Z arbitrarily large. This usually occurs when a constraint is left out of the model.

3.3-1.

Proportionality: It is fair to assume the amount of work and money spent and the profit earned are directly proportional to the fraction of partnership purchased in either venture.

Additivity: The profit as well as time and money requirements for one venture should not affect the profit or time and money requirements of the other venture. This assumption is reasonably satisfied.

Divisibility: Because both friends will allow purchase of any fraction of a full partnership, divisibility is a reasonable assumption.

Certainty: Because we don't know how accurate the friends' profit estimates are, this is a more doubtful assumption. We should conduct sensitivity analysis after finding the optimal solutions for the current values of the profits.

3.3-2.

Proportionality: OK, since if either variable is fixed, the objective value grows in proportion to the increase in the other variable.

Additivity: Not OK, since activities interact. For example, the objective value with $(x_1, x_2) = (1, 1)$ isn't equal to the objective value with $(x_1, x_2) = (1, 0)$ plus the objective value with $(x_1, x_2) = (0, 1)$.

Divisibility: Not OK, since activity levels are not allowed to be fractional.

Certainty: OK, since data is given as accurate.

3.4-1

a) Proportionality: OK, since beam effects on tissue types are proportional to beam strength.
Additivity: OK, since it was stated that effects from multiple beams are additive.
Divisibility: OK, since beam strength can be any fractional level.
Certainty: Due to the complicated analysis required to estimate the data on radiation absorbtion in different tissue types, sensitivity analysis should be used.

b)
Proportionality: OK, as long as there is no set up cost associated with planting a crop.
Additivity: OK, as long as crops do not interact.
Divisibility: OK, since acres are divisible.
Certainty: OK, since data can be accurately determined.

c)
Proportionality: OK, since set up costs were considered.
Additivity: OK, since it was stated that there is no interaction.
Divisibility: OK, since methods can be used at fractional levels
Certainty: Data is hard to estimate so it could easily be stochastic. Sensitivity analysis should be used.

3.4-2.*a*) Reclaiming solid wastes

Proportionality: The amalgamation & treatment processes are unlikely to be proportional. There are bound to be set up costs. (Treating 1,000 lbs. of material will not cost the same as treating 10 lbs. of material 100 ties.)

Additivity: Probably O.K., though it is possible that there is some interaction between treatments of materials (e.g., if *A* is treated after *B*, the machines do not need to be cleaned out . . .)

Divisibility: Probably O.K., unless selling/buying materials can only be done in batches (of 100 lbs., say).

Certainty: Selling/buying prices may change; costs of treatment & amalgamation are, most likely, crude estimates and may also change.

3.4-2. *b*) Personnel Scheduling

Proportionality: O.K. It is possible that *some* costs are not proportional to # of agents hired (e.g., benefits, working space, . . .), but for the most part, this assumption is satisfied.

Additivity: O.K. Same, pretty much, as proportionality.

Divisibility: Clearly, one cannot hire a fraction of an agent.

Certainty: The "min. # of agents needed" is suspect. Perhaps 45 agents will suffice instead of 48, for example, for a nominal fee. Also, does an agent in one shift do the same amount of work as one from another (or even the same) shift?

(*c*) Distribution goods through a distribution network

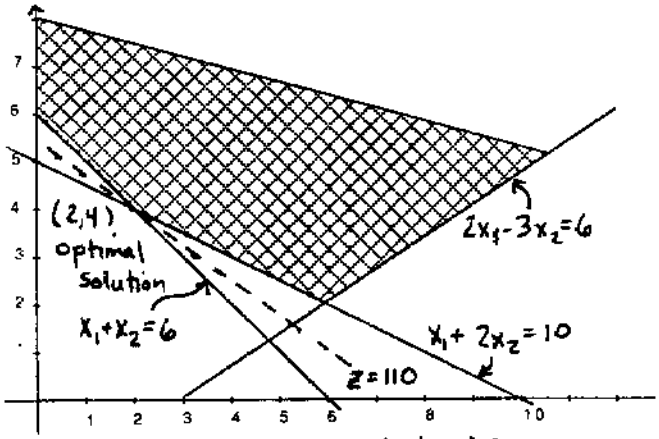
Proportionality: As in (*a*), there is probably a "set-up" cost for delivery (i.e., delivering 50 units one at a time will cost much more than delivering all 50 at the same time).

Additivity: Probably O.K., but it is possible that 2 routes may be "combined" to lower costs (e.g., if $x_{F2\text{-}OC} = x_{OC\text{-}W2} = 50$, the truck *may* be able to deliver 50 units directly from F2 to W2, without stopping at DC, saving some money). Another question is whether F1 and F2 produce equivalent units.

Divisibility: It seems that one cannot deliver a fraction of a unit.

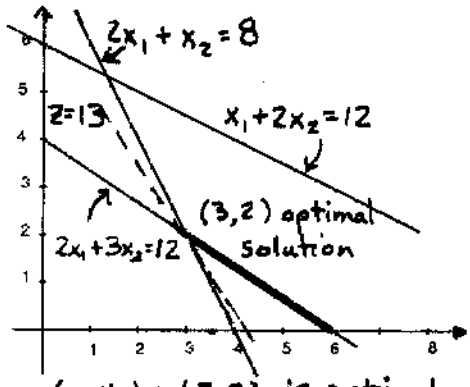
Certainty: Shipping costs are probably approximations and are subject to change, as are amounts produced. Even the capacities may depend on available daily trucking force, weather, etc. As in any problem, sensitivity analysis should be done.

3.4-3.



$(x_1, x_2) = (2,4)$ is optimal with $Z = 110$

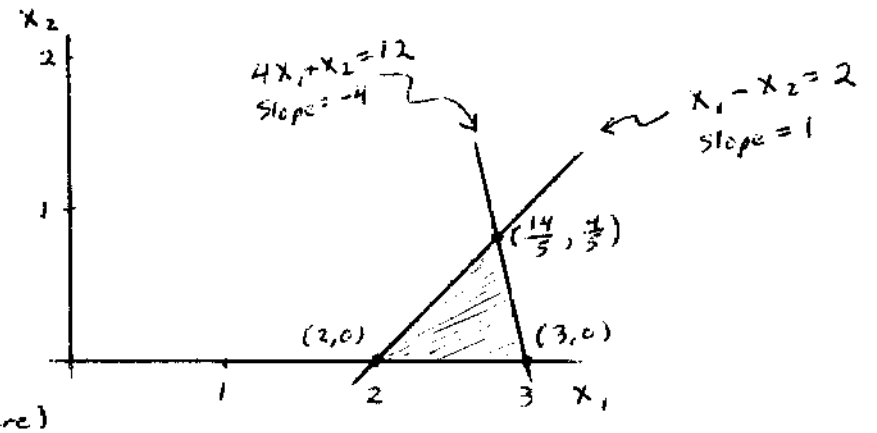
3.4-4.



$(x_1, x_2) = (3,2)$ is optimal with $Z = 13$

3.4-5

Feasible region:



Since $c_2 > 0$ ($c_2 = 2$, here)

If slope of objective $= -\frac{c_1}{c_2} > 1$, then (2,0) is optimal

or $c_1 < -2 \Rightarrow (x_1^*, x_2^*) = (2,0)$

If $1 > -\frac{c_1}{c_2} > -12$, then $(\frac{14}{5}, \frac{4}{5})$ is optimal

or $-2 < c_1 < 24 \Rightarrow (x_1^*, x_2^*) = (\frac{14}{5}, \frac{4}{5})$

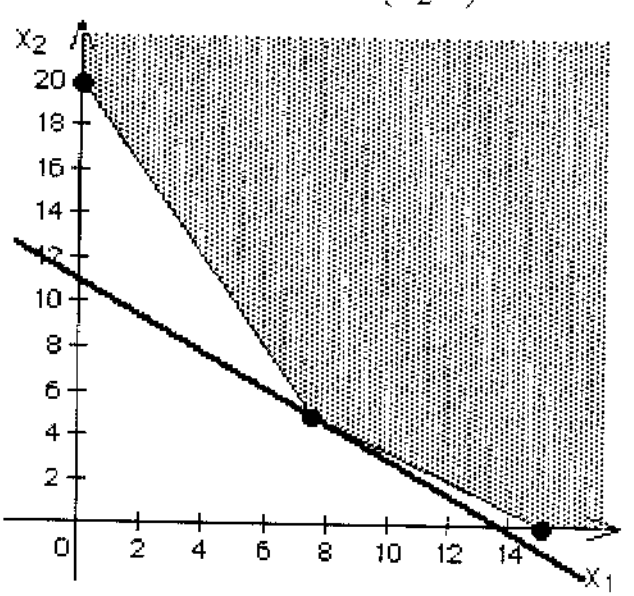
If $-\frac{c_1}{c_2} < -12$ (or $c_1 > 24$), then (3,0) is optimal

Of course, if $c_1 = -2$, then both (2,0) and $(\frac{14}{5}, \frac{4}{5})$ are optimal (as are all convex combinations of these),

and if $c_1 = 24$, $(\frac{14}{5}, \frac{4}{5})$, (3,0), and all convex combinations are optimal.

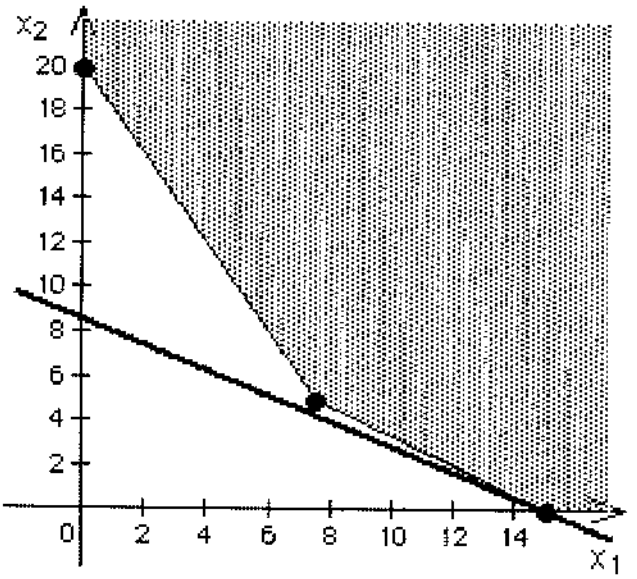
3.4-6

a) Optimal Solution: $(x_1, x_2) = \left(7\frac{1}{2}, 5\right)$ and $C = 550$.

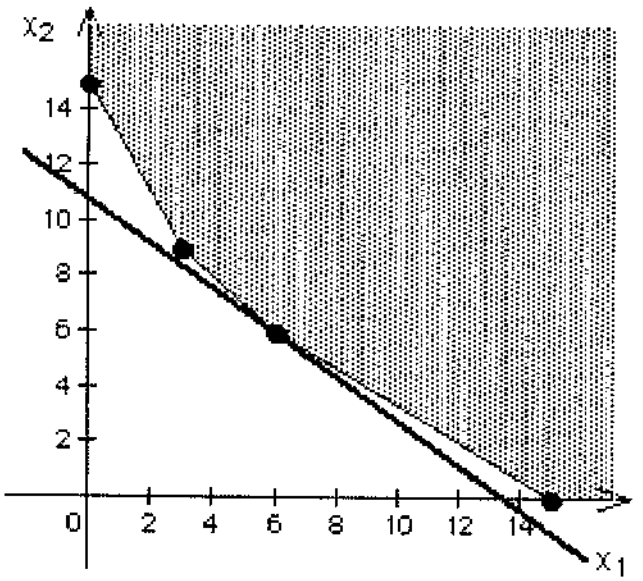


3.4-6

b) Optimal Solution: $(x_1, x_2) = (15, 0)$ and $C = 600$.



c) Optimal Solution: $(x_1, x_2) = (6, 6)$ and $C = 540$.



3.4-7 a)

Minimize $C = 4S + 2P$,

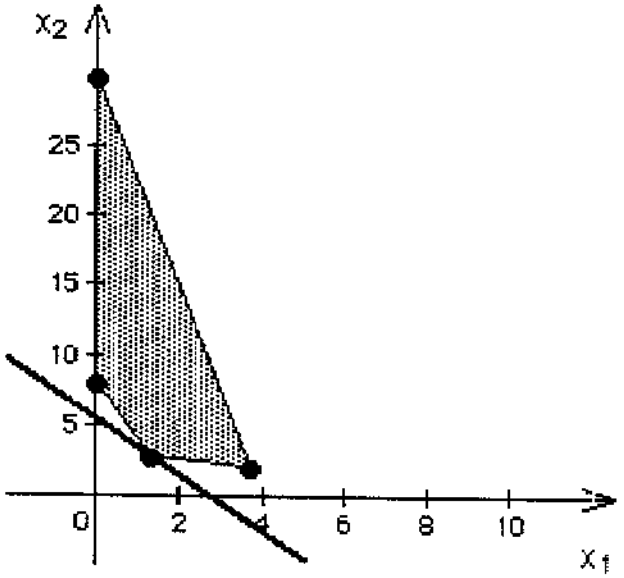
subject to $5S + 15P \geq 50$

$20S + 5P \geq 40$

$15S + 2P \leq 60$

and $S \geq 0, \ P \geq 0.$

b) Optimal Solution: $(S,P)=(x_1,x_2)=(1.3,2.9)$ and $C=10.91$.



c)

	Contribution Per Unit				
	Steak	Potato	Totals		Level
Carbohydrate	5	15	50	≥	50
Protein	20	5	40	≥	40
Fat	15	2	24.91	≤	60
Unit Cost	4	2	**$ 10.91**		
Solution	**1.3**	**2.9**			

3.4-8

a)

Minimize $C=0.4A+0.8B$,

subject to $800A+1000B\geq 8000$

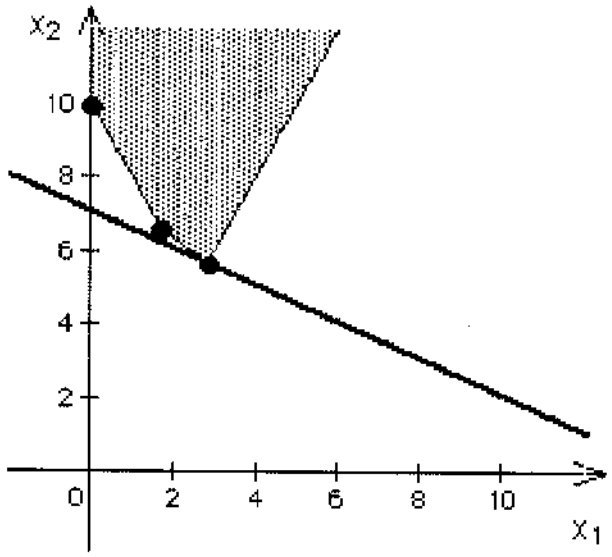
$140A+70B\geq 700$

$\frac{2}{3}A-\frac{1}{3}B\leq 0$

and $A\geq 0,\ B\geq 0.$

3.4-8

b) Optimal Solution: $(A, B) = (x_1, x_2) = (2.85, 5.72)$ and $C = 5.72$.



3.4-9

a)

Let x_{ij} = the amount of space leased in month i for a period of j months

for $i = 1,...,5$ and $j = 1,...,6-i$.

Minimize $C = 650(x_{11} + x_{21} + x_{31} + x_{41} + x_{51}) + 1000(x_{12} + x_{22} + x_{32} + x_{42})$

$+1350(x_{13} + x_{23} + x_{33}) + 1600(x_{14} + x_{24}) + 1900x_{15}$

subject to $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \geq 30,000$

$x_{12} + x_{13} + x_{14} + x_{15} + x_{21} + x_{22} + x_{23} + x_{24} \geq 20,000$

$x_{13} + x_{14} + x_{15} + x_{22} + x_{23} + x_{24} + x_{31} + x_{32} + x_{33} \geq 40,000$

$x_{14} + x_{15} + x_{23} + x_{24} + x_{32} + x_{33} + x_{41} + x_{42} \geq 10,000$

$x_{15} + x_{24} + x_{33} + x_{42} + x_{51} \geq 50,000$

and $x_{ij} \geq 0$, for $i = 1,...,5$ and $j = 1,...,6-i$.

3.4-9 b)

	A	B	C	D	E	F	G	H	I
1		Contribution Toward Required Amount							
2	Month	1-1	1-2	1-3	1-4	1-5	2-1	2-2	2-3
3	1	1	1	1	1	1			
4	2		1	1	1	1	1	1	1
5	3			1	1	1		1	1
6	4				1	1			1
7	5					1			
8	Unit Cost	$ 650	$ 1,000	$ 1,350	$ 1,600	$ 1,900	$ 650	$ 1,000	$ 1,350
9	Solution	**0**	**0**	**0**	**0**	**30000**	**0**	**0**	**0**

	J	K	L	M	N	O	P	Q	R	S
1	Contribution Toward Required Amount									Resource
2	2-4	3-1	3-2	3-3	4-1	4-2	5-1	Totals		Available
3								30000	≥	30000
4	1							30000	≥	20000
5	1	1	1	1				40000	≥	40000
6	1		1	1	1	1		30000	≥	10000
7	1			1		1	1	50000	≥	50000
8	$ 1,600	$ 650	$ 1,000	$ 1,350	$ 650	$ 1,000	$ 650	**76499994**		
9	**0**	**10000**	**0**	**0**	**0**	**0**	**20000**			

Data cells: B3:P8 and S3:S7
Changing cells: B9:P9
Target cell: Q8
Output cells: Q3:Q7

	Q
3	=SUMPRODUCT(B3:P3,B9:P9)
4	=SUMPRODUCT(B4:P4,B9:P9)
5	=SUMPRODUCT(B5:P5,B9:P9)
6	=SUMPRODUCT(B6:P6,B9:P9)
7	=SUMPRODUCT(B7:P7,B9:P9)
8	**=SUMPRODUCT(B8:P8,B9:P9)**

3.4-10

a) Let f_1 = number of full-time consultants working the morning shift (8 a.m.-4 p.m.),
f_2 = number of full-time consultants working the afternoon shift (12 p.m.-8 p.m.),
f_3 = number of full-time consultants working the evening shift (4 p.m.-midnight),
p_1 = number of part-time consultants working the first shift (8 a.m.-12 p.m.),
p_2 = number of part-time consultants working the second shift (12 p.m.-4 p.m.),
p_3 = number of part-time consultants working the third shift (4 p.m.-8 p.m.),
p_4 = number of part-time consultants working the fourth shift (8 p.m.-midnight).

Minimize $C = (\$14 / \text{hour})(8 \text{ hours})[f_1 + f_2 + f_3] +$

$(\$5 / \text{hour})(4 \text{ hours})[p_1 + p_2 + p_3 + p_4]$,

subject to
$$f_1 + p_1 \geq 4$$
$$f_1 + f_2 + p_2 \geq 8$$
$$f_2 + f_3 + p_3 \geq 10$$
$$f_3 + p_4 \geq 6$$
$$f_1 \geq 2p_1$$
$$f_1 + f_2 \geq 2p_2$$
$$f_2 + f_3 \geq 2p_3$$
$$f_3 \geq 2p_4$$

and $f_1 \geq 0,\ f_2 \geq 0,\ f_3 \geq 0,\ p_1 \geq 0,\ p_2 \geq 0,\ p_3 \geq 0,\ p_4 \geq 0.$

b)

	Benefit Contribution Per Unit of Each Activity									Minimum
Time	FT 1	FT 2	FT 3	PT 1	PT 2	PT 3	PT 4	Totals		Required
8 am-12 noon	1	0	0	1	0	0	0	4	≥	4
	1	0	0	-2	0	0	0	4	≥	0
noon-4 pm	1	1	0	0	1	0	0	8	≥	8
	1	1	0	0	-2	0	0	8	≥	0
4 pm-8 pm	0	1	1	0	0	1	0	10	≥	10
	0	1	1	0	0	-2	0	10	≥	0
8 pm-midnight	0	0	1	0	0	0	1	6	≥	6
	0	0	1	0	0	0	-2	6	≥	0
Unit Cost	14	14	14	12	12	12	12	**$ 196.00**		
Solution	**4**	**4**	**6**	**0**	**0**	**0**	**0**			

3.4-11

a) Let

$x_{F1\text{-}C1}$ = number of units shipped from Factory 1 to Customer 1,

$x_{F1\text{-}C2}$ = number of units shipped from Factory 1 to Customer 2,

$x_{F1\text{-}C3}$ = number of units shipped from Factory 1 to Customer 3,

$x_{F2\text{-}C1}$ = number of units shipped from Factory 2 to Customer 1,

$x_{F2\text{-}C2}$ = number of units shipped from Factory 2 to Customer 2.

$x_{F2\text{-}C3}$ = number of units shipped from Factory 2 to Customer 3.

$$\text{Minimize } C = 600x_{F1-C1} + 800x_{F1-C2} + 700x_{F1-C3} + 400x_{F2-C1} + 900x_{F2-C2} + 600x_{F2-C3}$$

subject to

$$x_{F1-C1} + x_{F1-C2} + x_{F1-C3} = 400$$

$$x_{F2-C1} + x_{F2-C2} + x_{F2-C3} = 500$$

$$x_{F1-C1} + x_{F2-C1} = 300$$

$$x_{F1-C2} + x_{F2-C2} = 200$$

$$x_{F1-C3} + x_{F2-C3} = 400$$

and

$$x_{F1-C1} \geq 0,\ x_{F1-C2} \geq 0,\ x_{F1-C3} \geq 0,$$

$$x_{F2-C1} \geq 0,\ x_{F2-C2} \geq 0,\ x_{F2-C3} \geq 0.$$

b)

Contribution Toward Required Amount Per Unit Shipped

Requirement	Shipping Lane F1-C1	F2-C2	F1-C3	F2-C1	F2-C2	F2-C3	Totals		Required Amount
F1 Amount	1	1	1	0	0	0	400	=	400
F2 Amount	0	0	0	1	1	1	500	=	500
C1 Amount	1	0	0	1	0	0	300	=	300
C2 Amount	0	1	0	0	1	0	200	=	200
C3 Amount	0	0	1	0	0	1	400	=	400
Unit Cost	600	800	700	400	200	400	**$ 410,000**		
Solution	**300**	**0**	**100**	**0**	**200**	**300**			

3.4-12
a)

Let

$x_{M1\text{-}S1}$ = number of units shipped from Mine 1 to Storage 1,
$x_{M1\text{-}S2}$ = number of units shipped from Mine 1 to Storage 2,
$x_{M2\text{-}S1}$ = number of units shipped from Mine 2 to Storage 1,
$x_{M2\text{-}S2}$ = number of units shipped from Mine 2 to Storage 2,
$x_{S1\text{-}P}$ = number of units shipped from Storage 1 to the Plant,
$x_{S2\text{-}P}$ = number of units shipped from Storage 2 to the Plant.

$$\text{Minimize } C = 2000x_{M1\text{-}S1} + 1700x_{M1\text{-}S2} + 1600x_{M2\text{-}S1} + 1100x_{M2\text{-}S2} + 400x_{S1\text{-}P} + 800x_{S2\text{-}P}$$

subject to

$$x_{M1\text{-}S1} + x_{M1\text{-}S2} = 40$$
$$x_{M2\text{-}S1} + x_{M2\text{-}S2} = 60$$
$$x_{M1\text{-}S1} + x_{M2\text{-}S1} - x_{S1\text{-}P} = 0$$
$$x_{M1\text{-}S2} + x_{M2\text{-}S2} - x_{S2\text{-}P} = 0$$
$$x_{S1\text{-}P} + x_{S2\text{-}P} = 100$$
$$x_{M1\text{-}S1} \le 30, \quad x_{M1\text{-}S2} \le 30$$
$$x_{M2\text{-}S1} \le 50, \quad x_{M2\text{-}S2} \le 50$$
$$x_{S1\text{-}P} \le 70, \quad x_{S2\text{-}P} \le 70$$

and

$$x_{M1\text{-}S1} \ge 0, \quad x_{M1\text{-}S2} \ge 0, \quad x_{M2\text{-}S1} \ge 0,$$
$$x_{M2\text{-}S2} \ge 0, \quad x_{S1\text{-}P} \ge 0, \quad x_{S2\text{-}P} \ge 0.$$

b)

	Contribution Toward Required Amount Per Unit Shipped								
Requirement	M1-S1	M1-S2	M2-S1	M2-S2	S1-P	S2-P	Totals		Required Amount
M1 Amount	1	1	0	0	0	0	40	=	40
M2 Amount	0	0	1	1	0	0	60	=	60
S1 Amount	1	0	1	0	-1	0	0	=	0
S2 Amount	0	1	0	1	0	-1	0	=	0
P Amount	0	0	0	0	1	1	100	=	100
Capacity	30	30	50	50	70	70			
Unit Cost	2000	1700	1600	1100	400	800	**$ 194,000**		
Solution	**10**	**30**	**10**	**50**	**70**	**30**			

3.4-13 a)

$$A_1 + B_1 + R_1 = 60{,}000$$
$$A_2 + B_2 + C_2 + R_2 = R_1$$
$$A_3 + B_3 + R_3 = R_2 + 1.40A_1$$
$$A_4 + R_4 = R_3 + 1.40A_2 + 1.70B_1$$
$$D_5 + R_5 = R_4 + 1.40A_3 + 1.70B_2$$

b)

Let A_t = amount invested in investment A at the beginning of year t
B_t = amount invested in investment A at the beginning of year t
C_t = amount invested in investment A at the beginning of year t
D_t = amount invested in investment A at the beginning of year t
R_t = amount not invested at the beginning of year t.

Maximize $P = 1.40A_5 + 1.70B_5 + 1.90C_5 + 1.30D_5 + R_5$

subject to
$$A_1 + B_1 + R_1 = 60{,}000$$
$$A_2 + B_2 + C_2 - R_1 + R_2 = 0$$
$$-1.40A_1 + A_3 + B_3 - R_2 + R_3 = 0$$
$$-1.40A_2 + A_4 - 1.70B_1 - R_3 + R_4 = 0$$
$$-1.40A_3 - 1.70B_2 + D_5 - R_4 + R_5 = 0$$

and $A_t \geq 0,\ B_t \geq 0,\ C_t \geq 0,\ D_t \geq 0,\ R_t \geq 0.$

c)

Contribution Toward Required Amount Per Unit

	Investment									Remainder							Required
Year	A1	A2	A3	A4	B1	B2	B3	C2	D5	R1	R2	R3	R4	R5	Totals		Amount
1	1	0	0	0	1	0	0	0	0	1	0	0	0	0	60000	=	60000
2	0	1	0	0	0	1	0	1	0	-1	1	0	0	0	0	=	0
3	-1.4	0	1	0	0	0	1	0	0	0	-1	1	0	0	0	=	0
4	0	-1.4	0	1	-1.7	0	0	0	0	0	0	-1	1	0	-1.33577E-12	=	0
5	0	0	-1.4	0	0	-1.7	0	0	1	0	0	0	-1	1	1.45519E-11	=	0
Unit Profit	0	0	0	1.4	0	0	1.7	1.9	1.3	0	0	0	0	1	**$ 152,880**		
Solution	**60000**	**0**	**84000**	**0**	**0**	**0**	**0**	**0**	**117600**	**0**	**0**	**0**	**0**	**0**			

3.4-14 a)

Let x_1 = amount of Alloy 1 used,
x_2 = amount of Alloy 2 used,
x_3 = amount of Alloy 3 used,
x_4 = amount of Alloy 4 used,
x_5 = amount of Alloy 5 used.

Minimize $C = 22x_1 + 20x_2 + 25x_3 + 24x_4 + 27x_5$

subject to
$$60x_1 + 25x_2 + 45x_3 + 20x_4 + 50x_5 = 40$$
$$10x_1 + 15x_2 + 45x_3 + 50x_4 + 40x_5 = 35$$
$$30x_1 + 60x_2 + 10x_3 + 30x_4 + 10x_5 = 25$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0.$

3.4-14
b)

Requirement	Contribution Toward Required Amount: Alloy 1	Alloy 2	Alloy 3	Alloy 4	Alloy 5	Totals		Required Amount
% tin	60	25	45	20	50	40	=	40
% zinc	10	15	45	50	45	35	=	35
% lead	30	60	10	30	10	25	=	25
% total	1	1	1	1	1	1	=	1
Unit Cost	22	20	25	24	27	**$ 23.46**		
Solution	**0.0435**	**0.2826**	**0.6739**	**0**	**3E-15**			

3.4-15
a) Let

x_{P1L} = number of large units produced at Plant 1,
x_{P1M} = number of medium units produced at Plant 1.
x_{P1S} = number of small units produced at Plant 1,
x_{P2L} = number of large units produced at Plant 2,
x_{P2M} = number of medium units produced at Plant 2.
x_{P2S} = number of small units produced at Plant 2,
x_{P3L} = number of large units produced at Plant 3,
x_{P3M} = number of medium units produced at Plant 3.
x_{P3S} = number of small units produced at Plant 3.

$$\text{Maximize } P = 420x_{P1L} + 360x_{P1M} + 300x_{P1S} + 420x_{P2L} + 360x_{P2M} + 300x_{P2S} + 420x_{P3L} + 360x_{P3M} + 300x_{P3S}$$

subject to

$$x_{P1L} + x_{P1M} + x_{P1S} \le 750$$
$$x_{P2L} + x_{P2M} + x_{P2S} \le 900$$
$$x_{P3L} + x_{P3M} + x_{P3S} \le 450$$
$$20x_{P1L} + 15x_{P1M} + 12x_{P1S} \le 13000$$
$$20x_{P2L} + 15x_{P2M} + 12x_{P2S} \le 12000$$
$$20x_{P3L} + 15x_{P3M} + 12x_{P3S} \le 5000$$
$$x_{P1L} + x_{P2L} + x_{P3L} \le 900$$
$$x_{P1M} + x_{P2M} + x_{P3M} \le 1200$$
$$x_{P1S} + x_{P2S} + x_{P3S} \le 750$$
$$\frac{1}{750}x_{P1L} + \frac{1}{750}x_{P1M} + \frac{1}{750}x_{P1S} - \frac{1}{900}x_{P2L} - \frac{1}{900}x_{P2M} - \frac{1}{900}x_{P2S} = 0$$
$$\frac{1}{750}x_{P1L} + \frac{1}{750}x_{P1M} + \frac{1}{750}x_{P1S} - \frac{1}{450}x_{P3L} - \frac{1}{450}x_{P3M} - \frac{1}{450}x_{P3S} = 0$$

and

$$x_{P1L} \ge 0,\ x_{P1M} \ge 0,\ x_{P1S} \ge 0,\ x_{P2L} \ge 0,\ x_{P2M} \ge 0,$$
$$x_{P2S} \ge 0,\ x_{P3L} \ge 0,\ x_{P3M} \ge 0,\ x_{P3S} \ge 0.$$

3.4-15

b)

Resource	Resource Usage Per Unit of Each Activity P1-L	P1-M	P1-S	P2-L	P2-M	P2-S	P3-L	P3-M	P3-S	Totals		Resource Available
Capacity P1	1	1	1	0	0	0	0	0	0	694.444	≤	750
Capacity P2	0	0	0	1	1	1	0	0	0	833.333	≤	900
Capacity P3	0	0	0	0	0	0	1	1	1	416.667	≤	450
Space P1	20	15	12	0	0	0	0	0	0	13000	≤	13000
Space P2	0	0	0	20	15	12	0	0	0	12000	≤	12000
Space P3	0	0	0	0	0	0	20	15	12	5000	≤	5000
Sales P1	1	0	0	1	0	0	1	0	0	516.667	≤	900
Sales P2	0	1	0	0	1	0	0	1	0	844.444	≤	1200
Sales P3	0	0	1	0	0	1	0	0	1	583.333	≤	750

Requirement	Contribution Toward Required Amount Activity P1-L	P1-M	P1-S	P2-L	P2-M	P2-S	P3-L	P3-M	P3-S	Totals		Required Amount
%P1=%P2	0.00133	0.00133	0.00133	-0.0011	-0.0011	-0.0011	0	0	0	0	=	0
%P1=%P3	0.00133	0.00133	0.00133	0	0	0	-0.0022	-0.0022	-0.0022	0	=	0
Unit Profit	420	360	300	420	360	300	420	360	300	**$ 696,000**		
Solution	**516.667**	**177.778**	**0**	**0**	**666.667**	**166.667**	**0**	**0**	**416.667**			

3.4-16 a)

Let

x_{1F} = number of tons of cargo type 1 stowed in the front compartment,
x_{1C} = number of tons of cargo type 1 stowed in the center compartment
x_{1B} = number of tons of cargo type 1 stowed in the back compartment,
x_{2F} = number of tons of cargo type 2 stowed in the front compartment,
x_{2C} = number of tons of cargo type 2 stowed in the center compartment
x_{2B} = number of tons of cargo type 2 stowed in the back compartment,
x_{3F} = number of tons of cargo type 3 stowed in the front compartment,
x_{3C} = number of tons of cargo type 3 stowed in the center compartment
x_{3B} = number of tons of cargo type 3 stowed in the back compartment,
x_{4F} = number of tons of cargo type 4 stowed in the front compartment,
x_{4C} = number of tons of cargo type 4 stowed in the center compartment
x_{4B} = number of tons of cargo type 4 stowed in the back compartment.

Maximize $P = 320x_{1F} + 320x_{1C} + 320x_{1B} + 400x_{2F} + 400x_{2C} + 400x_{2B} +$
$360x_{3F} + 360x_{3C} + 360x_{3B} + 290x_{4F} + 290x_{4C} + 400x_{4B}$

subject to
$$x_{1F} + x_{2F} + x_{3F} + x_{4F} \le 12$$
$$x_{1C} + x_{2C} + x_{3C} + x_{4C} \le 18$$
$$x_{1B} + x_{2B} + x_{3B} + x_{4B} \le 10$$
$$x_{1F} + x_{1C} + x_{1B} \le 20$$
$$x_{2F} + x_{2C} + x_{2B} \le 16$$
$$x_{3F} + x_{3C} + x_{3B} \le 25$$
$$x_{4F} + x_{4C} + x_{4B} \le 13$$
$$500x_{1F} + 700x_{2F} + 600x_{3F} + 400x_{4F} \le 7000$$
$$500x_{1C} + 700x_{2C} + 600x_{3C} + 400x_{4C} \le 9000$$
$$500x_{1B} + 700x_{2B} + 600x_{3B} + 400x_{4B} \le 5000$$
$$\frac{1}{12}x_{1F} + \frac{1}{12}x_{2F} + \frac{1}{12}x_{3F} + \frac{1}{12}x_{4F} - \frac{1}{18}x_{1C} - \frac{1}{18}x_{2C} - \frac{1}{18}x_{3C} - \frac{1}{18}x_{4C} = 0$$
$$\frac{1}{12}x_{1F} + \frac{1}{12}x_{2F} + \frac{1}{12}x_{3F} + \frac{1}{12}x_{4F} - \frac{1}{10}x_{1B} - \frac{1}{10}x_{2B} - \frac{1}{10}x_{3B} - \frac{1}{10}x_{4B} = 0$$

and
$$x_{1F} \ge 0,\ x_{1C} \ge 0,\ x_{1B} \ge 0,\ x_{2F} \ge 0,\ x_{2C} \ge 0,\ x_{2B} \ge 0,$$
$$x_{3F} \ge 0,\ x_{3C} \ge 0,\ x_{3B} \ge 0,\ x_{4F} \ge 0,\ x_{4C} \ge 0,\ x_{4B} \ge 0.$$

3.4-16

b)

	Resource Usage Per Unit of Each Activity														Resource
Resource	1F	1C	1B	2F	2C	2B	3F	3C	3B	4F	4C	4B	Totals		Available
Front Wt.	1	0	0	1	0	0	1	0	0	1	0	0	12	≤	12
Center Wt.	0	1	0	0	1	0	0	1	0	0	1	0	18	≤	18
Back Wt.	0	0	1	0	0	1	0	0	1	0	0	1	10	≤	10
Cargo 1 Wt.	1	1	1	0	0	0	0	0	0	0	0	0	15	≤	20
Cargo 2 Wt.	0	0	0	1	1	1	0	0	0	0	0	0	12	≤	16
Cargo 3 Wt.	0	0	0	0	0	0	1	1	1	0	0	0	0	≤	25
Cargo 4 Wt.	0	0	0	0	0	0	0	0	0	1	1	1	13	≤	13
Space Front	500	0	0	700	0	0	600	0	0	400	0	0	7000	≤	7000
Space Center	0	500	0	0	700	0	0	600	0	0	400	0	9000	≤	9000
Space Back	0	0	500	0	0	700	0	0	600	0	0	400	5000	≤	5000

	Contribution Toward Required Amount														Required
Requirement	1F	1C	1B	2F	2C	2B	3F	3C	3B	4F	4C	4B	Totals		Amount
%F=%C	0.0833	-0.0556	0	0.0833	-0.0556	0	0.0833	-0.0556	0	0.0833	-0.0556	0	0	=	0
%F=%B	0.0833	0	-0.1	0.0833	0	-0.1	0.0833	0	-0.1	0.0833	0	-0.1	0	=	0
Unit Profit	320	320	320	400	400	400	360	360	360	290	290	290	$ 13,330		
Solution	**0**	**5**	**10**	**7.33333**	**4.167**	**0.000**	**0**	**0**	**0**	**4.66667**	**8.333**	**0.000**			

3.4-17 a)

Let M = number of men's gloves to produce per week,
W = number of women's gloves to produce per week,
C = number of children's gloves to produce per week,
F = number of full-time workers to employ,
PT = number of part-time workers to employ.

Maximize $P = 8M + 10W + 6C - 13(40)F - 10(20)PT$

subject to $2M + 1.5W + C \le 5000$

$30M + 45W + 40C \le 40(60)F + 20(60)PT$

$F \ge 20$

$F \ge 2PT$

and $M \ge 0,\ W \ge 0,\ C \ge 0,\ F \ge 0,\ PT \ge 0.$

b)

	Resource Usage Per Unit of Each Activity							Resource
Resource	Men's	Women's	Children's	Full-Time	Part-Time	Totals		Available
Leather	2	1.5	1	0	0	5000	≤	5000
Labor	30	45	40	-2400	-1200	0	≤	0
Full-Time Employees	0	0	0	1	0	25	≥	20
Employee Ratio	0	0	0	1	-2	0	≥	0
Unit Profit	8	10	6	-520	-200	**4500**		
Solution	**2500**	**0**	**0**	**25**	**12.5**			

3.4-18

a) Let X_{ij} = # hours operator i is assigned to work of day j

(i = KC, OH, HB, SC, KS, NK ; j = M, T, W, Th, F)

$$\text{Minimize } Z = 10(X_{KC,M} + X_{KC,W} + X_{KC,F}) + 10.1(X_{OH,T} + X_{OH,Th})$$
$$+9.9(X_{HB,M} + X_{HB,T} + X_{HB,W} + X_{HB,F})$$
$$+9.8(X_{SC,M} + X_{SC,T} + X_{SC,W} + X_{SC,F})$$
$$+10.8(X_{KS,M} + X_{KS,W} + X_{KS,Th}) + 11.3(X_{NK,Th} + X_{NK,F})$$

subject to

$X_{KC,M} \le 6$ $X_{KC,W} \le 6$ $X_{KC,F} \le 6$

$X_{OH,M} \le 6$ $X_{OH,Th} \le 6$

$X_{HB,M} \le 4$ $X_{HB,T} \le 8$ $X_{HB,W} \le 4$ $X_{HB,F} \le 4$

$X_{SC,M} \le 5$ $X_{SC,T} \le 5$ $X_{SC,W} \le 5$ $X_{SC,F} \le 5$

$X_{KS,M} \le 3$ $X_{KS,W} \le 3$ $X_{KS,Th} \le 3$

$X_{NK,M} \le 6$ $X_{NK,F} \le 2$

$$X_{KC,M} + X_{KC,W} + X_{KC,F} \ge 8$$
$$X_{OH,T} + X_{OH,Th} \ge 8$$
$$X_{HB,M} + X_{HB,T} + X_{HB,W} + X_{HB,F} \ge 8$$
$$X_{SC,M} + X_{SC,T} + X_{SC,W} + X_{SC,F} \ge 8$$
$$X_{KS,M} + X_{KS,W} + X_{KS,Th} \ge 7$$
$$X_{NK,Th} + X_{NK,F} \ge 7$$
$$X_{KC,M} + X_{HB,M} + X_{SC,M} + X_{KS,M} = 14$$
$$X_{OH,T} + X_{HB,T} + X_{SC,T} = 14$$
$$X_{KC,W} + X_{HB,W} + X_{SC,W} + X_{KS,W} = 14$$
$$X_{OH,Th} + X_{KS,Th} + X_{NK,Th} = 14$$
$$X_{KC,F} + X_{HB,F} + X_{SC,F} + X_{NK,F} = 14$$
$$X_{ij} \ge 0 \quad \forall i, j$$

3.4-18 b)

Resource	KC,M	KC,W	KC,F	DH,Tu	DH,Th	HB,M	HB,Tu	HB,W	HB,F	SC,M	SC,Tu	SC,W	SC,F	KS,M	KS,W	KS,Th	NK,Th	NK,F	Totals		Resource Available
KC Knowledge	1	1	1																9	≥	8
DH Knowledge				1	1														8	≥	8
HB Knowledge						1	1	1	1										19	≥	8
SC Knowledge										1	1	1	1						20	≥	8
KS Knowledge														1	1	1			7	≥	7
NK Knowledge																	1	1	7	≥	7
Mon Hours	1					1				1				1					14	≥	14
Tues Hours				1			1				1								14	≥	14
Wed Hours		1						1				1			1				14	≥	14
Thurs Hours					1											1	1		14	≥	14
Fri Hours			1						1				1					1	14	≥	14
Availability KC,M	1																		4	≤	6
Availability KC,W		1																	2	≤	6
Availability DH,Tu				1															2	≤	6
Availability DH,TH					1														6	≤	6
Availability HB,M						1													4	≤	4
Availability HB,Tu							1												7	≤	8
Availability HB,W								1											4	≤	4
Availability HB,F									1										4	≤	4
Availability SC,M										1									5	≤	5
Availability SC,Tu											1								5	≤	5
Availability SC,W												1							5	≤	5
Availability SC,F													1						5	≤	5
Availability KS,M														1					1	≤	3
Availability KS,W															1				3	≤	3
Availability KS,Th																1			3	≤	8
Availability NK,Th																	1		5	≤	8
Availability NK,F																		1	2	≤	2
Unit Cost	10	10	10	10.1	10.1	9.8	9.9	9.9	9.9	9.8	9.8	9.8	9.8	10.8	10.8	10.8	11.3	11.3	710		
Solution	4	2	3	2	6	4	7	4	4	5	5	5	5	1	3	3	5	2			

3.4-19

a) Let S = Tablespoons of strawberry flavoring,
CR = Tablespoons of cream,
V = Tablespoons of vitamin supplement,
A = Tablespoons of artificial sweetener,
T = Tablespoons of thickening agent,

Minimize $C = 10S + 8CR + 25V + 15A + 6T$

subject to
$$50S + 100CR + 120A + 80T \geq 380$$
$$50S + 100CR + 120A + 80T \leq 420$$
$$S + 75CR + 30T \leq 0.2(50S + 100C + 120A + 80T)$$
$$20S + 50V + 2T \geq 50$$
$$S \geq 2A$$
$$3S + 8CR + V + 2A + 25T = 15$$

and $S \geq 0,\ CR \geq 0,\ V \geq 0,\ A \geq 0,\ T \geq 0.$

Resource	Strawberry	Cream	Vitamin	Sweetener	Thickener	Totals		Available
Min Calories	50	100	0	120	80	380	≥	380
Max Calories	50	100	0	120	80	380	≤	420
Fat	-9	55	0	-24	14	-73	≤	0
Vitamin	20	0	50	0	2	128	≥	50
Taste	1	0	0	-2	0	0	≥	0
Thickness	3	8	1	2	25	15	=	15
Unit Cost	10	8	25	15	6	**90**		
Solution	**3.455**	**0**	**1**	**2**	**0**			

3.4-20 a) Let B = slices of bread,
P= Tablespoons of peanut butter,
S = Tablespoons of strawberry jelly,
G = graham crackers,
M = cups of milk,
J = cups of juice.

Minimize $C = 5B + 4P + 7S + 8G + 15M + 35J$

subject to
$$70B + 100P + 50S + 60G + 150M + 100J \geq 400$$
$$70B + 100P + 50S + 60G + 150M + 100J \leq 600$$
$$10B + 75P + 20G + 70M \leq .3(70B + 100P + 50S + 60G + 150M + 100J)$$
$$3S + 2M + 120J \geq 60$$
$$3B + 4P + G + 8M + J \geq 12$$
$$B = 2$$
$$P \geq 2S$$
$$M + J \geq 1$$

and $B \geq 0,\ P \geq 0,\ S \geq 0,\ G \geq 0,\ M \geq 0,\ J \geq 0.$

b)

	Resource Usage Per Unit of Each Activity								Resource
Resource	Bread	PB	Jelly	Crackers	Milk	Juice	Totals		Available
Min Calories	70	100	50	60	150	100	600	≥	400
Max Calories	70	100	50	60	150	100	600	≤	600
Fat	-11	45	-15	2	25	-30	-145	≤	0
Vitamin C	0	0	3	0	2	120	514	≥	60
Protein	3	4	0	1	8	1	12	≥	12
Bread	1	0	0	0	0	0	2	=	2
PB&J	0	1	-2	0	0	0	0	≥	0
Liquid	0	0	0	0	1	1	4	≥	1
Unit Cost	5	4	7	8	15	35	**163**		
Solution	**2**	**0**	**0**	**0**	**0**	**4**			

3. 5-1. a) The two factors which often hinder the use of optimization models by managers are cultural differences and response time. Cultural differences cause managers and model developers to often have a hard time understanding each other. Response time is often slow due to the time to translate, formulate and solve the manager's problem using optimization systems.

b) As stated in the article, "[b]ased on unit profits, the company in the past emphasized the manufacture of thicker plywoods (TYPEA), but the optimization procedure showed that in fact thinner plywoods (TYPEB) were more profitable. This product mix change has increased the overall profitability of the company by 20%."

c) The chapter before "Conclusions" describes this: "With the success of this application, management is now eager to use optimization for other problems, too. Since Ponderosa uses timber for products other than plywood, they intend to explore the optimum allocation of this raw material between different products. Raw material and inventory management is another potential area of application. Also, the conversion of the optimization model into the financial planning language will now facilitate the integration of financial models with production models."

d) The Xerox of the appendices:

APPENDIX A. MATHEMATICAL FORMULATION OF THE PROBLEM

X_{it} = Amount of product i in period t $(i = 1, \ldots, n)$, $(t = 1, \ldots, T)$
C_{it} = Contribution margin of product i in period t
Y_{jt} = Amount of veneer sheet type j produced in period t $(j = 1, \ldots, n_1)$
Z_{kt} = Amount of green veneer type k produced in period t $(k = 1, \ldots, n_2)$
C_{pt} = Consumption of log type p in period t $(p = 1, \ldots, n_3)$
F_{pt} = Final inventory of log type p in period t
S_{pt} = Supply of log type p in period t
P_i = Pressing hours required per unit of product i
G_i = Polishing hours required per unit of product i
A_t = Total polishing hours available in period t
B_t = Total pressing hours available in period t
D_{ij} = Amount of type j veneer sheet required per unit of product i
E_{pT} = Final inventory requirement coefficients for log type p in period T
Q_{kp} = Green veneer type k yield per unit of log type p
H_{kj} = Green veneer type k required per unit of veneer sheet of type j
U_{it} = Upper limit on market demand for product i in period t
L_{it} = Lower limit on market demand for product i in period t

Objective

$$\text{Maximize} \sum_{i,t} C_{it} X_{it}.$$

Constraints

Polishing capacity limited:

$$\sum_i G_i X_{it} \leq A_t, \quad t = 1, \ldots, T$$

Pressing capacity limited:

$$\sum_i P_i X_{it} \leq B_t, \quad t = 1, \ldots, T$$

Veneer sheet required:

$$\sum_i D_{ij} X_{it} = Y_{jt}, \quad j = 1, \ldots, n_1, \quad t = 1, \ldots, T$$

Material balance on log:

$$C_{pt} + F_{pt} - F_{pt-1} = S_{pt}, \quad p = 1, \ldots, n_3, \quad t = 1, \ldots, T$$

Final inventory composition requirement:

$$\sum_p E_{pT} F_{pT} = 0$$

Green veneer yield from logs:

$$\sum_p Q_{kp} C_{pt} = Z_{kt}, \quad k = 1, \ldots, n_2, \quad t = 1, \ldots, T$$

Green veneers required:

$$\sum_j H_{kj} Y_{jt} = Z_{kt}, \quad k = 1, \ldots, n_2, \quad t = 1, \ldots, T$$

Market constraints:

$$L_{it} \leq X_{it} \leq U_{it}.$$

3.5-1. d) (cont'd)

APPENDIX B. STRUCTURE OF THE LP MODEL

Vector Rows	Products by Grade and Thickness	Veneer Sheet Production	Green Veneer Production	Log Consumption	Final Log Inventory	Right Hand Side
Market Constraints	I					≤Limits Forecast by Sales
Veneer Requirements	X1	−I				=0
Pressing & Polishing Constraints	X2					≤ Production Capacity
Green Veneer Requirement		X3	−I +I			=0
Green Veneer Generated			−I	X4		=0
Material Balance				I	I	= Initial Inventory + Log Supply
Final Inventory Composition Requirements					X5	=0
Objective Function	Unit Contribution Margin					= Maximize Contribution Margin

3.5-2. a) The shift schedules at airports and reservations offices were done by hand prior to this study. (see paragraph 2, col 2, p. 42)

b) The project requirements (see bottom of col. 1, p. 42) were:
i) to determine the needs for increased manpower,
ii) to identify excess manpower for reallocation,
iii) to reduce the time required for preparing schedules,
iv) to make manpower allocation more day- and time-sensitve, and
v) to quantify the cost associated with scheduling.

c) United Airlines dealt with the integrality problem by using "[a] heuristic rounding algorithm similar to that described in Henderson and Berry [1979, see article for reference] ... incorporated in the Report Module ... and serv[ing] to covert the Schduling Module's fractionated LP solution into a workable shift schedule." (see p. 45)

d) Flexibility was necessary to "[satisfy] the group culture at each office [which was] essential in garnering field support. As a result, office-specified input variables, such as the number of start times, the preferred shift lengths, the length of breaks, preferred days off combinations, and so forth, became an intergral part of SMPS. This versatility gave office managers the luxury of evaluating schedules incorporating different input parameters but identical manpower requirements." (p. 47)

e) Benefits included: (p.48)
i) significant labor cost savings,
ii) improved customer service,
iii) improved emplyee schedules,
iv) quantified manpower planning and evaluation,
(pp. 48-49 describes these in more detail)

3.5-3. a) During the years preceding this OR study, "the price of crude oil increased tenfold ... as the short-term interest rates more than tripled ... This meant that the cost associated with financing the working capital employed in the refining and marketing industry increased more than 30 fold during that time ... Con-sequently, it [became] vastly more important to maintain tight control over required working capital." (p. 2)

b) "Citgo's distribution network of pipelines, tankers, and barges span[ned] the eastern two-third of the United States. The addition of the Southland 7-Eleven store increase[d] their marketing and distribution network to all of the 48 contiguous United States." (p. 4)

c) An 11-week planning horizon, partitioned into 6 one-week periods and 1 five-week period, was used. Also, "the model [was] partitioned into time zones, and replications of the model [were] employed to represent the multiple time periods." (p. 6)

d) Citgo used "a medium-sized computer, an IBM 4381 and mod." Typical run times for "model generation, solution and reports [were] respectively two minutes, half a minute, and seven minutes."

3,5-3. e) The four types of model users were the product managers, the pricing manager, the product traders, and the budget manager. "Product manager compare[d] the model recommendations to the actual operational decisions to determine the existence and cause of discrepancies. They also use[d] the model's what-if capabilities to generate economically viable alternatives to current and forecasted operations ... The pricing manager use[d] the model in several ways. He use[d] one set of reports to set ranges for terminal prices for each product ... In addition, the pricing manager use[d] the wholesale report to help set prices and reommended volumes for bulk sales made to reduce excess inventories ... [Product] traders use[d] the Volume Summary Report and the Infeasibility Report to determine which side of the trading board they should be on for each product ... They [could] also use the model's what-if capabilities to determine the sensitivity of spot prices to the required purchases or sales volumes as prices fluctuate[d] during the week ... [T] he budget manager use[d] the Financial Summary Report to generate various components of the monthly and quarterly budgets." (p. 9)

f) The major reports generated by the SDM system are listed on page 8 of the article. They are:

i) Infeasibility report,
ii) In-transit, Terminal, Exchange, Inventory reports,
iii) Spot Recommendation report,
iv) Purchases, Sales, Trades reports,
v) Wholesale report,
vi) Volume Summary report,
vii) Financial Summary report.

g) "The education of the users was a challenge because both organizational responsibilities and people within the organizations flucutuated." (p. 10) "Another major implementation challenge concerned the collection, validation and correction of input data for the model." (p. 11) "A third implementation challenge concerned the forecasting sales volumes and wholesales prices. Citgo forcasted only total volumes for monthly and quarterly budgets. The SDM systems, on the other hand, required volume and price forecasts to be detailed by terminal, by product, by line of business, and by week." (p. 11)

h) "The major benefit realized from the SDM Model was the reduction in Citgo's product inventory with no drop in service levels ... Another direct benefit of implementing the model was the improvement in operational decision making: improvements in coordination, pricing and purchasing decisions, as well as the incorporation of the 'new concepts' into the decision process itself ... In addition to the direct benefits, several indirect benefits result[ed] from the modeling process and implementation ... One such indirect benefit [was] the establishment of a corporate data base which provide[d] common, up-to-date, on-line operational information for current decision support. Another such benefit [was] the utilization of a single forcast throughout the different departments, thus keeping the entire organization focused and traveling down a single path. A third benefit [was] the closed-loop planning process fostered by the continual feedback provided by the product manager when comparing actual decision to model-recommended decision ... A fourth benefit [was] the increased interdepartmental communication the model's use has fostered at Citgo ... A final benefit [was] the insight gained from the modeling process itself." (pp. 15-16)

3.6-1

a)

Maximize $P = 20x_1 + 30x_2$,

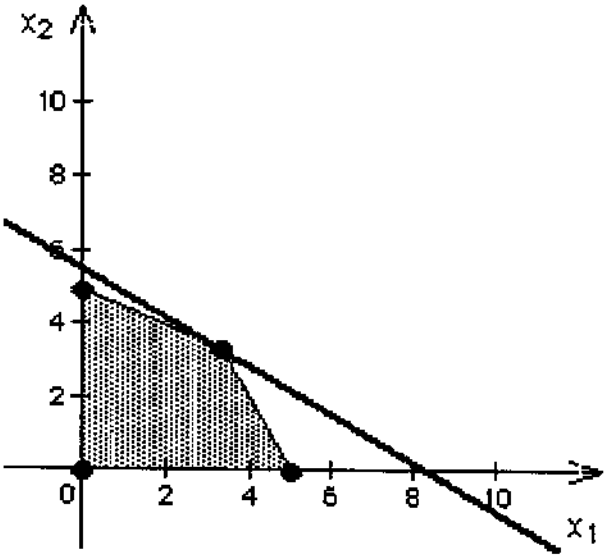
subject to $2x_1 + x_2 \le 10$

$3x_1 + 3x_2 \le 20$

$2x_1 + 4x_2 \le 20$

and $x_1 \ge 0,\ x_2 \ge 0.$

b) Optimal Solution: $(x_1, x_2) = \left(3\frac{1}{3}, 3\frac{1}{3}\right)$ and $P = 166.67$.



c, e)

Resource	Resource Usage Per Unit of Each Activity Activity 1	Activity 2	Totals		Resource Available
1	2	1	10	≤	10
2	3	3	20	≤	20
3	2	4	20	≤	20
Unit Profit	20	30	**$ 166.67**		
Solution	**3.333**	**3.333**			

d)

(x_1, x_2)	Feasible?	P	
(2,2)	Yes	$100	
(3,3)	Yes	$150	
(2,4)	Yes	$160	Best
(4,2)	Yes	$140	
(3,4)	No		
(4,3)	No		

3.6-2

a)

Maximize $P = 50A + 40B + 30C$,

subject to $.02A + .03B + .05C \le 40$

$.05A + .02B + .04C \le 40$

and $A \ge 0,\ B \ge 0,\ C \ge 0.$

b & d)

Resource	Resource Usage Per Unit of Each Activity Part A	Part B	Part C	Totals		Resource Available
Machine 1	0.02	0.03	0.05	40	≤	40
Machine 2	0.05	0.02	0.04	40	≤	40
Unit Profit	50	40	30	**$ 61,818.18**		
Solution	**363.636**	**1090.909**	**0**			

c)

(x_1, x_2, x_3)	Feasible?	P	
(500,500,300)	No		
(350,1000,0)	Yes	$57,500	
(400,1000,0)	Yes	$60,000	Best

Many answers are possible.

3.6-3

a)

Minimize $C = 60x_1 + 50x_2$,

subject to $5x_1 + 3x_2 \ge 60$

$2x_1 + 2x_2 \ge 30$

$7x_1 + 9x_2 \ge 126$

and $x_1 \ge 0,\ x_2 \ge 0.$

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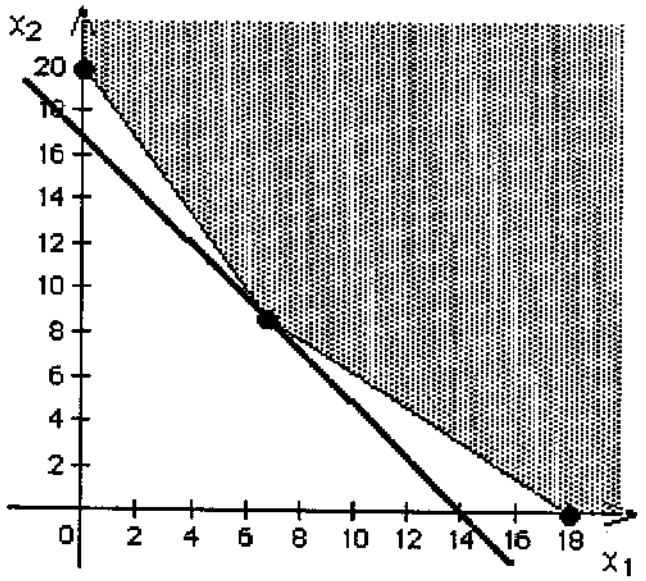
3.6-3 Optimal Solution: $(x_1, x_2) = (6.75, 8.75)$ and $C = 842.50$.

b)



c & e)

Benefit	Benefit Contribution Per Unit of Each Activity: Activity 1	Activity 2	Totals		Minimum Level
1	5	3	60	≥	60
2	2	2	31	≥	30
3	7	9	126	≥	126
Unit Cost	60	50	**$ 842.50**		
Solution	**6.75**	**8.75**			

d)

(x_1, x_2)	Feasible?	C	
(7,7)	No		
(7,8)	No		
(8,7)	No		
(8,8)	Yes	$880	Best
(8,9)	Yes	$930	
(9,8)	Yes	$940	

3.6-4

a)

Minimize $C = 84C + 72T + 60A$,

subject to $90C + 20T + 40A \geq 200$

$30C + 80T + 60A \geq 180$

$10C + 20T + 60A \geq 150$

and $C \geq 0,\ T \geq 0,\ A \geq 0.$

b & e)

Nutritional Ingredient	Kilogram of Corn	Kilogram of Tankage	Kilogram of Alfalfa	Totals		Minimum Level
Carbohydrates	90	20	40	200	≥	200
Proteins	30	80	60	180	≥	180
Vitamins	10	20	60	157	≥	150
Unit Cost	84	72	60	**$ 242**		
Solution	**1**	**0**	**2**			

c) $(x_1, x_2, x_3) = (1,2,2)$ is a feasible solution with a daily cost of \$348.00. This diet will provide 210 kg of carbohydrates, 310 kg of protein, and 170 kg of vitamins daily.

d) Answers will vary.

3.6-5

a)

Minimize $C = x_1 + x_2 + x_3$,

subject to $2x_1 + x_2 + .5x_3 \geq 400$

$.5x_1 + .5x_2 + x_3 \geq 100$

$1.5x_2 + 2x_3 \geq 300$

and $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

b & e)

Year	Benefit Contribution Per Unit of Each Asset: Asset 1	Asset 2	Asset 3	Totals		Minimum Cash Flow Required
5	2	1	0.5	400	≥	400
10	0.5	0.5	1	150	≥	100
20	0	1.5	2	300	≥	300
Unit Cost	1	1	1	**$ 300**		
Solution	**100**	**200**	**0**			

c) $(x_1, x_2, x_3) = (100,100,200)$ is a feasible solution. This would generate \$400 million in 5 years, \$300 million in 10 years, and \$550 million in 20 years. The total invested will be \$400 million.

d) Answers will vary.

3.7-1

a -

Notation

i: products

j: months

k: plants

l: processes

m: regions

X_{ijklm}: amount produced, of product i, in month j, in plant k, using process l, to be sold in region m.

D_{ijm}: Demand of product i in month j in region m

C_{ikl}: unit production cost of product i in plant k using process l

R_{ikl}: production rate of product i in plant k using process l.

S_{im} amount stored to be sold in March, of product i in region m.

P_i: Selling price of product i

$T_{i,k,m}$: transportation cost of product i, produced in plant k, to be sold in region m.

A_j: days available for production in month j.

L: storage limit

M_i: storage cost (per unit) of product i.

(CONT'D)

3.7-1 a) (CONT'D)

The objective is to:

MAX Profit = Revenue - Production Cost
- Inventory Cost
- Transportation Cost

So $$\text{Profit} = \sum_i P_i \left(\sum_{j,k,l,m} x_{ijklm} \right) - \sum_{iklm} C_{ikl} \left(\sum_{j,m} x_{ijklm} \right)$$

$$- \sum_i M_i \left(\sum_m S_{im} \right) - \sum_{i,k,m} T_{ikm} \left(\sum_{j,l} x_{ijklm} \right)$$

Subject to the constraints:

$$\sum_{k,l} x_{ijklm} - S_{im} \leq D_{ijm} \text{ ; For } j = \text{Feb.}$$
$$i = 1,2$$
$$m = 1,2$$

$$\sum_{k,l} x_{ijklm} + S_{im} \leq D_{ijm} \text{ , For } j = \text{March}$$
$$i = 1,2$$
$$m = 1,2$$

$$\sum_i S_{im} \leq L \quad \text{for } m = 1,2$$

$$\sum_{i,l} \frac{1}{R_{ikl}} \left(\sum_m x_{ijklm} \right) \leq A_j \quad \text{for } j = \text{Feb, March and } k = 1,2.$$

$$x_{i,j,k,l,m} \geq 0 \quad \text{for } i = 1,2 \text{ ; } j = F, M \text{ ; } k = 1,2 \text{ ; } l = 1,2 \text{ ; } m = 1,2.$$

3.7-1
b)

Quantity produced to be sold in the same region with process 1

Product	Plant 1		Plant2	
	February	March	February	March
1	0	0	0	0
2	2400	2760	3200	3680

Quantity produced to be sold in the other region with process 1

Product	Plant 1		Plant2	
	February	March	February	March
1	0	0	0	0
2	0	0	0	0

Quantity produced to be sold in the same region with process 2

Product	Plant 1		Plant2	
	February	March	February	March
1	0	0	0	0
2	0	0	0	0

Quantity produced to be sold in the other region with process 2

Product	Plant 1		Plant2	
	February	March	February	March
1	0	0	0	0
2	0	0	0	0

Days available		Storage Limit
February	20	1000
March	23	

		Total Profit
Revenue	1348480	$333,680
Prod Cost	1014800	
Transp Cost	0	
Storage Cost	0	

Demand

Product	Region1		Region2	
	February	March	February	March
1	3600	6300	4900	4200
2	4500	5400	5100	6000

Production costs

Product	Plant 1		Plant2	
	Process 1	Process 2	Process 1	Process 2
1	$ 62	$ 59	$ 61	$ 65
2	$ 78	$ 85	$ 89	$ 86

Production rates

Product	Plant 1		Plant2	
	Process 1	Process 2	Process 1	Process 2
1	100	140	130	110
2	120	150	160	130

Transportation Costs

Region	Product 1		Product 2	
	1	2	1	2
1	0	9	0	7
2	9	0	7	0

Product	Revenue
1	$83
2	$112

Product	Storage cost
1	$ 3
2	$ 4

Stored quantity

Region	Product 1	Product 2
1	0	0
2	0	0

Demand satisfied

Product	Plant 1		Plant2	
	February	March	February	March
1	0	0	0	0
2	2400	2760	3200	3680

Capacity used

	Plant1	Plant2
Feb	20	20
Mar	23	23

Amount stored

Plant1	0
Plant2	0

3-47

3-51

c-

```
{   Problem 3.7-1 in MPL   }

TITLE
    ManufacturingProblem;

INDEX
    product = (pr1,pr2);
    month = (feb,mar);
    plant = (pl1,pl2);
    process = (ps1,ps2);
    region = (r1,r2);

DATA
    demand[product,month,region] := (3600,4900,
                                     6300,4200,
                                     4500,5100,
                                     5400,6000);
    days[month] := (20,23);
    storagecost[product] := (3,4);
    prodcost[product,plant,process] := (62,59,
                                        67,65,
                                        78,85,
                                        89,86);
    rate[product,plant,process] := (100,140,
                                    130,110,
                                    120,150,
                                    160,130);
    price[product] := (83,112);
    transpcost[product,plant,region] := (0,9,
                                         9,0,
                                         0,7,
                                         7,0);

DECISION VARIABLES
    Volume[product,month,plant,process,region];
    Store[product,region];

MACRO

    Revenues := SUM(product,month,plant,process,region: price*Volume);
    ProductionCost := SUM(product,plant,process,month,region: prodcost*Volume);
    TransportationCost := SUM(product,plant,region,month,process: transpcost*Volume);
    StorageCost := SUM(product,region: storagecost*Store);

MODEL

    MAX  TotalProfit = Revenues - ProductionCost - TransportationCost - StorageCost;

SUBJECT TO
    SalesFeb[product,region,month] where(month=feb) : SUM(plant,process: Volume - Store) <= demand;
    SalesMar[product,region,month] where(month=mar) : SUM(plant,process: Volume + Store) <= demand;
    StorageLimit[region] : SUM(product: Store) <= 1000;
    Capacity[plant,month] : SUM(product,process,region: Volume/rate) <= days;

END
```

(CONT'D)

3.7-1 c) cont'd)

SOLUTION RESULT

Optimal solution found

MAX TotalPro = 333680.0000

MACROS

Macro Name	Values
Revenues	1348480.0000
ProductionCost	1014800.0000
TransportationCost	0.0000
StorageCost	0.0000

DECISION VARIABLES

VARIABLE Volume[product,month,plant,process,region] :

product	month	plant	process	region	Activity	Reduced Cost
pr1	feb	pl1	ps1	r1	0.0000	-19.8000
pr1	feb	pl1	ps1	r2	0.0000	-28.8000
pr1	feb	pl1	ps2	r1	0.0000	-5.1429
pr1	feb	pl1	ps2	r2	0.0000	-14.1429
pr1	feb	pl2	ps1	r1	0.0000	-15.3077
pr1	feb	pl2	ps1	r2	0.0000	-6.3077
pr1	feb	pl2	ps2	r1	0.0000	-24.4545
pr1	feb	pl2	ps2	r2	0.0000	-15.4545
pr1	mar	pl1	ps1	r1	0.0000	-19.8000
pr1	mar	pl1	ps1	r2	0.0000	-28.8000
pr1	mar	pl1	ps2	r1	0.0000	-5.1429
pr1	mar	pl1	ps2	r2	0.0000	-14.1429
pr1	mar	pl2	ps1	r1	0.0000	-15.3077
pr1	mar	pl2	ps1	r2	0.0000	-6.3077
pr1	mar	pl2	ps2	r1	0.0000	-24.4545
pr1	mar	pl2	ps2	r2	0.0000	-15.4545
pr2	feb	pl1	ps1	r1	2400.0000	0.0000
pr2	feb	pl1	ps1	r2	0.0000	-7.0000
pr2	feb	pl1	ps2	r1	0.0000	-0.2000
pr2	feb	pl1	ps2	r2	0.0000	-7.2000
pr2	feb	pl2	ps1	r1	0.0000	-7.0000
pr2	feb	pl2	ps1	r2	3200.0000	0.0000
pr2	feb	pl2	ps2	r1	0.0000	-9.3077
pr2	feb	pl2	ps2	r2	0.0000	-2.3077
pr2	mar	pl1	ps1	r1	2760.0000	0.0000
pr2	mar	pl1	ps1	r2	0.0000	-7.0000
pr2	mar	pl1	ps2	r1	0.0000	-0.2000
pr2	mar	pl1	ps2	r2	0.0000	-7.2000
pr2	mar	pl2	ps1	r1	0.0000	-7.0000
pr2	mar	pl2	ps1	r2	3680.0000	0.0000
pr2	mar	pl2	ps2	r1	0.0000	-9.3077
pr2	mar	pl2	ps2	r2	0.0000	-2.3077

(cont'd)

3.7-1 c) CONT'D]

VARIABLE Store[product,region] :

product	region	Activity	Reduced Cost
pr1	r1	0.0000	-3.0000
pr1	r2	0.0000	-3.0000
pr2	r1	0.0000	-4.0000
pr2	r2	0.0000	-4.0000

d)

```
MODEL:
! Problem 3.7-1 - Lingo Version;
SETS:
PRODUCT/PR1 PR2/: PRICE, STORAGECOST;
MONTH/FEB MAR/: DAYS;
PLANT/PL1 PL2/;
PROCESS/PS1 PS2/;
REGION/R1 R2/;
LINK1(PRODUCT,MONTH,PLANT,PROCESS,REGION): VAR;
LINK2(PRODUCT,MONTH,REGION): DEMAND;
LINK3(PRODUCT,PLANT,PROCESS): PRODCOST;
LINK4(PRODUCT,PLANT,PROCESS): RATE;
LINK5(PRODUCT,REGION): STORE;
LINK6(PRODUCT,PLANT,REGION): TRANSPCOST;
ENDSETS

!OBJECTIVE FUNCTION;
MAX = @SUM(PRODUCT(I): PRICE(I)*@SUM(MONTH(J): @SUM(PLANT(K): @SUM(PROCESS(L):
@SUM(REGION(M): VAR(I,J,K,L,M))))) - @SUM(LINK3(I,K,L): PRODCOST(I,K,L)*@SUM(MONTH(J):
@SUM(REGION(M): VAR(I,J,K,L,M)))) - @SUM(PRODUCT(I): STORAGECOST(I)*@SUM(REGION(M):
STORE(I,M))) - @SUM(LINK6(I,K,M): TRANSPCOST(I,K,M)*@SUM(MONTH(J): @SUM(PROCESS(L):
VAR(I,J,K,L,M))));

!CONSTRAINTS;
@FOR(PRODUCT(I): @FOR(REGION(M): @SUM(PLANT(K): @SUM(PROCESS(L): VAR(I,FEB,K,L,M))) -
STORE(I,M) <= DEMAND(I,FEB,M)));
@FOR(PRODUCT(I): @FOR(REGION(M): @SUM(PLANT(K): @SUM(PROCESS(L): VAR(I,MAR,K,L,M))) +
STORE(I,M) <= DEMAND(I,MAR,M)));
@FOR(REGION(M): @SUM(PRODUCT(I): STORE(I,M))<=1000);
@FOR(PLANT(K): @FOR(MONTH(J): @SUM(PRODUCT(I): @SUM(PROCESS(L):
(1/RATE(I,K,L))*@SUM(REGION(M): VAR(I,J,K,L,M)))) <= DAYS(J)));

!DATA PART;
DATA:
DEMAND = 3600 4900
         6300 4200
         4500 5100
         5400 6000;
DAYS = 20 23;
STORAGECOST = 3 4;
PRODCOST = 62 59
           61 65
           78 85
           89 86;
RATE = 100 140
       130 110
       120 150
       160 130;
PRICE = 83 112;
TRANSPCOST = 0 9
             9 0
             0 7
             7 0;
ENDDATA
END
```

3.7-1 d) (CONT'D)

Global optimal solution found at step: 8
Objective value: 333680.0

Variable	Value
VAR(P1, FEB, P1, P1, R1)	0.0000000
VAR(P1, FEB, P1, P1, R2)	0.0000000
VAR(P1, FEB, P1, P2, R1)	0.0000000
VAR(P1, FEB, P1, P2, R2)	0.0000000
VAR(P1, FEB, P2, P1, R1)	0.0000000
VAR(P1, FEB, P2, P1, R2)	0.0000000
VAR(P1, FEB, P2, P2, R1)	0.0000000
VAR(P1, FEB, P2, P2, R2)	0.0000000
VAR(P1, MAR, P1, P1, R1)	0.0000000
VAR(P1, MAR, P1, P1, R2)	0.0000000
VAR(P1, MAR, P1, P2, R1)	0.0000000
VAR(P1, MAR, P1, P2, R2)	0.0000000
VAR(P1, MAR, P2, P1, R1)	0.0000000
VAR(P1, MAR, P2, P1, R2)	0.0000000
VAR(P1, MAR, P2, P2, R1)	0.0000000
VAR(P1, MAR, P2, P2, R2)	0.0000000
VAR(P2, FEB, P1, P1, R1)	2400.000
VAR(P2, FEB, P1, P1, R2)	0.0000000
VAR(P2, FEB, P1, P2, R1)	0.0000000
VAR(P2, FEB, P1, P2, R2)	0.0000000
VAR(P2, FEB, P2, P1, R1)	0.0000000
VAR(P2, FEB, P2, P1, R2)	3200.000
VAR(P2, FEB, P2, P2, R1)	0.0000000
VAR(P2, FEB, P2, P2, R2)	0.0000000
VAR(P2, MAR, P1, P1, R1)	2760.000
VAR(P2, MAR, P1, P1, R2)	0.0000000
VAR(P2, MAR, P1, P2, R1)	0.0000000
VAR(P2, MAR, P1, P2, R2)	0.0000000
VAR(P2, MAR, P2, P1, R1)	0.0000000
VAR(P2, MAR, P2, P1, R2)	3680.000
VAR(P2, MAR, P2, P2, R1)	0.0000000
VAR(P2, MAR, P2, P2, R2)	0.0000000
STORE(P1, R1)	0.0000000
STORE(P1, R2)	0.0000000
STORE(P2, R1)	0.0000000
STORE(P2, R2)	0.0000000

3.7-2

a)

```
MAX

    50x1+20x2+25x3 ;

SUBJECT TO

    9x1+3x2+5x3<=500 ;
    5x1+4x2<=350;
    3x1+2x3<=150;
    x3<=20;

END
```

Variable Name	Activity
x1	26.1905
x2	54.7619
x3	20.0000

b)

```
max = 50*x1+20*x2+25*x3;
9*x1+3*x2+5*x3<=500;
5*x1+4*x2<=350;
3*x1+2*x3<=150;
x3<=20;
x1>=0; x2>=0; x3>=0;
```

Global optimal solution found at step: 4
Objective value: 2904.762

Variable	Value
X1	26.19048
X2	54.76190
X3	20.00000

3.7-3

a)

```
TITLE
    TransportationProblem;

INDEX
    supply = (Wh1,Wh2);
    dest   = (C1,C2,C3);

DATA
    MaxCapacity[supply]  := (400,500);
    Required[dest]       := (300,200,400);

    ShippingCost[supply,dest] := (600,800,700,
                  400,900,600);


DECISION VARIABLES
    VolumeShipped[supply,dest]  ->  ""


MODEL

    MIN  TotalCost = SUM(supply,dest: ShippingCost * VolumeShipped);

SUBJECT TO

    Capacity[supply] : SUM(dest: VolumeShipped)    =  MaxCapacity ;
    Demand[dest]     : SUM(supply: VolumeShipped)  =  Required ;

END
```

5)

```
MODEL:
! Problem 3.7-3 - Lingo Version;
SETS:
      FACTORIES /F1 F2/: CAPACITY;
      CUSTOMERS /C1 C2 C3/: DEMAND;
      LINKS(FACTORIES, CUSTOMERS): COST, VOLUME;
ENDSETS
[OBJECTIVE] MIN = @SUM(LINKS(I,J):COST(I,J)*VOLUME(I,J));
!DEMAND CONSTRAINTS;
@FOR(CUSTOMERS(J): @SUM(FACTORIES(I): VOLUME(I,J))=DEMAND(J));
!SUPPLY CONSTRAINTS;
@FOR(FACTORIES(I): @SUM(CUSTOMERS(J):VOLUME(I,J))=CAPACITY(I));


!HERE IS THE DATA;
DATA:
CAPACITY = 400 500;
DEMAND = 300 200 400;
COST = 600 800 700
       400 200 400;
ENDDATA
END
```

```
Global optimal solution found at step:                 2
Objective value:                              410000.0

                    Variable           Value

                VOLUME( F1, C1)        300.0000
                VOLUME( F1, C2)       0.0000000
                VOLUME( F1, C3)        100.0000
                VOLUME( F2, C1)       0.0000000
                VOLUME( F2, C2)        200.0000
                VOLUME( F2, C3)        300.0000
```

3.1-4
a)

```
TITLE
    TransportationProblem;

INDEX
    Factories = (F1,F2,F3);
    Models    = (L,M,S);

DATA
    Capacity[Factories]   := (750,900,450);
    Space[Factories]      := (13000,12000,5000);
    Sales[Models]         := (900,1200,750);
    Size[Models]          := (20,15,12);
    Profit[Factories,Models] := (420,360,300,
                420,360,300,
                420,360,300);


DECISION VARIABLES

    Volume[Factories,Models]  ->  ""

MODEL

    MAX  TotalProfit = SUM(Factories,Models: Profit * Volume);

SUBJECT TO

    Cap[Factories] : SUM(Models: Volume)     <=  Capacity ;
    Sal[Models]    : SUM(Factories: Volume)  <=  Sales ;
    Spa[Factories] : SUM(Models: Size*Volume)  <= Space;
    Ratio1 : SUM(Models: Volume[F1]/Capacity[F1]-Volume[F2]/Capacity[F2])  = 0;
    Ratio2 : SUM(Models: Volume[F1]/Capacity[F1]-Volume[F3]/Capacity[F3])  = 0;

END
```

SOLUTION RESULT

Optimal solution found

MAX TotalPro = 696000.0000

DECISION VARIABLES

VARIABLE Volume[Factories,Models] :

Factories	Models	Activity
F1	L	516.6667
F1	M	177.7778
F1	S	0.0000
F2	L	0.0000
F2	M	666.6667
F2	S	166.6667
F3	L	0.0000
F3	M	0.0000
F3	S	416.6667

3.7-4

b)

```
MODEL:
! Problem 3.7-4 - Lingo Version;
SETS:
      FACTORIES /F1 F2 F3/: CAPACITY, SPACE, SALES;
      MODELS /L M S/: DEMAND, SIZE;
      LINKS(FACTORIES, MODELS): PROFIT, VOLUME;
ENDSETS

[OBJECTIVE] MAX = @SUM(LINKS(I,J):PROFIT(I,J)*VOLUME(I,J));

!CAPACITY CONSTRAINTS;
@FOR(FACTORIES(I): @SUM(MODELS(J):VOLUME(I,J))<=CAPACITY(I));

!SPACE CONSTRAINTS;
@FOR(FACTORIES(I): @SUM(MODELS(J):SIZE(J)*VOLUME(I,J))<=SPACE(I));

!SALES CONSTRAINTS;
@FOR(MODELS(J): @SUM(FACTORIES(I):VOLUME(I,J))<=SALES(J));

!PROPORTIONALITY CONSTRAINTS;
@SUM(MODELS(J):VOLUME(1,J)/CAPACITY(1)-VOLUME(2,J)/CAPACITY(2))=0;
@SUM(MODELS(J):VOLUME(1,J)/CAPACITY(1)-VOLUME(3,J)/CAPACITY(3))=0;


!HERE IS THE DATA;
DATA:
CAPACITY = 750 900 450;
SPACE = 13000 12000 5000;
SALES = 900 1200 750;
DEMAND = 900 1200 750;
SIZE = 20 15 12;
PROFIT = 420 360 300
         420 360 300
         420 360 300;
ENDDATA
END
```

```
        Global optimal solution found at step:            16
        Objective value:                          696000.0

             VOLUME( F1, L)        516.6667
             VOLUME( F1, M)        177.7778
             VOLUME( F1, S)       0.0000000
             VOLUME( F2, L)       0.0000000
             VOLUME( F2, M)        666.6667
             VOLUME( F2, S)        166.6667
             VOLUME( F3, L)       0.0000000
             VOLUME( F3, M)       0.0000000
             VOLUME( F3, S)        416.6667
```

4.7-5
a)

```
TITLE
    TransportationProblem;

INDEX
    student = (KC,OH,HB,SC,KS,NK);
    day = (M,TU,W,TH,F);

DATA

    Wage[student]       :=(10,10.1,9.9,9.8,10.8,11.3);
    Gender[student]         := (0,0,0,0,1,1);
    Available[student,day] := (6,0,6,0,6,
                    0,6,0,6,0
                    4,8,4,0,4
                    5,5,5,0,5
                    3,0,3,8,0
                    0,0,0,6,2);


DECISION VARIABLES

    Work[student,day]  ->  ""

MODEL

    MIN  TotalCost = SUM(student,day: Wage * Work);

SUBJECT TO

    TimeConstraint[student,day] : Work <= Available ;
    MinimumWork0[student] where(Gender=0) : SUM(day: Work) >=8 ;
    MinimumWork1[student] where(Gender=1) : SUM(day: Work) >=7 ;
    AlwaysOpen[day] : SUM(student: Work) = 14 ;
END
```

MIN TotalCos = 709.6000

VARIABLE Work[student,day] :

student	day	Activity
KC	M	4.0000
KC	TU	0.0000
KC	W	2.0000
KC	TH	0.0000
KC	F	3.0000
OH	M	0.0000
OH	TU	2.0000
OH	W	0.0000
OH	TH	6.0000
OH	F	0.0000
HB	M	4.0000
HB	TU	7.0000
HB	W	4.0000
HB	TH	0.0000
HB	F	4.0000
SC	M	5.0000
SC	TU	5.0000
SC	W	5.0000
SC	TH	0.0000
SC	F	5.0000
KS	M	1.0000
KS	TU	0.0000
KS	W	3.0000
KS	TH	3.0000
KS	F	0.0000
NK	M	0.0000
NK	TU	0.0000
NK	W	0.0000
NK	TH	5.0000
NK	F	2.0000

4.7-5
b)

```
MODEL:
! Problem 3.7-5 - Lingo Version;
SETS:
      STUDENTS /KC OH HB SC KS NK/: WAGE, GENDER;
      DAYS /M TU W TH F/;
      LINKS(STUDENTS, DAYS): AVAILABLE, WORK;
ENDSETS

[OBJECTIVE] MIN = @SUM(LINKS(I,J):WAGE(I)*WORK(I,J));

!TIME CONSTRAINTS;
@FOR(LINKS(I,J): WORK(I,J)<=AVAILABLE(I,J));

!MINIMUM WORK CONSTRAINTS;
@FOR(STUDENTS(I) GENDER(I) #EQ# 0: @SUM(LINKS(I,J):WORK(I,J))>=8);
@FOR(STUDENTS(I) GENDER(I) #EQ# 1: @SUM(LINKS(I,J):WORK(I,J))>=7);

!ALWAYS OPEN CONSTRAINTS;
@FOR(DAYS(J): @SUM(LINKS(I,J): WORK(I,J))=14);

!HERE IS THE DATA;
DATA:
WAGE = 10 10.1 9.9 9.8 10.8 11.3;
GENDER = 0 0 0 0 1 1;
AVAILABLE=6 0 6 0 6
          0 6 0 6 0
          4 8 4 0 4
          5 5 5 0 5
          3 0 3 8 0
          0 0 0 6 2;
ENDDATA
END
```

WORK(KC, M)	2.000000	WORK(SC, M)	5.000000
WORK(KC, TU)	0.0000000	WORK(SC, TU)	5.000000
WORK(KC, W)	3.000000	WORK(SC, W)	5.000000
WORK(KC, TH)	0.0000000	WORK(SC, TH)	0.0000000
WORK(KC, F)	4.000000	WORK(SC, F)	5.000000
WORK(OH, M)	0.0000000	WORK(KS, M)	3.000000
WORK(OH, TU)	2.000000	WORK(KS, TU)	0.0000000
WORK(OH, W)	0.0000000	WORK(KS, W)	2.000000
WORK(OH, TH)	6.000000	WORK(KS, TH)	2.000000
WORK(OH, F)	0.0000000	WORK(KS, F)	0.0000000
WORK(HB, M)	4.000000	WORK(NK, M)	0.0000000
WORK(HB, TU)	7.000000	WORK(NK, TU)	0.0000000
WORK(HB, W)	4.000000	WORK(NK, W)	0.0000000
WORK(HB, TH)	0.0000000	WORK(NK, TH)	6.000000
WORK(HB, F)	4.000000	WORK(NK, F)	1.000000

3.7-6 a)

```
MODEL

    MIN  84c+72t+60a;

SUBJECT TO

90c+20t+40a>=200;
30c+80t+60a>=180;
10c+20t+60a>=150;

END
```

SOLUTION RESULT

Optimal solution found

MIN Z = 241.7143

DECISION VARIABLES

PLAIN VARIABLES

Variable Name	Activity
c	1.1429
t	0.0000
a	2.4286

b)

```
! Problem 3.7-6 - Lingo Version;

[OBJECTIVE] MIN = 84*C+72*T+60*A;

!CONSTRAINTS;

90*C+20*T+40*A>=200;
30*C+80*T+60*A>=180;
10*C+20*T+60*A>=150;
```

Global optimal solution found at step: 8
Objective value: 241.7143

Variable	Value
C	1.142857
T	0.0000000
A	2.428571

3.7-7 a)

```
MODEL

    MIN  x1+x2+x3;

SUBJECT TO

2x1+x2+0.5x3>=400;
0.5x1+0.5x2+x3>=100;
1.5x2+2x3>=300;

END
```

SOLUTION RESULT

Optimal solution found

MIN Z = 300.0000

Variable Name	Activity
x1	100.0000
x2	200.0000
x3	0.0000

3.7-7 b)

```
! Problem 3.7-7 - Lingo Version;

[OBJECTIVE] MIN = X+Y+Z;

!CONSTRAINTS;

2*X+Y+0.5*Z>=400;
0.5*X+0.5*Y+Z>=100;
1.5*Y+2*Z>=300;
```

```
Global optimal solution found at step:                 8
Objective value:                                300.0000

                    Variable           Value
                           X        100.0000
                           Y        200.0000
                           Z        0.0000000
```

```
Global optimal solution found at step:                21
Objective value:                                709.6000
```

3.7-8

a) let

i : 1,...,10 denote mills

j : 1,...,1000 denote customers

k : 1,...,5 type of paper

l : 1,2,3 machines

m : 1,...,4 materials

Now define:

$x_{i,j,k,l}$: amount of paper k to be made at papermill i on machine type l shipped to customer j.

$P_{i,k,l}$: Production cost per unit made in mill i using paper k and machine l

$T_{i,j,k}$: Transportation cost (per unit) for paper k from mill i to customer j

$D_{j,k}$: Demand of paper k by customer j

$F_{k,l,m}$: number of units of material m needed to produce 1 unit of paper type k on machine l.

$\bar{C}_{i,m}$: units of material m available in mill i

$\hat{F}_{k,l}$: capacity units of machine type l that will produce 1 unit of paper type K.

$\hat{C}_{i,l}$: capacity units of machine type l available in paper mill i

(CONTD)

(CONT'D)

with these definitions our LP becomes:

$$\text{minimize} \sum_{i,k,l} P_{i,k,l} \left(\sum_j x_{i,j,k,l} \right) + \sum_{i,j,k} T_{i,j,k} \left(\sum_l x_{i,j,k,l} \right)$$

subject to:

Demand: $\sum_{i,l} x_{i,j,k,l} \geq D_{j,k}$ $\quad j: 1,\ldots,10000$, $k: 1,\ldots,5$

Raw Material:

$$\sum_{k,l} \bar{F}_{k,l,u} \left(\sum_j x_{i,j,k,l} \right) \leq \bar{C}_{i,u}$$

$i: 1,\ldots,10$, $u: 1,\ldots,4$

Capacity:

$$\sum_k \hat{F}_{k,l} \left(\sum_j x_{i,j,k,l} \right) \leq \hat{C}_{i,l}$$

$i: 1,\ldots,10$, $l: 1,2,3$

$$x_{i,j,k,l} \geq 0 \quad \forall i,j,k,l.$$

Notice that

$\sum_l x_{i,j,k,l}$: amount of paper k shipped to customer j from mill i.

and

$\sum_j x_{i,j,k,l}$: amount of paper k made with machine l in mill i

b) There are 5,070 functional constraints and 150,000 variables.

c)

```
|   Problem 3.7-8   |

TITLE
    PaperManufacturing;

INDEX
    mill = 1..10;
    customer = 1..1000;
    machine = 1..3;
    material = 1..4;
    paper = 1..5;

DATA
    Required[customer,paper] = DATAFILE(Required.dat);
    Rate1[paper,machine,material] = DATAFILE(Rate1.dat);
    RawMaterial[mill,material] = DATAFILE(RawMaterial.dat);
    Rate2[paper,machine] = DATAFILE(Rate2.dat);
    MaxCapacity[mill,machine] = DATAFILE(MaxCapacity.dat);
    ProdCost[mill,paper,machine] = DATAFILE(ProdCost);
    TranspCost[mill,customer,paper] = DATAFILE(TranspCost);

DECISION VARIABLES
    Quantity[mill,customer,machine,paper]  ->  ""

MODEL

    MIN  TotalCost = SUM(mill,customer,machine,paper: ProdCost * Quantity)
        + SUM(mill,customer,machine,paper: TranspCost * Quantity);

SUBJECT TO

    Demand[customer,paper] : SUM(mill,machine: Quantity) >= Required ;
    Supply[mill,material] : SUM(customer,paper,machine: Rate1 * Quantity) <= RawMaterial;
    Capacity[mill,machine] : SUM(customer,paper: Rate2 * Quantity) < MaxCapacity ;
END
```

d)

```
MODEL:
! Problem 3.7-8 - Lingo Version;
SETS:
MILLS /1..10/;
CUSTOMERS /1..1000/;
MACHINES /1..3/;
MATERIALS /1..4/;
PAPER /1..5/;
LINK1(CUSTOMERS,PAPER): DEMAND;
LINK2(PAPER, MACHINES, MATERIALS): RATE1;
LINK3(MILLS, MATERIALS): CAPACITY1;
LINK4(PAPER, MACHINES): RATE2;
LINK5(MILLS, MACHINES): CAPACITY2;
LINK6(MILLS, PAPER, MACHINES): PROD_COST;
LINK7(MILLS, CUSTOMERS, PAPER): TRANSP_COST;
LINK8(MILLS, CUSTOMERS, PAPER, MACHINES): QUANTITY;
ENDSETS

!OBJECTIVE IS TO MINIMIZE PRODUCTION COST + TRANSPORTATION COST;
MIN = @SUM(LINK6(I,K,L):PROD_COST(I,K,L) * @SUM(CUSTOMERS(J): QUANTITY(I,J,K,L))) +
      @SUM(LINK7(I,J,K):TRANSP_COST * @SUM(MACHINES(L): QUANTITY(I,J,K,L)));

!DEMAND CONSTRAINTS;
@FOR(LINK1(J,K): @SUM(MILLS(I): @SUM(MACHINES(L): QUANTITY(I,J,K,L))) >= DEMAND(J,K));

!RAW MATERIALS SUPPLY CONSTRAINTS;
@FOR(LINK3(I,M): @SUM(PAPER(K): @SUM(MACHINES(L): RATE1(K,L,M)*@SUM(CUSTOMERS(J):
QUANTITY(I,J,K,L)))) <= CAPACITY1(I,M));

!CAPACITY SUPPLY CONSTRAINTS;
@FOR(LINK5(I,L): @SUM(PAPER(K): RATE2(K,L) * @SUM(CUSTOMERS(J): QUANTITY(I,J,K,L))) <=
CAPACITY2(I,L));

!READ DATA FROM AN EXCEL FILE;
DATA:
DEMAND, RATE1, CAPACITY1, RATE2, CAPACITY2, PROD_COST, TRANSP_COST =
@WKX('C:\LINGO\DATA.WK4','DEMAND','RATE1','CAPACITY1','RATE2','CAPACITY2','PROD_COST','TRA
NSP_COST');
ENDDATA
END
```

Cases

3-1 a) In this case, we have two decision variables: one variable to determine the number of Family Thrillseekers we should assemble and one variable to determine the number of Classy Cruisers we should assemble. We also have the following three constraints:

1. The plant has a maximum of 48,000 labor hours. Each Thrillseeker requires six labor hours, and each Cruiser requires 10.5 labor hours. The sum of the total number of labor hours required to assemble all Thrillseekers and all Cruisers must be less than or equal to 48,000 hours.

2. The plant has a maximum of 20,000 doors available. Each Thrillseeker requires four doors, and each Cruiser requires two doors. The sum of the total number of doors required to assemble all Thrillseekers and all Cruisers must be less than or equal to 20,000 doors.

3. Because the demand for Cruisers is limited to 3,500 cars, the decision variable for the number of Cruisers we should assemble must be less than or equal to 3,500.

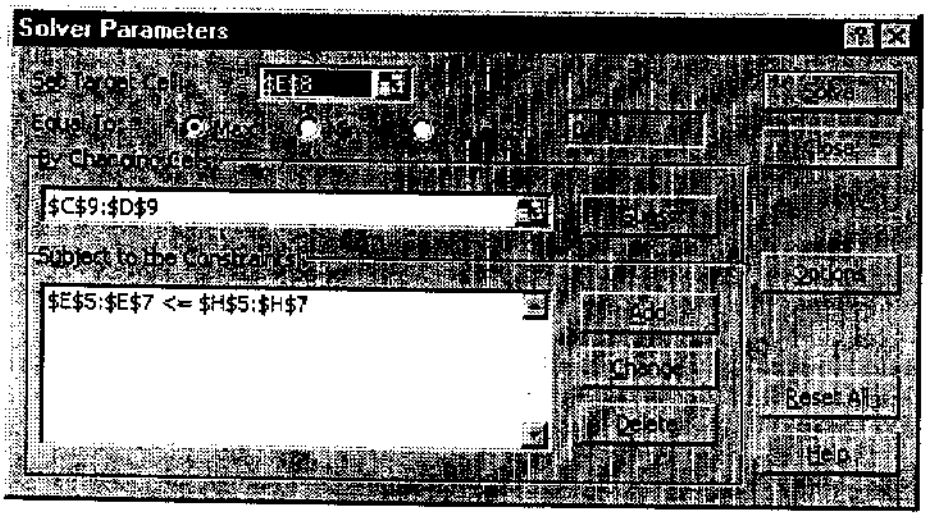
The formulas used in the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	=SUMPRODUCT(C5:D5,C9:D9)	<	=	48000
6		Doors	4	2	=SUMPRODUCT(C6:D6,C9:D9)	<	=	20000
7		Cruiser Demand	0	1	=SUMPRODUCT(C7:D7,C9:D9)	<	=	3500
8		Profit ($thousands)	3.6	5.4	**=SUMPRODUCT(C8:D8,C9:D9)**			
9		Solution	**3800**	**2400**				

The values used in the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	48000	<	=	48000
6		Doors	4	2	20000	<	=	20000
7		Cruiser Demand	0	1	2400	<	=	3500
8		Profit ($thousands)	3.6	5.4	**26640**			
9		Solution	**3800**	**2400**				

We specify the following Solver settings.



Finally, throughout this case we use the following solver options.

Rachel's plant should assemble 3,800 Thrillseekers and 2,400 Cruisers to obtain a maximum profit of $26,640,000.

b) In part (a) above, we observed that the Cruiser demand constraint was not binding. Therefore, raising the demand for the Cruiser will not change the optimal solution. The marketing campaign should not be undertaken.

c) The new value of the right-hand side of the labor constraint becomes 48,000 * 1.25 = 60,000 labor hours. All formulas and Solver settings used in part (a) remain the same. The values for the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	56250	<	=	60000
6		Doors	4	2	20000	<	=	20000
7		Cruiser Demand	0	1	3500	<	=	3500
8		Profit ($thousands)	3.6	5.4	**30600**			
9		Solution	**3250**	**3500**				

Rachel's plant should now assemble 3,250 Thrillseekers and 3,500 Cruisers to achieve a maximum profit of $30,600,000.

d) Using over time labor increases the profit by $30,600,000 – $26,640,000 = $3,960,000. Rachel should therefore be willing to pay at most $3,960,000 extra for overtime labor beyond regular time rates.

e) The value of the right-hand side of the Cruiser demand constraint is 3,500 * 1.20 = 4,200 cars. The value of the right-hand side of the labor hour constraint is 48,000 * 1.25 = 60,000 hours. All formulas and Solver settings used in part (a) remain the same. Ignoring the costs of the advertising campaign and overtime labor, the values for the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	60000	<	=	60000
6		Doors	4	2	20000	<	=	20000
7		Cruiser Demand	0	1	4000	<	=	4200
8		Profit ($thousands)	3.6	5.4	**32400**			
9		Solution	**3000**	**4000**				

Rachel's plant should produce 3,000 Thrillseekers and 4,000 Cruisers for a maximum profit of $32,400,000. This profit excludes the costs of advertising and using overtime labor.

f) The advertising campaign costs $500,000. In the solution to part (e) above, we used the maximum overtime labor available, and the maximum use of overtime labor costs $1,600,000. Thus, our solution in part (e) required an extra $500,000 + $1,600,000 = $2,100,000. We perform the following cost/benefit analysis:

Profit in part (e):	$32,400,000
– Advertising and overtime costs:	$ 2,100,000
	$30,300,000

We compare the $30,300,000 profit with the $26,640,000 profit obtained in part (a) and conclude that the decision to run the advertising campaign and use overtime labor is a very wise, profitable decision.

g) Because we consider this question independently, the values of the right-hand sides for the Cruiser demand constraint and the labor hour constraint are the same as those in part (a). We now change the profit for the Thrillseeker from 3.6 to 2.8 in the problem formulation. All formulas and Solver settings used in part (a) remain the same. The values for the problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	48000	<	=	48000
6		Doors	4	2	14500	<	=	20000
7		Cruiser Demand	0	1	3500	<	=	3500
8		Profit ($thousands)	2.8	5.4	**24150**			
9		Solution	**1875**	**3500**				

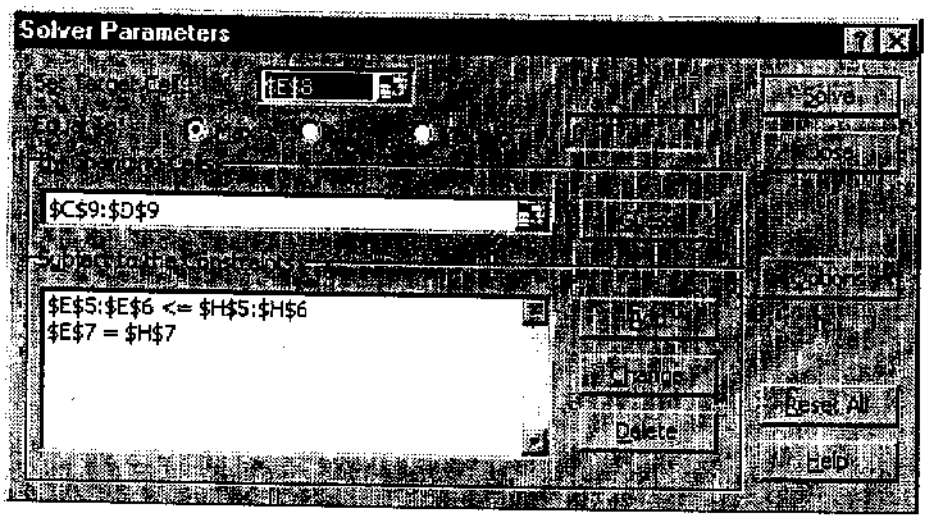
Rachel's plant should assemble 1,875 Thrillseekers and 3,500 Cruisers to obtain a maximum profit of $24,150,000.

h) Because we consider this question independently, the profit for the Thrillseeker remains the same as the profit specified in part (a). The labor hour constraint changes. Each Thrillseeker now requires 7.5 hours for assembly. All formulas and Solver settings used in part (a) remain the same. The values for the new problem formulation follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	7.5	10.5	48000	<	=	48000
6		Doors	4	2	13000	<	=	20000
7		Cruiser Demand	0	1	3500	<	=	3500
8		Profit ($thousands)	3.6	5.4	**24300**			
9		Solution	**1500**	**3500**				

Rachel's plant should assemble 1,500 Thrillseekers and 3,500 Cruisers for a maximum profit of $24,300,000.

i) Because we consider this question independently, we use the problem formulation used in part (a). In this problem, however, the number of Cruisers assembled has to be strictly equal to the total demand. We use the following new Solver settings:



The formulas used in the problem formulation remain the same as those used in part (a). The values used in the problem follow.

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	6	10.5	48000	<	=	48000
6		Doors	4	2	14500	<	=	20000
7		Cruiser Demand	0	1	3500		=	3500
8		Profit ($thousands)	3.6	5.4	**25650**			
9		Solution	**1875**	**3500**				

The new profit is $25,650,000, which is $26,640,000 – $25,650,000 = $990,000 less than the profit obtained in part (a). This decrease in profit is less than $2,000,000, so Rachel should meet the full demand for the Cruiser.

j) We now combine the new considerations described in parts (f), (g), and (h). In part (f), we decided to use both the advertising campaign and the overtime labor. The advertising campaign raises the demand for the Cruiser to 4,200 sedans, and the overtime labor increases the labor hour capacity of the plant to 60,000 labor hours. In part (g), we decreased the profit generated by a Thrillseeker to $2,800. In part (h), we increased the time to assemble a Thrillseeker to 7.5 hours. Including the increased demand for Cruisers, the increased plant capacity, the decreased unit profit for a Thrillseeker, and the increased time to assemble a Thrillseeker, the new problem is formulated as follows:

	A	B	C	D	E	F	G	H
3			Thrillseeker	Cruiser	Totals			Right-Hand
4		Constraint						Side
5		Labor Hours	7.5	10.5	60000	<	=	60000
6		Doors	4	2	16880	<	=	20000
7		Cruiser Demand	0	1	4200	<	=	4200
8		Profit ($thousands)	2.8	5.4	**28616**			
9		Solution	**2120**	**4200**				

The formulas and Solver settings used for this problem are the same as those used in part (a). Rachel's plant should assemble 2,120 Thrillseekers and 4,200 Cruisers for a maximum profit of $28,616,000 – $2,100,000 = $26,516,000.

3-2 a) We want to determine the amount of potatoes and green beans Maria should purchase to minimize ingredient costs. We have two decision variables: one variable to represent the amount (in pounds) of potatoes Maria should purchase and one variable to represent the amount (in pounds) of green beans Maria should purchase. We also have constraints on nutrition, taste, and weight.

Nutrition Constraints

1. We first need to ensure that the dish has 180 grams of protein. We are told that 100 grams of potatoes have 1.5 grams of protein and 10 ounces of green beans have 5.67 grams of protein. Since we have decided to measure our decision variables in pounds, however, we need to determine the grams of protein in one pound of each ingredient.

We perform the following conversion for potatoes:

$$100\text{g of potatoes}\left(\frac{1\text{ lb}}{453.6\text{g}}\right) = 0.220459\text{ lb of potatoes}$$

$$\frac{1.5\text{g of protein}}{0.22046\text{ lb of potatoes}} = \frac{6.804\text{g of protein}}{1\text{ lb of potatoes}}$$

We perform the following conversion for green beans:

$$10\text{ oz of green beans}\left(\frac{28.35\text{ g}}{1\text{ oz}}\right) = 283.5\text{ g of green beans}$$

$$283.5\text{ g of green beans}\left(\frac{1\text{ lb}}{453.6\text{ g}}\right) = 0.625\text{ lb of green beans}$$

$$\frac{5.67\text{ g of protein}}{0.625\text{ lb of green beans}} = \frac{9.072\text{ g of protein}}{1\text{ lb of green beans}}$$

The total grams of protein in the potatoes and green beans Maria purchases for the casserole must be greater than or equal to 180 grams.

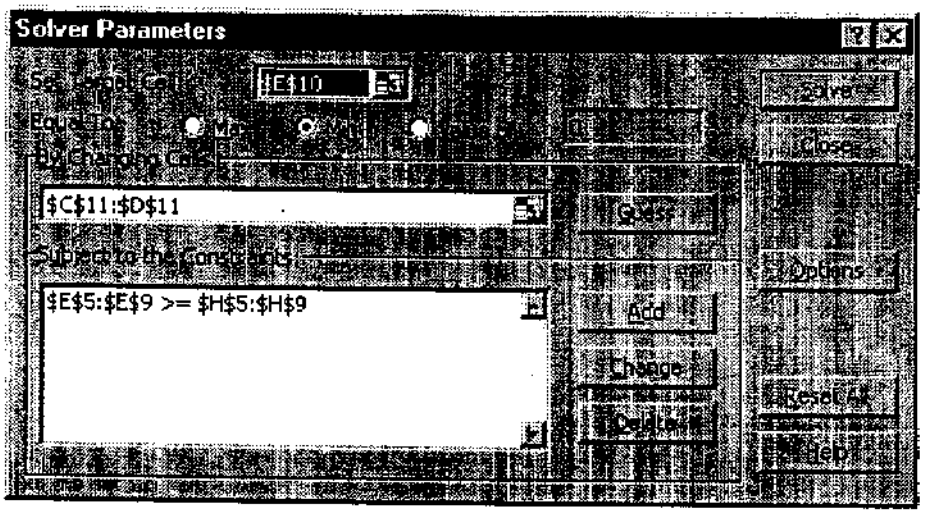
[redacted]

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	=SUMPRODUCT(C5:D5,C11:D11)	>	=	180
6		Iron (mg)	1.3608	5.4432	=SUMPRODUCT(C6:D6,C11:D11)	>	=	80
7		Vitamin C (mg)	54.432	45.36	=SUMPRODUCT(C7:D7,C11:D11)	>	=	1050
8		Taste	5	-6	=SUMPRODUCT(C8:D8,C11:D11)	>	=	0
9		Amount (lb)	1	1	=SUMPRODUCT(C9:D9,C11:D11)	>	=	22.046
10		Cost (per lb)	0.4	1	**=SUMPRODUCT(C10:D10,C11:D11)**			
11		Solution (lb)	**13.5667**	**11.3056**				

The values for the problem and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	194.8717949	>	=	180
6		Iron (mg)	1.3608	5.4432	80	>	=	80
7		Vitamin C (mg	54.432	45.36	1251.282051	>	=	1050
8		Taste	5	-6	0	>	=	0
9		Amount (lb)	1	1	24.87224709	>	=	22.046
10		Cost (per lb)	0.4	1	**16.73223895**			
11		Solution (lb)	**13.567**	**11.306**				

The Solver settings used to solve the problem follow.



Finally, throughout this case we use the following Solver options.

Maria should purchase 13.567 lb of potatoes and 11.306 lb of green beans to obtain a minimum cost of $16.73.

b) The taste constrint changes. The new constraint is now.

$$\frac{\text{pounds of potatoes}}{\text{pounds of green beans}} > \frac{1}{2}$$

2 (pounds of potatoes) > 1 (pounds of green beans)

The formulas and Solver settings used to solve the problem remain the same as in part (a). The values for the problem and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	180	>	=	180
6		Iron (mg)	1.3608	5.4432	80	>	=	80
7		Vitamin C (mg	54.432	45.36	1110	>	=	1050
8		Taste	2	-1	8.450911229	>	=	0
9		Amount (lb)	1	1	22.4132863	>	=	22.046
10		Cost (per lb)	0.4	1	**16.2404468**			
11		Solution (lb)	**10.288**	**12.125**				

Maria should purchase 10.288 lb of potatoes and 12.125 lb of green beans to obtain a minimum cost of $16.24.

c) The right-hand side of the iron constraint changes from 80 mg to 65 mg. The formulas and Solver settings used in the problem remain the same as in part (a). The values for the new problem formulation and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	180	>	=	180
6		Iron (mg)	1.3608	5.4432	65	>	=	65
7		Vitamin C (mg	54.432	45.36	1222.5	>	=	1050
8		Taste	5	-6	31.04791299	>	=	0
9		Amount (lb)	1	1	23.79115226	>	=	22.046
10		Cost (per lb)	0.4	1	**14.31143445**			
11		Solution (lb)	**15.800**	**7.992**				

Maria should purchase 15.8 lb of potatoes and 7.992 lb of green beans to obtain a minimum cost of $14.31.

d) The iron requirement remains 65 mg. We need to change the price per pound of green beans from \$1.00 per pound to \$0.50 per pound. The formulas and Solver settings used in the problem remain the same as in part (a). The values for the new problem formulation and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Green Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	9.072	180	>	=	180
6		Iron (mg)	1.3608	5.4432	73.89473684	>	=	65
7		Vitamin C (mg	54.432	45.36	1155.789474	>	=	1050
8		Taste	5	-6	0	>	=	0
9		Amount (lb)	1	1	22.97410192	>	=	22.046
10		Cost (per lb)	0.4	0.5	**10.2339181**	3		
11		Solution (lb)	**12.531**	**10.443**				

Maria should purchase 12.531 lb of potatoes and 10.443 lb of green beans to obtain a minimum cost of \$10.23.

e) We still have two decision variables: one variable to represent the amount (in pounds) of potatoes Maria should purchase and one variable to represent the amount (in pounds) of lima beans Maria should purchase. To determine the grams of protein in one pound of lima beans, we perform the following conversion:

$$\frac{22.68 \text{ g of protein}}{0.625 \text{ lb of lima beans}} = \frac{36.288 \text{ g of protein}}{1 \text{ lb of lima beans}}$$

To determine the milligrams of iron in one pound of lima beans, we perform the following conversion:

$$\frac{6.804 \text{ mg of iron}}{0.625 \text{ lb of lima beans}} = \frac{10.8864 \text{ mg of iron}}{1 \text{ lb of lima beans}}$$

Lima beans contain no vitamin C, so we do not have to perform a measurement conversion for vitamin C.

We change the decision variable from green beans to lima beans and insert the new parameters for protein, iron, vitamin C, and cost. The formulas and Solver settings used in the problem remain the same as in part (a). The values for the new problem formulation and solution follows.

	A	B	C	D	E	F	G	H
3			Potatoes	Lima Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	36.288	260.4166667	>=		180
6		Iron (mg)	1.3608	10.8864	65	>=		65
7		Vitamin C (mg)	54.432	0	1050	>=		1050
8		Taste	5	-6	75.094	>=		0
9		Amount (lb)	1	1	22.84961052	>=		22.046
10		Cost (per lb)	0.4	0.6	**9.85174162**	3		
11		Solution (lb)	**19.290**	**3.559**				

Maria should purchase 19.29 lb of potatoes and 3.559 lb of lima beans to obtain a minimum cost of $9.85.

f) Edson takes pride in the taste of his casserole, and the optimal solution from above does not seem to preserve the taste of the casserole. First, Maria forces Edson to use lima beans instead of green beans, and lima beans are not an ingredient in Edson's original recipe. Second, although Edson places no upper limit on the ratio of potatoes to beans, the above recipe uses an over five to one ratio of potatoes to beans. This ratio seems unreasonable since such a large amount of potatoes will overpower the taste of beans in the recipe.

g) We only need to change the values on the right-hand side of the iron and vitamin C constraints. The formulas and Solver settings used in the problem remain the same as in part (a). The values used in the new problem formulation and solution follow.

	A	B	C	D	E	F	G	H
3			Potatoes	Lima Beans	Totals			Right-Hand
4		Constraint						Side
5		Protein (g)	6.804	36.288	428.571803	≥	=	180
6		Iron (mg)	1.3608	10.8864	120	>	=	120
7		Vitamin C (mg)	54.432	0	685.723282	≥	=	500
8		Taste	5	-6	6.300	>	=	0
9		Amount (lb)	1	1	22.046	>	=	22.046
10		Cost (per lb)	0.4	0.6	**10.7080406**	1		
11		Solution (lb)	**12.598**	**9.448**				

Maria should purchase 12.598 lb of potatoes and 9.448 lb of lima beans to obtain a minimum cost of $10.71.

3-3 a) The number of operators that the hospital needs to staff the call center during each two-hour shift can be found in the following table:

	A	B	C	D	E	F
1	work	average number of	number of calls from	number of calls from	number of operators	number of operators
2	shift	calls per hour	English speakers	Spanish speakers	speaking English	speaking Spanish
3	7am to 9am	40	32	8	6	2
4	9am to 11am	85	68	17	12	3
5	11am to 1pm	70	56	14	10	3
6	1pm to 3pm	95	76	19	13	4
7	3pm to 5pm	80	64	16	11	3
8	5pm to 7pm	35	28	7	5	2
9	7pm to 9pm	10	8	2	2	1

For example, the average number of phone calls per hour during the shift from 7am to 9am equals 40. Since, on average, 80% of all phone calls are from English speakers, there is an average number of 32 phone calls per hour from English speakers during that shift. Since one operator takes, on average, 6 phone calls per hour, the hospital needs 32/6 = 5.333 English-speaking operators during that shift. The hospital cannot employ fractions of an operator and so needs 6 English-speaking operators for the shift from 7am to 9am.

b) The problems of determining how many Spanish-speaking operators and English-speaking operators Lenny needs to hire to begin each shift are independent. Therefore we can formulate two smaller linear programming models instead of one large model. We are going to have one model for the scheduling of the Spanish-speaking operators and another one for the scheduling of the English-speaking operators.

Lenny wants to minimize the operating costs while answering all phone calls. For the given scheduling problem we make the assumption that the only operating costs are the wages of the employees for the hours that they answer phone calls. The wages for the hours during which they perform paperwork are paid by other cost centers. Moreover, it does not matter for the callers whether an operator starts his or her work day with phone calls or with paperwork. For example, we do not need to distinguish between operators who start their day answering phone calls at 9am and operators who start their day with paperwork at 7am, because both groups of operators will be answering phone calls at the same time. And only this time matters for the analysis of Lenny's problem.

We define the decision variables according to the time when the employees have their first shift of answering phone calls. For the scheduling problem of the English-speaking operators we have 7 decision variables. First, we have 5 decision variables for full-time employees.

The number of operators having their first shift on the phone from 7am to 9am.
The number of operators having their first shift on the phone from 9am to 11am.
The number of operators having their first shift on the phone from 11am to 1pm.
The number of operators having their first shift on the phone from 1pm to 3pm.
The number of operators having their first shift on the phone from 3pm to 5pm.

In addition, we define 2 decision variables for part-time employees.

The number of part-time operators having their first shift from 3pm to 5pm.
The number of part-time operators having their first shift from 5pm to 7pm.

The unit cost coefficients in the objective function are the wages operators earn while they answer phone calls. All operators who have their first shift on the phone from 7am to 9am, 9am to 11am, or 11am to 1pm finish their work on the phone before 5pm. They earn 4*S10 = $40 during their time answering phone calls. All operators who have their first shift on the phone from 1pm to 3pm or 3pm to 5pm have one shift on the phone before 5pm and another one after 5pm. They earn 2*$10+2*$12 = $44 during their time answering phone calls. The second group of part-time operators, those having their first shift from 5pm to 7pm, earn 4*$12 = $48 during their time answering phone calls.

There are 7 constraints, one for each two-hour shift during which phone calls need to be answered. The right-hand sides for these constraints are the number of operators needed to ensure that all phone calls get answered in a timely manner. On the left-hand side we determine the number of operators on the phone during any given shift. For example, during the 11am to 1pm shift the total number of operators answering phone calls equals the sum of the number of operators who started answering calls at 7am and are currently in their second shift of the day and the number of operators who started answering calls at 11am.

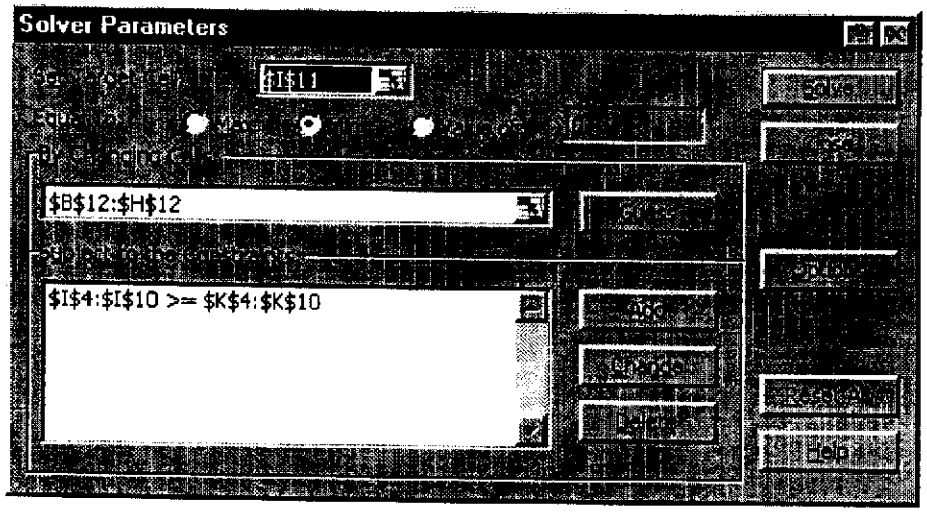
The following spreadsheet describes the entire problem formulation for the English-speaking employees:

	A	B	C	D	E	F	G	H	I	J	K
1											
2	Shifts of	Number of operators whose first shift of answering phone calls in English is from									Required number
3	phone operators	7am to 8am	8am to 11am	11am to 1 pm	1pm to 3pm	3pm to 5pm	5pm to 5pm (P)	5pm to 9pm (P)	Totals		of operators
4	7am to 8am	1	0	0	0	0	0	0	6	>=	6
5	8am to 11am	0	1	0	0	0	0	0	13	>=	12
6	11am to 1 pm	1	0	1	0	0	0	0	10	>=	10
7	1pm to 3pm	0	1	0	1	0	0	0	13	>=	13
8	3pm to 5pm	0	0	1	0	1	1	0	11	>=	11
9	5pm to 7pm	0	0	0	1	0	1	1	5	>=	5
10	7pm to 9pm	0	0	0	0	1	0	1	2	>=	2
11	Unit cost	40	40	40	44	44	44	48	1228	=	Total cost
12	Solution	6	13	4	0	2	5	0			

The following formulas are used in the problem formulation:

	I
1	
2	
3	Totals
4	=SUMPRODUCT(B4:H4,B12:H12)
5	=SUMPRODUCT(B5:H5,B12:H12)
6	=SUMPRODUCT(B6:H6,B12:H12)
7	=SUMPRODUCT(B7:H7,B12:H12)
8	=SUMPRODUCT(B8:H8,B12:H12)
9	=SUMPRODUCT(B9:H9,B12:H12)
10	=SUMPRODUCT(B10:H10,B12:H12)
11	**=SUMPRODUCT(B11:H11,B12:H12)**
12	

The solver appears as follows:

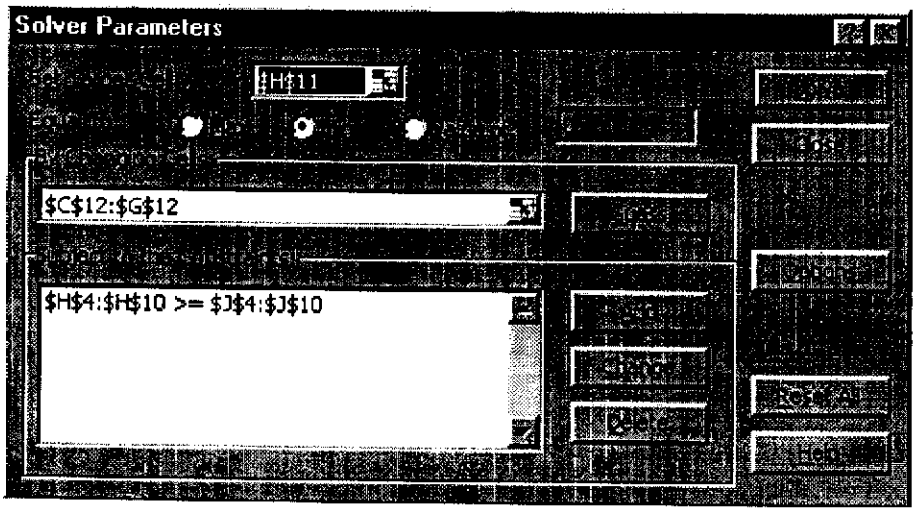


Throughout this analysis we use the following solver options:

The linear programming model for the Spanish-speaking employees can be developed in a similar fashion.

	A	B	C	D	E	F	G	H	I	J
1										
2		Shifts of	Number of operators whose first shift of answering phone calls in Spanish is from							Required number
3		phone operators	7am to 9am	9am to 11am	11am to 1 pm	1pm to 3pm	3pm to 5pm	Totals		of operators
4		7am to 9am	1	0	0	0	0	2	>=	2
5		9am to 11am	0	1	0	0	0	3	>=	3
6		11am to 1 pm	1	0	1	0	0	4	>=	3
7		1pm to 3pm	0	1	0	1	0	5	>=	4
8		3pm to 5pm	0	0	1	0	1	3	>=	3
9		5pm to 7pm	0	0	0	1	0	2	>=	2
10		7pm to 9pm	0	0	0	0	1	1	>=	1
11		Unit cost	40	40	40	44	44	412	=	Total cost
12		Solution	2	3	2	2	1			

	H
1	
2	
3	Totals
4	=SUMPRODUCT(C4:G4,C12:G12)
5	=SUMPRODUCT(C5:G5,C12:G12)
6	=SUMPRODUCT(C6:G6,C12:G12)
7	=SUMPRODUCT(C7:G7,C12:G12)
8	=SUMPRODUCT(C8:G8,C12:G12)
9	=SUMPRODUCT(C9:G9,C12:G12)
10	=SUMPRODUCT(C10:G10,C12:G12)
11	**=SUMPRODUCT(C11:G11,C12:G12)**
12	

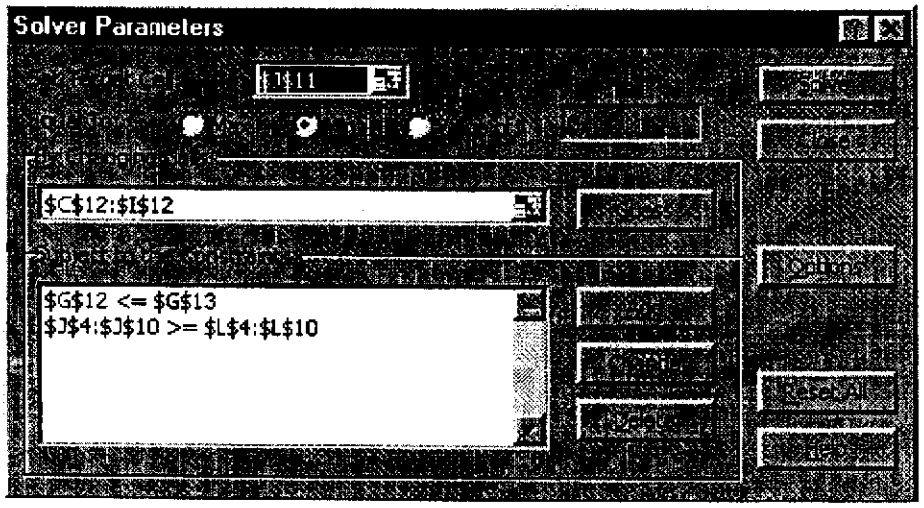


c) Lenny should hire 25 full-time English-speaking operators. Of these operators, 6 have their first phone shift from 7am to 9am, 13 from 9am to 11am, 4 from 11am to 1pm, and 2 from 3pm to 5pm. Lenny should also hire 5 part-time operators who start their work at 3pm. In addition, Lenny should hire 10 Spanish-speaking operators. Of these operators, 2 have their first shift on the phone from 7am to 9am, 3 from 9am to 11am, 2 from 11am to 1pm and 1pm to 3pm, and 1 from 3pm to 5pm. The total (wage) cost of running the calling center equals $1640 per day.

d) The restriction that Lenny can find only one English-speaking operator who wants to start work at 1pm affects only the linear programming model for English-speaking operators. This restriction does not put a bound on the number of operators who start their first phone shift at 1pm because those operators can start work at 11am with paperwork. However, this restriction does put an upper bound on the number of operators having their first phone shift from 3pm to 5pm. The new worksheet appears as follows.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		Shifts of	Number of operators whose first shift of answering phone calls in English is from									Required number
3		phone operators	7am to 9am	9am to 11am	11am to 1 pm	1pm to 3pm	3pm to 5pm	3pm to 5pm (P)	5 pm to 7pm (P)	Totals		of operators
4		7am to 9am	1	0	0	0	0	0	0	6	>=	6
5		9am to 11am	0	1	0	0	0	0	0	13	>=	12
6		11am to 1 pm	1	0	1	0	0	0	0	12	>=	10
7		1pm to 3pm	0	1	0	1	0	0	0	13	>=	13
8		3pm to 5pm	0	0	1	0	1	1	0	11	>=	11
9		5pm to 7pm	0	0	0	1	0	1	1	5	>=	5
10		7pm to 9pm	0	0	0	0	1	0	1	2	>=	2
11		Unit cost	40	40	40	44	44	44	48	1268	=	Total cost
12		Solution	6	13	6	0	1	4	1			
13		Upper bounds					1					

The Solver dialogue box displays the additional constraint.



Lenny should hire 26 full-time English-speaking operators. Of these operators, 6 have their first phone shift from 7am to 9am, 13 from 9am to 11am, 6 from 11am to 1pm, and 1 from 3pm to 5pm. Lenny should also hire 4 part-time operators who start their work at 3pm and 1 part-time operator starting work at 5pm. The hiring of Spanish-speaking operators is unaffected. The new total (wage) costs equal $1680 per day.

e) For each hour, we need to divide the average number of calls per hour by the average processing speed, which is 6 calls per hour. The number of bilingual operators that the hospital needs to staff the call center during each two-hour shift can be found in the following table:

	A	B	C
1	work	average number of	number of operators
2	shift	calls per hour	speaking English
3	7am to 9am	40	7
4	9am to 11am	85	15
5	11am to 1pm	70	12
6	1pm to 3pm	95	16
7	3pm to 5pm	80	14
8	5pm to 7pm	35	6
9	7pm to 9pm	10	2

f) The linear programming model for Lenny's scheduling problem can be found in the same way as before, only that now all operators are bilingual.

	A	B	C	D	E	F	G	H	I	J	K
1											
2	Shifts of	Number of operators whose first shift of answering phone calls in both languages is from									Required number
3	phone operators	7am to 9am	9am to 11am	11am to 1pm	1pm to 3pm	3pm to 5pm	3pm to 5pm (P)	5pm to 7pm (P)	Totals		of operators
4	7am to 9am	1	0	0	0	0	0	0	7	>=	7
5	9am to 11am	0	1	0	0	0	0	0	16	>=	15
6	11am to 1 pm	1	0	1	0	0	0	0	13	>=	12
7	1pm to 3pm	0	1	0	1	0	0	0	16	>=	16
8	3pm to 5pm	0	0	1	0	1	1	0	14	>=	14
9	5pm to 7pm	0	0	0	1	0	1	1	6	>=	6
10	7pm to 9pm	0	0	0	0	1	0	1	2	>=	2
11	Unit cost	40	40	40	44	44	44	48	**1512**	=	Total cost
12	Solution	**7**	**16**	**6**	**0**	**2**	**6**	**0**			

(The formulas and the solver dialogue box are identical to those in part (b).)

Lenny should hire 31 full-time bilingual operators. Of these operators, 7 have their first phone shift from 7am to 9am, 16 from 9am to 11am, 6 from 11am to 1pm, and 2 from 3pm to 5pm. Lenny should also hire 6 part-time operators who start their work at 3pm. The total (wage) cost of running the calling center equals $1512 per day.

g) The total cost of part (f) is $1512 per day; the total cost of part (b) is $1640. Lenny could pay an additional $1640-$1512 = $128 in total wages to the bilingual operators without increasing the total operating cost beyond those for the scenario with only monolingual operators. The increase of $128 represents a percentage increase of 128/1512 = 8.466%.

h) Creative Chaos Consultants has made the assumption that the number of phone calls is independent of the day of the week. But maybe the number of phone calls is very different on a Monday than it is on a Friday. So instead of using the same number of average phone calls for every day of the week, it might be more appropriate to determine whether the day of the week affects the demand for phone operators. As a result Lenny might need to hire more part-time employees for some days with an increased calling volume.

Similarly, Lenny might want to take a closer look at the length of the shifts he has scheduled. Using shorter shift periods would allow him to "fine tune" his calling centers and make it more responsive to demand fluctuations.

Lenny should investigate why operators are able to answer only 6 phone calls per hour. Maybe additional training of the operators could enable them to answer phone calls quicker and so increase the number of phone calls they are able to answer in an hour.

Finally, Lenny should investigate whether it is possible to have employees switching back and forth between paperwork and answering phone calls. During slow times phone operators could do some paperwork while they are sitting next to a phone, while in times of sudden large call volumes employees who are scheduled to do paperwork could quickly switch to answering phone calls.

Lenny might also want to think about the installation of an automated answering system that gives callers a menu of selections. Depending upon the caller's selection, the call is routed to an operator who specializes in answering questions about that selection.

4.7-8 c) (CONT'D)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$3	Variables	11	0	5	1E+30	0.363636
$D$3	Variables	0	-0.33333333	4	0.33333333	1E+30
$E$3	Variables	3	0	-1	2.66666667	1.333333
$F$3	Variables	0	-0.66666667	3	0.66666667	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$5	Constraints	24	0.66666667	24	12	132
$H$6		36	1	36	1E+30	12

4.7-9. a)

Resource	Resource Usage Per Unit of Each Activity: Activity 1	Activity 2	Totals		Resource Available
A	5	4	16.25	≤	20
B	6	9	30	≤	30
C	2	5	15	≤	15
Unit Profit	20	30	**$ 100.00**		
Solution	**1.25**	**2.5**			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Solution Activity 1	1.25	0.000	20	0	8
$C$7	Solution Activity 2	2.5	0.000	30	20	0

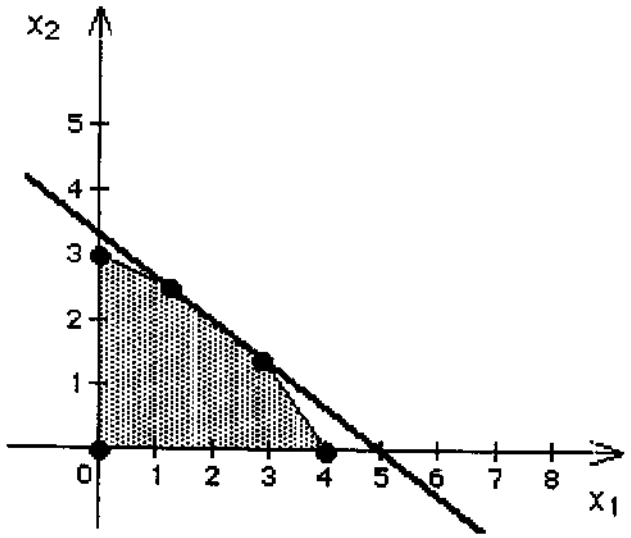
Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$3	A Totals	16.25	0.000	20	1E+30	3.75
$D$4	B Totals	30	3.333	30	2.647058823	3
$D$5	C Totals	15	0	15	1.666666667	2.142857143

b) The sensitivity report indicates that the problem has other optimal solutions

[redacted]

d) Optimal Solution: $(x_1, x_2) = (2.857, 1.429)$, (1.25, 2.5) and all points on the connecting line. Profit = \$100 million.



4.1-10 a)

Resource	Resource Usage Per Unit of Each Activity: Activity 1	Activity 2	Totals		Resource Available
A	15	5	300	≤	300
B	10	6	240	≤	240
C	8	12	300	≤	450
Unit Profit	500	300	**\$ 12,000**		
Solution	**15**	**15**			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Solution Activity 1	15	0.00	500	400	0
$C$7	Solution Activity 2	15	0.0	300	0	133.3333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$3	A Totals	300	0.00	300	60	83.33333333
$D$4	B Totals	240	50	240	42.85714286	40
$D$5	C Totals	300	0	450	1E+30	150

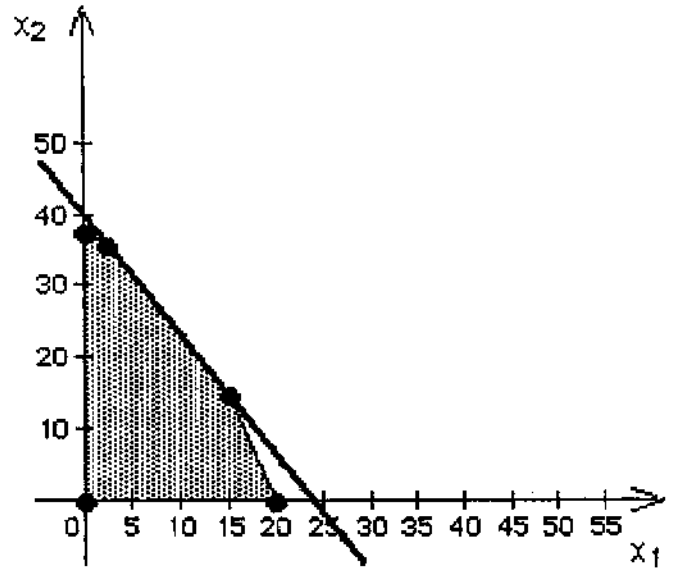
4.7-10.

b) The sensitivity report indicates that the problem has other optimal solutions because the allowable decrease of Activity 1 and the allowable increase of Activity 2 are 0.

	Resource Usage Per Unit of Each Activity				Resource
Resource	Activity 1	Activity 2	Totals		Available
A	15	5	217	≤	300
B	10	6	240	≤	240
C	8	12	450	≤	450
Unit Profit	500	301	**$ 12,036**		
Solution	**2.5**	**35.833**			

c) The other optimal solutions will be located on the line segment connecting the two optimal solutions found in parts a) and b).

d) Optimal Solution: $(x_1, x_2) = (15,15)$, (2.5, 35.833) and all points on the connecting line. Profit = $12,000.



Cases

4-1 a) The fixed design and fashion costs are sunk costs and therefore should not be considered when setting the production now in July. Since the velvet shirts have a positive contribution to covering the sunk costs, they should be produced or at least considered for production according to the linear programming model. Had Ted raised these concerns before any fixed costs were made, then he would have been correct to advise against designing and producing the shirts. With a contribution of $22 and a demand of 6000 units, maximum expected profit will be only $132,000. This amount will not be enough to cover the $500,000 in fixed costs directly attributable to this product.

b) The following insight greatly simplifies the analysis of the problem. The production processes of the various clothing items are not all linked together. We can separate the clothing items according to the materials that are used in their production and instead of one large linear programming problem we can formulate 4 smaller problems.

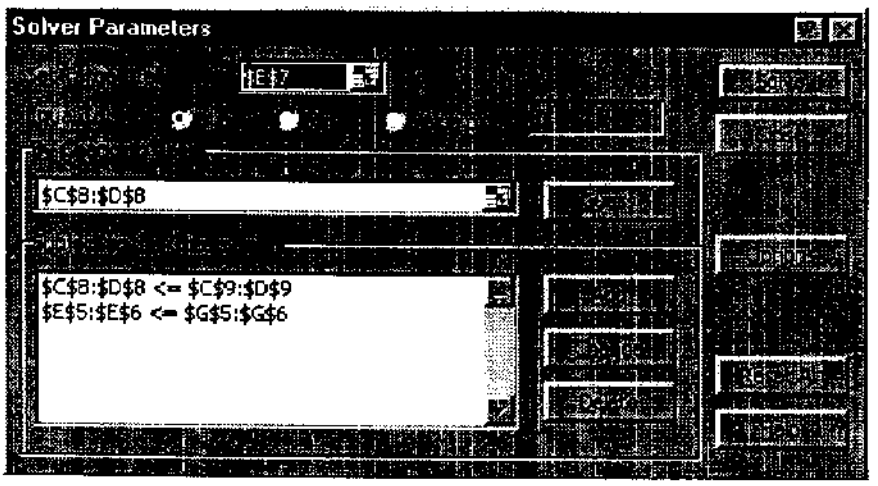
We use the term net contribution of a sales item to describe the difference between its total revenues and variable costs. The net contribution does not reflect any part of the fixed costs.

The cashmere sweater is the only item consisting of cashmere. The net contribution of one cashmere sweater equals $450 - $150 – 1.5*$60 = $210. TrendLines can sell at most 4000 sweaters and has 9000 yards of cashmere as raw material. It is optimal to produce 4000 sweaters using 6000 yards of cashmere yielding a net contribution of 4000*$210 = $840,000.

The silk blouse and camisole are the only items using silk and no other materials are used for these items. We can determine the optimal production amounts of these two items through a simple linear program. The first constraint models the resource limitation in the production process that Katherine has ordered 18,000 yards of silk. The second constraint models the production condition that whenever a silk blouse is produced automatically also a silk camisole is produced. Finally we must include the stated upper bounds on the number of silk items we can sell.

	A	B	C	D	E	F	G
1							
2							
3			Activity				
4		Constraint	silk blouse	silk camisole	Totals		Constraint RHS
5		silk	1.5	0.5	18000	<=	18000
6		production	1	-1	-8000	<=	0
7		unit profit	60.5	53.5	**1226000**		
8		Solution	**7000**	**15000**			
9		Maximum	12000	15000			

	E
3	
4	Totals
5	=SUMPRODUCT(C5:D5,C8:D8)
6	=SUMPRODUCT(C6:D6,C8:D8)
7	**=SUMPRODUCT(C7:D7,C8:D8)**
8	

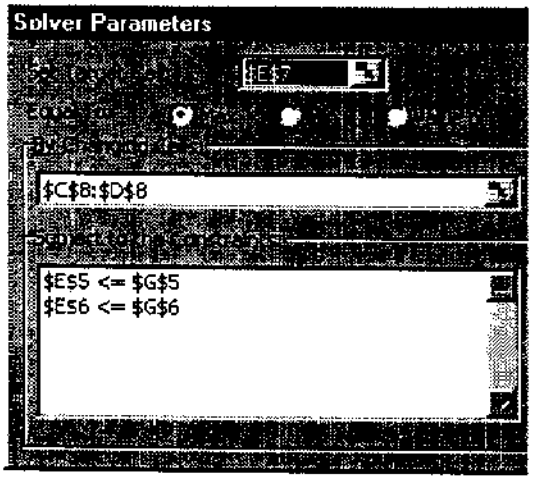


Throughout this case we use the following solver options.

TrendLines should produce 7000 silk blouses and 15000 silk camisoles yielding a net contribution of $1,226,000.

We can determine the optimal production plan for the items made from cotton in a similar fashion. There are no demand limitations for the cotton items.

	A	B	C	D	E	F	G
1							
2							
3			Activity				
4		Constraint	cotton sweater	cotton m-s	Totals		Constraint RHS
5		wool	1.5	0.5	30000	<=	30000
6		production	1	-1	-60000	<=	0
7		unit profit	66.25	33.75	**2025000**		
8		Solution	**0**	**60000**			

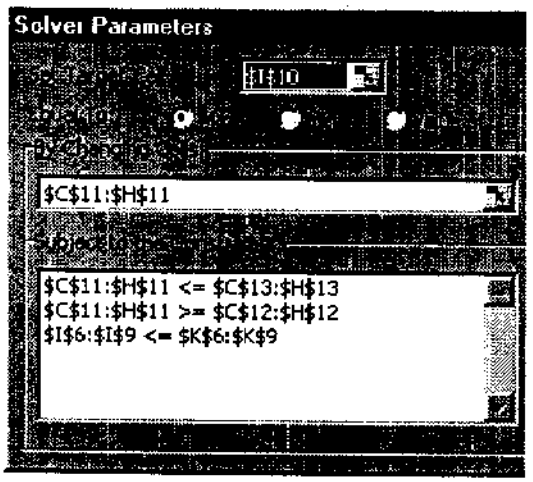


TrendLines should produce 60000 cotton mini-skirts but no cotton sweaters yielding a net contribution of $2,025,000.

It remains to develop a linear programming problem for determining the optimal production quantities of the tailored wool slacks, the tailored skirt, the wool blazer, the velvet pants and shirts, and the button-down blouse. We include four constraints for the resource limitations on wool, velvet, rayon, and acetate. Upper and lower bounds are given for many items. When there is no lower bound, we insert 0, when there is no upper bound, we determine a safe upper bound as a consequence of the resource limitations.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail.wool slacks	tail.skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		woo	3	0	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	155.25	136	22	26.625	**2771933.333**		
11		Solution	**4200**	**8066.666667**	**5000**	**0**	**6000**	**9244.444444**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	20000	5000	5500	6000	20000			

	I
4	
5	Totals
6	=SUMPRODUCT(C6:H6,C11:H11)
7	=SUMPRODUCT(C7:H7,C11:H11)
8	=SUMPRODUCT(C8:H8,C11:H11)
9	=SUMPRODUCT(C9:H9,C11:H11)
10	**=SUMPRODUCT(C10:H10,C11:H11)**
11	



TrendLines should produce 4200 wool slacks, 8066.67 skirts, 5000 wool blazers, no velvet pants, 6000 velvet shirts, and 9244.44 button-down blouses. The net contribution of these items equals $2,771,933.33. (Of course, TrendLines cannot produce two-thirds of a skirt, so the actual solution should be integer. You will learn about integer programming in chapter 8.)

The net contribution of all clothing items equals S840,000 + $1,226,00 + $ 2,025,000 + $2,771,933.33 = $6,862,933.33. So far we have not considered the sunk costs for the three fashion shows and the designers which total $8,960,000. The total profit equals $6,862,933.33 - $8,960,000 = -$2,097,066.67. So, TrendLines actually loses almost $2.1 million.

c) If velvet cannot be sent back to the textile wholesaler, then the whole quantity will be considered as a sunk cost and therefore added to the fixed costs. The objective function coefficients of items using velvet will no longer include the material cost. The objective function coefficients of the velvet pants and shirts are now \$175 and \$40, respectively.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail,wool slacks	tail,skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	20000	<=	20000
10		Unit profit	110	143.25	155.25	172	40	26.625	2983822.22		
11		Solution	**4200**	**3177.777778**	**5000**	**3666.666667**	**6000**	**15762.96296**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	20000	5000	5500	6000	20000			

The production plan changes considerably. TrendLines should produce 4200 wool slacks, 3177.77 skirts, 5000 wool blazers, 3666.67 velvet pants, 6000 velvet shirts, and 15762.92 button-down blouses. The production decisions for all other items are unaffected by the change. The net contribution of all clothing items equals \$840,000 + \$1,226,00 + \$ 2,025,000 + \$2,983,822.22 = \$7,074,822.22. The sunk costs now include the material cost for velvet and total \$9,200,000. The loss equals \$9,200,000 - \$7,074,822.22 = \$2,125,177.78.

d) When TrendLines cannot return the velvet to the wholesaler, the costs for velvet cannot be recovered. These cost are no longer variable cost but now are sunk cost. As a consequence the increased net contribution of the velvet items makes them more attractive to produce. This way the revenues from selling these items can contribute to the recovery of at least some of the fixed costs. Instead of zero TrendLines produces now 3666.67 velvet pants. These pants also require some acetate and thus their production affects the production plan for all other items. Since it is not optimal to make full use of the ordered velvet in part (b) it comes as no surprise that the loss in part (c) is even bigger than in part (b).

e) The unit contribution of a wool blazer changes to \$75.25.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail,wool slacks	tail,skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	20100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	9000	<=	20000
10		Unit profit	110	143.25	75.25	138	22	26.625	2436933.333		
11		Solution	**4200**	**10066.66667**	**3000**	**0**	**6000**	**6577.777778**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	20000	5000	5500	6000	20000			

TrendLines should produce 4200 wool slacks, 10066.67 skirts, the minimum of 3000 wool blazers, no velvet pants, 6000 velvet shirts, and 6577.78 button-down blouses. The production decisions for all other items are unaffected by the change. The net contribution of all clothing items equals \$840,000 + \$1,226,00 + \$ 2,025,000 + \$2,436,933.33 = \$6,527,933.33. The loss equals \$8,960,000 - \$6,527,933.33= \$2,432,066.67.

f) The right-hand-side of the acetate constraint changes.

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3			Resource Usage Per Unit of Each Activity								
4			Activity								Resource
5		Resource	tail.wool slacks	tail.skirt	wool blazer	velvet pants	velvet shirt	b.-d. blouse	Totals		Available
6		wool	3	0	2.5	0	0	0	25100	<=	45000
7		acetate	2	1.5	1.5	2	0	0	38000	<=	38000
8		rayon	0	2	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	155.25	136	22	26.625	**3490266.667**		
11		Solution	**4200**	**14733.33333**	**5000**	**0**	**6000**	**355.5555556**			
12		Minimum	4200	2800	3000	0	0	0			
13		Maximum	7000	15000	5000	5500	6000	20000			

TrendLines should produce 4200 wool slacks, 14733.33 skirts, the minimum of 5000 wool blazers, no velvet pants, 6000 velvet shirts, and 355.55 button-down blouses. The production decisions for all other items are unaffected by the change. The net contribution of all clothing items equals $840,000 + $1,226,00 + $ 2,025,000 + $3,490,266.67 = $7,581,266.67. The loss equals $8,960,000 - $7,581,266.67 = $1,378,733.33.

g) The net contribution of one cashmere sweater sold in the November sale equals 0.6*$450 - $150 – 1.5*$60 = $30. After producing 4000 sweaters to be sold in September and October TrendLines has 3000 yards of cashmere as raw material left. It is optimal to produce 2000 more sweaters using the remaining 3000 yards of cashmere yielding an additional contribution of 2000*$30 = $60,000.

For the three linear programming problems determining the production plans for all other clothing items we need to include new decision variables representing the number of clothing items that are sold during the November sale. Clearly TrendLines does not want to produce items with a negative net contribution. Therefore, we need to consider only those clothing items that have a positive net contribution after taking the sales price into account.

	A	B	C	D	E	F	G	H
1								
2								
3			Activity					
4		Constraint	silk blouse	silk camisole	silk camisole(sale)	Totals		Constraint RHS
5		silk	1.5	0.5	0.5	18000	<=	18000
6		production	1	-1	-1	-8000	<=	0
7		unit profit	60.5	53.5	5.5	**1226000**		
8		Solution	**7000**	**15000**	**0**			
9		Maximum	12000	15000	36000			

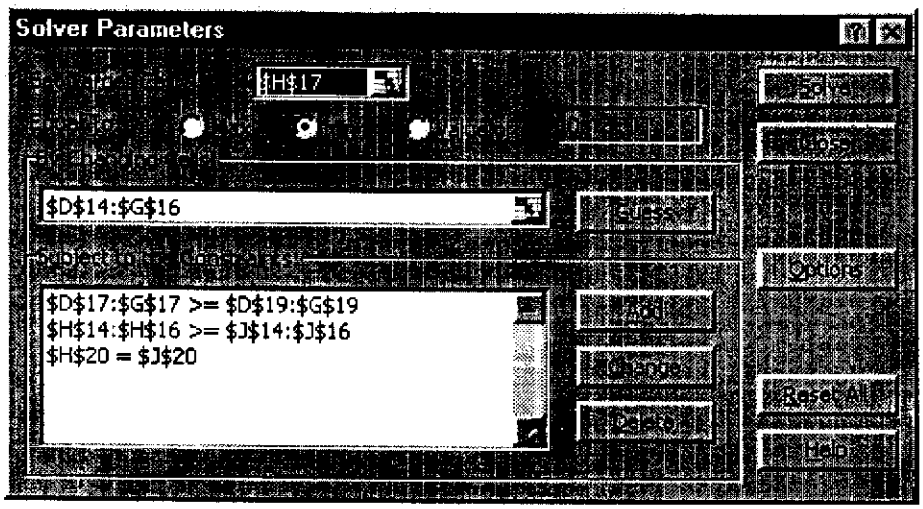
	A	B	C	D	E	F	G	H	I
1									
2									
3			Activity						
4		Constraint	cotton sweater	sweater(sale)	cotton m-s	m-s (sale)	Totals		Constraint RHS
5		wool	1.5	1.5	0.5	0.5	30000	<=	30000
6		production	1	1	-1	-1	-60000	<=	0
7		unit profit	66.25	14.25	33.75	3.75	**2025000**		
8		Solution	**0**	**0**	**60000**	**0**			

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2													
3			Resource Usage Per Unit of Each Activity										
4			Activity										
5		Resource	tail.wool slacks	tail.skirt	skirt (sale)	wool blazer	blazer (sale)	velvet pants	velvet shirt	b.-d. blouse	Totals		Resource Available
6		wool	3	0	0	2.5	2.5	0	0	0	25000	<=	45000
7		acetate	2	1.5	1.5	1.5	1.5	2	0	0	28000	<=	28000
8		rayon	0	2	2	0	0	0	0	1.5	30000	<=	30000
9		velvet	0	0	0	0	0	3	1.5	0	9000	<=	20000
10		unit profit	110	143.25	35.25	155.25	27.25	136	22	26.625	**2771933.333**		
11		Solution	**4200**	**8066.666667**	**0**	**5000**	**0**	**0**	**6000**	**9244.444444**			
12		Minimum	4200	2800	0	3000	0	0	0	0			
13		Maximum	7000	15000	15000	5000	20000	5500	6000	20000			

It only pays to produce 2000 more Cashmere sweaters. The production plan for all other items is the same as in part (b). The sale of the Cashmere sweaters reduces the loss by $60,000 to $2,037,066.67.

4-2 a) We define 12 decision variables, one for each age group surveyed in each region. Rob's restrictions are easily modeled as constraints. For example, his condition that at least 20 percent of the surveyed customers have to be from the first age group requires that the sum of the variables for the age group "18 to 25" across all three regions is at least 400. All his other requirements are modeled similarly. Finally, the sum of all variables has to equal 2000, because that is the number of customers Rob wants to have interviewed.

	B	C	D	E	F	G	H	I	J
2			Cost per Person						
3			Age Group						
4			18 to 25	26 to 40	41 to 50	51 and over			
5		Silicon Valley	$4.75	$6.50	$6.50	$5.00			
6	Region	Big Cities	$5.25	$5.75	$6.25	$6.25			
7		Small Towns	$6.50	$7.50	$7.50	$7.25			
11			Number of People Surveyed						
12			Age Group						
13			18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions	
14		Silicon Valley	600	0	0	300	900	>=	300
15	Region	Big Cities	0	550	150	0	700	>=	700
16		Small Towns	250	0	150	0	400	>=	400
17	Totals		850	550	300	300	$11,200	=	Total Cost
18			>=	>=	>=	>=	$12,880.00	=	Bid
19	Survey restrictions		400	550	300	300	Total Surveys		
20							2000	=	2000
22		Formula in Cell H14:	=SUM(D14:G14)						
23		Formula in Cell H15:	=SUM(D15:G15)						
24		Formula in Cell H16:	=SUM(D16:G16)						
25		Formula in Cell D17:	=SUM(D14:D16)						
26		Formula in Cell E17:	=SUM(E14:E16)						
27		Formula in Cell F17:	=SUM(F14:F16)						
28		Formula in Cell G17:	=SUM(G14:G16)						
29		Formula in Cell H20:	=SUM(D14:G16)						
30		Formula in Cell H17:	=SUMPRODUCT(D5:G7,D14:G16)						
31		Formula in Cell H18:	=1.15*H17						

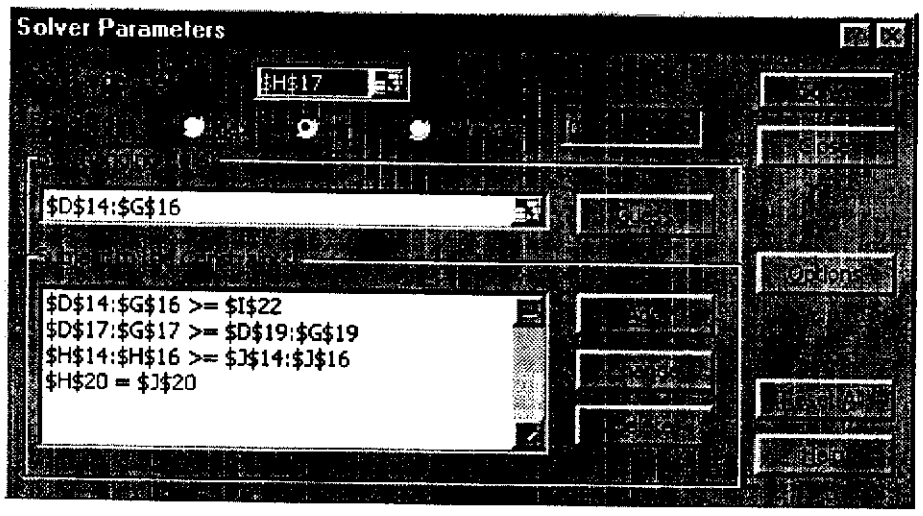


The cost of conducting the survey meeting all constraints imposed by AmeriBank incurs cost of $11,200. The mix of customers is displayed in the spreadsheet above.

b) Sophisticated Surveys will submit a bid of 1.15*$11200 = $12,880.

c) We need to include the new lower-bound constraint on all variables.

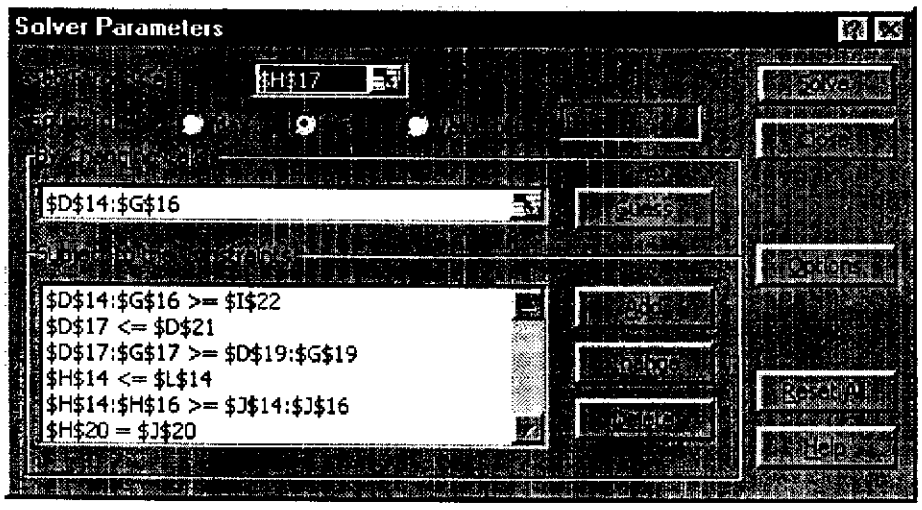
	A	B	C	D	E	F	G	H	I	J	K
1											
2				Cost per Person							
3				Age Group							
4				18 to 25	26 to 40	41 to 50	51 and over				
5			Silicon Valley	$4.75	$6.50	$6.50	$5.00				
6		Region	Big Cities	$5.25	$5.75	$6.25	$6.25				
7			Small Towns	$6.50	$7.50	$7.50	$7.25				
8											
9											
10											
11				Number of People Surveyed							
12				Age Group							
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions		
14			Silicon Valley	600	50	50	200	900	>=	300	
15		Region	Big Cities	150	450	50	50	700	>=	700	
16			Small Towns	100	50	200	50	400	>=	400	
17		Totals		850	550	300	300	11387.5	=	Total Cost	
18				>=	>=	>=	>=	$13,095.62	=	Bid	
19		Survey restrictions		400	550	300	300	Total Surveys			
20								2000	=	2000	
21								Minimum value for each variable			
22			Formula in cell H14:	"=SUM(D14:G14)"					50		
23			Formula in cell H15:	"=SUM(D15:G15)"							
24			Formula in cell H16:	"=SUM(D16:G16)"							
25			Formula in cell D17:	"=SUM(D14:D16)"							
26			Formula in cell E17:	"=SUM(E14:E16)"							
27			Formula in cell F17:	"=SUM(F14:F16)"							
28			Formula in cell G17:	"=SUM(G14:G16)"							
29			Formula in cell H20:	"=SUM(D14:G16)"							
30			Formula in cell H17:	"=SUMPRODUCT(D5:G7,D14:G16)"							
31			Formula in cell H18:	"=1.15*H17"							



The new requirement increases the bid to $13,095.62.

d) We include upper bounds on the total number of people surveyed in Silicon Valley and from the age group of 18 to 25 year-olds.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2				Cost per Person								
3				Age Group								
4				18 to 25	26 to 40	41 to 50	51 and over					
5			Silicon Valley	$4.75	$6.50	$6.50	$5.00					
6		Region	Big Cities	$5.25	$5.75	$6.25	$6.25					
7			Small Towns	$6.50	$7.50	$7.50	$7.25					
8												
9												
10												
11				Number of People Surveyed								
12				Age Group								
13				18 to 25	26 to 40	41 to 50	51 and over	Totals	Survey restrictions			
14			Silicon Valley	**100**	**50**	**50**	**450**	650	>=	300	<=	650
15		Region	Big Cities	**400**	**450**	**50**	**50**	950	>=	700		
16			Small Towns	**100**	**50**	**200**	**50**	400	>=	400		
17		Totals		600	550	300	550	**$11,575**	=	Total Cost		
18				>=	>=	>=	>=	$13,311.25	=	Bid		
19	Survey restrictions			400	550	300	300	Total Surveys				
20				<=				2000	=	2000		
21				600				Minimum value for each variable				
22									50			
23												
24			Formula in cell H14:	"=SUM(D14:G14)"								
25			Formula in cell H15:	"=SUM(D15:G15)"								
26			Formula in cell H16:	"=SUM(D16:G16)"								
27			Formula in cell D17:	"=SUM(D14:D16)"								
28			Formula in cell E17:	"=SUM(E14:E16)"								
29			Formula in cell F17:	"=SUM(F14:F16)"								
30			Formula in cell G17:	"=SUM(G14:G16)"								
31			Formula in cell H20:	"=SUM(D14:G16)"								
32			Formula in cell H17:	"=SUMPRODUCT(D5:G7,D14:G16)"								
33			Formula in cell H18:	"=1.15*H17"								



The new requirements increase the bid to $13,311.25.

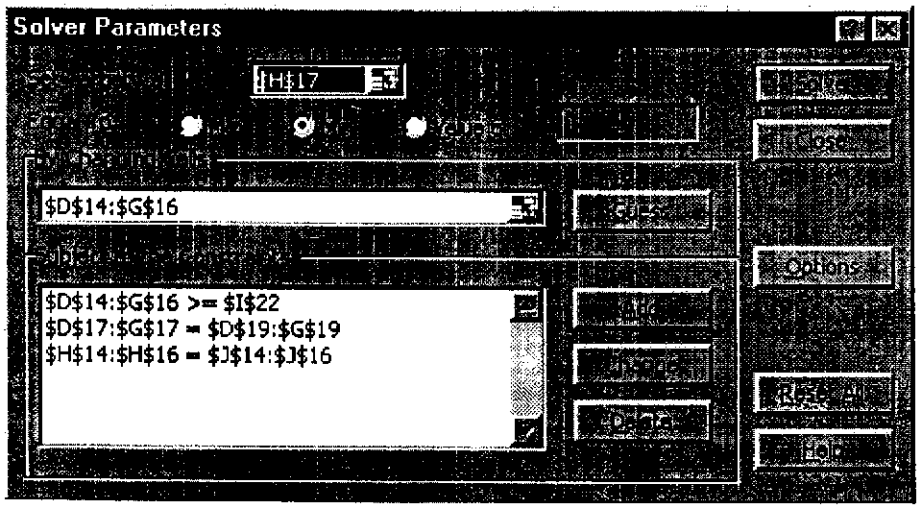
e) The three cost factors for the age group "18 to 25" are changed.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2				Cost per Person								
3				Age Group								
4				18 to 25	26 to 40	41 to 50	51 and over					
5			Silicon Valley	$6.50	$6.50	$6.50	$5.00					
6		Region	Big Cities	$6.75	$5.75	$6.25	$6.25					
7			Small Towns	$7.00	$7.50	$7.50	$7.25					
8												
9												
10												
11				Number of People Surveyed								
12				Age Group								
13				18 to 25	26 to 40	41 to 50	51 and over	Totals		Survey restrictions		
14			Silicon Valley	**50**	**50**	**50**	**500**	650	>=	300	<=	650
15		Region	Big Cities	**100**	**600**	**200**	**50**	950	>=	700		
16			Small Towns	**250**	**50**	**50**	**50**	400	>=	400		
17		Totals		400	700	300	600	**$ 12,025**	=	Total Cost		
18				>=	>=	>=	>=	$13,828.75	=	Bid		
19		Survey restrictions		400	550	300	300	Total Surveys				
20				<=				2000	=	2000		
21				600				Minimum value for each variable				
22									50			
23												
24			Formula in cell H14:	"=SUM(D14:G14)"								
25			Formula in cell H15:	"=SUM(D15:G15)"								
26			Formula in cell H16:	"=SUM(D16:G16)"								
27			Formula in cell D17:	"=SUM(D14:D16)"								
28			Formula in cell E17:	"=SUM(E14:E16)"								
29			Formula in cell F17:	"=SUM(F14:F16)"								
30			Formula in cell G17:	"=SUM(G14:G16)"								
31			Formula in cell H20:	"=SUM(D14:G16)"								
32			Formula in cell H17:	"=SUMPRODUCT(D5:G7,D14:G16)"								
33			Formula in cell H18:	"=1.15*H17"								

With the new cost factors the bid increases to $13,828.75.

f) We eliminate all lower and upper bounds on the age groups and regions and replace them with Rob's strict requirements. These requirements also ensure that exactly 2000 people are surveyed so that we can drop that constraint too.

	A	B	C	D	E	F	G	H	I	J	K
1											
2				Cost per Person							
3				Age Group							
4				18 to 25	26 to 40	41 to 50	51 and over				
5			Silicon Valley	$6.50	$6.50	$6.50	$5.00				
6		Region	Big Cities	$6.75	$5.75	$6.25	$6.25				
7			Small Towns	$7.00	$7.50	$7.50	$7.25				
8											
9											
10											
11				Number of People Surveyed							
12				Age Group							
13				18 to 25	26 to 40	41 to 50	51 and over	Totals		Survey restrictions	
14			Silicon Valley	**50**	**50**	**50**	**250**	400	=	400	
15		Region	Big Cities	**50**	**600**	**300**	**50**	1000	=	1000	
16			Small Towns	**400**	**50**	**50**	**100**	600	=	600	
17		Totals		500	700	400	400	**$ 12,475**	=	Total Cost	
18				=	=	=	=	$14,346.25	=	Bid	
19	Survey restrictions			500	700	400	400				
20											
21								Minimum value for each variable			
22			Formula in cell H14:	=SUM(D14:G14)"					50		
23			Formula in cell H15:	=SUM(D15:G15)"							
24			Formula in cell H16:	=SUM(D16:G16)"							
25			Formula in cell D17:	=SUM(D14:D16)"							
26			Formula in cell E17:	=SUM(E14:E16)"							
27			Formula in cell F17:	=SUM(F14:F16)"							
28			Formula in cell G17:	=SUM(G14:G16)"							
29			Formula in cell H20:	=SUM(D14:G16)"							
30			Formula in cell H17:	=SUMPRODUCT(D5:G7,D14:G16)"							
31			Formula in cell H18:	"=1.15*H17"							



Rob's strict requirements increase the cost of the survey by $450. The new bid of Sophisticated Surveys is $14,346.25.

4-3

a & b)

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	422.222222	177.777778	600	=	600
Area 3	0	227.777778	322.222222	550	=	550
Area 4	350	0	0	350	=	350
Area 5	366.666667	0	133.333333	500	=	500
Area 6	83.3333333	0	366.666667	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 555,555.56

Grade Constraints:

	School 1	School 2	School 3
6th Graders	269.333333	368.555556	339.111111
7th Graders	258	362.111111	300.888889
8th Graders	242.666667	369.333333	360
30% of Total	240	330	300
36% of Total	288	396	360

c) The recommendation to the school board is to assign students to schools as shown in the above solution section of the spreadsheet. Quantities that are not integers must be rounded since partial students cannot be sent.

d) The following solution decreases total bussing costs by over $135,000 but violates the grade constraints that were imposed. Solutions will vary and those than satisfy the grade constraints will be likely to increase the total bussing costs.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	0	0	550	550	=	550
Area 4	350	0	0	350	=	350
Area 5	500	0	0	500	=	500
Area 6	0	0	450	450	=	450
Total	850	1050	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 420,000.00

Grade Constraints:

	School 1	School 2	School 3
6th Graders	293	366	378
7th Graders	310	339	302
8th Graders	247	345	380
30% of Total	255	315	300
36% of Total	306	378	360

e) The number of students assigned from each area to each school changes to the solution shown below and the total bussing cost is reduced by almost $162,000.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	0
4	350	0.28	0.4	0.32	0	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	0	0	550	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.181818	0	181.818182	500	=	500
Area 6	131.818182	50	268.181818	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 393,536.36

Grade Constraints:

	School 1	School 2	School 3
6th Graders	268.909091	383	327.090909
7th Graders	285.090909	353	312.909091
8th Graders	248	364	360
30% of Total	240	330	300
36% of Total	288	396	360

f) The number of students assigned from each area to each school changes to the solution shown below and the total bussing cost is reduced by over $215,000.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	0	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0 3	0.32	0.38	600	0	0
4	350	0.28	0 4	0.32	0	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	0	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	38.7096771	411.290323	0	450	=	450
Area 2	0	236.559139	363.440861	600	=	600
Area 3	0	77.95699	472.04301	550	=	550
Area 4	350	0	0	350	=	350
Area 5	435.483871	0	64.5161288	500	=	500
Area 6	75.3064517	374.193548	0	450	=	450
Tota	900	1100	900			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 340,053.76

Grade Constraints:

	School 1	School 2	School 3
6th Graders	306	359.752688	301.247312
7th Graders	324	352.247312	274.752688
8th Graders	270	378	324
30% of Total	270	330	270
36% of Total	324	396	324

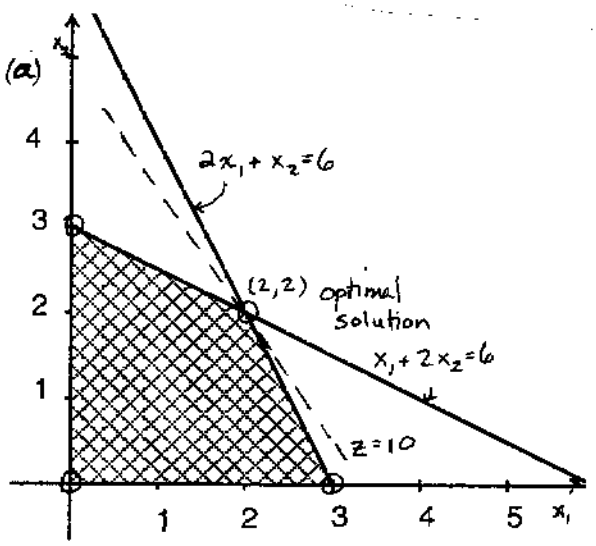
g)

Option	Cost	# students walking 1 to 1.5 miles	# students walking more than 1.5 miles
current	$555,555.56	0	0
1	$393,636.36	900	0
2	$340,053.76	900	491

h) Answers will vary.

5.1-1.

(a)



$(x_1, x_2) = (2,2)$ is optimal with $z = 10$

(c) Maximize $z = 3x_1 + 2x_2$

subject to: $2x_1 + x_2 + x_3 = 6$

$x_1 + 2x_2 + x_4 = 6$

$x_i \geq 0$ for $i = 1, 2, 3, 4$.

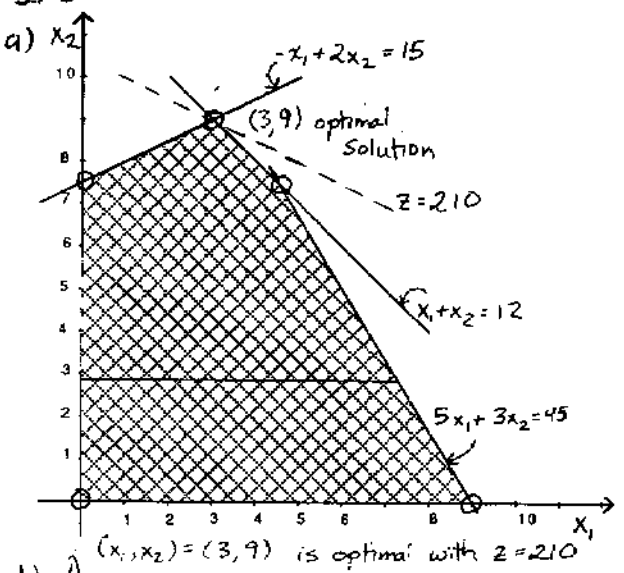
b) – d)

Defining Equations	CP	Feasible?	Basic Solution	Indicating Var's:	Equations
$x_1 = 0$ $x_2 = 0$	(0,0)	Yes	(0,0,6,6)	x_1 x_2	$x_3 = 6$ $x_4 = 6$
$x_1 = 0$ $x_1 + 2x_2 = 6$	(0,3)	Yes	(0,3,3,0)	x_1 x_4	$x_2 + x_3 = 6$ $2x_2 = 6$
$x_1 = 0$ $2x_1 + x_2 = 6$	(0,6)	No	(0,6,0,-6)	x_1 x_3	$x_2 = 6$ $2x_2 + x_4 = 6$
$x_2 = 0$ $x_1 + 2x_2 = 6$	(6,0)	No	(6,0,-6,0)	x_2 x_4	$2x_1 + x_3 = 6$ $x_1 = 6$
$x_2 = 0$ $2x_1 + x_2 = 6$	(3,0)	Yes	(3,0,0,3)	x_2 x_3	$2x_1 = 6$ $x_1 + x_4 = 6$
$2x_1 + x_2 = 6$ $x_1 + 2x_2 = 6$	(2,2)	Yes	(2,2,0,0)	x_3 x_4	$2x_1 + x_2 = 6$ $x_1 + 2x_2 = 6$

5.1-1.e)

Step #	CPF Soln	Deleted Defining Eqn.	Added Defining Eqn	Deleted Ind. Var.	Added Ind. Var.
1	(0, 0)	$x_1 = 0$	$2x_1 + x_2 = 6$	x_1	x_3
2	(3, 0)	$x_2 = 0$	$x_1 + 2x_2 = 6$	x_2	x_4
3	(2, 2)	Optimal			

5.1-2

a) 

$(x_1, x_2) = (3, 9)$ is optimal with $Z = 210$

(c) Maximize $Z = 10x_1 + 20x_2$

subject to

$$-x_1 + 2x_2 + x_3 = 15$$
$$x_1 + x_2 + x_4 = 12$$
$$5x_1 + 3x_2 + x_5 = 45$$

$x_i \geq 0$ for $i = 1, 2, 3, 4, 5$.

b)-d)

Defining Eqns	CP	Feasible?	Basic Soln	Ind. Variables	Equations
$x_1 = 0$ $x_2 = 0$	(0,0)	Yes	(0,0,15,12,45)	x_1 x_2	$x_3 = 15$ $x_4 = 12$ $x_5 = 45$
$x_1 = 0$ $-x_1 + 2x_2 = 15$	(0,7.5)	Yes	(0,7.5,0,4.5,22.5)	x_1 x_3	$2x_2 = 15$ $x_2 + x_4 = 12$ $3x_2 + x_5 = 45$
$x_1 = 0$ $x_1 + x_2 = 15$	(0,12)	No	(0,12,-9,0,9)	x_1 x_4	$2x_2 + x_3 = 15$ $x_2 = 12$ $3x_2 + x_5 = 45$
$x_1 = 0$ $5x_1 + 3x_2 = 45$	(0,15)	No	(0,15,-15,-3,0)	x_1 x_5	$2x_2 + x_3 = 15$ $x_2 + x_4 = 12$ $3x_2 = 45$
$x_2 = 0$ $-x_1 + 2x_2 = 15$	(-15,0)	No	(-15,0,0,3,120)	x_2 x_3	$-x_1 = 15$ $x_1 + x_4 = 12$ $5x_1 + x_5 = 45$
$x_2 = 0$ $x_1 + x_2 = 12$	(12,0)	No	(12,0,27,0,-15)	x_2 x_4	$-x_1 + x_3 = 15$ $x_1 = 12$ $5x_1 + x_5 = 45$
$x_2 = 0$ $5x_1 + 3x_2 = 45$	(9,0)	Yes	(9,0,24,3,0)	x_2 x_5	$-x_1 + x_3 = 15$ $x_1 + x_4 = 12$ $5x_1 = 45$
$-x_1 + 2x_2 = 15$ $x_1 + x_2 = 12$	(3,9)	Yes	(3,9,0,0,3)	x_3 x_4	$-x_1 + 2x_2 = 15$ $x_1 + x_2 = 12$ $5x_1 + 3x_2 + x_5 = 45$
$-x_1 + 2x_2 = 15$ $5x_1 + 3x_2 = 45$	$(\frac{45}{13}, \frac{120}{13})$	No	$(\frac{45}{13}, \frac{126}{13}, 0, -\frac{9}{13}, 0)$	x_3 x_5	$-x_1 + 2x_2 = 15$ $x_1 + x_2 + x_4 = 12$ $5x_1 + 3x_2 = 45$
$x_1 + x_2 = 12$ $5x_1 + 3x_2 = 45$	(4.5, 7.5)	Yes	(4.5, 7.5, 3.5, 0, 0)	x_4 x_5	$-x_1 + 2x_2 + x_3 = 15$ $x_1 + x_2 = 12$ $5x_1 + 3x_2 = 45$

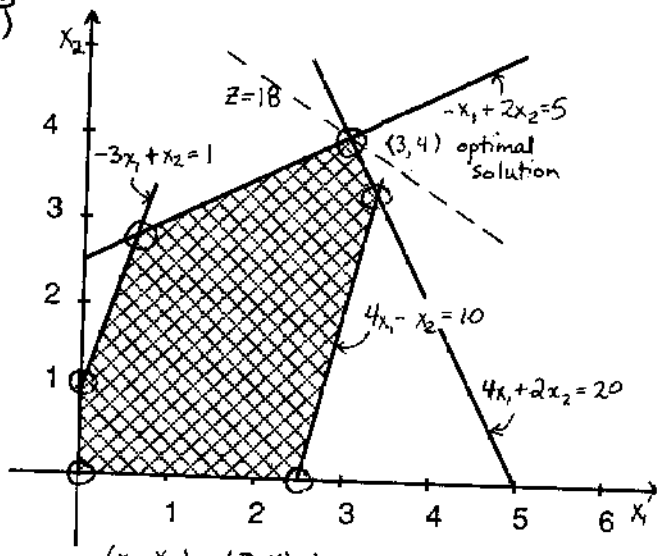
5.1-2

e)

Step #	CPF Soln	Deleted Defining Eqn.	Added Defining Eqn.	Deleted Ind. Var.	Added Ind. Var
1	(0, 0)	$x_2 = 0$	$-x_1 + 2x_2 = 15$	x_2	x_3
2	(0, 7.5)	$x_1 = 0$	$x_1 + x_2 = 12$	x_1	x_4
3	(3, 9)	Optimal			

5.1-3

a)



$(x_1, x_2) = (3,4)$ is optimal with $z = 18$

b)

CPF Solns	Defining Eqns	BF Solutions	Nonbasic var's	Z
(0,0)	$x_1 = 0$ $x_2 = 0$	(0, 0, 1, 20, 10, 5)	x_1 & x_2	0
(0,1)	$x_1 = 0$ $-3x_1 + x_2 = 1$	(0, 1, 0, 18, 11, 3)	x_1 & x_3	3
(0.6, 2.8)	$-3x_1 + x_2 = 1$ $-x_1 + 2x_2 = 5$	(0.6, 2.8, 0, 12, 10.4, 0)	x_3 & x_6	9.6
(3, 4) *	$-x_1 + 2x_2 = 5$ $4x_1 + 2x_2 = 20$	(3, 4, 6, 0, 2, 0)	x_4 & x_6	18 *
(3.33, 3.33)	$4x_1 + 2x_2 = 20$ $4x_1 - x_2 = 10$	(3.33, 3.33, 7.67, 0, 0, 1.67)	x_4 & x_5	16.67
(2.5, 0)	$4x_1 - x_2 = 10$ $x_2 = 0$	(2.5, 0, 8.5, 10, 0, 7.5)	x_2 & x_5	5

* Optimal

5.1-3. c)

CP Infeas Soln's	Defining Eqns	BI Infeas Solns	Non-basic Var's
$(-\frac{1}{3}, 0)$	$-3x_1+x_2=1$ $x_2=0$	$(-\frac{1}{3}, 0, 0, 21\frac{1}{3}, 11\frac{1}{3}, 4\frac{2}{3})$	x_3 x_2
$(-5, 0)$	$-x_1+2x_2=5$ $x_2=0$	$(-5, 0, -14, 40, 30, 0)$	x_6 x_2
$(0, 10)$	$4x_1+2x_2=20$ $x_1=0$	$(0, 10, -9, 0, 20, -15)$	x_4 x_1
$(0, \frac{5}{2})$	$-x_1+2x_2=5$ $x_1=0$	$(0, \frac{5}{2}, -\frac{3}{2}, 15, 12\frac{1}{2}, 0)$	x_6 x_1
$(\frac{9}{5}, \frac{32}{5})$	$4x_1+2x_2=20$ $-3x_1+x_2=1$	$(\frac{9}{5}, \frac{32}{5}, 0, 0, \frac{46}{5}, -6)$	x_4 x_3
$(11, 34)$	$-3x_1+x_2=1$ $4x_1-x_2=10$	$(11, 34, 0, -92, 0, -52)$	x_3 x_5
$(\frac{25}{7}, \frac{30}{7})$	$4x_1-x_2=10$ $-x_1+2x_2=5$	$(\frac{25}{7}, \frac{30}{7}, \frac{52}{7}, \frac{-20}{7}, 0, 0)$	x_5 x_6
$(5, 0)$	$4x_1+2x_2=20$ $x_2=0$	$(5, 0, 16, 0, -10, 10)$	x_4 x_2
$(0, -10)$	$4x_1-x_2=10$ $x_1=0$	$(0, -10, 11, 40, 0, 25)$	x_5 x_1

All sets yield a solution.

5.1-4. a) $(x_1, x_2, x_3) = (10, 0, 0)$

b) $x_2 = 0$

$x_3 = 0$

$x_1 - x_2 + 2x_3 = 10$

5.1-5

(a)

Corner Point Feasible Solution	Defining Equations
(0,0,0)	$x_1=0$, $x_2=0$, $x_3=0$
(4,0,0)	$x_1=4$, $x_2=0$, $x_3=0$
(4,2,0)	$x_1=4$, $x_1+x_2=6$, $x_3=0$
(2,4,0)	$x_2=4$, $x_1+x_2=6$, $x_3=0$
(0,4,0)	$x_1=0$, $x_2=4$, $x_3=0$
(0,4,2)	$x_1=0$, $x_2=4$, $-x_1+2x_3=4$
(2,4,3)	$x_1+x_2=6$, $x_2=4$, $-x_1+2x_3=4$
(4,2,4)	$x_1+x_2=6$, $x_1=4$, $-x_1+2x_3=4$
(4,0,4)	$x_2=0$, $x_1=4$, $-x_1+2x_3=4$
(0,0,2)	$x_2=0$, $x_1=0$, $-x_1+2x_3=4$

(b)
$$x_1+x_2=6$$
$$x_2=0$$
$$-x_1+2x_3=4$$

(c)
$$x_1=4$$
$$x_1=0$$
$$x_2=0$$

The system is inconsistent.

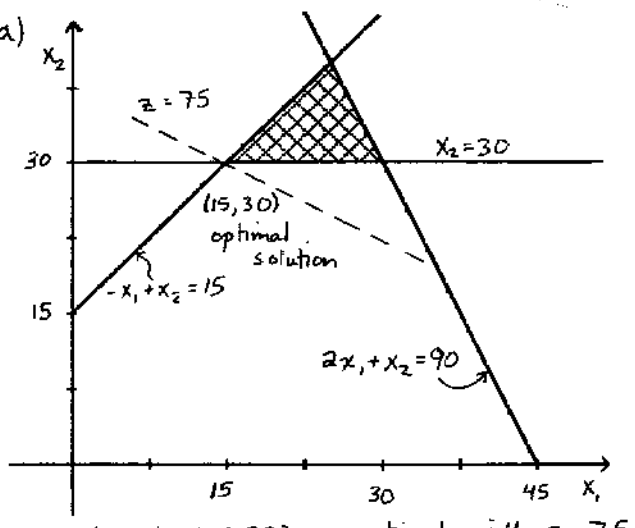
5.1-6

(a) & (b)

Defining Equations	Corner Point	Feasible?	Basic Solution $(y_1, y_2, y_3, y_4, y_5)$	Nonbasic Variables
$y_1=0$, $y_2=0$, $y_3=0$	(0,0,0)	No	(0,0,0,-3,-5)	y_1, y_2, y_3
$y_1=0$, $y_2=0$, $y_1+3y_3=3$	(0,0,1)	No	(0,0,1,0,-3)	y_1, y_2, y_4
$y_1=0$, $y_2=0$, $2y_2+2y_3=5$	(0,0,2.5)	Yes	(0,0,2.5,4.5,0)	y_1, y_2, y_5
$y_1=0$, $y_1+3y_3=3$, $y_3=0$	No Solution			y_1, y_3, y_4
$y_1=0$, $2y_2+2y_3=5$, $y_3=0$	(0,2.5,0)	No	(0,2.5,0,-3,0)	y_1, y_3, y_5
$y_2=0$, $y_1+3y_3=3$, $y_3=0$	(3,0,0)	No	(3,0,0,0,-5)	y_2, y_3, y_4
$y_2=0$, $2y_2+2y_3=5$, $y_3=0$	No Solution			y_2, y_3, y_5
$y_2=0$, $y_1+3y_3=3$, $2y_2+2y_3=5$	(-4.5, 0, 2.5)	No	(-4.5,0,2.5,0,0)	y_2, y_4, y_5
$y_3=0$, $y_1+3y_3=3$, $2y_2+2y_3=5$	(3,2.5,0)	Yes	(3,2.5,0,0,0)	y_3, y_4, y_5
$y_1=0$, $y_1+3y_3=3$, $2y_2+2y_3=5$	(0,1.5,1)	Yes	(0,1.5,1,0,0)	y_1, y_4, y_5

S.1-7

(a)



$(x_1, x_2) = (15, 30)$ is optimal with $z = 75$

(b)

Corner Point	Defining Equations	Basic Feasible Solution $(x_1, x_2, x_3, x_4, x_5)$	Nonbasic Variables
(15, 30)	$x_2 = 30$, $-x_1 + x_2 = 15$	(15, 30, 0, 30, 0)	x_3 and x_5
(30, 30)	$x_2 = 30$, $2x_1 + x_2 = 90$	(30, 30, 15, 0, 0)	x_4 and x_5
(25, 40)	$-x_1 + x_2 = 15$, $2x_1 + x_2 = 90$	(25, 40, 0, 0, 10)	x_3 and x_4

S1-8

(a) and (b)

Defining Equations	Corner Point	Feasible?	Basic Solution $(x_1, x_2, x_3, x_4, x_5)$	Nonbasic Variables
$x_1 = 0$, $x_2 = 0$	(0, 0)	No	(0, 0, -10, 6, -6)	x_1 and x_2
$x_1 = 0$, $2x_1 + x_2 = 10$	(0, 10)	No	(0, 10, 0, -14, 4)	x_1 and x_3
$x_1 = 0$, $-3x_1 + 2x_2 = 6$	(0, 3)	No	(0, 3, -7, 0, -3)	x_1 and x_4
$x_1 = 0$, $x_1 + x_2 = 6$	(0, 6)	No	(0, 6, -4, -6, 0)	x_1 and x_5
$x_2 = 0$, $2x_1 + x_2 = 10$	(5, 0)	No	(5, 0, 0, 21, -11)	x_2 and x_3
$x_2 = 0$, $-3x_1 + 2x_2 = 6$	(-2, 0)	No	(-2, 0, -14, 0, -8)	x_2 and x_4
$x_2 = 0$, $x_1 + x_2 = 6$	(6, 0)	Yes	(6, 0, 2, 24, 0)	x_2 and x_5
$2x_1 + x_2 = 10$, $-3x_1 + 2x_2 = 6$	(2, 6)	Yes	(2, 6, 0, 0, 8)	x_3 and x_4
$2x_1 + x_2 = 10$, $x_1 + x_2 = 6$	(4, 2)	Yes	(4, 2, 0, 14, 0)	x_3 and x_5
$-3x_1 + 2x_2 = 6$, $x_1 + x_2 = 6$	(1.2, 4.8)	No	(1.2, 4.8, -2.8, 0, 0)	x_4 and x_5

5.1-9 a) and b)

Defining Equations	Corner Point	Feasible?	Basic Solution $(x_1, x_2, x_3, x_4, x_5, x_6)$	Nonbasic Variables
$x_1=0$, $x_2=0$	(0,0)	Yes	(0,0, 10,60, 18,44)	x_1 and x_2
$x_1=0$, $x_2=10$	(0,10)	Yes	(0,10, 0,10, 8, 34)	x_1 and x_3
$x_1=0$, $2x_1+5x_2=60$	(0,12)	No	(0,12,-2, 0, 6, 32)	x_1 and x_4
$x_1=0$, $x_1+x_2=18$	(0,18)	No	(0,18,-8, -30, 0, 26)	x_1 and x_5
$x_1=0$, $3x_1+x_2=44$	(0, 44)	No	(0, 44, -34, -160, -26, 0)	x_1 and x_6
$x_2=0$, $x_2=10$	No Solution			x_2 and x_3
$x_2=0$, $2x_1+5x_2=60$	(30,0)	No	(30,0, 10, 0,-12,-46)	x_2 and x_4
$x_2=0$, $x_1+x_2=18$	(18,0)	No	(18,0, 10, 24, 0, -10)	x_2 and x_5
$x_2=0$, $3x_1+x_2=44$	(14.67,0)	Yes	(14.67,0, 10, 30.67, 3.33,0)	x_2 and x_6
$x_2=10$, $2x_1+5x_2=60$	(5, 10)	Yes	(5, 10, 0, 0, 3, 19)	x_3 and x_4
$x_2=10$, $x_1+x_2=18$	(8, 10)	No	(8, 10, 0, -6, 0, 10)	x_3 and x_5
$x_3=10$, $3x_1+x_2=44$	(11.33, 10)	No	(11.33, 10, 0, -12.67, -3.33, 0)	x_3 and x_6
$2x_1+5x_2=60$, $x_1+x_2=18$	(10, 8)	Yes	(10, 8, 2, 0, 0, 6)	x_4 and x_5
$2x_1+5x_2=60$, $3x_1+x_2=44$	(12.31, 7.08)	No	(12.31, 7.08, 2.92, 0, -1.38, 0)	x_4 and x_6
$x_1+x_2=18$, $3x_1+x_2=44$	(13, 5)	Yes	(13, 5, 5, 9, 0, 0)	x_5 and x_6

5.1-10 a) If the feasible region is unbounded then there may be no optimal solution.

b) An optimal solution may contain all points on a line segment between two corner points.

c) If an adjacent corner point has an equal objective function value then all the points on the connecting line segment will also be optimal.

5.1-11 a) False, (p. 5-10) Property 1: ⓐ If there is exactly one optimal solution, then it must be a CPF. ⓑ If there are multiple optimal solutions, then at least 2 must be adjacent CPF Soln's. We can take a convex combination of 2 optimal CPF Soln's to get a non-CPF optimal solution, for example.

b) False, the number of CPF solutions is at most $\frac{(m+n)!}{m!\,n!} = \binom{m+n}{n}$ (p. 5-12).

c) False, Property 3 (p. 5-13). If this statement were true, it would mean no matter which CPF we started on, it or one of its adjacent CPF solutions would be optimal! (which, of course, is not the case)

5.1-12. (*a*) True. By property 1(*a*), there must be multiple solutions (since this optimal solution is not a CPF). This then means there are an infinite number of optimal solutions since any convex combination of optimal solutions is also optimal.
(*b*) True. The points on the line segment connecting x^* and x^{**} can be represented by $x = \alpha Z^* + (1 - \alpha)x^{**}$, α ranging from 0 to 1. Let $Z^* = Z^{**}$ represent the objective function values for x^* and x^{**}. Then the objective function value at x would be (by linearity) $Z = \alpha Z^* + (1 - \alpha)Z^{**} = \alpha Z^* + (1 - \alpha) Z^* = Z^*$. So the objective function value is the same at all these points. Furthermore, since the feasible region is convex, all these points must be feasible.
(*c*) False. The simultaneous solution of any set of n constraint boundary equations may be infeasible, or it may not even exist (e.g., $x_1 = 2$, $x_1 = 4$).

5.1-13. (*a*). True. If there are *no* optimal solutions, then the problem must have no feasible solutions *or* the objective value can be increased indefinitely (Chap. 3). The former is not the case (assumed in the problem) and the latter cannot be true since the feasible region is *bounded.* Thus, there must be at least one optimal solution.
(*b*) False. If a solution is optimal, it need not be a BF solution. A convex combination of BF solutions can give an optimal but not Basic (i.e. CP) solution. However, it is true that if optimal solutions exist, then at least one of them must be a BF solution. This follows straight from Property 1 (since BF solutions $\Rightarrow$ CPF solutions).
(*c*) True. Since BF solutions correspond to CPF solutions, this follows directly from Property 2.

5.1-14

$x_1 = 0$
$2x_1 + x_2 + 3x_3 = 60$
$3x_1 + 3x_2 + 5x_3 = 120$

} This system has solution $(x_1, x_2, x_3) = (0, 15, 15)$

5.1-15

Since $x_2 > 0$ and $x_3 > 0$, $x_2 = 0$ and $x_3 = 0$ cannot be part of the three boundary equations. Therefore, the boundary equations are: $x_1 = 0$

$2x_1 + x_2 + x_3 = 20$

$3x_1 + x_2 + 2x_3 = 30$

The solution to this set of equation (and, therefore, the optimal) is: $(x_1, x_2, x_3) = (0, 10, 10)$

5.1-16

Since $x_1 > 0$ and $x_3 > 0$, $x_1 = 0$ and $x_3 = 0$ can't be part of the three boundary equations. If $x_2 = 0$, then $x_1 + x_3 \leq 24$ and $3x_1 + 3x_3 \leq 60$ or, equivalently, $x_1 + x_3 \leq 24$ and $x_1 + x_3 \leq 20$. This implies the second constraint must have some slack and can't be a boundary equation. Therefore, the boundary equations are: $x_2 = 0$

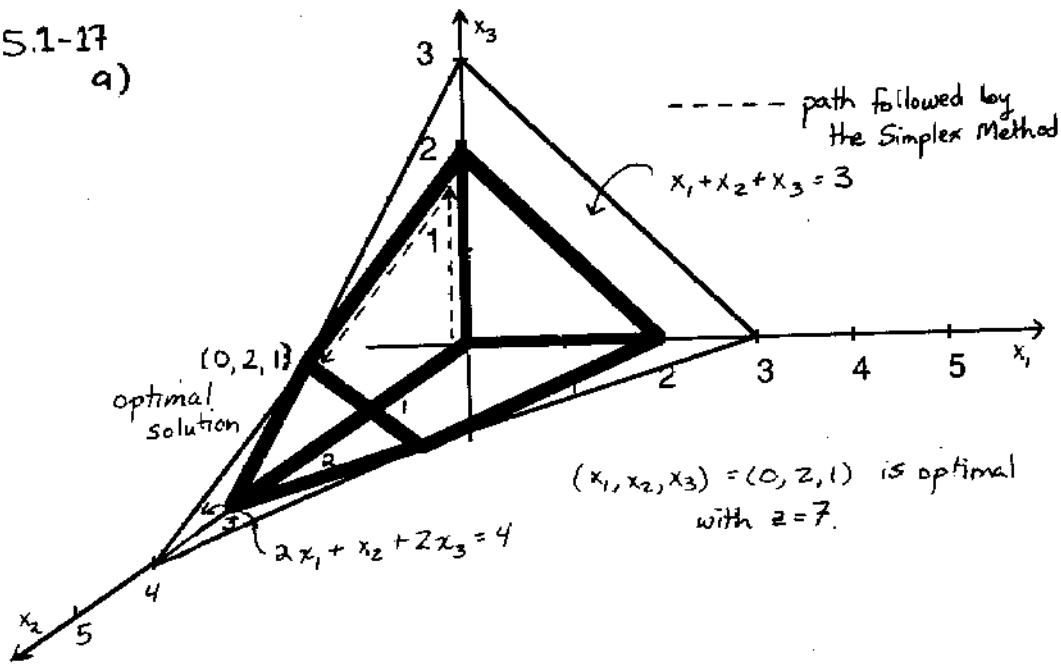
$x_1 + 3x_2 + 2x_3 = 30$

$3x_1 + 5x_2 + 3x_3 = 60$.

The optimal solution is $(x_1, x_2, x_3) = (10, 0, 10)$

5.1-17

a)



$(x_1, x_2, x_3) = (0, 2, 1)$ is optimal with $z = 7$.

5.1-17

(b) The path is chosen because moving along the edges chosen provides the greatest increase in the objective value for a unit move in the chosen direction for any possible edge at each vertex/decision point.

(c)

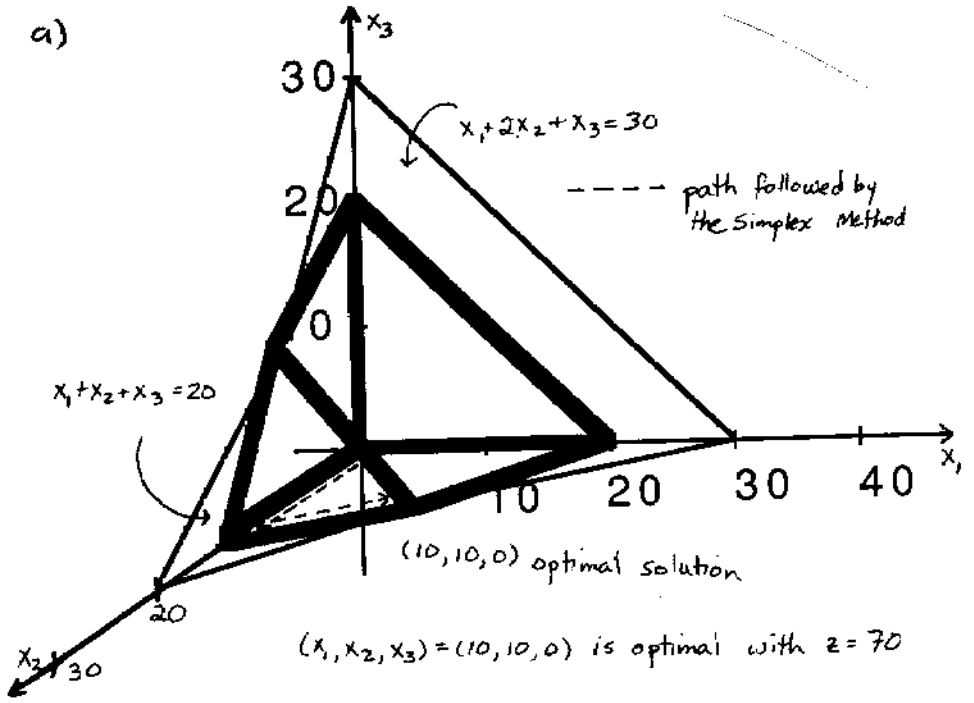
Edge	Constraint Boundary Equations	End Points	Additional Constraint
1	$x_2=0$, $x_1=0$	$(0,0,0)$	$x_3=0$
		$(0,0,2)$	$2x_1+x_2+2x_3=4$
2	$2x_1+x_2+2x_3=4$, $x_1=0$	$(0,0,2)$	$x_2=0$
		$(0,2,1)$	$x_1+x_2+x_3=3$

(d) and (e)

Corner Point	Defining Equations	Basic Feasible Solution (x_1,x_2,x_3,x_4,x_5)	Nonbasic Variables
$(0,0,0)$	$x_1=0$, $x_2=0$, $x_3=0$	$(0,0,0,4,3)$	x_1, x_2, x_3
$(0,0,2)$	$x_1=0$, $x_2=0$, $2x_1+x_2+2x_3=4$	$(0,0,2,0,1)$	x_1, x_2, x_4
$(0,2,1)$	$x_1=0$, $2x_1+x_2+2x_3=4$, $x_1+x_2+x_3=3$	$(0,0,2,0,1)$	x_1, x_4, x_5

The nonbasic variables, having value zero, are equivalent to indicating variables. They indicate that their associated inequality constraints are actually equalities. These associated equalities are the defining equations.

5.1-18 a)



$(x_1, x_2, x_3) = (10, 10, 0)$ is optimal with $z = 70$

(b) The path is chosen because moving along these edges provides the greatest increase in the objective value for a unit move in the chosen direction for any possible edge at each vertex / decision point.

(c)

Edge	Constraint Boundary Equations	End Points	Additional Constraint
1	$x_1=0$, $x_3=0$	(0,0,0)	$x_2=0$
		(0,15,0)	$x_1+2x_2+x_3=30$
2	$x_3=0$, $x_1+2x_2+x_3=30$	(0,15,0)	$x_1=0$
		(10,10,0)	$x_1+x_2+x_3=20$

(d) and (e)

Corner Point	Defining Equations	Basic Feasible Solution $(x_1, x_2, x_3, x_4, x_5)$	Nonbasic Variables
(0,0,0)	$x_1=0$, $x_2=0$, $x_3=0$	(0,0,0,20,30)	x_1, x_2, x_3
(0,15,0)	$x_1=0$, $x_3=0$, $x_1+2x_2+x_3=30$	(0, 15, 0, 5, 0)	x_1, x_3, x_5
(10,10,0)	$x_3=0$, $x_1+2x_2+x_3=30$, $x_1+x_2+x_3=20$	(10, 10, 0, 0, 0)	x_3, x_4, x_5

The nonbasic variables, having value zero, are equivalent to indicating variables. They indicate that their associated inequality constraints are actually equalities. These associated equalities are the defining equations.

5.1-19

(a) For the objective function Maximize $z = x_3$, both corner points $(4,2,4)$ and $(4,0,4)$ provide the maximum value $z = 4$

(b) For the objective function Maximize $z = -x_1 + 2x_3$, all the corner points, $(0,0,2)$, $(4,0,4)$, $(4,2,4)$, $(2,4,3)$ and $(0,4,2)$ provide the maximum value $z = 4$

5.1-20

(a) Geometrically each constraint is a plane and the points feasible for a given (inequality) constraint form a half-space. The line segment defined by any two feasible points must lie entirely on the feasible side of the plane and, therefore, all points on the line segment are feasible implying that the set of solutions for any one constraint is a convex set.

(b) Because the points in the feasible region of the linear program satisfy all constraints simultaneously, it must be the case that for any two feasible points the points on the line segment joining them must also satisfy each constraint (from part (a)). Therefore, the set of solutions that satisfy all constraints simultaneously is a convex set.

5.1-21

For the objective function Maximize $z = 3x_1 + 4x_2 + 3x_3$, starting at the point $(0,0,0)$ the first edge chosen would be that connecting to $(0,4,0)$ because moving along that edge increases z faster than any other edge.

From $(0,4,0)$ the edge that increases z fastest connects to either $(0,4,2)$ or $(2,4,0)$. From either of these the edge with fastest increase connects to $(2,4,3)$. From $(2,4,3)$ the only edge that will provide an increase in z connects to the optimal solution $(4,2,4)$

5.1-22 a)

Original Constraint	Boundary Equation	Indicating Variable
$x_1 \geq 0$	$x_1 = 0$	x_1
$x_2 \geq 0$	$x_2 = 0$	x_2
$x_3 \geq 0$	$x_3 = 0$	x_3
$x_1 + x_4 = 4$	$x_1 = 4$	x_4
$x_2 + x_5 = 4$	$x_2 = 4$	x_5
$x_1 + x_2 + x_6 = 6$	$x_1 + x_2 = 6$	x_6
$-x_1 + 2x_3 + x_7 = 4$	$-x_1 + 2x_3 = 4$	x_7

(b)

Corner Point Feasible Solution	Defining Equations	Basic Feasible Solutions $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$	Nonbasic Variables
(2, 4, 3)	$x_1 + x_2 = 6$, $x_2 = 4$, $-x_1 + 2x_3 = 4$	(2, 4, 3, 2, 0, 0, 0)	x_5, x_6, x_7
(4, 2, 4)	$x_1 + x_2 = 6$, $-x_1 + 2x_3 = 4$, $x_1 = 4$	(4, 2, 4, 0, 2, 0, 0)	x_4, x_6, x_7
(0, 4, 2)	$x_1 = 0$, $x_2 = 4$, $-x_1 + 2x_3 = 4$	(0, 4, 2, 4, 0, 2, 0)	x_1, x_5, x_7
(2, 4, 0)	$x_3 = 0$, $x_1 + x_2 = 6$, $x_2 = 4$	(2, 4, 0, 2, 0, 0, 6)	x_3, x_5, x_6

(c) Because the sets of defining equations of (4, 2, 4), (0, 4, 2) and (2, 4, 0) differ from the set of defining equations of (2, 4, 3) by only one equation, they are adjacent to (2, 4, 3). Since the sets of defining equations of (4, 2, 4), (0, 4, 2) and (2, 4, 0) differ by more than one equation they are not adjacent.

If we substitute "nonbasic variables" for "defining equations" and "variable" for "equation," we see the same statement is true.

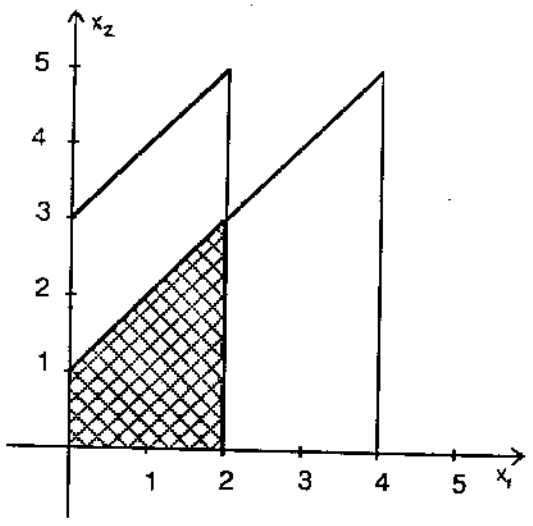
5.1-23

(a) x_5 enters

(b) x_4 leaves

(c) (4, 2, 4, 0, 2, 0, 0)

5.1-24



(a) The optimal solution is:

$$\begin{pmatrix} x_3 \\ x_1 \\ x_5 \end{pmatrix} = B^{-1}b = \frac{1}{27}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix}\begin{pmatrix} 180 \\ 270 \\ 180 \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \\ 50 \end{pmatrix}$$

$$z = c^T x = (8, 4, 6, 3, 9)\begin{pmatrix} 30 \\ 0 \\ 50 \\ 0 \\ 50 \end{pmatrix} = 990$$

(b) The shadow prices are:

$$c_B B^{-1} = \frac{1}{27}(6, 8, 9)\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 1.33 \\ 1 \\ 2.67 \end{pmatrix}$$

5.2-1 (a) The optimal solution is:

$$\begin{pmatrix} x_3 \\ x_1 \\ x_5 \end{pmatrix} = B^{-1}b = \frac{1}{27}\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix}\begin{pmatrix} 180 \\ 270 \\ 180 \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \\ 50 \end{pmatrix}$$

$$Z = c^T x = (8, 4, 6, 3, 9)\begin{pmatrix} 30 \\ 0 \\ 50 \\ 0 \\ 50 \end{pmatrix} = 990$$

(b) The shadow prices are:

$$c_B B^{-1} = \frac{1}{27}(6, 8, 9)\begin{pmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 1.33 \\ 1 \\ 2.67 \end{pmatrix}$$

5.2-2

$c = (5, 8, 7, 4, 6, 0, 0)$, $A = \begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$

$c_B = (0,0)$ $-c = (-5, -8, -7, -4, -6, 0, 0)$ so x_2 enters

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ so x_7 leaves the basis

Iteration 1: $\eta = \begin{pmatrix} -3/5 \\ 1/5 \end{pmatrix}$ $B_{NEW}^{-1} = \begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix}$ $x_B = \begin{pmatrix} x_6 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$c_B = (0,8)$ Revised Row 0: $(0, 8/5)\begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix} - (5, 8, 7, 4, 6, 0, 0)$

$= (-1/5, 0, -3/5, -4/5, 2/5, 0, 8/5)$

so x_4 enters the basis

Revised x_4 coefficients: $\begin{pmatrix} 1 & -3/5 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix}$ so x_6 leaves the basis

Iteration 2: $\eta = \begin{pmatrix} 5/4 \\ -1/2 \end{pmatrix}$ $B_{NEW}^{-1} = \begin{pmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{pmatrix}$ $x_B = \begin{pmatrix} x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5/4 & -3/4 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5 \end{pmatrix}$

$c_B = (4,8)$ Revised Row 0: $(1,1)\begin{pmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{pmatrix} - (5, 8, 7, 4, 6, 0, 0)$

$= (0, 0, 0, 0, 0, 1, 1)$

Current Solution is optimal

$(x_1, x_2, x_3, x_4, x_5) = (0, 5, 0, 5/2, 0)$ $z = 8 \cdot 5 + 4 \cdot 5/2 = 50$

5.2-3

$c = (4, 3, 6, 0, 0)$, $A = \begin{pmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 30 \\ 40 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 40 \end{pmatrix} = \begin{pmatrix} 30 \\ 40 \end{pmatrix}$

$c_B = (0,0)$ $-c = (-4, -3, -6, 0, 0)$ so x_3 enters the basis

Revised x_3 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ so x_4 leaves the basis

Iteration 1: $\eta = \begin{pmatrix} 1/3 \\ -1 \end{pmatrix}$ $B_{NEW}^{-1} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 40 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$

5.2-3 (cont')

$c_B = (6, 0)$ Revised Row 0 : $(2,0)\begin{pmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{pmatrix} - (4, 3, 6, 0, 0)$

$= (2, -1, 0, 2, 0)$ so x_2 enters the basis.

Revised x_2 coefficients : $\begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$ so x_5 leaves the basis.

Iteration 2 : $\eta = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $B^{-1}_{NEW} = \begin{pmatrix} 2/3 & -1/3 \\ -1 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 40 \end{pmatrix} = \begin{pmatrix} 20/3 \\ 10 \end{pmatrix}$

$c_B = (6, 3)$ Revised Row 0 : $(1,1)\begin{pmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{pmatrix} - (4, 3, 6, 0, 0)$

$= (1, 0, 0, 1, 1)$

Current Solution is optimal.

$(x_1, x_2, x_3) = (0, 10, \frac{20}{3})$ $\quad z = 10 \cdot 3 + \frac{20}{3} \cdot 6 = 70$

5.2-4

For Corner Point (0,0): $x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$

$x_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$ Row 0 = $(-3, -2, 0, 0)$

For Corner Point (3,0): $x_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$

$x_B = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\quad c_B = (3, 0)$

Row 0 $= \left(\frac{3}{2}, 0\right)\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} - (3, 2, 0, 0) = \left(0, -\frac{1}{2}, \frac{3}{2}, 0\right)$

For Corner Point (2,2): $x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$

$x_B = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\quad c_B = (3, 2)$

Row 0 : $(3, 2)\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} - (3, 2, 0, 0) = (0, 0, 1/3, 1/3)$

$(x_1, x_2) = (2, 2)$ is optimal with $z = 3 \cdot 2 + 2 \cdot 2 = 10$

5.2-5

$c = (1, 2, 0, 0)$ $\quad A = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

$c_B = (0, 0)$ $\quad -c = (-1, -2, 0, 0)$ so x_2 enters the basis

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ so x_3 leaves the basis

Iteration 1: $\eta = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}$ $\quad B_{NEW}^{-1} = \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}$ $\quad x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix}$

$c_B = (2, 0)$ $\quad$ Revised Row 0: $(2/3, 0)\begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} - (1, 2, 0, 0)$

$= (-1/3, 0, 2/3, 0)$ x_1 enters the basis

Revised x_1 coefficients: $\begin{pmatrix} 1/3 & 0 \\ -1/3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ x_4 leaves the basis

Iteration 2: $\eta = \begin{pmatrix} -1/2 \\ 3/2 \end{pmatrix}$ $\quad B_{NEW}^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$ $\quad x_B = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$c_B = (2, 1)$ $\quad$ Revised Row 0: $(1/2, 1/2)\begin{pmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} - (1, 2, 0, 0)$

$= (0, 0, 1/2, 1/2)$

Current Solution is optimal

$(x_1, x_2) = (2, 2)$ $\quad z = 2 \cdot 1 + 2 \cdot 2 = 6$

5.2-6

$c = (2, -2, 3, 0, 0, 0)$ $\quad A = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad x_B = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$

$c_B = (0, 0, 0)$ $\quad -c = (-2, 2, -3, 0, 0, 0)$ x_3 enters the basis

Revised x_3 coefficients: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ x_5 leaves the basis.

Iteration 1: $\eta = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$ $\quad B_{NEW}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$ $\quad x_B = \begin{pmatrix} x_4 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}$

$c_B = (0, 3, 0)$ $\quad$ Revised Row 0: $(0, 3, 0)\begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} - (2, -2, 3, 0, 0, 0)$

$= (4, -1, 0, 0, 3, 0)$ x_2 enters the basis

5.2-6 (cont')

Revised x_2 coefficients: $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ x_4 leaves the basis.

Iteration 2: $\eta = \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \end{pmatrix}$ $B^{-1}_{NEW} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -2 & -1 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$c_B = (-2, 3, 0)$ Revised Row 0: $(\frac{1}{2}, \frac{5}{2}, 0)\begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} - (2, -2, 3, 0, 0, 0)$

$= (\frac{5}{2}, 0, 0, \frac{1}{2}, \frac{5}{2}, 0)$

Current Solution is optimal

$(x_1, x_2, x_3) = (0, 1, 3)$ $z = -2 \cdot 1 + 3 \cdot 3 = 7$

5.2-7

a) $c = (10, 20, 0, 0, 0)$ $A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix}$

$c_B = (0, 0, 0)$ $-c = (-10, -20, 0, 0, 0)$ x_2 enters the basis

Revised x_2 coefficients: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ x_3 leaves the basis

Iteration 1: $\eta = \begin{pmatrix} 1/2 \\ -1/2 \\ -3/2 \end{pmatrix}$ $B^{-1}_{NEW} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix} = \begin{pmatrix} 15/2 \\ 9/2 \\ 45/2 \end{pmatrix}$

$c_B = (20, 0, 0)$ Revised Row 0: $(10, 0, 0)\begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{pmatrix} - (10, 20, 0, 0, 0)$

$= (-20, 0, 10, 0, 0)$ x_1 enters the basis

Revised x_1 coefficients: $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/2 \\ 13/2 \end{pmatrix}$ x_4 leaves the basis.

Iteration 2: $\eta = \begin{pmatrix} 1/3 \\ 2/3 \\ -13/3 \end{pmatrix}$ $B^{-1}_{NEW} = \begin{pmatrix} 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 2/3 & -13/3 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_2 \\ x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ -1/3 & 2/3 & 0 \\ 2/3 & -13/3 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ 12 \\ 45 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 3 \end{pmatrix}$

$c_B = (20, 10, 0)$ Revised Row 0: $(10/3, 40/3, 0)\begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{pmatrix} - (10, 20, 0, 0, 0)$

$= (0, 0, 10/3, 40/3, 0)$

Current Solution is optimal

$(x_1, x_2) = (3, 9)$ $z = 10 \cdot 3 + 20 \cdot 9 = 210$

5.2-7(b) $c = (5, 4, -1, 3, 0, 0)$ $A = \begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$

Iteration 0: $B = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 24 \\ 36 \end{pmatrix}$

$c_B = (0, 0)$ $\quad -c = (-5, -4, 1, -3, 0, 0)$ x_1 enters the basis.

Revised x_1 coefficients: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ x_5 leaves the basis.

Iteration 1: $\eta = \begin{pmatrix} 1/3 \\ -1 \end{pmatrix}$ $B^{-1}_{NEW} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}$ $x_B = \begin{pmatrix} x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$

$c_B = (5, 0)$ Revised Row 0: $(5/3, 0)\begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix} - (5, 4, -1, 3, 0, 0)$

$= (0, -2/3, -4, -4/3, 5/3, 0)$ x_3 enters

Revised x_3 coefficients: $\begin{pmatrix} 1/3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ x_6 leaves the basis.

Iteration 2: $\eta = \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}$ $B^{-1}_{NEW} = \begin{pmatrix} 1/12 & 1/4 \\ -1/4 & 1/4 \end{pmatrix}$ $x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/12 & 1/4 \\ -1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 24 \\ 36 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$

$c_B = (5, -1)$ Revised Row 0: $(2/3, 1)\begin{pmatrix} 3 & 2 & -3 & 1 & 1 & 0 \\ 3 & 3 & 1 & 3 & 0 & 1 \end{pmatrix} - (5, 4, -1, 3, 0, 0)$

$= (0, 1/3, 0, 2/3, 2/3, 1)$

Current Solution is optimal

$(x_1, x_2, x_3, x_4) = (11, 0, 3, 0)$ $\quad z = 11 \cdot 5 + 3(-1) = 52$

5.3-1

(a) $B^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$. The final constraint columns for (x_1, x_2, x_3) will be

$$B^{-1}\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

$c_B = (-1, 0, 2)$. The final objective coefficients for (x_1, x_2, x_3) will be

$$(-1, 0, 2)\begin{pmatrix} 5 & 1 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix} - (1, -1, 2) = (2, 0, 0)$$

The right-hand side is $B^{-1}b = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 5 \\ 11 \end{pmatrix}$. $z = (-1, 0, 2)\begin{pmatrix} 14 \\ 5 \\ 11 \end{pmatrix} = 8$

Final Tableau becomes:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	2	0	0	1	1	0	8
X2	1	0	5	1	0	1	3	0	14
X6	2	0	2	0	0	0	1	1	5
X3	3	0	4	0	1	1	2	0	11

(b) The Defining Equations are:

$$2x_1 - 2x_2 + 3x_3 = 5$$
$$x_1 + x_2 - x_3 = 3$$
$$x_1 = 0$$

5.3-2

(a) The final constraint columns for (x_1, x_2, x_3, x_4) will be:

$$B^{-1}\begin{pmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

The final objective coefficients for (x_1, x_2, x_3, x_4) will be:

$$-c + c_B B^{-1} A = -(4,3,1,2) + (3,2)\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 3 & 1 \end{pmatrix} = (3,0,2,0)$$

The right-hand side is $B^{-1}b = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. $Z = (3,2)\begin{pmatrix} 1 \\ 3 \end{pmatrix} = 9$

The final tableau becomes:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	3	0	2	0	1	1	9
X2	1	0	1	1	-1	0	1	-1	1
X4	2	0	2	0	3	1	-1	2	3

(b) The Defining Equations are:

$$4x_1 + 2x_2 + x_3 + x_4 = 5$$
$$3x_1 + x_2 + 2x_3 + x_4 = 4$$
$$x_1 = 0$$
$$x_3 = 0$$

5.3-3

The final constraint columns for (x_1, x_2, x_3) will be:

$$B^{-1}\begin{pmatrix} 2 & 2 & 1/2 \\ -4 & -2 & -3/2 \\ 1 & 2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 1/2 \\ -4 & -2 & -3/2 \\ 1 & 2 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The final objective coefficients for (x_1, x_2, x_3) will be:

$$-(6,1,2) + (0,2,6)\begin{pmatrix} 0 & 4 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (0,7,0)$$

The right-hand side will be $B^{-1}b = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$

$$Z = (0,2,6)\begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix} = 6$$

The final tableau becomes:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	7	0	2	0	2	6
X5	1	0	0	4	0	1	1	2	7
X3	2	0	0	4	1	-2	0	4	0
X1	3	0	1	0	0	1	0	-1	1

5.3-4 (a) The final constraint coefficients for (x_1, x_2, x_3) will be:

$$B^{-1}\begin{pmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ -\frac{3}{2} & 1 & 0 \end{pmatrix}.$$

The final objective coefficients will be:

$$-(1,-1,2) + (0,2,-1)\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ -\frac{3}{2} & 1 & 0 \end{pmatrix} = \left(\tfrac{3}{2}, 0, 0\right)$$

The right-hand side will be: $B^{-1}b = \begin{pmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 15 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$

$$Z = (0, 2, -1)\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} = 5$$

The final tableau becomes:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	1.5	0	0	0	1.5	0.5	5
X4	1	0	1	0	0	1	-1	-2	5
X3	2	0	0.5	0	1	0	0.5	0.5	3
X2	3	0	-1.5	1	0	0	-0.5	0.5	1

(b) The Defining Equations are:

$$2x_1 - x_2 + x_3 = 2$$
$$-x_1 + x_2 + x_3 = 4$$
$$x_1 = 0$$

5.3-5 (a) The constraint coefficients for (x_1, x_2, x_3) will be:

$$B^{-1}\begin{pmatrix} 8 & 2 & 3 \\ 4 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{16} & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 \\ -\frac{3}{8} & \frac{1}{4} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 2 & 3 \\ 4 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{9}{16} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 1 \end{pmatrix}$$

The objective coefficients for (x_1, x_2, x_3) will be:

$$-(20, 6, 8) + (20, 6, 0, 0)\begin{pmatrix} 1 & 0 & \frac{9}{16} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & -\frac{1}{8} \\ 0 & 0 & 1 \end{pmatrix} = \left(0, 0, -\tfrac{5}{4}\right)$$

The right-hand side is: $B^{-1}b = \begin{pmatrix} \frac{3}{16} & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 \\ -\frac{3}{8} & \frac{1}{4} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 200 \\ 100 \\ 50 \\ 20 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ 0 \\ 20 \end{pmatrix}$

$$Z = C_B^r X_B = (20, 6, 0, 0)\begin{pmatrix} 25 \\ 0 \\ 0 \\ 20 \end{pmatrix} = 500$$

5.3-5. a) (cont')

The tableau is:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	-1.25	2.25	0.5	0	0	500
X1	1	0	1	0	0.563	0.188	-0.13	0	0	25
X2	2	0	0	1	-0.75	-0.25	0.5	0	0	0
X6	3	0	0	0	-0.13	-0.38	0.25	1	0	0
X7	4	0	0	0	1	0	0	0	1	20

(b) The Revised Simplex Method would generate the reduced costs for Row 0 and then the revised column for x_3

(c) The Defining Equations are:

$$8x_1 + 2x_2 + 3x_3 = 200$$
$$4x_1 + 3x_2 = 100$$
$$x_3 = 0$$

$2x_1 + x_3 = 50$ is also a "tight" constraint

5.3-6. $B^{-1} = \begin{pmatrix} \frac{1}{12} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$ The final right-hand side will be:

$$B^{-1}b = \begin{pmatrix} \frac{1}{12} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 120 \\ 150 \end{pmatrix} = \begin{pmatrix} 55 \\ 15 \end{pmatrix}$$

$$Z^* = (6, 5, -1, 4)\begin{pmatrix} 55 \\ 0 \\ 15 \\ 0 \end{pmatrix} = 315$$

Also Z^* can be calculated by the reduced objective coefficients:

$$Z^* = \left(\frac{3}{4}, \frac{5}{4}\right)\begin{pmatrix} 120 \\ 150 \end{pmatrix} = 315$$

$b_1^* = 55,\ b_2^* = 15,\ Z^* = 315$

5.3-7. a) $[-c_1, -c_2, -c_3 \mid 0\ 0 \mid 0] + [\frac{3}{5}, \frac{4}{5}]\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 0 & b \\ 2 & 1 & 3 & 0 & 1 & 2b \end{array}\right] = [\frac{7}{11}, 0, 0, \frac{3}{5}, \frac{4}{5}, Z^*]$

Or:

$$[-c_1, -c_2, -c_3 \mid 0\ 0 \mid 0] + \left[\frac{11}{5}, 2, 3, \frac{3}{5}, \frac{4}{5}, \frac{11}{5}b\right] = [\tfrac{7}{11}, 0, 0, \tfrac{3}{5}, \tfrac{4}{5}, Z^*]$$

So $c_1 = \frac{11}{5} - \frac{7}{11} = \frac{3}{2}$

$c_2 = 2$

$c_3 = 3$

b) $B^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$ $\quad B^{-1}b = b^* \Leftrightarrow \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} b \\ 2b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1/5 \\ 3/5 \end{pmatrix} b = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow b = 5$$

5.3-7, c) Using (a), $Z^* = C_B b^* = (C_2, C_3)\begin{pmatrix}1\\3\end{pmatrix} = (2,3)\begin{pmatrix}1\\3\end{pmatrix} = 11$.

Using (b), $Z^* = \tilde{C}_B b = \left(\frac{3}{5}, \frac{4}{5}\right)\begin{pmatrix}b\\2b\end{pmatrix} = \left(\frac{3}{5}, \frac{4}{5}\right)\begin{pmatrix}5\\10\end{pmatrix} = 11$.

5.3-8 First iteration: $\frac{5}{2}$ of row 2 is added to row 0

(i.e. premultiply $A_0 = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right| \begin{matrix}4\\12\\18\end{matrix}\Big]$ by $[0, \frac{5}{2}, 0]$

and add to row 0)

Second iteration; row 3 is added to row 0

(i.e. premultiply $A_1 = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 3 & 0 & 0 & -1 & 1 \end{array}\right| \begin{matrix}4\\6\\6\end{matrix}\Big] = \begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&-1&1\end{bmatrix} A_0$

by $[0,0,1]$ and add to row 0)

Therefore, the final row 0 is:

Init row 0 $+ [0, \frac{5}{2}, 0] A_0 + [0,0,1]\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&-1&1\end{bmatrix} A_0$

Final row 0 $= [-3, -5 \,|\, 0, 0, 0 \,|\, 0] + [0, \frac{3}{2}, 1] A_0$ as desired

$$\left([0, \tfrac{5}{2}, 0] + [0,0,1]\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&-1&1\end{bmatrix} = [0, \tfrac{1}{2}, 1] \right)$$

5.3-9. a) Use the columns corresponding to artificial variables in exactly the same way a slack column would have been used.

Note: The shadow price of this column may be positive or negative.

b) For the reversed inequalities, use the negative of the column corresponding to the slack variable (the artificial column may be discarded) in exactly the same formulae.

c) Same as (b)

d) No change, use slack and artificial variables as above.

5.3-10. For the tableau see the solution to problem 18 in chapter 4. The columns that will contain S^* are those corresponding to x_5 and x_6 since those columns contain the identity in the initial tableau.

5.3-11.

(a) The final constraint columns for $(x_1, x_2, x_3, x_4, x_6)$ are:

$$B^{-1}\begin{pmatrix} 1 & 4 & 2 & -1 & 0 \\ 3 & 2 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3/10 & -1/10 \\ -2/10 & 2/5 \end{pmatrix}\begin{pmatrix} 1 & 4 & 2 & -1 & 0 \\ 3 & 2 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3/5 & -3/10 & 1/10 \\ 1 & 0 & -2/5 & 3/10 & -2/5 \end{pmatrix}$$

The final objective coefficients for $(x_1, x_2, x_3, x_4, x_6)$ are:

$$(-4M+2, -6M+3, -2M+2, M, M) - (-6M+3, -4M+2)\begin{pmatrix} 0 & 1 & 3/5 & -3/10 & 1/10 \\ 1 & 0 & -2/5 & 3/10 & -2/5 \end{pmatrix}$$

$$= (0, 0, 1, \tfrac{1}{2}, \tfrac{1}{2})$$

The right-hand side is: $B^{-1}b = \begin{pmatrix} 3/10 & -1/10 \\ -2/10 & 2/5 \end{pmatrix}\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 4/5 \end{pmatrix}$

$$Z = -14M + c_B^T x_B = -14M + (-6M+3, -4M+2)\begin{pmatrix} 9/5 \\ 4/5 \end{pmatrix} = 7$$

The final tableau is:

Bas Var	Eq No	Z	X1	X2	Coefficient of X3	X4	x_6	$\bar{x}_5$	$\bar{x}_7$	Right side
								1M	1M	
Z	0	-1	0	0	1	0.5	0.5	-0.5	-0.5	-7
X2	1	0	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
X1	2	0	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8

b) The constraints in the original tableau are $(A \mid I \mid I \mid b)$ with the second identity matrix corresponding to the artificial variables. When we pre-multiply by M, we get:

$$(A^* \mid S^* \mid L^* \mid b^*) = M(A \mid I \mid I \mid b) = (MA \mid M \mid M \mid Mb)$$

and we have $M = S^* = L^*$. For this problem $M = \begin{pmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{pmatrix}$

The original Row 0 $t = (c + e^T AM \mid Me^T \mid 0 \mid -Me^T b)$

and in the final tableau

$$t^* = t + vT = (z^* + c \mid -y^* \mid Me^T - y^* \mid z^*)$$

$$= (c + e^T AM \mid Me^T \mid 0 \mid -Me^T b) + v^T(A \mid I \mid I \mid b)$$

5.3-11 b) (cont') $t^* = c + e^T AM + v^T A \mid Me^T + v \mid v \mid -Me^T b + vb)$

We can see $v = -y^* + Me^T$

For this problem $v = \left(-\frac{1}{2} + M, \; -\frac{1}{2} + M\right)$

(c) If $t = (2, 3, 2, 0, M, 0, M, 0)$, then in matrix notation

$t = (c \mid 0 \mid Me^T \mid Me^T b)$.

$t^* = t + v^T T = (z^* + c \mid y^* \mid Me^T - y^* \mid Z^*)$

$= (c \mid 0 \mid Me^T \mid Me^T b) + v^T (A \mid I \mid I \mid b)$

$= c + v^T A \mid v \mid Me^T + v \mid Me^T b + v^T b)$.

We conclude $v = -y^*$. For this problem $v = \left(-\frac{1}{2}, -\frac{1}{2}\right)$

(d) The defining equations are $x = Mb$ or $M^{-1}x = b$.

- or -

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 8 \\ 3x_1 + 2x_2 &= 6 \\ x_3 &= 0. \end{aligned}$$

5.3-12

(a) The final constraint columns for (x_1, x_2, x_3) are:

$$B^{-1}\begin{pmatrix} 1 & 3 & 2 \\ 1 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 1 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & -2 & 2 \end{pmatrix}$$

The final objective coefficients for (x_1, x_2, x_3) are:

$$-(2, 4, 3) + (2, 0)\begin{pmatrix} 1 & 3 & 2 \\ 0 & -2 & 2 \end{pmatrix} = (0, 2, 1)$$

The right-hand side is $B^{-1}b = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$. $z = c_B^T x_B = (2, 0)\begin{pmatrix} 20 \\ 10 \end{pmatrix} = 40$

The final tableau is:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X_5	X_4	X6	Right side
							1M	1N	
Z	0	1	0	2	1	0	2	0	40
X1	1	0	1	3	2	0	1	0	20
X5	2	0	0	-2	2	1	1	-1	10

(b) The Defining Equations are:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 20 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

S.3-13a) $-2x_1 + 2x_2 + x_3 + x_4 = 10$

$3x_1 + x_2 - x_3 + x_5 = 20$

$x_1 - x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = -5$

$4x_2 + \frac{1}{2}x_3 + \frac{3}{2}x_4 + x_5 = 35$

$x_1 + 3x_2 + x_4 + x_5 = 30$

$8x_2 + x_3 + 3x_4 + 2x_5 = 70$

$(x_1, x_2, x_3) = (30, 0, 70)$ is optimal

$z = 30 \cdot 3 + 70 \cdot 2 = 230$

(b) If we add (-3) of the 1st row from the result in (a) and (-2) of the 2nd row from the result in (a) to our original objective, we get:

$3x_1 + 7x_2 + 2x_3$

$-3x_1 - 9x_2 - 3x_4 - 3x_5$

$-16x_2 - 2x_3 - 6x_4 - 4x_5$

$-18x_2 - 9x_4 - 7x_5$

The shadow prices are 9 and 7

(c) The Defining Equations are:

$-2x_1 + 2x_2 + x_3 = 10$

$3x_1 + x_2 - x_3 = 20$

$x_2 = 0$

(d) $B = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$. $x_B = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 30 \\ 70 \end{pmatrix}$

$y^* = (3, 2) \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = (9, 7)$

Revised Row 0 $= (9, 7) \begin{pmatrix} -2 & 2 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 & 1 \end{pmatrix} - (3, 7, 2, 0, 0) = (0, 18, 0, 9, 7)$

so, the current solution is optimal.

(e) The final tableau is:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	18	0	9	7	230
X1	1	0	1	3	0	1	1	30
X3	2	0	0	8	1	3	2	70

Chapter 6: Duality Theory and Sensitivity Analysis

6.1-1(a)

	x_1	x_2	
y_1	1	0	≤ 4
y_2	1	3	≤ 15
y_3	2	1	≤ 10
	IV	IV	
	3	2	

Dual: Minimize $4y_1 + 15y_2 + 10y_3$

subject to: $y_1 + y_2 + 2y_3 \geq 3$

$3y_2 + y_3 \geq 2$

$y_i \geq 0$ for $i = 1, 2, 3$

(b)

	x_1	x_2	x_3	x_4	
y_1	3	2	-3	1	≤ 24
y_2	3	3	1	3	≤ 36
	IV	IV	IV	IV	
	5	4	-1	3	

Dual: Minimize $24y_1 + 36y_2$

subject to: $3y_1 + 3y_2 \geq 5$

$2y_1 + 3y_2 \geq 4$

$-3y_1 + y_2 \geq -1$

$y_1 + 3y_2 \geq 3$

$y_1 \geq 0, y_2 \geq 0$

6.1-2 (a) Minimize $15y_1 + 12y_2 + 45y_3$

subject to: $-y_1 + y_2 + 5y_3 \geq 10$

$2y_1 + y_2 + 3y_3 \geq 20$

$y_i \geq 0$ for $i = 1, 2, 3$

(b) Minimize $4y_1 + 2y_2 + 12y_3$

subject to: $-y_1 + 2y_2 + y_3 \geq 2$

$y_1 - y_2 + y_3 \geq -2$

$y_1 + y_2 + 3y_3 \geq 3$

$y_i \geq 0$ for $i = 1, 2, 3$.

6.1-3. (a)

	x_1	x_2	x_3	x_4	
y_1	1	-2	4	3	≤ 20
y_2	-4	6	5	-4	≤ 40
y_3	2	-3	3	8	≤ 50
	IV	IV	IV	IV	
	5	1	3	4	

Dual: Minimize $20y_1 + 40y_2 + 50y_3$

subject to
$$y_1 - 4y_2 + 2y_3 \ge 5$$
$$-2y_1 + 6y_2 - 3y_3 \ge 1$$
$$4y_1 + 5y_2 + 3y_3 \ge 3$$
$$3y_1 - 4y_2 + 8y_3 \ge 4$$
$$y_i \ge 0 \text{ for } i = 1, 2, 3.$$

(b) The dual problem has no feasible solutions.

6.1-4. (a) Apply the simplex method to the dual of the problem, since the dual has fewer constraints (not including nonnegativity constraints). We expect the simplex method will pass through fewer basic feasible solutions

(b) Apply the simplex method to the primal problem, since it has fewer constraints (not including nonnegativity constraints). We expect the simplex method will pass through fewer basic feasible solutions.

6.1-5. (a) Dual is Minimize $12y_1 + y_2$

subject to:
$$y_1 + 2y_2 \ge -1$$
$$y_1 \ge -2$$
$$2y_1 - y_2 \ge -1$$
$$y_1 \ge 0, y_2 \ge 0$$

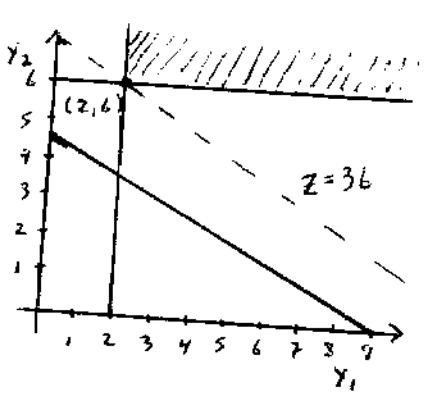
(b) It is clear from the dual problem that $(y_1, y_2) = (0, 0)$ is the optimal dual solution. By strong duality $z = 0 \le 0$.

6.1-6. a) Dual is

Minimize $w = 3y_1 + 5y_2$

subject to
$$y_1 \ge 2$$
$$y_2 \ge 6$$
$$y_1 + 2y_2 \ge 9$$
$$y_1 \ge 0, y_2 \ge 0$$

b)



Optimal. $(y_1^*, y_2^*) = (2, 6)$

So shadow prices for resource 1 and 2 are $y_1^* = 2$, $y_2^* = 6$.

6.1-6. c)

Optimal Solution

Value of the
Objective Function: Z = 36

Variable	Value
X_1	3
X_2	5
X_3	0

Constraint	Slack or Surplus	Shadow Price
1	0	2
2	0	6

Sensitivity Analysis

Objective Function Coefficient

Current Value	Allowable Range To Stay Optimal Minimum	Maximum
2	0	$+\infty$
6	3.5	$+\infty$
9	$-\infty$	14

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
3	0	$+\infty$
5	0	$+\infty$

6.1-7. a) Dual is

Minimize $w = 6y_1 + 4y_2$

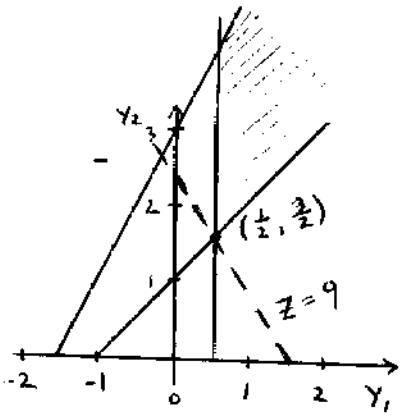
subject to $2y_1 \geq 1$

$2y_1 - y_2 \geq -3$

$-2y_1 + 2y_2 \geq 2$

$y_1 \geq 0,\ y_2 \geq 0$

b)



Optimal: $(y_1^*, y_2^*) = (\frac{1}{2}, \frac{3}{2})$

Shadow prices for resources 1 and 2 are $y_1^* = \frac{1}{2}$, $y_2^* = \frac{3}{2}$ resp.

c) **Optimal Solution**

Value of the
Objective Function: Z = 9

Variable	Value
X_1	5
X_2	0
X_3	2

Constraint	Slack or Surplus	Shadow Price
1	0	0.5
2	0	1.5

Sensitivity Analysis

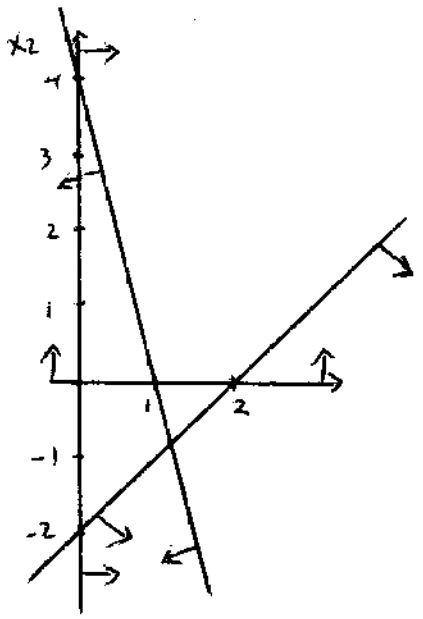
Objective Function Coefficient

Current Value	Allowable Range To Stay Optimal Minimum	Maximum
1	0	$+\infty$
-3	$-\infty$	-0.5
2	-1	7

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
6	-4	$+\infty$
4	0	$+\infty$

6.1-8. a)



There is no feasible region

b) Dual is

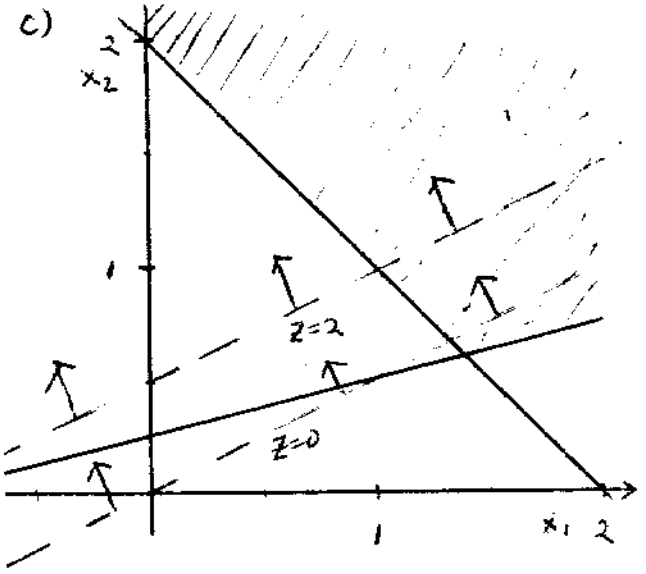
Minimize $w = -Y_1 + 4Y_2$

subject to $-Y_1 + 4Y_2 \geq 1$

$Y_1 + Y_2 \geq 2$

$Y_1 \geq 0,\ Y_2 \geq 0$

c)



We see that Z can be increased infinitely

One possible solution: $(0, y_2)$ let $y_2 \to \infty$.

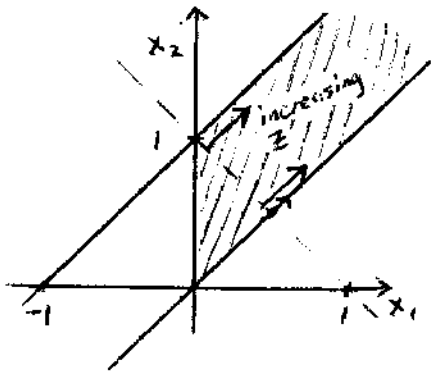
6.1-9. Primal: Max $Z = x_1 + x_2$

subject to: $-x_1 + x_2 \leq 1$

$x_1 - x_2 \leq 0$

$x_1 \geq 0,\ x_2 \geq 0$

(Let $x_1 = x_2 = c$ and $c \to \infty$
$Z = 2c$ is unbounded)



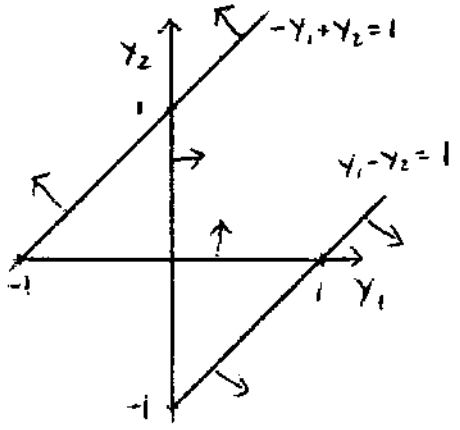
Dual: Min $w = Y_1$

subject to: $-Y_1 + Y_2 \geq 1$

$Y_1 - Y_2 \geq 1$

$Y_1 \geq 0,\ Y_2 \geq 0$

(No feasible points)



6.1-10. Primal: Max $Z = X_1$

subject to: $X_1 - X_2 \le 0$

$-X_1 + X_2 \le -1$

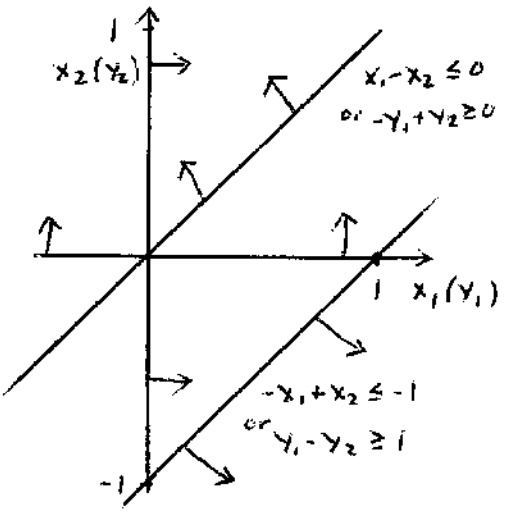
$X_1 \ge 0, X_2 \ge 0$

Dual: Min $w = -Y_2$

subject to: $Y_1 - Y_2 \ge 1$

$-Y_1 + Y_2 \ge 0$

$Y_1 \ge 0, Y_2 \ge 0$



The graph to the right shows that neither the primal nor the dual have any feasible points. (Both feasible regions are the same)

6.1-11. Primal: Max $Z = X_1 + X_2$

subject to: $X_1 \le -1$

$X_1 + X_2 \le 0$

$X_1 \ge 0, X_2 \ge 0$

Clearly infeasible (what is X_1?)

Dual: Min $w = -Y_1$

subject to: $Y_1 + Y_2 \ge 1$

$Y_2 \ge 1$

$Y_1 \ge 0, Y_2 \ge 0$

Clearly, $(c, 1)$, $c \to \infty$ is feasible, unbounded

6.1-12. Let X° and Y° be a primal feasible point and a dual feasible point, respectively. Then by weak duality,

$$-\infty < cX^\circ \le Y^\circ b < \infty$$

Furthermore, for <u>any</u> primal feasible point X and <u>any</u> dual feasible point y,

$$cX \le Y^\circ b \quad \text{and} \quad cX^\circ \le y b$$

This means that the primal problem <u>cannot</u> be unbounded (since it is bounded above by $Y^\circ b$), and, similarly, the dual problem cannot be unbounded (since it is bounded below by cX°). Therefore, since the primal problem (and the dual problem) has feasible solution and the objective is not unbounded, it must have an optimal solution.

6.1-13. a) From primal, $Ax \le b, x \ge 0$, and from dual, $y^TA \ge c, y \ge 0$. Thus,

$$y^TA - c^T \ge 0, \; x \ge 0, \text{ so } (y^TA - c^T)x \ge 0$$
$$b - Ax \ge 0, \; y \ge 0, \text{ so } y^T(b - Ax) \ge 0$$

In other words,

$$y^TAx \ge c^Tx \quad \text{and} \quad y^Tb \ge y^TAx$$

So $y^Tb \ge y^TAx \ge c^Tx$,

which is weak duality.

b) There are many ways to prove this. The simplest is by contradiction: Assume primal z can be increased indefinitely and the dual problem <u>does</u> have a feasible solution. By above argument, we see that $c^Tx \le \tilde{y}^Tb$ for all primal feasible x, given $\tilde{y}$ is a dual feasible solution. But this means z is bounded above (contradiction). Thus, if z is unbounded (can be increased indefinitely), then dual must not have any feasible solutions.

6.1-14. Primal: Max $z = cx$ subject to $Ax \le b$, $x \ge 0$

Dual: Min $v = yb$ subject to $yA \ge c$, $y \ge 0$

Since changing b to $\bar{b}$ keeps the dual feasible region unchanged, y^* must be feasible for the new problem. Let $\bar{y}$ be the optimal solution for the new dual. Then clearly $\bar{y}\bar{b} \le y^*\bar{b}$ since $\bar{y}$ is optimal. Further, by strong duality, we know that $c\bar{x} = \bar{y}\bar{b}$. Thus, $c\bar{x} = \bar{y}\bar{b} \le y^*\bar{b}$.

6.1-15. a) True. If A is an $n \times m$ matrix (n rows, m columns), then, in standard form, the number of functional constraints for the primal is n, for the dual is m. The number of variables in the primal is m, in the dual is n. So for both the sum is $m+n$.

b) False. This cannot be true since the weak and strong duality theorems imply that the primal and dual objective function values are the same <u>only</u> at optimality.

c) False. If the primal problem has an unbounded objective function, the dual problem must be infeasible (since by weak duality, if the dual has a feasible point, $\tilde{y}$, the objective value of the primal $z = cx \le \tilde{y}b$.)

6.2-1. a) Iter 0: Since all coefficients are zero, at the current solution (0,0), the 3 resources (production time per week at plant 1, 2 & 3) are "free goods." That is, having more doesn't increase our objective.

Iter 1: $[0, \frac{5}{2}, 0]$ Now resource 2 has been entirely used up and contributes $\frac{5}{2}$ to profit per unit of resource. Since this is positive it is worthwhile to continue fully using this resource.

Iter 2: $[0, \frac{3}{2}, 1]$ Resources 2 & 3 are used up and contributes a positive unit to profit. Resource 1 is a free good while resources 2 & 3 contribute $\frac{3}{2}$ and 1 per unit of resource, resp.

b) Iter 0: $[-3, -5]$ Both activities 1 & 2 (# batches of product 1 & 2 produced per week) can be initiated to give a more profitable allocation of the resources. That is, the current contribution to profits of the resources required to produce 1 batch of product 1 or 2 is smaller than the unit profit per batch of product 1 or 2, resp.

Iter 1: $[-3, 0]$ Again activity 1 can be initiated to give a more profitable use of resources, but activity 2 is already being produced (or the resources are being used just as well in other activities).

Iter 2: $[0, 0]$ Both activities are being produced (or the resources are being used just as profitably elsewhere).

c) Iter 1: Since activities 1 & 2 can be initiated to increase profit (given same amount of resources), we choose to increase one of these. We choose activity 2 as the entering activity (basic variable) since it increases the profit by 5 for every unit of product 2 produced (as opposed to 3 for product 1).

Iter 2: Only activity 1 can be initiated for more profit, so we do so.

Iter 3: Both activity 1 & 2 are being used. Furthermore, since the coefficients for x_3, x_4, x_5 are non-negative, it is not worthwhile to cut back on the use of any of the resources. Thus, we must be at the optimal solution.

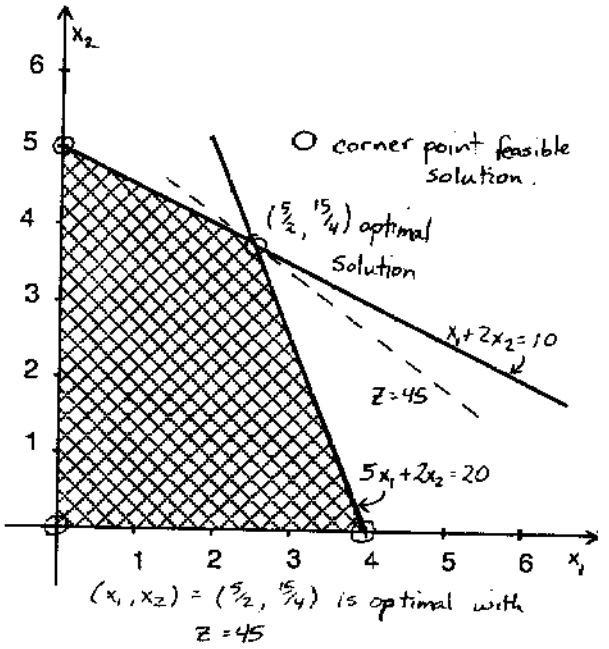
6.3-1. (a) Minimize $v = 20y_1 + 10y_2$

subject to: $5y_1 + y_2 \geq 6$

$2y_1 + 2y_2 \geq 8$

$y_1 \geq 0, y_2 \geq 0$

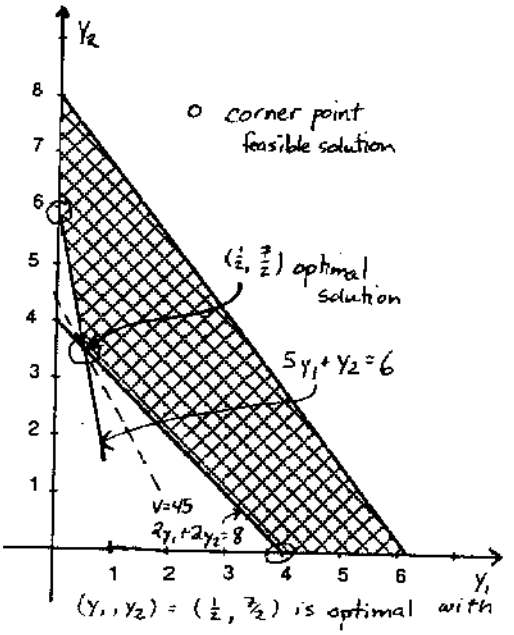
6.3-1. (b) Primal



$(x_1, x_2) = (\frac{5}{2}, \frac{15}{4})$ is optimal with $Z = 45$

Corner Point infeasible solutions are:
(0,10) and (10,0).

Dual



$(y_1, y_2) = (\frac{1}{2}, \frac{7}{2})$ is optimal with $V = 45$

Corner Point infeasible solutions are:
(0,4), (0,0), $(\frac{6}{5}, 0)$.

(c)

Primal Basic Solution (x_1, x_2, x_3, x_4)	Feasible?	Z	Dual Basic Solution (y_1, y_2, y_3, y_4)	Feasible?
(0, 5, 10, 0)	Yes	40	(0, 4, -2, 0)	No
(0, 0, 20, 10)	Yes	0	(0, 0, -6, -8)	No
(4, 0, 0, 6)	Yes	24	$(\frac{6}{5}, 0, 0, -\frac{28}{5})$	No
$(\frac{5}{2}, \frac{15}{4}, 0, 0)$	Yes	45	$(\frac{1}{2}, \frac{7}{2}, 0, 0)$	Yes
(0, 10, 0, -10)	No	80	(4, 0, 14, 0)	Yes
(10, 0, -30, 0)	No	60	(0, 6, 0, 4)	Yes

(d)

	Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
	Z	0	1	-6	-8	0	0	0
0	X3	1	0	5	2	1	0	20
	X4	2	0	1	2*	0	1	10

Solutions

Primal	Dual
(0, 0, 20, 10)	(0, 0, -6, -8)

	Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
	Z	0	1	-2	0	0	4	40
1	X3	1	0	4*	0	1	-1	10
	X2	2	0	0.5	1	0	0.5	5

Primal	Dual
(0, 5, 10, 0)	(0, 4, -2, 0)

6.3-1. d) (cont')

	Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	Right side
	Z	0	1	0	0	0.5	3.5	45
2	X1	1	0	1	0	0.25	-0.25	2.5
	X2	2	0	0	1	-0.13	0.625	3.75

$(5/2, 15/4, 0, 0)$ $(1/2, 7/2, 0, 0)$

6.3-2. (a) Minimize $v = 8y_1 + 4y_2$

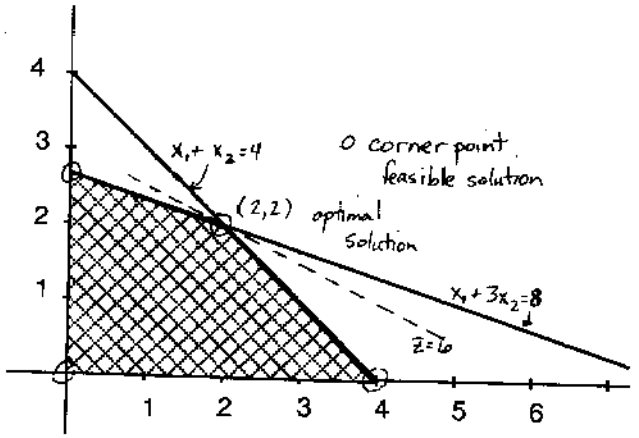
subject to:

$y_1 + y_2 \geq 1$

$3y_1 + y_2 \geq 2$

$y_1 \geq 0 \quad y_2 \geq 0$

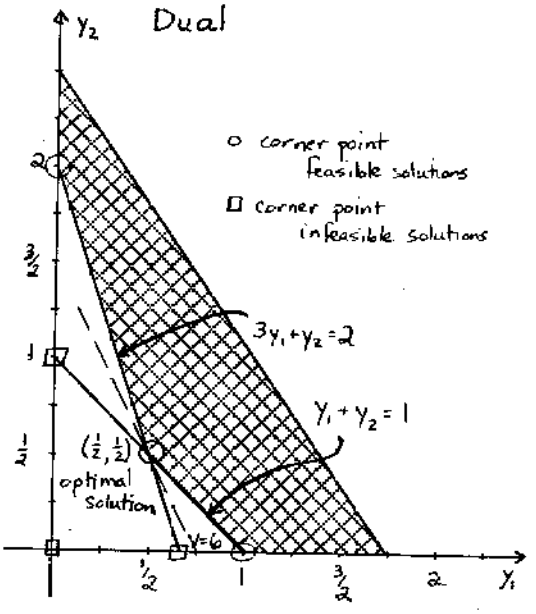
(b) Primal



$(x_1, x_2) = (2, 2)$ is optimal with $z = 6$

Corner Point infeasible solutions are:
(8,0) and (0,4)

Dual



$(y_1, y_2) = (\frac{1}{2}, \frac{1}{2})$ is optimal with $v = 6$

(c)

Primal Basic Solution (x_1, x_2, x_3, x_4)	Feasible?	z	Dual Basic Solution (y_1, y_2, y_3, y_4)	Feasible?
$(4, 0, 4, 0)$	Yes	4	$(0, 1, 0, -1/3)$	No
$(0, 0, 8, 4)$	Yes	0	$(0, 0, -1, -2)$	No
$(0, 8/3, 0, 4/3)$	Yes	$16/3$	$(2/3, 0, -1/3, 0)$	No
$(2, 2, 0, 0)$	Yes	6	$(1/2, 1/2, 0, 0)$	Yes
$(0, 4, -2, 0)$	No	8	$(0, 2, -1, 0)$	Yes
$(8, 0, 0, -4)$	No	8	$(1, 0, 0, 1)$	Yes

6.3-2. d)

	Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
	Z	0	1	-1	-2	0	0	0
0	X3	1	0	1	3*	1	0	8
	X4	2	0	1	1	0	1	4

	Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
	Z	0	1	-0.33	0	0.667	0	5.333
1	X2	1	0	0.333	1	0.333	0	2.667
	X4	2	0	0.667*	0	-0.33	1	1.333

	Bas Var	Eq No	Z	X1	X2	X3	X4	Right side
	Z	0	1	0	0	0.5	0.5	6
2	X2	1	0	0	1	0.5	-0.5	2
	X1	2	0	1	0	-0.5	1.5	2

Solutions	Primal	Dual
0	$(0,0,8,4)$	$(0,0,-1,-2)$
1	$(0, \frac{8}{3}, 0, \frac{4}{3})$	$(\frac{2}{3}, 0, -\frac{1}{3}, 0)$
2	$(2,2,0,0)$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$

6.3-3.

Nonbasic Primal Variables	Associated Dual Variables	Nonbasic Dual Variables
x_1, x_2	y_4, y_5	y_1, y_2, y_3
x_1, x_4	y_4, y_2	y_1, y_3, y_5
x_4, x_5	y_2, y_3	y_1, y_4, y_5
x_3, x_5	y_1, y_3	y_2, y_4, y_5
x_2, x_3	y_5, y_1	y_2, y_3, y_4
x_1, x_5	y_4, y_3	y_1, y_2, y_5
x_3, x_4	y_1, y_2	y_3, y_4, y_5
x_2, x_5	y_5, y_3	y_1, y_2, y_4

In all cases Complementary Slackness holds: $x_1 y_4 = 0$, $x_2 y_5 = 0$, $x_3 y_1 = 0$, $x_4 y_2 = 0$, $x_5 y_3 = 0$.

6.3-4. If either the primal or the dual has a degenerate optimal basic feasible solution, then the other may have multiple solutions.

For example, in the problem Maximize $3x_1$

subject to $a_{11}x_1 + x_2 = 0$

$-2x_1 + x_3 = 1$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

If $a_{11} > 0$, we can pivot and get an alternative optimal solution to the dual problem.

If $a_{11} \leq 0$, we cannot.

6.3-4. (cont') The converse is true, however. If a problem has multiple optimal solutions, then two of them must be adjacent corner points. To move from the tableau of one solution to that of the other requires exactly one pivot. Let's say x_j enters and x_k leaves. A partial tableau is below:

	x_j	RHS
	$\bar{c}_j$	
x_k	$\bar{a}_{kj}$	$\bar{b}_k$

$\bar{a}_{kj}$ must be positive and $\bar{b}_k \geq 0$.

If $\bar{b}_k > 0$, then $\bar{c}_j = 0$ or z would change value with the pivot.

If $\bar{b}_k = 0$, then x_j pivots in at value zero and the resulting tableau represents the same corner point, contradicting the assumption that the two optimal solutions are distinct.

6.3-5. a) Minimize $v = y_1$

subject to:

$$y_1 \geq 2$$
$$-y_1 \geq -4$$
$$y_1 \geq 0$$

The optimal solution is $v = y_1 = 2$

(b) $(y_1, y_2, y_3) = (2, 0, 2)$ is the optimal basic feasible solution for the dual. By Complementary Slackness, $y_1 x_3 = y_2 x_1 = y_3 x_2 = 0$, so $x_2 = x_3 = 0$. Since $x_1 - x_2 + x_3 = 1$, $(x_1, x_2, x_3) = (1, 0, 0)$ is optimal for the primal

(c) For $c_1 > 4$ the dual will have no feasible solution. Therefore, the primal objective function will be unbounded.

6.3-6. a) Minimize $w = 10y_1 + 10y_2$

subject to:

$$y_1 + 3y_2 \geq 2$$
$$2y_1 + 3y_2 \geq 7$$
$$y_1 + 2y_2 \geq 4$$
$$y_1 \geq 0,\ y_2 \geq 0$$

(b) $(0, \frac{5}{2})$ is feasible for the dual problem. Using Weak Duality, $w = 10 \cdot 0 + 10 \cdot \frac{5}{2} = 25 \geq z$ so the optimal primal objective value is less than 25.

6.3-6.

(c)

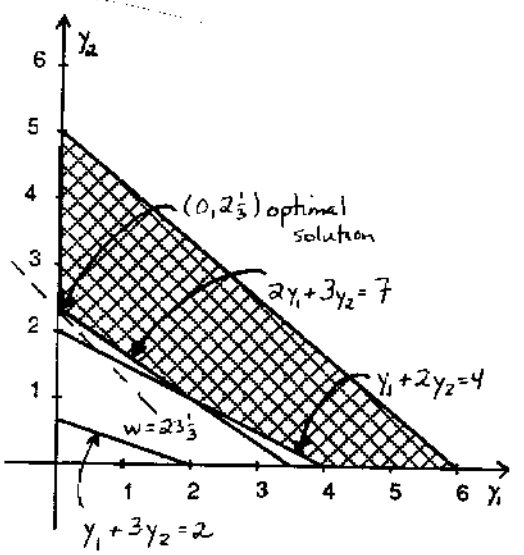
Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-7	-4	0	0	0
X4	1	0	1	2	1	1	0	10
X5	2	0	3	3*	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	5	0	0.667	0	2.333	23.33
X4	1	0	-1	0	-0.33*	1	-0.67	3.333
X2	2	0	1	1	0.667	0	0.333	3.333

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	3	0	0	2	1	30
X3	1	0	3	0	1	-3	2	-10
X2	2	0	-1	1	0	2	-1	10

The primal basic solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 10, -10, 0, 0)$ which is not feasible. The dual basic solution is $(y_1, y_2, z_1 - c_1, z_2 - c_2, z_3 - c_3) = (2, 1\frac{1}{3}, 0, 0)$

(d)



$(y_1, y_2) = (0, 2\frac{1}{3})$ is optimal with $w = 23\frac{1}{3}$.

From the dual solution y_2, y_3 and y_5 are basic, therefore, x_3, x_5 and x_1 are nonbasic primal variables. x_2 and x_4 are basic.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-7	-4	0	0	0
X4	1	0	1	2	1	1	0	10
X5	2	0	3	3*	2	0	1	10

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	5	0	0.667	0	2.333	23.33
X4	1	0	-1	0	-0.33*	1	-0.67	3.333
X2	2	0	1	1	0.667	0	0.333	3.333

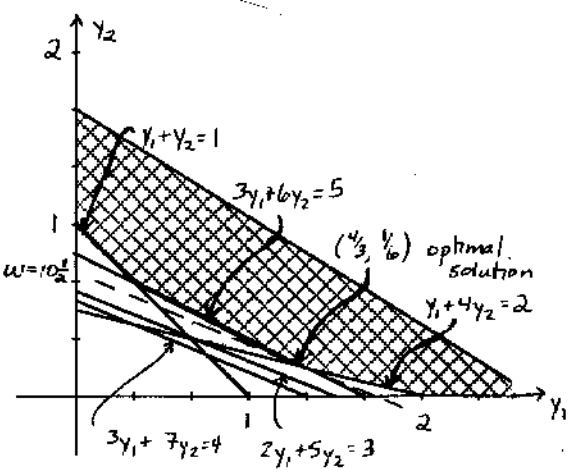
$(x_1, x_2, x_3, x_4, x_5) = (0, 3\frac{1}{3}, 0, 3\frac{1}{3}, 0)$ is the primal optimal basic solution with $Z = 23\frac{1}{3}$

6.3-7. a) Minimize $w = 6y_1 + 15y_2$

subject to:

$$y_1 + 4y_2 \geq 2$$
$$3y_1 + 6y_2 \geq 5$$
$$2y_1 + 5y_2 \geq 3$$
$$3y_1 + 7y_2 \geq 4$$
$$y_1 + y_2 \geq 1$$
$$y_1 \geq 0, \quad y_2 \geq 0$$

(b)



$(y_1, y_2) = (\frac{4}{3}, \frac{1}{6})$ is optimal with $w = 10\frac{1}{2}$.

(c) $(z_1 - c_1)$ and $(z_2 - c_2)$ are nonbasic in the dual, so x_1 and x_2 must be basic in the optimal primal solution.

(d)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	-2	-5	-3	-4	-1	0	0	0
X6	1	0	1	3*	2	3	1	1	0	6
X7	2	0	4	6	5	7	1	0	1	15

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	-0.33	0	0.333	1	0.667	1.667	0	10
X2	1	0	0.333	1	0.667	1	0.333	0.333	0	2
X7	2	0	2*	0	1	1	-1	-2	1	3

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	0.5	1.167	0.5	1.333	0.167	10.5
X2	1	0	0	1	0.5	0.833	0.5	0.667	-0.17	1.5
X1	2	0	1	0	0.5	0.5	-0.5	-1	0.5	1.5

$(x_1, x_2) = (\frac{3}{2}, \frac{3}{2})$ is optimal with $z = 10\frac{1}{2}$.

(e) The Defining Equations are:

$$x_1 + 3x_2 + 2x_3 + 3x_4 + x_5 = 6$$
$$4x_1 + 6x_2 + 5x_3 + 7x_4 + x_5 = 15$$
$$x_3 = 0$$
$$x_4 = 0$$
$$x_5 = 0$$

Which have solution

$$(x_1, x_2, x_3, x_4, x_5) = (\tfrac{3}{2}, \tfrac{3}{2}, 0, 0, 0)$$

6.3-8. a) Minimize $v = 10y_1 + 20y_2$

Subject to: $-2y_1 + 3y_2 \geq 3$

$2y_1 + y_2 \geq 7$

$y_1 - y_2 \geq 2$

$y_1 \geq 0,\ y_2 \geq 0$

(c) The Defining Equations are:

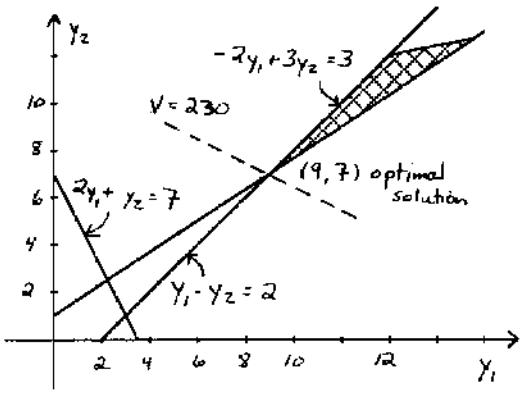
$$-2y_1 + 3y_2 - y_3 = 3$$
$$2y_1 + y_2 - y_4 = 7$$
$$y_1 - y_2 - y_5 = 2$$
$$y_3 = 0$$
$$y_5 = 0$$

which have solution

$(y_1, y_2, y_3, y_4, y_5) = (9, 7, 0, 18, 0)$

(b) Because x_2, x_4 and x_5 are nonbasic in the optimal primal solution, y_1, y_2 and y_4 will be basic in the optimal dual solution.

(d)



$(y_1, y_2) = (9, 7)$ is optimal with $v = 230$

6.3-9 a) Minimize $v = 10y_1 + 60y_2 + 18y_3 + 44y_4$

Subject to: $2y_2 + y_3 + 3y_4 \geq 2$

$y_1 + 5y_2 + y_3 + y_4 \geq 1$

$y_i \geq 0$ for $i = 1, 2, 3, 4$

(b) The Defining Equations for $(x_1, x_2) = (13, 5)$ are:

$$x_1 + x_2 = 18$$
$$3x_1 + x_2 = 44$$

We conclude y_3 and y_4 must be basic for the optimal dual solution and y_1, y_2, y_5 and y_6 nonbasic.

(c) The basic variables for the primal optimal solution are x_1, x_2, x_3 and x_4. We can procede by introducing x_1 and x_2 into the basis from the initial tableau.

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-2	-1	0	0	0	0	0
X3	1	0	0	1	1	0	0	0	10
X4	2	0	2	5	0	1	0	0	60
X5	3	0	1	1	0	0	1	0	18
X6	4	0	3*	1	0	0	0	1	44

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	-0.33	0	0	0	0.667	29.33
X3	1	0	0	1	1	0	0	0	10
X4	2	0	0	4.333	0	1	0	-0.67	30.67
X5	3	0	0	0.667*	0	0	1	-0.33	3.333
X1	4	0	1	0.333	0	0	0	0.333	14.67

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0	0	0.5	0.5	31
X3	1	0	0	0	1	0	-1.5	0.5	5
X4	2	0	0	0	0	1	-6.5	1.5	9
X2	3	0	0	1	0	0	1.5	-0.5	5
X1	4	0	1	0	0	0	-0.5	0.5	13

$(x_1, x_2, x_3, x_4, x_5, x_6) = (13, 5, 5, 9, 0, 0)$ is optimal with $Z = 31$.

The basic dual solution is:

$(y_1, y_2, y_3, y_4, y_5, y_6) = (0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0)$

6.3-9. (d) The Defining Equations for the optimal solution are:

$$2y_2 + y_3 + 3y_4 = 2$$
$$y_1 + 5y_2 + y_3 + y_4 = 1$$
$$y_1 = 0$$
$$y_2 = 0$$

which are satisfied by the dual solution given in part (c).

3-10. (a) The optimal solution for the dual will correspond to the revised row 0 computed by the Revised Simplex Method to determine optimality.

(b) The complementary basic solution corresponds to the revised row 0 as well.

6.4-1. a) Dual: Minimize $Y_0 = 10y_1 + 2y_2$

subject to:
$$y_1 + 2y_2 = 1$$
$$2y_1 + y_2 \geq 1$$
$$y_2 \leq 0 \quad (y_1 \text{ unconstrained in sign})$$

b) Standard form: Maximize $Z = x_1^+ - x_1^- + x_2$

subject to:
$$x_1^+ - x_1^- + 2x_2 \leq 10$$
$$-x_1^+ + x_1^- - 2x_2 \leq -10$$
$$-2x_1^+ + 2x_1^- - x_2 \leq -2$$
$$x_1^+ \geq 0, \; x_1^- \geq 0, \; x_2 \geq 0$$

Dual of Standard form:

Minimize $Y_0 = 10y_1 - 10y_2 - 2y_3$

subject to:
$$\left.\begin{aligned} y_1 - y_2 - 2y_3 &\geq 1 \\ -y_1 + y_2 + 2y_3 &\geq -1 \end{aligned}\right\} \Leftrightarrow y_1 - y_2 - 2y_3 = 1$$
$$2y_1 - 2y_2 - y_3 \geq 1$$
$$y_1 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0$$

Let $y_1' = y_1 - y_2$ (so y_1' is unrestricted in sign)

and $y_2' = -y_3$ (so $y_2' \leq 0$)

Then the dual is: Min $Y_0 = 10y_1' + 2y_2'$

subject to
$$y_1' + 2y_2' = 1$$
$$2y_1' + y_2' \geq 1$$
$$y_2' \leq 0 \; (y_1' \text{ unrestricted in sign})$$

as given in (a)

6.4-2. a) Since $\{Ax = b\}$ is equivalent to $\left\{\begin{bmatrix} A \\ -A \end{bmatrix} x = \begin{bmatrix} b \\ -b \end{bmatrix}\right\}$, changing the primal functional constraints from $Ax \le b$ to $Ax = b$ changes the dual to:

Minimize $Y_c = [\tilde{y}^T, \tilde{w}^T]\begin{bmatrix} b \\ -b \end{bmatrix}$

subject to: $[\tilde{y}^T, \tilde{w}^T]\begin{bmatrix} A \\ -A \end{bmatrix} \ge c$

$\tilde{y}, \tilde{w} \ge 0$

where $\tilde{y}, \tilde{w}$ are simply the first and last halfs of the vector normally called y

This is equivalent to

$$\text{Min } Y_c = (\tilde{y} - \tilde{w})^T b$$
$$\text{subject to: } (\tilde{y} - \tilde{w})^T A \ge c$$
$$\tilde{y}, \tilde{w} \ge 0$$

Now, letting $y = \tilde{y} - \tilde{w}$, so y is unrestricted in sign.

The dual is: $\text{Min } Y_c = yb$

subject to: $yA \ge c$

(y unrestricted in sign)

So the only difference is that the non-negativity constraints $Y \ge 0$ have been deleted.

b) Similarly (as in (a)), $\{Ax \ge b\}$ is equivalent to $\{-Ax \le -b\}$

So the dual of

$\text{Max } Z = cx$
s.t. $Ax \ge b$
$x \ge 0$

is

Dual: $\text{Min } Y_c = \tilde{y}(-b)$
s.t. $\tilde{y}(-A) \ge c$
$\tilde{y} \ge 0$

Letting $y = -\tilde{y}$,

Dual: $\text{Min } Y_c = yb$
s.t. $yA \ge c$
$y \le 0$

So $Y \ge 0$ is replaced with $Y \le 0$ in dual.

c) Primal:

$\text{Max } Z = cx$
s.t. $Ax \le b$
(x unrestr.)

$\Leftrightarrow$

Primal

$\text{Max } Z = cx^+ - cx^-$
s.t. $Ax^+ - Ax^- \le b$
$x^+ \ge 0, x^- \ge 0$

$\Leftrightarrow$

Dual:

$\text{Min } Y_c = yb$
s.t. $yA \ge c$
$y(-A) \ge -c$
$y \ge 0$

$\Leftrightarrow$

Dual:

$\text{Min } Y_c = yb$
s.t. $yA = c$
$y \ge 0$

So $yA \ge c$ is simply replaced by $yA = c$.

6.4-3. Dual is: Maximize $V = 8y_1 + 6y_2$

subject to:

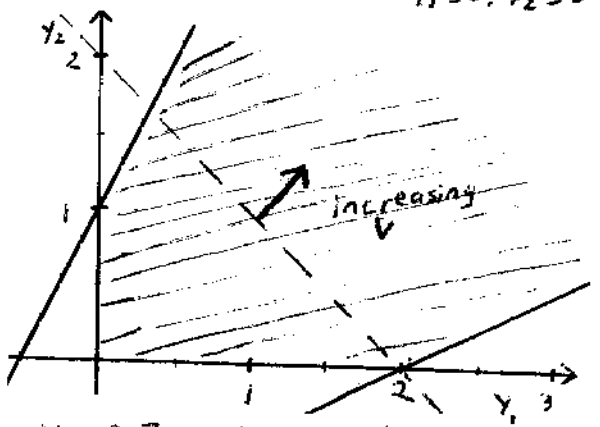
$$y_1 + 3y_2 \le 2$$
$$4y_1 + 2y_2 \le 3$$
$$2y_1 \le 1$$
$$y_1 \ge 0,\ y_2 \ge 0$$

6.4-4. a) Dual is: Maximize $V = y_1 + y_2$

subject to:

$$-2y_1 + y_2 \le 1$$
$$y_1 - 2y_2 \le 2$$
$$y_1 \ge 0,\ y_2 \ge 0$$

b)



Since V can be increased indefinitely, the primal problem cannot have any feasible solutions. (This can be shown by weak duality, keeping in mind that the primal and dual are "reversed" in this problem)

6.4-5. $\min Y_0 = 2.7y_1 + 6y_2 + 6y_3'$

s.t. $0.3y_1 + 0.5y_2 + 0.6y_3' \ge -0.4$

$0.1y_1 + 0.5y_2 + 0.4y_3' \ge -0.5$

$y_1 \ge 0,\ y_3' \le 0$ (y_2 unconstrained in sign)

is equivalent to:

$\max -Y_0 = -2.7y_1 - 6y_2 - 6y_3'$ $\qquad \min Z \iff \max(-Z)$

s.t. $0.3y_1 + 0.5y_2 + 0.6y_3' \ge -0.4$

$0.1y_1 + 0.5y_2 + 0.4y_3' \ge -0.5$

$y_1 \ge 0,\ y_3' \le 0$ (y_2 unconstrained)

which is equivalent to:

$\max Y_0' = 2.7y_1' + 6y_2' + 6y_3$

s.t. $-0.3y_1' - 0.5y_2' - 0.6y_3 \ge -0.4$

$-0.1y_1' - 0.5y_2' - 0.4y_3 \ge -0.5$

$y_1' \le 0,\ y_3 \ge 0$ (y_2' unconstrained)

Let $y_1' = -y_1$, $y_2' = -y_2$, $y_3 = -y_3'$

which is equivalent to:

$\max Y_0' = 2.7y_1' + 6y_2' + 6y_3$

s.t. $0.3y_1' + 0.5y_2' + 0.6y_3 \le 0.4$

$0.1y_1' + 0.5y_2' + 0.4y_3 \le 0.5$

$y_1' \le 0,\ y_3 \ge 0$ (y_2' unconstrained)

$\sum_{j=1}^{n} a_{ij}x_j \ge b_i \iff -\sum_{j=1}^{n} a_{ij}x_j \le -b_i$

as desired.

6.4-6. a) Minimize $Z = 3x_1 + 2x_2$

subject to:
$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Dual: Maximize $V = 10y_1 + 6y_2 + 6y_3$

subject to
$$2y_1 - 3y_2 + y_3 \leq 3$$
$$y_1 + 2y_2 + y_3 \leq 2$$
$$y_1 \geq 0,\ y_2 \leq 0,\ y_3 \geq 0$$

b) Maximize $Z = 2x_1 + 5x_2 + 3x_3$

subject to:
$$x_1 - 2x_2 + x_3 \geq 20$$
$$2x_1 + 4x_2 + x_3 = 50$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

Dual: Minimize $V = 20y_1 + 50y_2$

subject to:
$$y_1 + 2y_2 \geq 2$$
$$-2y_1 + 4y_2 \geq 5$$
$$y_1 + y_2 \geq 3$$
$$y_1 \leq 0 \quad (y_2 \text{ unconstrained})$$

c) Max $Z = -2x_1 + x_2 - 4x_3 + 3x_4$

s.t.
$$x_1 + x_2 + 3x_3 + 2x_4 \leq 4$$
$$x_1 - x_3 + x_4 \geq -1$$
$$2x_1 + x_2 \leq 2$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 2$$
$$x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

(x_1 unconstrained in sign)

Dual: Min $V = 4y_1 - y_2 + 2y_3 + 2y_4$

s.t.
$$y_1 + y_2 + 2y_3 + y_4 = -2$$
$$y_1 + y_3 + 2y_4 \geq 1$$
$$3y_1 - y_2 + y_4 \geq -4$$
$$2y_1 + y_2 + 2y_4 \geq 3$$
$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$$

(y_4 unconstrained in sign)

6.4-7. a) Dual: Minimize $V = 300y_1 + 300y_2$

subject to:
$$2y_1 + 8y_2 \geq 4$$
$$3y_1 + y_2 \geq 2$$
$$4y_1 + y_2 \geq 3$$
$$2y_1 + 5y_2 \geq 5$$

(y_1, y_2 unrestricted in sign)

b) Max $Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$

s.t.
$$2x_1 + 3x_2 + 4x_3 + 2x_4 = 300$$
$$8x_1 + x_2 + x_3 + 5x_4 = 300$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

(To standard form) →

Max $Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$

s.t.
$$2x_1 + 3x_2 + 4x_3 + 2x_4 \leq 300$$
$$-2x_1 - 3x_2 - 4x_3 - 2x_4 \leq -300$$
$$8x_1 + x_2 + x_3 + 5x_4 \leq 300$$
$$-8x_1 - x_2 - x_3 - 5x_4 \leq -300$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

↓ Finding Dual

Dual:

Min $V = 300y_1 - 300y_2 + 300y_3 - 300y_4$

s.t.
$$2y_1 - 2y_2 + 8y_3 - 8y_4 \geq 4$$
$$3y_1 - 3y_2 + y_3 - y_4 \geq 2$$
$$4y_1 - 4y_2 + y_3 - y_4 \geq 3$$
$$2y_1 - 2y_2 + 5y_3 - 5y_4 \geq 5$$
$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0$$

← Let $y_1' = y_1 - y_2$, $y_2' = y_3 - y_4$

Dual:

Min $V = 300y_1' + 300y_2'$

s.t.
$$2y_1' + 8y_2' \geq 4$$
$$3y_1' + y_2' \geq 2$$
$$4y_1' + y_2' \geq 3$$
$$2y_1' + 5y_2' \geq 5$$

same as in (a)

(y_1', y_2' unrestricted in sign)

6.4-8, (a) Minimize $v = 120y_1 + 80y_2 + 100y_3$

subject to:
$$y_2 - 3y_3 = -1$$
$$3y_1 - y_2 + y_3 = 2$$
$$y_1 - 4y_2 + 2y_3 = 1$$
$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.$$

(b) Standard form for the primal is:

Maximize $z = -x_1' + x_1'' + 2x_2' - 2x_2'' + x_3' - x_3''$

subject to:
$$3x_2' - 3x_2'' + x_3' - x_3'' \leq 120$$
$$x_1' - x_1'' - x_2' + x_2'' - 4x_3' + 4x_3'' \leq 80$$
$$-3x_1' + 3x_1'' + x_2' - x_2'' + 2x_3' - 2x_3'' \leq 100$$
$$x_1' \geq 0,\ x_1'' \geq 0,\ x_2' \geq 0,\ x_2'' \geq 0,$$
$$x_3' \geq 0,\ x_3'' \geq 0.$$

The dual is: Minimize $v = 120y_1 + 80y_2 + 100y_3$

subject to:
$$y_2 - 3y_3 \geq -1$$
$$-y_2 + 3y_3 \geq 1$$
$$3y_1 - y_2 + y_3 \geq 2$$
$$-3y_1 + y_2 - y_3 \geq -2$$
$$y_1 - 4y_2 + 2y_3 \geq 1$$
$$-y_1 + 4y_2 - 2y_3 \geq -1$$
$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$$

By combining constraints 1 and 2, 3 and 4 and 5 and 6 we get the form in part (a).

6.4-9. The Wyndor dual in standard form is:

Maximize $(-v) = -4y_1 - 12y_2 - 18y_3$

subject to:
$$-y_1 - 3y_3 \leq -3$$
$$-2y_2 - 2y_3 \leq -5$$
$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0$$

The dual of this dual is:

Minimize $(-z) = -3x_1 - 5x_2$

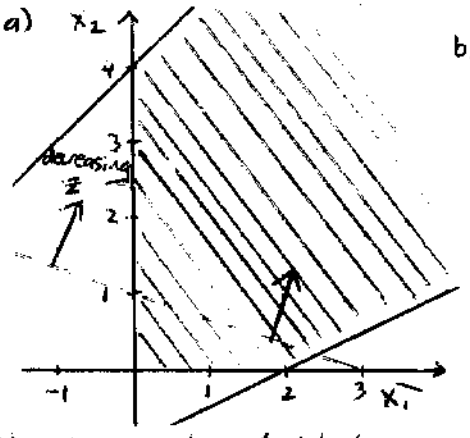
subject to:
$$-x_1 \geq -4$$
$$-2x_2 \geq -12$$
$$-3x_1 - 2x_2 \geq -18$$
$$x_1 \geq 0,\ x_2 \geq 0$$

This is equivalent to the primal:

Maximize $z = 3x_1 + 5x_2$

subject to:
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0,\ x_2 \geq 0$$

6.4-10. a)



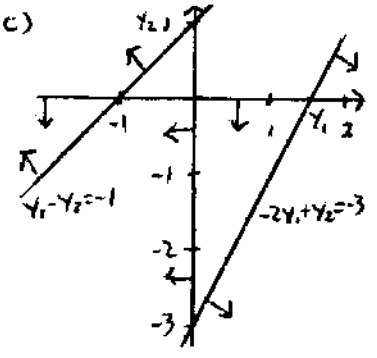
Objective is unbounded below.

b) Dual:

Maximize $2y_1 + 4y_2$

subject to:
$$y_1 - y_2 \leq -1$$
$$-2y_1 + y_2 \leq -3$$
$$y_1 \leq 0,\ y_2 \leq 0$$

c)



As can be seen by the figure, the dual has no feasible solutions.

6.5-1. a) Since x_1 was non-basic, changing the coefficients of x_1 does not affect feasibility. To check optimality, we check dual feasibility:

The first dual constraint becomes

$$0y_1 + 5y_2 \geq -2$$

which clearly must be true ($y_2 \geq 0$).

Therefore, the current basic solution remains optimal.

b) Adding a new variable does not affect primal feasibility (if we let $x_6 = 0$). To check for optimality, we check dual feasibility:

The constraint (corresponding to x_6) in the dual is:

$$3y_1 + 5y_2 \geq 10$$

(Note: we are assuming that $x_6 \geq 0$. If one assumes x_6 is unrestricted, the constraint would become $3y_1 + 5y_2 = 10$.)

$y_1 = 5, y_2 = 0$. So $3(5) + 5(0) = 15 \geq 10$

and the current basic solution (with $x_6 = 0$) remains optimal.

6.5-2. a) Since x_3 was non-basic, changing the coefficients of x_3 does not affect primal feasibility. To check optimality, we see if the dual solution is still feasible:

The third dual constraint becomes

$$3y_1 + y_2 - 2y_3 \geq 2$$

$y_1 = 0, y_2 = \frac{3}{2}, y_3 = \frac{1}{2}$, So $3(0) + \frac{3}{2} - 2(\frac{1}{2}) = \frac{1}{2} \neq 2$

Thus, the current basic solution though feasible, is not optimal.

b) If we let $x_8 = 0$, adding the new variable does not affect primal feasibility. We check dual feasibility:

The dual constraint (corresponding to x_8) is

$$-2y_1 + y_2 + 2y_3 \geq -1$$

or $-2(0) + \frac{3}{2} + 2(\frac{1}{2}) = \frac{5}{2} \geq -1$

So the current basic solution (with $x_8 = 0$) remains optimal.

6.5-3. a) Since x_3 is non-basic, primal feasibility still holds. To check for optimality, we look at the dual constraint (corresponding to x_3):

$$3y_1 - 2y_2 \geq -2 \quad (y_1 = 0, y_2 = 2 \Rightarrow 3(0) - 2(2) = -4 \neq -2)$$

Since this constraint does not hold, the current basic solution is not optimal.

6.5-3. b) If we let $X_4 = 0$, primal feasibility still holds. We need only check to see that the dual constraint corresponding to X_4 still holds:

$$Y_1 + 2Y_2 \geq 3$$

or $\quad 0 + 2(2) = 4 \geq 3$

So, since dual feasibility holds, the current basic solution remains feasible and optimal.

6.5-4. Since X_3 was non-basic, changing the coefficients of X_3 does not affect primal feasibility. To check for optimality, we must see if the complementary basic solution for the dual problem remains feasible:

The third dual constraint becomes

$$3Y_1 + 2Y_2 + Y_3 \geq 4 \quad (Y_1 = 1,\ Y_2 = 1,\ Y_3 = 0)$$

or $3(1) + 2(1) + 0 = 5 \geq 4$

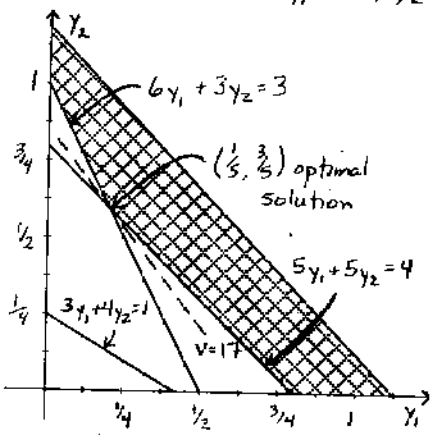
So the current basic solution remains optimal.

6.6-1. (a) $(X_1, X_2, X_3) = (5/3, 0, 3)$. $z = 17$

(b) Minimize $v = 25y_1 + 20y_2$

subject to:
$$6y_1 + 3y_2 \geq 3$$
$$3y_1 + 4y_2 \geq 1$$
$$5y_1 + 5y_2 \geq 4$$
$$y_1 \geq 0,\ y_2 \geq 0.$$

(c)



$(\frac{1}{5}, \frac{3}{5})$ is optimal with $v = 17$

(d) Since the new dual constraint

$2y_1 + 3y_2 \geq 3$

is violated for $y_1^* = \frac{1}{5}$, $y_2^* = \frac{3}{5}$

the current solution is no longer optimal.

(e) New x_2 column is:

$$\begin{pmatrix} 1/3 & -1/3 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 4/5 \end{pmatrix}$$

(f) The new primal variable adds a constraint to the dual,

$3y_1 + 2y_2 \geq 2$, which is not satisfied by $(y_1, y_2) = (\frac{1}{5}, \frac{3}{5})$. Therefore, the previous solution is no longer optimal.

(g) $\bar{c}_{NEW} = (\frac{1}{5}, \frac{3}{5})\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 2 = -\frac{1}{5}$

New column $= \begin{pmatrix} 1/3 & -1/3 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/5 \end{pmatrix}$.

6.6-2.

(a) $\left.\begin{matrix}\Delta b_1 = -10 \\ \Delta b_2 = 0\end{matrix}\right\} \quad \Delta z = \left(\frac{1}{5}, \frac{3}{5}\right)\begin{pmatrix}-10\\0\end{pmatrix} = -2$

$\Delta b_1^* = \left(\frac{1}{3}, -\frac{1}{3}\right)\begin{pmatrix}-10\\0\end{pmatrix} = -\frac{10}{3}$

$\Delta b_2^* = \left(-\frac{1}{5}, \frac{2}{5}\right)\begin{pmatrix}-10\\0\end{pmatrix} = 2$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	2	0	0.2	0.6	15
X1	1	0	1	-0.33	0	0.333	-0.33	-1.67
X3	2	0	0	1	1	-0.2	0.4	5

The current basic solution $(-\frac{5}{3}, 0, 5, 0, 0)$ is infeasible and super optimal.

(b) $\left.\begin{matrix}\Delta b_1 = 0 \\ \Delta b_2 = -15\end{matrix}\right\} \quad \Delta z = \left(\frac{1}{5}, \frac{3}{5}\right)\begin{pmatrix}0\\-15\end{pmatrix} = -9$

$\Delta b_1^* = \left(\frac{1}{3}, -\frac{1}{3}\right)\begin{pmatrix}0\\-15\end{pmatrix} = 5$

$\Delta b_2^* = \left(-\frac{1}{5}, \frac{2}{5}\right)\begin{pmatrix}0\\-15\end{pmatrix} = -6$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	2	0	0.2	0.6	8
X1	1	0	1	-0.33	0	0.333	-0.33	6.667
X3	2	0	0	1	1	-0.2	0.4	-3

The current basic solution $(\frac{20}{3}, 0, -3, 0, 0)$ is infeasible and superoptimal.

(c) $\Delta c_2 = 3 \quad \Big\} \quad \Delta(z_2^* - c_2) = -3$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	-1	0	0.2	0.6	17
X1	1	0	1	-0.33	0	0.333	-0.33	1.667
X3	2	0	0	1	1	-0.2	0.4	3

The current basic solution $(\frac{5}{3}, 0, 3, 0, 0)$ is feasible, but not optimal.

(d) $\Delta c_3 = -1 \quad \Big\} \quad \Delta(z_3^* - c_3) = 1$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	2	1	$\frac{1}{5}$	$\frac{3}{5}$	17
X1	1	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
X3	2	0	0	1	1	$-\frac{1}{5}$	$\frac{2}{5}$	3

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	0	$\frac{2}{5}$	$\frac{1}{5}$	14
X1	1	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
X3	2	0	0	1	1	$-\frac{1}{5}$	$\frac{2}{5}$	3

The current basic solution $(\frac{5}{3}, 0, 3, 0, 0)$ is feasible and optimal.

(e) $\Delta a_{22} = -3 \Rightarrow \Delta(z_2^* - c_2) = \left(\frac{1}{5} \quad \frac{3}{5}\right)\begin{pmatrix}0\\-3\end{pmatrix} = -\frac{9}{5}$

$\Delta a_{12}^* = \left(\frac{1}{3}, -\frac{1}{3}\right)\begin{pmatrix}0\\-3\end{pmatrix} = 1$

$\Delta a_{22}^* = \left(-\frac{1}{5}, \frac{2}{5}\right)\begin{pmatrix}0\\-3\end{pmatrix} = -\frac{6}{5}$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0.2	0	0.2	0.6	17
X1	1	0	1	0.667	0	0.333	-0.33	1.667
X3	2	0	0	-0.2	1	-0.2	0.4	3

The current basic solution $(\frac{5}{3}, 0, 3, 0, 0)$ is feasible and optimal.

(f) $\Delta a_{11} = 4 \Rightarrow \Delta(z_1^* - c_1) = \left(\frac{1}{5}, \frac{3}{5}\right)\begin{pmatrix}4\\0\end{pmatrix} = \frac{4}{5}$

$\Delta a_{11}^* = \left(\frac{1}{3}, -\frac{1}{3}\right)\begin{pmatrix}4\\0\end{pmatrix} = \frac{4}{3}$

$\Delta a_{21}^* = \left(-\frac{1}{5}, \frac{2}{5}\right)\begin{pmatrix}4\\0\end{pmatrix} = -\frac{4}{5}$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	$\frac{4}{5}$	2	0	$\frac{1}{5}$	$\frac{3}{5}$	17
X1	1	0	$\frac{7}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
X3	2	0	$-\frac{4}{5}$	1	1	$-\frac{1}{5}$	$\frac{2}{5}$	3

6.6-2.
(f) (cont.)

Tableau in Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	2.11	0	0.09	0.71	16.43
X1	1	0	1	0.14	0	0.14	-0.14	0.71
X3	2	0	0	0.89	1	-0.09	0.29	3.57

The current basic solution (0.71, 0, 3.57, 0, 0) is feasible and optimal.

6.6-3. (a) In the dual problem,

$\Delta b_1 = -2$, $\Delta b_2 = 1$

$\Delta z^* = (1,1)\begin{pmatrix}-2\\1\end{pmatrix} = -1$

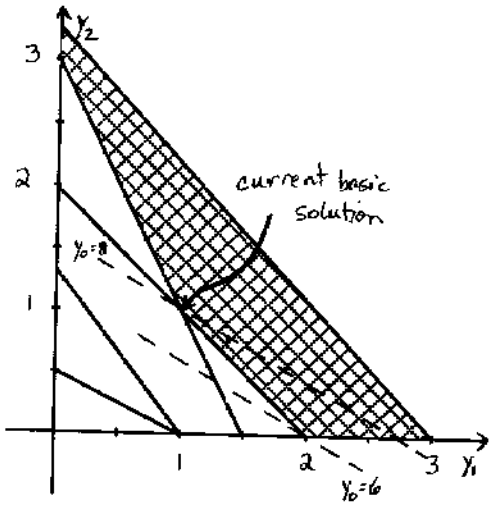
$\Delta b_1^* = (1,-1)\begin{pmatrix}-2\\1\end{pmatrix} = -3$

$\Delta b_2^* = (-1,2)\begin{pmatrix}-2\\1\end{pmatrix} = 4$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	3	0	2	0	1	1	8
X2	1	0	1	1	-1	0	1	-1	-2
X4	2	0	2	0	3	1	-1	2	7

From the tableau we can see the primal basic solution is feasible, but not optimal.



From the graph we can see the current basic solution is feasible, but not optimal

6.6-3. (b)

In the dual problem:

$\Delta c_1 = -1 \Rightarrow \Delta(z_1^* - c_1) = 1$

$\Delta c_2 = 2 \Rightarrow \Delta(z_2^* - c_2) = -2$

$\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$

$\Delta c_4 = 1 \Rightarrow \Delta(z_4^* - c_4) = -1$

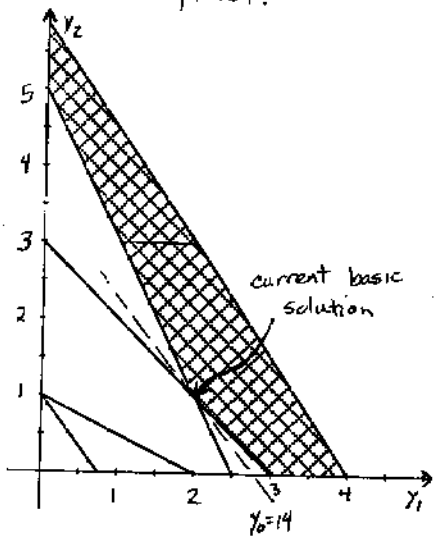
New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	4	-2	1	-1	1	1	9
X2	1	0	1	1	-1	0	1	-1	1
X4	2	0	2	0	3	1	-1	2	3

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	8	0	2	0	2	1	14
X2	1	0	1	1	-1	0	1	-1	1
X4	2	0	2	0	3	1	-1	2	3

The primal basic solution is both feasible and optimal.



From the graph we see the current basic solution is feasible and optimal.

6.6-3. (c) In the dual problem:

$\Delta c_1 = 3$, $\Delta a_{11} = -2$, $\Delta a_{21} = 1$

$\Delta(z_1^* - c_1) = -3 + (1,1)\begin{pmatrix}-2\\1\end{pmatrix} = -4$

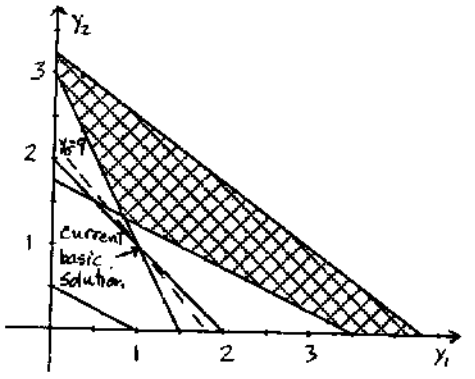
$\Delta a_{11}^* = (1,-1)\begin{pmatrix}-2\\1\end{pmatrix} = -3$

$\Delta a_{21}^* = (-1,2)\begin{pmatrix}-2\\1\end{pmatrix} = 4$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	0	2	0	1	1	9
X2	1	0	-2	1	-1	0	1	-1	1
X4	2	0	6	0	3	1	-1	2	3

The primal basic solution is infeasible, but satisfies optimality criterion.



From the graph, the current basic solution is infeasible and superoptimal.

(d) In the dual problem,

$\Delta c_2 = 7$, $\Delta a_{12} = 3$, $\Delta a_{22} = 1$

$\Delta(z_2^* - c_2) = -7 + (1,1)\begin{pmatrix}3\\1\end{pmatrix} = -3$

$\Delta a_{12}^* = (1,-1)\begin{pmatrix}3\\1\end{pmatrix} = 2$

$\Delta a_{22}^* = (-1,2)\begin{pmatrix}3\\1\end{pmatrix} = -1$

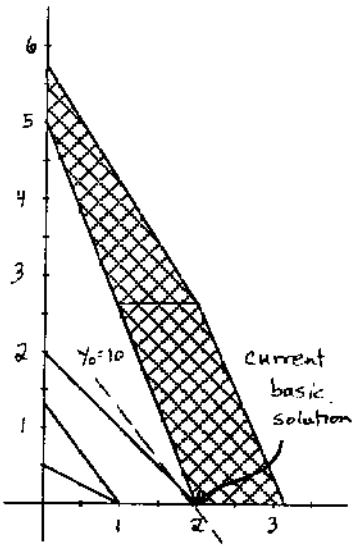
New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	3	-3	2	0	1	1	9
X2	1	0	1	3	-1	0	1	-1	1
X4	2	0	2	-1	3	1	-1	2	3

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	4	0	1	0	2	0	10
X2	1	0	0.333	1	-0.33	0	0.333	-0.33	0.333
X4	2	0	2.333	0	2.667	1	-0.67	1.667	3.333

The primal basic solution is feasible and optimal.



From the graph the current basic solution is feasible and optimal.

6.7-1.(a) $\Delta b_1 = 10$, $\Delta b_2 = 0$

$\Delta z^* = (5,0)\begin{pmatrix}10\\0\end{pmatrix} = 50$

$\Delta b_1^* = (1,0)\begin{pmatrix}10\\0\end{pmatrix} = 10$

$\Delta b_2^* = (-4,1)\begin{pmatrix}10\\0\end{pmatrix} = -40$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	150
X2	1	0	-1	1	3	1	0	30
X5	2	0	16	0	-2	-4	1	-30

The current basic solution is superoptimal, but infeasible.

(b) $\Delta b_1 = 0$, $\Delta b_2 = -20$

$\Delta z^* = (5,0)\begin{pmatrix}0\\-20\end{pmatrix} = 0$

$\Delta b_1^* = (1,0)\begin{pmatrix}0\\-20\end{pmatrix} = 0$

$\Delta b_2^* = (-4,1)\begin{pmatrix}0\\-20\end{pmatrix} = -20$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	16	0	-2	-4	1	-10

The current basic solution is superoptimal, but infeasible

(c) $\Delta b_1 = -10$, $\Delta b_2 = 10$

$\Delta b_1^* = (1,0)\begin{pmatrix}-10\\10\end{pmatrix} = -10$

$\Delta b_2^* = (-4,1)\begin{pmatrix}10\\10\end{pmatrix} = 50$

New Tableau: $\Delta z^* = (5,0)\begin{pmatrix}-10\\10\end{pmatrix} = -50$

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	50
X2	1	0	-1	1	3	1	0	10
X5	2	0	16	0	-2	-4	1	60

The current basic solution is feasible and optimal.

(d) $\Delta c_3 = -5 \Rightarrow \Delta(z_3^* - c_3) = 5$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	7	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	16	0	-2	-4	1	10

The current basic solution is feasible and optimal.

(e) $\Delta c_1 = 3$, $\Delta a_{11} = 1$, $\Delta a_{21} = -7$

$\Delta(z_1^* - c_1) = -3 + (5,0)\begin{pmatrix}1\\-7\end{pmatrix} = 2$

$\Delta a_{11}^* = (1,0)\begin{pmatrix}1\\-7\end{pmatrix} = 1$

$\Delta a_{21}^* = (-4,1)\begin{pmatrix}1\\-7\end{pmatrix} = -11$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	2	0	2	5	0	100
X2	1	0	0	1	3	1	0	20
X5	2	0	5	0	-2	-4	1	10

The current basic solution is feasible and optimal.

(f) $\Delta c_2 = 1$, $\Delta a_{12} = 1$, $\Delta a_{22} = 1$

$\Delta(z_2^* - c_2) = -1 + (5,0)\begin{pmatrix}1\\1\end{pmatrix} = 4$

$\Delta a_{12}^* = (1,0)\begin{pmatrix}1\\1\end{pmatrix} = 1$

$\Delta a_{22}^* = (-4,1)\begin{pmatrix}1\\1\end{pmatrix} = -3$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	4	2	5	0	100
X2	1	0	-1	2	3	1	0	20
X5	2	0	16	-3	-2	-4	1	10

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	2	0	-4	3	0	60
X2	1	0	-0.5	1	1.5	0.5	0	10
X5	2	0	14.5	0	2.5	-2.5	1	40

The current basic solution is feasible, but not optimal.

6.7-1. (g) $\Delta c_6 = 10$, $\Delta a_{16} = 3$, $\Delta a_{26} = 5$

$$\Delta(z_6^* - c_6) = -10 + (5,0)\begin{pmatrix}3\\5\end{pmatrix} = 5$$

$$\Delta a_{16}^* = (1,0)\begin{pmatrix}3\\5\end{pmatrix} = 3$$

$$\Delta a_{26}^* = (-4,1)\begin{pmatrix}3\\5\end{pmatrix} = -7$$

New Tableau:

Bas Var	Eq No	Z	Coefficients of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	5	0	5	100
X2	1	0	-1	1	3	1	0	3	20
X5	2	0	16	0	-2	-4	1	-7	10

The current basic solution is feasible and optimal

(h) New Tableau and Reduction to Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	5	0	0	100
X2	1	0	-1	1	3	1	0	0	20
X5	2	0	16	0	-2	-4	1	0	10
X6	3	0	2	3	5	0	0	1	50

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	5	0	0	100
X2	1	0	-1	1	3	1	0	0	20
X5	2	0	16	0	-2	-4	1	0	10
X6	3	0	5	0	-4	-3	0	1	-10

The current basic solution is superoptimal, but infeasible.

6.7-2. $\Delta b_1 = 20\theta$, $\Delta b_2 = -\theta$

$$\Delta z^* = (5,0)\begin{pmatrix}20\theta\\-\theta\end{pmatrix} = 100\theta$$

$$\Delta b_1^* = (1,0)\begin{pmatrix}20\theta\\-\theta\end{pmatrix} = 20\theta$$

$$\Delta b_2^* = (-4,1)\begin{pmatrix}20\theta\\-\theta\end{pmatrix} = -90\theta$$

Therefore, $z = 100 + 100\theta$.

we need $b_1^* \geq 0$ and $b_2^* \geq 0$

so $20 + 20\theta \geq 0$, $10 - 90\theta \geq 0$ $\Rightarrow -10 \leq \theta \leq 10/9$

(i) $\Delta c_1 = 0$, $\Delta a_{11} = 0$, $\Delta a_{21} = -2$

$$\Delta(z_1^* - c_1) = 0 + (5,0)\begin{pmatrix}0\\-2\end{pmatrix} = 0$$

$$\Delta a_{11}^* = (1,0)\begin{pmatrix}0\\-2\end{pmatrix} = 0$$

$$\Delta a_{21}^* = (-4,1)\begin{pmatrix}0\\-2\end{pmatrix} = -2$$

$\Delta c_2 = 0$, $\Delta a_{12} = 0$, $\Delta a_{22} = 1$

$$\Delta(z_2^* - c_2) = 0 + (5,0)\begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$\Delta a_{12}^* = (1,0)\begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$\Delta a_{22}^* = (-4,1)\begin{pmatrix}0\\1\end{pmatrix} = 1$$

$\Delta b_1 = 0$, $\Delta b_2 = 10$

$$\Delta z^* = (5,0)\begin{pmatrix}0\\10\end{pmatrix} = 0$$

$$\Delta b_1^* = (1,0)\begin{pmatrix}0\\10\end{pmatrix} = 0$$

$$\Delta b_2^* = (-4,1)\begin{pmatrix}0\\10\end{pmatrix} = 10$$

New tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	14	1	-2	-4	1	20

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	100
X2	1	0	-1	1	3	1	0	20
X5	2	0	15	0	-5	-5	1	0

6.7-3.

(a) $\Delta b_1 = 10$, $\Delta b_2 = 10$, $\Delta b_3 = -10$

$$\Delta Z^* = (0, \tfrac{3}{2}, \tfrac{1}{2}) \begin{pmatrix} 10 \\ 10 \\ -10 \end{pmatrix} = 10$$

$$\Delta b_1^* = (1, -1, -2) \begin{pmatrix} 10 \\ 10 \\ -10 \end{pmatrix} = 20$$

$$\Delta b_2 = (0, \tfrac{1}{2}, \tfrac{1}{2}) \begin{pmatrix} 10 \\ 10 \\ -10 \end{pmatrix} = 0$$

$$\Delta b_3 = (0, -\tfrac{1}{2}, \tfrac{1}{2}) \begin{pmatrix} 10 \\ 10 \\ -10 \end{pmatrix} = -10$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1.5	0	1.5	0.5	35
X4	1	0	0	0	1	1	-1	-2	30
X1	2	0	1	0	0.5	0	0.5	0.5	15
X2	3	0	0	1	-1.5	0	-0.5	0.5	-5

The current basic solution is superoptimal, but infeasible

(b) $\Delta a_{11} = -1$, $\Delta a_{21} = 1$, $\Delta a_{31} = -1$, $\Delta c_1 = -1$

$$\Delta a_{11}^* = (1, -1, -2) \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\Delta a_{21}^* = (0, \tfrac{1}{2}, \tfrac{1}{2}) \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\Delta a_{31}^* = (0, -\tfrac{1}{2}, \tfrac{1}{2}) \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = -1$$

$$\Delta(Z_1^* - c_1) = 1 + (0, \tfrac{3}{2}, \tfrac{1}{2}) \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 2$$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	2	0	1.5	0	1.5	0.5	25
X4	1	0	0	0	1	1	-1	-2	10
X1	2	0	1	0	0.5	0	0.5	0.5	15
X2	3	0	-1	1	-1.5	0	-0.5	0.5	5

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	0.5	0	0.5	-0.5	-5
X4	1	0	0	0	1	1	-1	-2	10
X1	2	0	1	0	0.5	0	0.5	0.5	15
X2	3	0	0	1	-1	0	0	1	20

The current basic solution is feasible, but not optimal.

(c) $\Delta c_3 = 1$, $\Delta a_{13} = 2$, $\Delta a_{23} = -1$, $\Delta a_{33} = -1$

$$\Delta(Z_3^* - c_3) = -1 + (0, \tfrac{3}{2}, \tfrac{1}{2}) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = -3$$

$$\Delta a_{13}^* = (1, -1, -2) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 5$$

$$\Delta a_{23}^* = (0, \tfrac{1}{2}, \tfrac{1}{2}) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = -1$$

$$\Delta a_{33}^* = (0, -\tfrac{1}{2}, +\tfrac{1}{2}) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 0$$

(c) (cont.)

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	-1.5	0	1.5	0.5	25
X4	1	0	0	0	6	1	-1	-2	10
X1	2	0	1	0	-0.5	0	0.5	0.5	15
X2	3	0	0	1	-1.5	0	-0.5	0.5	5

The current basic solution is feasible, but not optimal.

(d) $\Delta c_1 = 1$, $\Delta c_2 = -1$, $\Delta c_3 = 2$

$\Delta(Z_1^* - c_1) = -1$, $\Delta(Z_2^* - c_2) = 1$, $\Delta(Z_3^* - c_3) = -2$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	1	-0.5	0	1.5	0.5	25
X4	1	0	0	0	1	1	-1	-2	10
X1	2	0	1	0	0.5	0	0.5	0.5	15
X2	3	0	0	1	-1.5	0	-0.5	0.5	5

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1.5	0	2.5	0.5	35
X4	1	0	0	0	1	1	-1	-2	10
X1	2	0	1	0	0.5	0	0.5	0.5	15
X2	3	0	0	1	-1.5	0	-0.5	0.5	5

The current basic solution is feasible and optimal.

(e) New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	3/2	0	3/2	1/2	0	25
X4	1	0	0	0	1	1	-1	-2	0	10
X1	2	0	1	0	1/2	0	1/2	1/2	0	15
X2	3	0	0	1	-3/2	0	-1/2	1/2	0	5
X7	4	0	3	-2	1	0	0	0	1	30

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	3/2	0	3/2	1/2	0	25
X4	1	0	0	0	1	1	-1	-2	0	10
X1	2	0	1	0	1/2	0	1/2	1/2	0	15
X2	3	0	0	1	-3/2	0	-1/2	1/2	0	5
X7	4	0	0	0	-3/2	0	-5/2	-1/2	1	-5

The current basic solution is superoptimal, but infeasible.

6.7-3.

f) $\Delta c_8 = -1$, $\Delta a_{18} = -2$, $\Delta a_{28} = 1$, $\Delta a_{38} = 2$

$\Delta(z_8^* - c_8) = 1 + (0, \frac{3}{2}, \frac{1}{2})\begin{pmatrix}-2\\1\\2\end{pmatrix} = \frac{7}{2}$

$\Delta a_{18}^* = (1,-1,-2)\begin{pmatrix}-2\\1\\2\end{pmatrix} = -7$

$\Delta a_{28}^* = (0, \frac{1}{2}, \frac{1}{2})\begin{pmatrix}-2\\1\\2\end{pmatrix} = \frac{3}{2}$

$\Delta a_{38}^* = (0, -\frac{1}{2}, \frac{1}{2})\begin{pmatrix}-2\\1\\2\end{pmatrix} = \frac{1}{2}$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X8	Right side
Z	0	1	0	0	3/2	0	3/2	1/2	7/2	25
X4	1	0	0	0	1	1	-1	-2	-7	10
X1	2	0	1	0	1/2	0	1/2	1/2	3/2	15
X2	3	0	0	1	-3/2	0	-1/2	1/2	1/2	5

The current basic solution remains feasible and optimal.

6.7-4.

(a) $\Delta b_1 = -10$, $\Delta b_2 = 20$

$\Delta z^* = (0,2)\begin{pmatrix}-10\\20\end{pmatrix} = 40$

$\Delta b_1^* = (1,-1)\begin{pmatrix}-10\\20\end{pmatrix} = -30$

$\Delta b_2^* = (0,1)\begin{pmatrix}-10\\20\end{pmatrix} = 20$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	60
X4	1	0	0	-1	5	1	-1	-10
X1	2	0	1	4	-1	0	1	30

The current basic solution is superoptimal, but infeasible.

(b) $\Delta c_3 = 1$, $\Delta a_{13} = -1$, $\Delta a_{23} = -1$

$\Delta(z_3^* - c_3) = -1 + (0,2)\begin{pmatrix}-1\\-1\end{pmatrix} = -3$

$\Delta a_{13}^* = (1,-1)\begin{pmatrix}-1\\-1\end{pmatrix} = 0$

$\Delta a_{23}^* = (0,1)\begin{pmatrix}-1\\-1\end{pmatrix} = -1$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	-2	0	2	20
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-2	0	1	10

The current basic solution is feasible, but not optimal.

7-4. (c) $\Delta c_1 = 2$, $\Delta a_{11} = 2$, $\Delta a_{21} = 1$

$\Delta(z_1^* - c_1) = -2 + (0,2)\begin{pmatrix}2\\1\end{pmatrix} = 0$

$\Delta a_{11}^* = (1,-1)\begin{pmatrix}2\\1\end{pmatrix} = 1$

$\Delta a_{21}^* = (0,1)\begin{pmatrix}2\\1\end{pmatrix} = 1$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	20
X4	1	0	1	-1	5	1	-1	20
X1	2	0	2	4	-1	0	1	10

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	1	0	2	20
X4	1	0	0	-3	5.5	1	-1.5	15
X1	2	0	1	2	-0.5	0	0.5	5

The current basic solution is feasible and optimal.

(d) $\Delta c_6 = 3$, $\Delta a_{16} = 1$, $\Delta a_{26} = 2$

$\Delta(z_6^* - c_6) = -3 + (0,2)\begin{pmatrix}1\\2\end{pmatrix} = 1$

$\Delta a_{16}^* = (1,-1)\begin{pmatrix}1\\2\end{pmatrix} = -1$

$\Delta a_{26}^* = (0,1)\begin{pmatrix}1\\2\end{pmatrix} = 2$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	1	1	0	2	1	20
X4	1	0	0	-1	5	1	-1	-1	20
X1	2	0	1	4	-1	0	1	2	10

The current basic solution is feasible and optimal

(e) $\Delta c_1 = -1$, $\Delta c_2 = -2$, $\Delta c_3 = 1$

$\Delta(z_1^* - c_1) = 1$

$\Delta(z_2^* - c_2) = 2$

$\Delta(z_3^* - c_3) = -1$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	1	3	0	0	2	20
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-1	0	1	10

6.7-4

(e) (cont.) Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	-1	1	0	1	10
X4	1	0	0	-1	5	1	-1	20
X1	2	0	1	4	-1	0	1	10

The current basic solution is feasible, but not optimal

(f) New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	1	1	0	2	0	20
X4	1	0	0	-1	5	1	-1	0	20
X1	2	0	1	4	-1	0	1	0	10
X6	3	0	3	2	3	0	0	1	25

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	1	1	0	2	0	20
X4	1	0	0	-1	5	1	-1	0	20
X1	2	0	1	4	-1	0	1	0	10
X6	3	0	0	-10	6	0	-3	1	-5

The current basic solution is superoptimal, but infeasible.

(g) $\Delta a_{22} = -2$, $\Delta a_{23} = 3$

$\Delta(z_2^* - c_2) = (0,2)\begin{pmatrix}0\\-2\end{pmatrix} = -4$

$\Delta a_{12}^* = (1,-1)\begin{pmatrix}0\\-2\end{pmatrix} = 2$

$\Delta a_{22}^* = (0,1)\begin{pmatrix}0\\-2\end{pmatrix} = -2$

$\Delta(z_3^* - c_3) = (0,2)\begin{pmatrix}0\\3\end{pmatrix} = 6$

$\Delta a_{13}^* = (1,-1)\begin{pmatrix}0\\3\end{pmatrix} = -3$

$\Delta a_{23}^* = (0,1)\begin{pmatrix}0\\3\end{pmatrix} = 3$

$\Delta b_2 = 25$

$\Delta z^* = (0,2)\begin{pmatrix}0\\25\end{pmatrix} = 50$

$\Delta b_1^* = (1,-1)\begin{pmatrix}0\\25\end{pmatrix} = -25$

$\Delta b_2^* = (0,1)\begin{pmatrix}0\\25\end{pmatrix} = 25$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	-3	7	0	2	70
X4	1	0	0	1	2	1	-1	-5
X1	2	0	1	2	2	0	1	35

The current basic solution is neither feasible nor optimal.

6.7-5. $\Delta b_1 = 30$, $\Delta b_2 = -\theta$

$\Delta z^* = (0,2)\begin{pmatrix}30\\-\theta\end{pmatrix} = -2\theta$

$\Delta b_1^* = (1,-1)\begin{pmatrix}30\\-\theta\end{pmatrix} = 4\theta$

$\Delta b_2^* = (0,1)\begin{pmatrix}30\\-\theta\end{pmatrix} = -\theta$

$z^*(\theta) = 20 - 2\theta$

$(x_1, x_2, x_3, x_4, x_5) = (10-\theta, 0, 0, 20+4\theta, 0)$

which is feasible if

$-5 \le \theta \le 10$

6.7-6. (a) $\Delta b_1 = -5$, $\Delta b_2 = 1$, $\Delta b_3 = -2$

$\Delta z^* = (1,1,0)\begin{pmatrix}-5\\1\\-2\end{pmatrix} = -4$

$\Delta b_1^* = (1,3,0)\begin{pmatrix}-5\\1\\-2\end{pmatrix} = -2$

$\Delta b_2^* = (0,1,1)\begin{pmatrix}-5\\1\\-2\end{pmatrix} = -1$

$\Delta b_3^* = (1,2,0)\begin{pmatrix}-5\\1\\-2\end{pmatrix} = -3$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	2	1	1	0	14
X2	1	0	0	1	5	1	3	0	22
X6	2	0	0	0	2	0	1	1	6
X1	3	0	1	0	4	1	2	0	18

The current basic solution is feasible and optimal.

(b) $\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution remains feasible and optimal.

6.7-6.

c) $\Delta c_1 = 3 \Rightarrow \Delta(z_1^* - c_1) = -3$

New tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	0	2	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	6	2	3	0	39
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution is feasible and optimal.

(d) $\Delta c_3 = 3$, $\Delta a_{13} = 1$, $\Delta a_{23} = 1$, $\Delta a_{33} = 0$

$\Delta(z_3^* - c_3) = -3 + (1,1,0)\begin{pmatrix}1\\1\\0\end{pmatrix} = -1$

$\Delta a_{13}^* = (1,3,0)\begin{pmatrix}1\\1\\0\end{pmatrix} = 4$

$\Delta a_{23}^* = (0,1,1)\begin{pmatrix}1\\1\\0\end{pmatrix} = 1$

$\Delta a_{33}^* = (1,2,0)\begin{pmatrix}1\\1\\0\end{pmatrix} = 3$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	1	1	1	0	18
X2	1	0	0	1	9	1	3	0	24
X6	2	0	0	0	3	0	1	1	7
X1	3	0	1	0	7	1	2	0	21

The current basic solution remains feasible and optimal.

(e) $\Delta c_1 = -1$, $\Delta a_{11} = -2$, $\Delta a_{21} = -1$, $\Delta a_{31} = 2$

$\Delta(z_1^* - c_1) = 1 + (1,1,0)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -2$

$\Delta a_{11}^* = (1,3,0)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -5$

$\Delta a_{21}^* = (0,1,1)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = 1$

$\Delta a_{31}^* = (1,2,0)\begin{pmatrix}-2\\-1\\2\end{pmatrix} = -4$

$\Delta c_2 = -1$, $\Delta a_{12} = 1$, $\Delta a_{22} = 2$, $\Delta a_{32} = 3$

$\Delta(z_2^* - c_2) = 1 + (1,1,0)\begin{pmatrix}\frac{1}{2}\\\frac{1}{3}\\0\end{pmatrix} = 4$

$\Delta a_{12}^* = (1,3,0)\begin{pmatrix}\frac{1}{2}\\\frac{1}{3}\\0\end{pmatrix} = 7$

$\Delta a_{22}^* = (0,1,1)\begin{pmatrix}\frac{1}{2}\\\frac{1}{3}\\0\end{pmatrix} = 5$

$\Delta a_{32}^* = (1,2,0)\begin{pmatrix}\frac{1}{2}\\\frac{1}{3}\\0\end{pmatrix} = 5$.

(e) (cont.)

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	4	2	1	1	0	18
X2	1	0	-5	8	5	1	3	0	24
X6	2	0	1	5	2	0	1	1	7
X1	3	0	-3	5	4	1	2	0	21

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	-18	-7	-3	0	-114
X2	1	0	0	1	5	2	1	0	33
X6	2	0	0	0	-30	-13	-5	1	-206
X1	3	0	1	0	7	3	1	0	48

The current basic solution is neither feasible nor optimal.

(f) $\Delta c_1 = 3$, $\Delta c_2 = 2$, $\Delta c_3 = 2$

$\Delta(z_1^* - c_1) = -3$

$\Delta(z_2^* - c_2) = -2$

$\Delta(z_3^* - c_3) = -2$

New Tableau:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-3	-2	0	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

Proper Form:

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	22	6	13	0	129
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution is feasible and optimal.

(g) $\Delta a_{11} = -1 \Rightarrow \Delta(z_1^* - c_1) = (1,1,0)\begin{pmatrix}-1\\0\\0\end{pmatrix} = -1$

$\Delta a_{11}^* = (1,3,0)\begin{pmatrix}-1\\0\\0\end{pmatrix} = -1$

$\Delta a_{21}^* = (0,1,1)\begin{pmatrix}-1\\0\\0\end{pmatrix} = 0$

$\Delta a_{31}^* = (1,2,0)\begin{pmatrix}-1\\0\\0\end{pmatrix} = -1$

6.7-6. g) (cont')

$\Delta a_{12} = 1 \Rightarrow \Delta(z_2^* - c_2) = (1,1,0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

$\Delta a_{12}^* = (1,3,0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

$\Delta a_{22}^* = (0,1,1)\begin{pmatrix}1\\0\\0\end{pmatrix} = 0$

$\Delta a_{32}^* = (1,2,0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

$\Delta a_{13} = 2 \Rightarrow \Delta(z_3^* - c_3) = (1,1,0)\begin{pmatrix}2\\0\\0\end{pmatrix} = 2$

$\Delta a_{13}^* = (1,3,0)\begin{pmatrix}2\\0\\0\end{pmatrix} = 2$

$\Delta a_{23}^* = (0,1,1)\begin{pmatrix}2\\0\\0\end{pmatrix} = 0$

$\Delta a_{33}^* = (1,2,0)\begin{pmatrix}2\\0\\0\end{pmatrix} = 2$

$\Delta b_1 = -3 \Rightarrow \Delta z^* = (1,1,0)\begin{pmatrix}-3\\0\\0\end{pmatrix} = -3$

$\Delta b_1^* = (1,3,0)\begin{pmatrix}-3\\0\\0\end{pmatrix} = -3$

$\Delta b_2^* = (0,1,1)\begin{pmatrix}-3\\0\\0\end{pmatrix} = 0$

$\Delta b_3^* = (1,2,0)\begin{pmatrix}-3\\0\\0\end{pmatrix} = -3$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	-1	1	4	1	1	0	15
X2	1	0	-1	2	7	1	3	0	21
X6	2	0	0	0	2	0	1	1	7
X1	3	0	0	1	6	1	2	0	18

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	3	1	0	0	12
X2	1	0	0	1	6	1	2	0	18
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	5	1	1	0	15

The current basic solution is feasible and optimal.

(h) New Tableau

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	2	1	1	0	0	18
X2	1	0	0	1	5	1	3	0	0	24
X6	2	0	0	0	2	0	1	1	0	7
X1	3	0	1	0	4	1	2	0	0	21
X7	4	0	2	-1	3	0	0	0	1	60

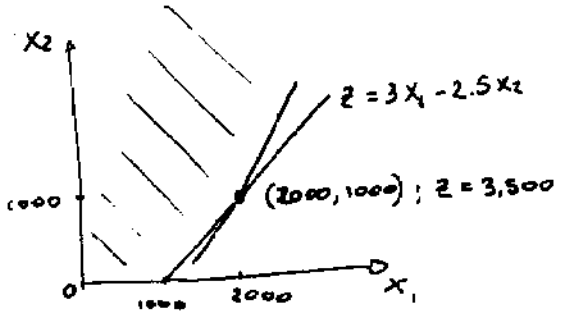
Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	2	1	1	0	0	18
X_2	1	0	0	1	5	1	3	0	0	24
X6	2	0	0	0	2	0	1	1	0	7
X1	3	0	1	0	4	1	2	0	0	21
X7	4	0	0	0	0	1	1	0	1	42

The current basic solution is feasible and optimal.

6.7-7

a)



max: $3x_1 - 2.5x_2$

subject to

$2x_1 - x_2 \leq 3000$

$x_1 - x_2 \leq 1000$

$x_1, x_2 \geq 0$

x_1: toys produced

x_2: subassemblies produced.

b)

Resource	Resource Usage Per Unit of Each Activity: Produce Toys	Produce Subassemblies	Totals		Resource Available
Sub A	2	-1	3000	≤	3000
Sub B	1	-1	1000	≤	1000
Unit Profit	3	-2.5	**$ 3,500.00**		
Solution	**2000**	**1000**			

c)

Unit Profit for Toys	Optimal Production Rates: Toys	Subassemblies	Total Profit
$2.00	1000	0	$2000
$2.50	1000	0	$2500
$3.00	2000	1000	$3500
$3.50	2000	1000	$4500
$4.00	2000	1000	$5500

The estimate of the unit profit for toys can decrease by somewhere between $0 and $0.50 before the optimal solution will change. We can not determine what will happen when the unit profit for toys increases since it is not shown in the table.

d)

Unit Profit for Subassemblies	Optimal Production Rates: Toys	Subassemblies	Total Profit
-$3.50	1000	0	$3000
-$3.00	1000	0	$3000
-$2.50	2000	1000	$3500
-$2.00	2000	1000	$4000
-$1.50	2000	1000	$4500

The estimate of the unit profit for subassemblies can decrease by somewhere between $0 and $0.50 before the optimal solution will change. We can not determine what will happen when the unit profit for toys increases since it is not shown in the table.

e) Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$6	Solution Produce Toys	2000	0	3	2	0.5
$C$6	Solution Produce Subassemblies	1000	0	-2.5	1	0.5

Constraints

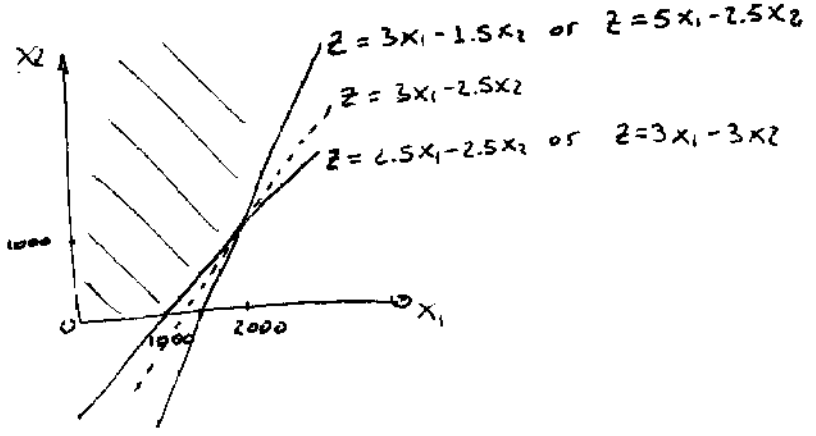
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$3	Sub A Totals	3000	0.5	3000	1E+30	1000
$D$4	Sub B Totals	1000	2	1000	500	1E+30

Based on the allowable increase and decrease in the Excel sensitivity report,

Current value of P_T: 3.
Allowable increase in P_T: 2. So, $P_T \leq 3 + 2 = 5$.
Allowable decrease in P_T: 0.5. So, $P_T \geq 3 - 0.5 = 2.5$.
Range of optimality for P_T: $2.5 \leq P_T \leq 5$.

Current value of P_S: -2.5.
Allowable increase in P_S: 1. So, $P_S \leq -2.5 + 1 = -1.5$.
Allowable decrease in P_S: 0.5. So, $P_S \geq -2.5 - 0.5 = -3$.
Range of optimality for P_S: $-3 \leq P_S \leq -1.5$

F)

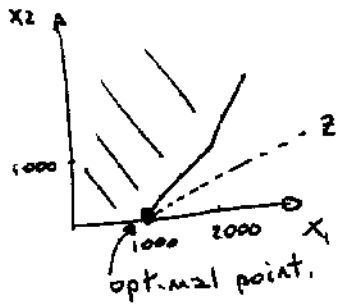


g)

Toys	Subassemblies -3.5	-3.00	-2.5	-2.00	-1.50
2.00				✓	✓
2.50				✓	✓
3.00				✓	
3.50	✓	✓		✓	
4.00	✓	✓			

✓ indicates that original solution is still optimal

h)

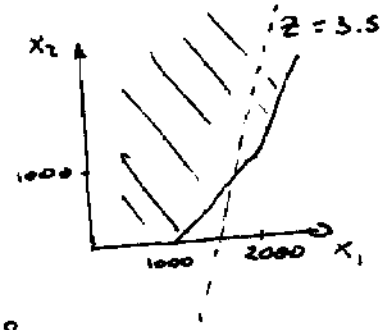


$Z = 2x_1 - 3.5x_2$

or $Z = 2.5x_1 - 3.5x_2$

or $Z = 2.0x_1 - 3x_2$

or $Z = 2.5x_1 - 3x_2$



$Z = 3.5x_1 - 1.5x_2$

or $Z = 4x_1 - 1.5x_2$

Problem is unbounded.

6.7-8

a) maximize $3x_1 - 2.5x_2$

x_1: toys produced

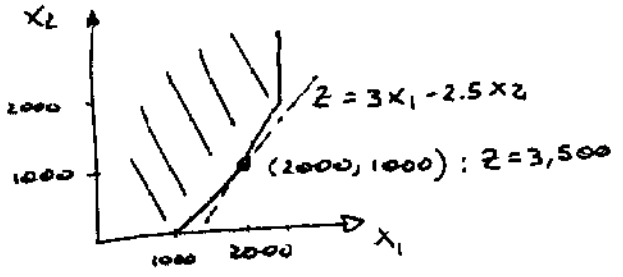
x_2: subassemblies produced.

Subject to

$2x_1 - x_2 \leq 3000$

$x_1 - x_2 \leq 1000$

$x_1 \leq 2500$



b)

Resource	Resource Usage Per Unit of Each Activity: Produce Toys	Produce Subassemblies	Totals		Resource Available
Sub A	2	-1	3000	≤	3000
Sub B	1	-1	1000	≤	1000
Max. Demand	1	0	2000	≤	2500
Unit Profit	3	-2.5	**$ 3,500.00**		
Solution	**2000**	**1000**			

c)

Resource	Resource Usage Per Unit of Each Activity: Produce Toys	Produce Subassemblies	Totals		Resource Available
Sub A	2	-1	3001	≤	3001
Sub B	1	-1	1000	≤	1000
Max. Demand	1	0	2001	≤	2500
Unit Profit	3	-2.5	**$ 3500.50**		
Solution	**2001**	**1001**			

The shadow price for subassembly A is $0.50, which is the maximum premium that the company should be willing to pay.

d)

Resource	Resource Usage Per Unit of Each Activity: Produce Toys	Produce Subassemblies	Totals		Resource Available
Sub A	2	-1	3000	≤	3000
Sub B	1	-1	1001	≤	1001
Max. Demand	1	0	1999	≤	2500
Unit Profit	3	-2.5	**$ 3,502.00**		
Solution	**1999**	**998**			

The shadow price for subassembly B is $2.00, which is the maximum premium that the company should be willing to pay.

e)

Supply Type A	Optimal Production Rates: Toys	Subassemblies	Total Profit
3000	2000	1000	$3500
3100	2100	1100	$3550
3200	2200	1200	$3600
3300	2300	1300	$3650
3400	2400	1400	$3700
3500	2500	1500	$3750
3600	2500	1500	$3750
3700	2500	1500	$3750
3800	2500	1500	$3750
3900	2500	1500	$3750
4000	2500	1500	$3750

The shadow price is still valid until the maximum supply of subassembly A is 3500.

f)

Supply Type B	Optimal Production Rates: Toys	Subassemblies	Total Profit
1000	2000	1000	$3500
1100	1900	800	$3700
1200	1800	600	$3900
1300	1700	400	$4100
1400	1600	200	$4300
1500	1500	0	$4500
1600	1500	0	$4500
1700	1500	0	$4500
1800	1500	0	$4500
1900	1500	0	$4500
2000	1500	0	$4500

The shadow price is still valid until the maximum supply of subassembly B is 1500.

g)

Adjustable Cells

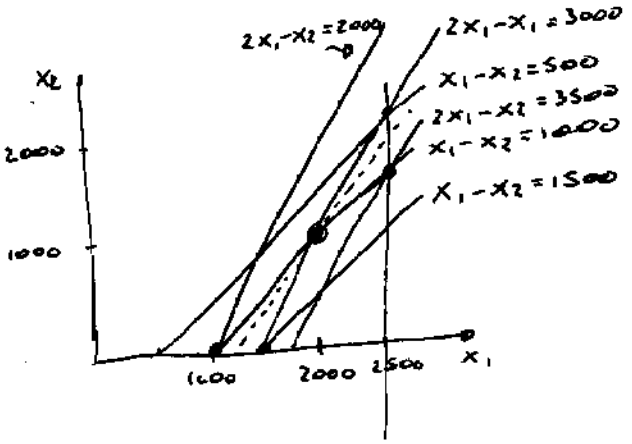
Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Solution Produce Toys	2000	0	3	2	0.5
$C$7	Solution Produce Subassemblies	1000	0	-2.5	1	0.5

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$3	Sub A Totals	3000	0.5	3000	500	1000
$D$4	Sub B Totals	1000	2	1000	500	500
$D$5	Max. Demand Totals	2000	0	2500	1E+30	500

As shown in the sensitivity report, the shadow price is $0.50 for subassembly A is $2.00 for subassembly B. According to the allowable increase and allowable decrease, the range of feasibility for the right-hand side of the subassembly A constraint is 2000 ≤ subassembly A ≤ 3500. The range of feasibility for the right-hand side of the subassembly B constraint is 500 ≤ subassembly B ≤ 1500.

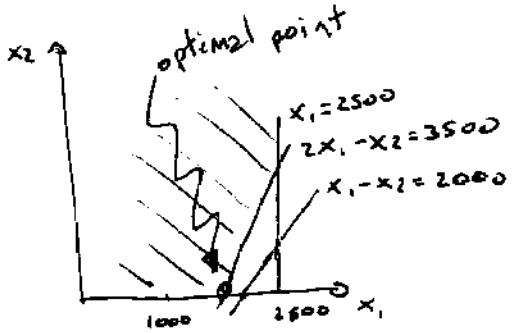
h)



i)

subasembly A	subasembly B	Shadow Prices Valid
3500	1500	✓
3500	2000	NO
4000	1500	✓
4000	2000	✓

j)



From the graph we see that the shadow price for subasembly B is zero, instead of #2.
So shadow prices don't predict any more the effect of changes in RHS.

6.7-9

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$11	Solution F1-DC	50	-200	300	200	1E+30
$C$11	Solution F2-DC	30	0	400	100	1E+30
$D$11	Solution F1-W1	30	0	700	1E+30	200
$E$11	Solution F2-W1	40	0	900	1E+30	100
$F$11	Solution DC-W1	30	0	200	200	1E+30
$G$11	Solution DC-W2	50	-100	400	100	1E+30

a) F2-DC, F2-W1, and DC-W2 have the smallest margins for error (100). The greatest effort in estimating the unit shipping costs should be placed on these shipping lanes.

b)

The range of optimality for C_{F1-DC} is $C_{F1-DC} \le 500$.

The range of optimality for C_{F2-DC} is $C_{F2-DC} \le 500$.

The range of optimality for C_{F1-W1} is $C_{F1-W1} \ge 500$.

The range of optimality for C_{F2-W1} is $C_{F2-W1} \ge 800$.

The range of optimality for C_{DC-W1} is $C_{DC-W1} \le 400$.

The range of optimality for C_{DC-W2} is $C_{DC-W2} \le 500$.

c) The range of optimality for each unit shipping cost indicates how much that shipping cost can change before you would want to change the shipping quantities used in the optimal solution.

d) Use the 100% rule for simultaneous changes in objective function coefficients. If the sum of the percentage changes does not exceed 100%, the optimal solution definitely will still be optimal. If the sum does exceed 100%, then we cannot be sure.

6.7-10 a) Optimal solution does not change.

b) Optimal solution does change to:

	Benefit Contribution Per Unit of Each Activity							Minimum
	Activity							Acceptable
Benefit	6am Shift	8am Shift	Noon Shift	4pm Shift	10pm Shift	Totals		Level
Serve 6-8am	1	0	0	0	0	48	≥	48
Serve 8-10am	1	1	0	0	0	79	≥	79
Serve 10-12am	1	1	0	0	0	79	≥	65
Serve 12-2pm	1	1	1	0	0	112	≥	87
Serve 2-4pm	0	1	1	0	0	64	≥	64
Serve 4-6pm	0	0	1	1	0	82	≥	73
Serve 6-8pm	0	0	1	1	0	82	≥	82
Serve 8-10pm	0	0	0	1	0	49	≥	43
Serve 10-12pm	0	0	0	1	1	64	≥	52
Serve 12-6am	0	0	0	0	1	15	≥	15
Unit Cost	170	160	175	170	195	**$ 30,150.00**		
Solution	**48**	**31**	**33**	**49**	**15**			

c) Optimal Solution will change to:

Benefit	Benefit Contribution Per Unit of Each Activity: 6am Shift	8am Shift	Noon Shift	4pm Shift	10pm Shift	Totals		Minimum Acceptable Level
Serve 6-8am	1	0	0	0	0	48	≥	48
Serve 8-10am	1	1	0	0	0	79	≥	79
Serve 10-12am	1	1	0	0	0	79	≥	65
Serve 12-2pm	1	1	1	0	0	112	≥	87
Serve 2-4pm	0	1	1	0	0	64	≥	64
Serve 4-6pm	0	0	1	1	0	82	≥	73
Serve 6-8pm	0	0	1	1	0	82	≥	82
Serve 8-10pm	0	0	0	1	0	49	≥	43
Serve 10-12pm	0	0	0	1	1	64	≥	52
Serve 12-6am	0	0	0	0	1	15	≥	15
Unit Cost	170	165	175	170	195	**$ 30,305.00**		
Solution	**48**	**31**	**33**	**49**	**15**			

d) Optimal Solution does not change.

e) Optimal Solution does not change.

a) to e) CONT'D).

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$15	Solution 6am Shift	48	0	170	1E+30	10
$C$15	Solution 8am Shift	31	0	160	10	160
$D$15	Solution Noon Shift	39	0	175	5	175
$E$15	Solution 4pm Shift	43	0	180	1E+30	5
$F$15	Solution 10pm Shift	15	0	195	1E+30	195

Part a)
Optimal solution does not change (within allowable increase of $10).

Part b)
Optimal solution does change (outside of allowable decrease of $5).

Part c)
By the 100% rule for simultaneous changes in the objective function, the optimal solution may or may not change.

C_{8AM}: $160 → $165. % of allowable increase $= 100\left(\frac{165-160}{10}\right) = 50\%$

C_{4PM}: $180 → $170. % of allowable decrease $= 100\left(\frac{180-170}{5}\right) = 200\%$

Sum = 250%.

(CONT'D)

(CONT'D)

Part d)

By the 100% rule for simultaneous changes in the objective function, the optimal solution does not change.

C_{6AM}: \$170 → \$166. % of allowable decrease = $100\left(\frac{170-166}{10}\right) = 50\%$

C_{8AM}: \$160 → \$164. % of allowable increase = $100\left(\frac{164-160}{10}\right) = 40\%$

C_{noon}: \$175 → \$171. % of allowable decrease = $100\left(\frac{175-171}{175}\right) = 2\%$

C_{4PM}: \$180 → \$184. % of allowable increase = $100\left(\frac{184-180}{\infty}\right) = 0\%$

C_{10PM}: \$195 → \$199. % of allowable increase = $100\left(\frac{199-195}{\infty}\right) = 0\%$

Sum = 92%.

Part e)

By the 100% rule for simultaneous changes in the objective function, the optimal solution may or may not change.

C_{6AM}: \$170 → \$173.40. % of allowable increase = $100\left(\frac{173.40-170}{\infty}\right) = 0\%$

C_{8AM}: \$160 → \$163.20. % of allowable increase = $100\left(\frac{163.20-160}{10}\right) = 32\%$

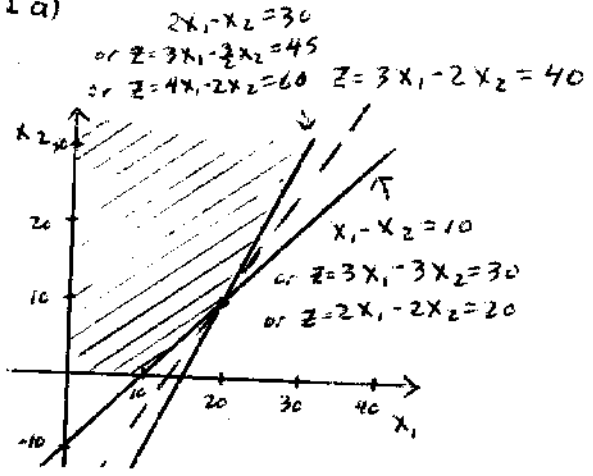
C_{noon}: \$175 → \$178.50. % of allowable increase = $100\left(\frac{178.50-175}{5}\right) = 70\%$

C_{4PM}: \$180 → \$183.60. % of allowable increase = $100\left(\frac{183.60-180}{\infty}\right) = 0\%$

C_{10PM}: \$195 → \$198.90. % of allowable increase = $100\left(\frac{198.9-195}{\infty}\right) = 0\%$

Sum = 102%.

6.7-11 a)



The allowable range to stay optimal for c_1 is $2 \le c_1 \le 4$,
for c_2 is $-3 \le c_2 \le -\frac{3}{2}$.

b) Increment c_1 by Δc_1 (so $c_1 = 3 + \Delta c_1$). This causes the coefficient of x_1 in row 0 of the final tableau to become $-\Delta c_1$. Adding Δc_1 times row 2 to row 0 will make this 0:

$$[-\Delta c_1, 0, 1, 1] + \Delta c_1[1, 0, 1, -1] = [0, 0, 1+\Delta c_1, 1-\Delta c_1]$$

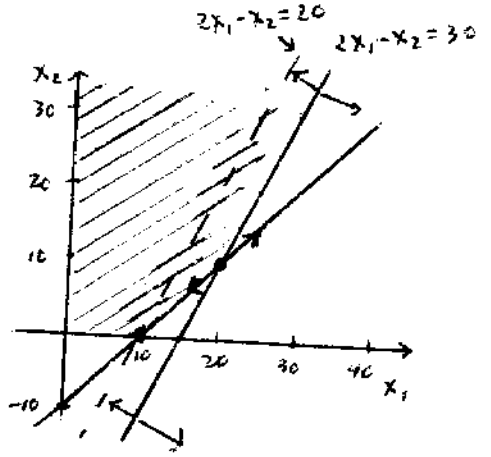
In order for this to stay optimal, we need $1 + \Delta c_1 \ge 0$ and $1 - \Delta c_1 \ge 0$ or $-1 \le \Delta c_1 \le 1$. Thus, the allowable range to stay optimal for c_1 is $3 - 1 \le c_1 \le 3 + 1$ or $2 \le c_1 \le 4$.

Similarly, incrementing c_2 by Δc_2 and adding Δc_2 times row 1 to the row 0 of the final tableau gives:

$$[0, -\Delta c_2, 1, 1] + \Delta c_2[0, 1, 1, -2] = [0, 0, 1+\Delta c_2, 1-2\Delta c_2]$$

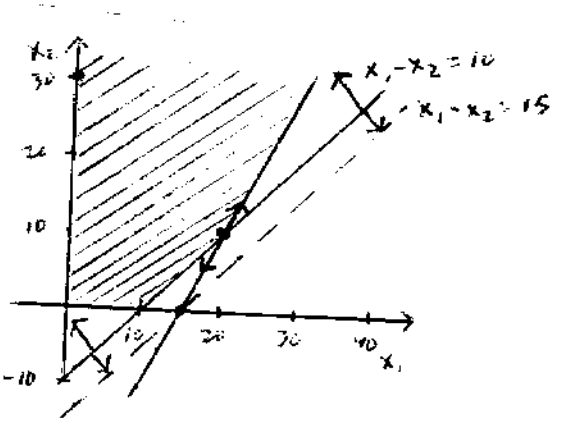
So to stay optimal, we need $-1 \le \Delta c_2 \le \frac{1}{2}$ or $-3 \le c_2 \le -\frac{3}{2}$.

c)



The allowable range to stay feasible for b_1 is

$20 \le b_1$



The allowable range to stay feasible for b_2 is

$b_2 \le 15$

d) Increment b_1 by Δb_1. The final right side will be

$$S^* \tilde{b} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 30+\Delta b_1 \\ 10 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} + \begin{pmatrix} \Delta b_1 \\ \Delta b_1 \end{pmatrix}$$

Thus, in order to be feasible, $\Delta b_1 \ge -10$

So the allowable range to feasible for b_1 is $b_1 \ge 20$.

Similarly incrementing b_2 by Δb_2, we get the final right side:

$$S^* \tilde{b} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 30 \\ 10+\Delta b_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} \Delta b_2$$

So $\Delta b_2 \le 5$ to stay feasible, or, in otherwords, the allowable range to stay feasible for b_2 is $b_2 \le 15$.

e)

```
MAX 3x1-2x2;

SUBJECT TO
2x1-x2<=30;
x1-x2<=10;
```

in MPL

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	2.0000	4.0000
x2	-2.0000	-3.0000	-1.5000

RANGES RHS

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	30.0000	20.0000	1E+020
c2	10.0000	-1E+020	15.0000

6.7-12 We increment b_i by Δb_i and check for feasibility for each i.

We can look at all three b_i's simultaneously:

$$b^* = S^* \bar{b} = \begin{pmatrix} 1 & 0 & 0 \\ -3/4 & 0 & 1/4 \\ 9/4 & 1 & -3/4 \end{pmatrix} \begin{pmatrix} 4+\Delta b_1 \\ 24+\Delta b_2 \\ 18+\Delta b_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3/2 \\ 39/2 \end{pmatrix} + \begin{pmatrix} 1 \\ -3/4 \\ 9/4 \end{pmatrix} \Delta b_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Delta b_2 + \begin{pmatrix} 0 \\ 1/4 \\ -3/4 \end{pmatrix} \Delta b_3$$

to remain feasible Δb_1 must satisfy (keeping $\Delta b_2 = \Delta b_3 = 0$)

$4 + \Delta b_1 \geq 0 \qquad \Delta b_1 \geq -4$

$3/2 - 3/4 \Delta b_1 \geq 0 \qquad \Delta b_1 \leq 2$

$\frac{39}{2} + \frac{9}{4} \Delta b_1 \geq 0 \qquad \Delta b_1 \geq -\frac{78}{9}$

or $-4 \leq \Delta b_1 \leq 2 \Rightarrow 0 \leq b_1 \leq 6$

Similarly, keeping $\Delta b_1 = \Delta b_3 = 0$, Δb_2 must satisfy

$$\frac{39}{2} + \Delta b_2 \geq 0 \Rightarrow \Delta b_2 \geq -\frac{39}{2}$$

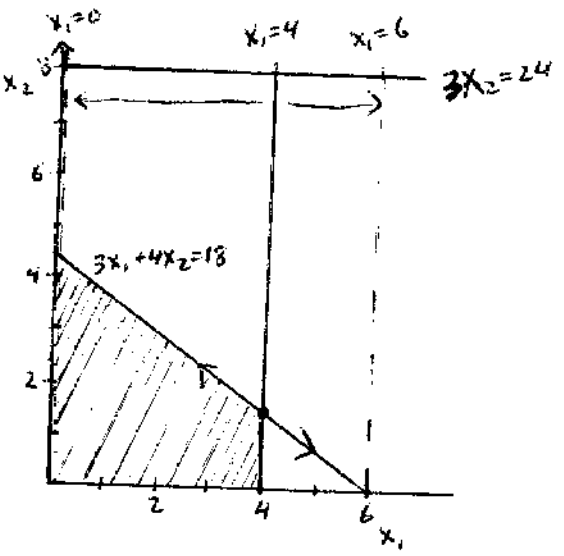
or $b_2 \geq \frac{9}{2}$

Finally, letting $\Delta b_1 = \Delta b_2 = 0$, Δb_3 must satisfy

$\frac{3}{2} + \frac{1}{4} \Delta b_3 \geq 0 \qquad \Delta b_3 \geq -6$

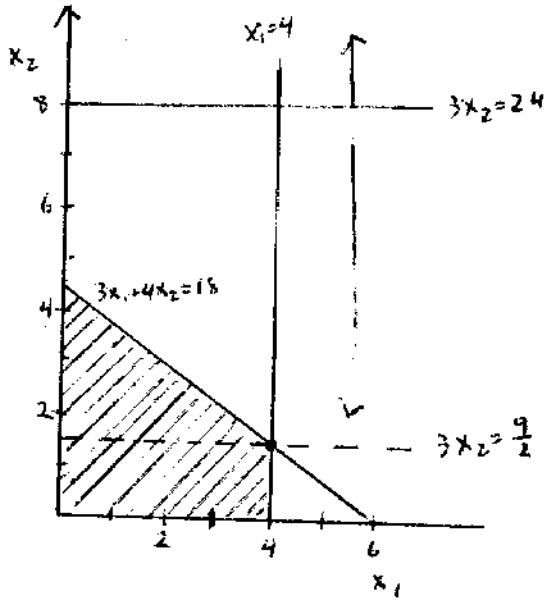
$\frac{39}{2} - \frac{3}{4} \Delta b_3 \geq 0 \qquad \Delta b_3 \leq 26$

or $12 \leq b_3 \leq 44$



The allowable range to stay feasible for b_1 is

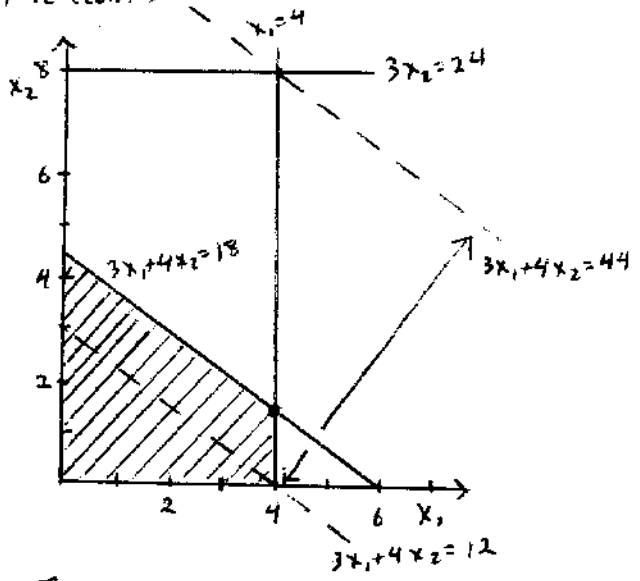
$0 \leq b_1 \leq 6$



The allowable range to stay feasible for b_2 is

$\frac{9}{2} \leq b_2$

6.7-12 (cont')



The allowable range to stay feasible for b_3 is

$$12 \le b_3 \le 44$$

6.7-14. (a) If b_1 or b_2 is changed,

$$(b_1^*, b_2^*) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} = \begin{pmatrix} \Delta b_1 + \Delta b_2 \\ \Delta b_1 + 2\Delta b_2 \end{pmatrix},$$

so both are sensitive parameters.

If c_1 changes, $\bar{c}_{1\,new} = \bar{c}_1 - \Delta c_1 = 8 - \Delta c_1$.

c_1 is not sensitive, since small changes in c_1 will not affect the optimal solution.

If c_2 or c_3 changes, $\bar{c}_{2\,new} = 0 - \Delta c_2$ and $\bar{c}_{3\,new} = 0 - \Delta c_3$ so the optimal objective value will change.

If a_{11} or a_{21} changes, it will only affect a_{11}^*, a_{22}^* and $\bar{c}_1$, so small changes won't affect the optimal solution. These parameters are not sensitive.

Changes in a_{12}, a_{13}, a_{22} and a_{23} will change B and also $B^{-1}b$. The optimal solution will change so these parameters are sensitive.

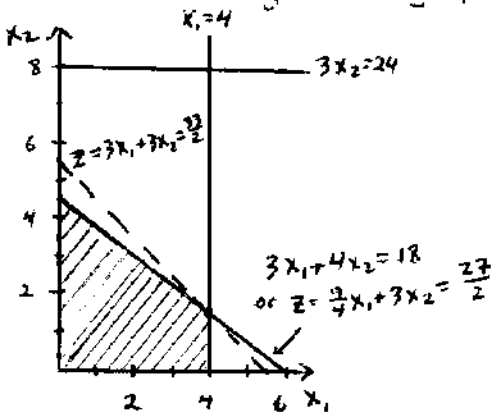
6.7-13 Algebraically: Increment c_1 by Δc_1 (so $c_1 = 3 + \Delta c_1$). The coefficient of x_1 in row 0 of the final tableau is $-\Delta c_1$. Adding Δc_1 times row 1 to row 0, we get:

$$\text{Final row } 0 = \left[-\Delta c_1, 0, \tfrac{3}{4}, 0, \tfrac{3}{4}\right] + \Delta c_1\left[1, 0, 1, 0, 0\right] = \left[0, 0, \tfrac{3}{4}+\Delta c_1, 0, \tfrac{3}{4}\right]$$

In order for this to stay optimal, we need $\frac{3}{4} + \Delta c_1 \ge 0$ or $\Delta c_1 \ge -\frac{3}{4}$

Thus, the allowable range to stay optimal for c_1 is $c_1 \ge \frac{9}{4}$.

Graphically:



The allowable range to stay optimal for c_1 is $c_1 \ge \frac{3}{4}$. (Note that no matter how large c_1 gets, $(4, \frac{3}{2})$ will be optimal as long as $c_1 \ge \frac{9}{4}$)

6.7-14 b) We've seen in (a) that changes in c_1 only affect $\bar{c}_1$. $\bar{c}_{1,new} = \bar{c}_1 - \Delta c_1 = 8 - \Delta c_1$. If $\Delta c_1 \leq 8$, the current solution is optimal. The allowable range for c_1 is:

$$c_1 \leq 11.$$

For changes in c_2, the current basis remains optimal (although the objective value may change) if:

Row 0 + Δc_2 Row 2 ≥ 0

This is equivalent to:

$$\left.\begin{array}{l} 8 + 5\Delta c_2 \geq 0 \\ 3 + \Delta c_2 \geq 0 \\ 4 + 2\Delta c_2 \geq 0 \end{array}\right\} \quad \Delta c_2 \geq -\tfrac{8}{5}$$

The allowable range for c_2 is

$$c_2 \geq -\tfrac{3}{5}$$

For changes in c_3, the current basis remains optimal (although the objective value may change) if:

Row 0 + Δc_3 Row 1 ≥ 0

Equivalently:

$$\left.\begin{array}{l} 8 + 3\Delta c_3 \geq 0 \\ 3 + \Delta c_3 \geq 0 \\ 4 + \Delta c_3 \geq 0 \end{array}\right\} \quad \Delta c_3 \geq -\tfrac{8}{3}$$

The allowable range for c_3 is:

$$c_3 \geq -\tfrac{2}{3}$$

(c) Changes in the right-hand side only affect the final right-hand side. We require $b^* \geq 0$.

For b_1 we need: $b^* + B^{-1}\begin{pmatrix}\Delta b_1 \\ 0\end{pmatrix} \geq 0$

-or- $\begin{pmatrix}30\\40\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\Delta b_1\\ 0\end{pmatrix} \geq 0$

-or- $\left.\begin{array}{l} 30 + \Delta b_1 \geq 0 \\ 40 + \Delta b_1 \geq 0 \end{array}\right\} \quad \Delta b_1 \geq -30$

The allowable range for b_1 is

$$b_1 \geq -10$$

For the current basis to remain optimal with b_2 changing we need:

$$b^* + B^{-1}\begin{pmatrix}0\\ \Delta b_2\end{pmatrix} \geq 0$$

-or- $\begin{pmatrix}30\\40\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\ \Delta b_2\end{pmatrix} \geq 0$

-or- $\left.\begin{array}{l} 30 + \Delta b_2 \geq 0 \\ 40 + 2\Delta b_2 \geq 0 \end{array}\right\} \quad \Delta b_2 \geq -20$

The allowable range for b_2 is:

$$b_2 \geq -10.$$

d) in MPL:

```
MAX 3x1+1x2+2x3;

SUBJECT TO
x1-x2+2x3<=20;
2x1+x2-x3<=10;
```

PLAIN VARIABLES

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	-1E+020	11.0000
x2	1.0000	-0.6000	1E+020
x3	2.0000	-0.6667	1E+020

RANGES RHS

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	20.0000	-10.0000	1E+020
c2	10.0000	-10.0000	1E+020

6.7-15 Increment C_2 by ΔC_2 (so $C_2 = 5 + \Delta C_2$). The resulting coefficient in the final tableau (for X_2 in row 0) is $-\Delta C_2$. Adding ΔC_2 times row 2 to row 0, we get:

Final row 0 $= \left[\frac{9}{2}, -\Delta C_2, 0, 0, \frac{5}{2}\right] + \Delta C_2\left[\frac{3}{2}, 1, 0, 0, \frac{1}{2}\right] = \left[\frac{9}{2} + \frac{3}{2}\Delta C_2, 0, 0, \frac{5}{2} + \frac{1}{2}\Delta C_2\right]$

Thus, to stay optimal, we need $\frac{9}{2} + \frac{3}{2}\Delta C_2 \geq 0$ and $\frac{5}{2} + \frac{1}{2}\Delta C_2 \geq 0$

In other words, $\Delta C_2 \geq -3$ or $C_2 \geq 2$.

So, the allowable range to stay optimal for C_2 is $C_2 \geq 2$.

Looking at figure 6.3, we see that if $C_2 = 2$, $Z = 3X_1 + 2X_2 = 18$ lies exactly on the constraint boundary. Thus, if C_2 were decreased any more, the optimal would be (4,3), not (0,9). On the other hand, increasing C_2 only makes the objective function "more horizontal" approaching $Z = X_2 = 9$, so for any $C_2 \geq 2$, the optimal remains (0,9).

6.7-16

(a) We need $b^* + \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \Delta b_1 \\ 0 \\ 0 \end{pmatrix} \geq 0$

-or- $(2,6,2) + (\Delta b_1, 0, 0) \geq 0$

So, $\Delta b_1 \geq -2$ and $b_1 \geq 2$.

For b_2 we require $b^* + \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ \Delta b_2 \\ 0 \end{pmatrix} \geq 0$

-or- $(2,6,2) + \left(\frac{1}{3}\Delta b_2, \frac{1}{2}\Delta b_2, -\frac{1}{3}\Delta b_2\right) \geq 0$

So, $-6 \leq \Delta b_2 \leq 6$. The allowable range for b_2 is: $6 \leq b_2 \leq 18$

For b_3 we need $b^* + \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \Delta b_3 \end{pmatrix} \geq 0$

-or- $(2,6,2) + \left(-\frac{1}{3}\Delta b_3, 0, \frac{1}{3}\Delta b_3\right) \geq 0$

So, $-6 \leq \Delta b_3 \leq 6$. The allowable range for b_3 is: $12 \leq b_3 \leq 24$.

(b) For changes in c_1 we need:

Row 0 + Δc_1 Row 3 ≥ 0

-or- $\frac{3}{2} - \frac{1}{3}\Delta c_1 \geq 0$

$1 + \frac{1}{3}\Delta c_1 \geq 0$

This implies $-3 \leq \Delta c_1 \leq \frac{9}{2}$. The allowable range for c_1 is: $0 \leq c_1 \leq \frac{15}{2}$.

For changes in c_2 we require:

Row 0 + Δc_2 Row 2 ≥ 0

-or- $\frac{3}{2} + \frac{1}{2}\Delta c_2 \geq 0$

This implies $-3 \leq \Delta c_2$. The allowable range for c_2 is: $2 \leq c_2$

(c)

```
MAX 3x1+5x2;

SUBJECT TO
x1<=4;
2x2<=12;
3x1+2x2<=18;
```

in MPL

RANGES RHS

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	4.0000	2.0000	1E-020
c2	12.0000	6.0000	18.0000
c3	18.0000	12.0000	24.0000

PLAIN VARIABLES

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	0.0000	7.5000
x2	5.0000	2.0000	1E+020

6.7-17. a) Increment b_1 by Δb_1 and calculate the new right side:

$$b^* = S^* \bar{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & -2/3 \end{pmatrix} \begin{pmatrix} 4+\Delta b_1 \\ 24 \\ 18 \\ 24 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Delta b_1$$

To stay feasible, $\Delta b_1 \geq -4$ or $b_1 \geq 0$

Similarly, incrementing b_2, b_3, b_4 by $\Delta b_2, \Delta b_3, \Delta b_4$ respectively, we get:

b_2: $b^* = \begin{pmatrix} 4 \\ 8 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Delta b_2 \geq 0 \Rightarrow \Delta b_2 \geq -8$

b_3: $b^* = \begin{pmatrix} 4 \\ 8 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Delta b_3 \geq 0 \Rightarrow \Delta b_3 \geq -2$

b_4: $b^* = \begin{pmatrix} 4 \\ 8 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \\ -2/3 \\ -2/3 \end{pmatrix} \Delta b_4 \geq 0 \Rightarrow \begin{array}{l} \Delta b_4 \geq -24 \\ \Delta b_4 \leq 12 \\ \Delta b_4 \leq 3 \end{array}$

So the allowable range to stay feasible for:

b_1 is $b_1 \geq 0$

b_2 is $b_2 \geq 16$

b_3 is $b_3 \geq 16$

b_4 is $0 \leq b_4 \leq 27$

b) Incrementing c_1 by Δc_1, the x_1 coefficient in row 0 of the final tableau becomes $\frac{1}{3} - \Delta c_1$. In order for this to remain optimal, $\frac{1}{3} - \Delta c_1 \geq 0$ or $\Delta c_1 \leq \frac{1}{3}$. Thus, the allowable range to stay optimal for c_1 is $c_1 \leq 3 + \frac{1}{3} = \frac{10}{3}$.

Incrementing c_2 by Δc_2, the x_2 coefficient in row 0 of the final tableau becomes $-\Delta c_2$. Using row 2 to eliminate this coefficient, we get:

Final row 0 $= \left[\frac{1}{3}, -\Delta c_2, 0, 0, 0, \frac{5}{3}\right] + \Delta c_2 \left[\frac{3}{3}, 1, 0, 0, 0, \frac{1}{3}\right] = \left[\frac{1}{3} + \frac{3}{3}\Delta c_2, 0, 0, 0, \frac{5}{3} + \frac{1}{3}\Delta c_2\right]$

In order to remain optimal, we need $\frac{1}{3} + \frac{3}{3}\Delta c_2 \geq 0$ and $\frac{5}{3} + \frac{1}{3}\Delta c_2 \geq 0$.

This means $\Delta c_2 \geq -\frac{1}{2}$ or $c_2 \geq \frac{9}{2}$.

c) in MPL

```
MAX 3x1+5x2;

SUBJECT TO
x1<=4;
2x2<=12;
3x1+2x2<=18;
2x1+3x2<=24;
```

(CONT'D)

c) (cont'd)
RANGES RHS

PLAIN CONSTRAINTS

Constraint Name	RHS Value	Lower Bound	Upper Bound
c1	4.0000	2.0000	1E+020
c2	12.0000	6.0000	14.4000
c3	18.0000	12.0000	21.0000
c4	24.0000	22.0000	1E+020

PLAIN VARIABLES

Variable Name	Coefficient	Lower Range	Upper Range
x1	3.0000	0.0000	7.5000
x2	5.0000	2.0000	1E+020

6.1-18

a) Optimal solution: produce no chocolate ice cream, 300 gallons of vanilla ice cream, and 75 gallons of banana ice cream. Total profit will be $341.25.

b) The optimal solution will change since $1.00 is outside the allowable range. The profit will go up, but how much can't be determined without resolving.

c) The optimal solution will not change since $0.92 is within the allowable range. Total profit will decrease by $2.25 (0.03 x 75) to $339.

d) The optimal solution will change. Since the change is within the allowable range, we can calculate the change in profit using the shadow price (1 x 3 = 3). The new profit will be $338.25.

e) This increase is outside of the allowable increase so the problem will have to be resolved to determine whether this is worthwhile.

f) The final value is 180 as shown in the totals column in the solution. The shadow price is 0 since we are using less milk than we have available. The R.H.Side value is 200 as given in the problem. The allowable increase is infinity since we are already using less than is available. The allowable decrease is 20 since the solution will change once the right-hand side drops below 180.

6.1-19

a) Let G = number of grandfather clocks produced
W = number of wall clocks produced

Maximize $P = 300G + 200W$,

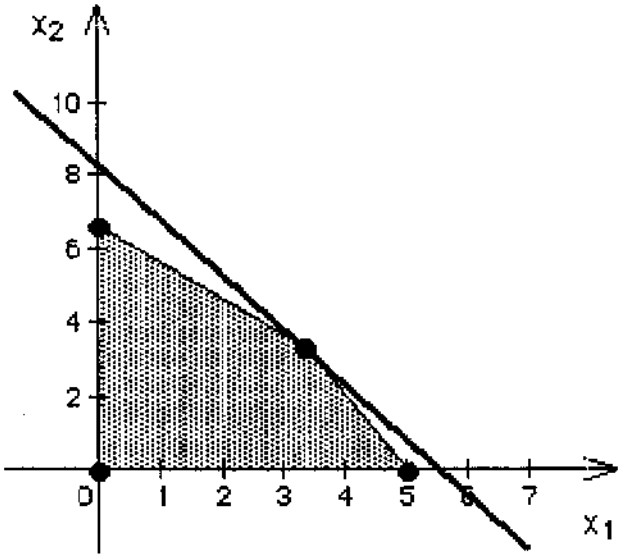
subject to $6G + 4W \le 40$,

$8G + 4W \le 40$,

$3G + 3W \le 20$,

and $G \ge 0,\ W \ge 0.$

6.1-18 b) Optimal Solution: $(G, W) = (x_1, x_2) = (3.333, 3.333)$ and $P = 1666.67$.



c & d)

Resource	Resource Usage Per Unit of Each Activity: Grandfather	Wall	Totals		Resource Available
David	6	4	33.333	≤	40
LaDeana	8	4	40.00	≤	40
Lydia	3	3	20	≤	20
Unit Profit	300	200	**$ 1,666.67**		
Solution	**3.333**	**3.333**			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$7	Solution Grandfather	3.333	0	300	100	100
$C$7	Solution Wall	3.333	0.000	200	100	50

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$3	David Totals	33.333	0	40	1E+30	6.667
$D$4	LaDeana Totals	40.00	25.00	40	13.333	13.333
$D$5	Lydia Totals	20	33.333	20	10	5

e) The optimal solution will remain the same since \$75 is within the allowable increase.

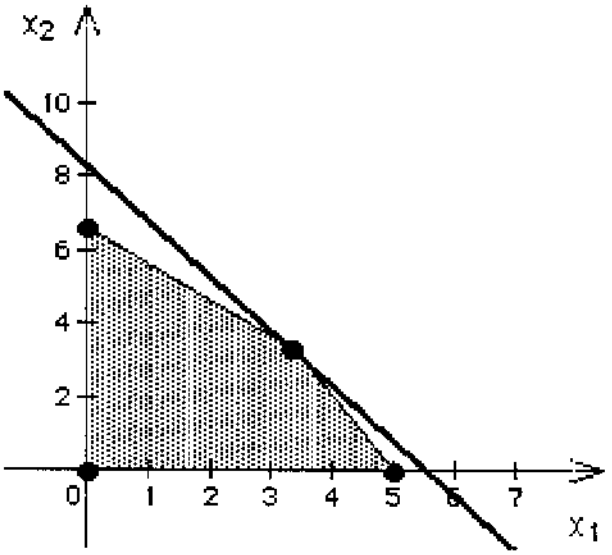
f) By the 100% rule for simultaneous changes in the objective function, the optimal solution may or may not change.

C_G: \$300 → \$375. % of allowable increase $= 100\left(\frac{375-300}{100}\right) = 75\%$

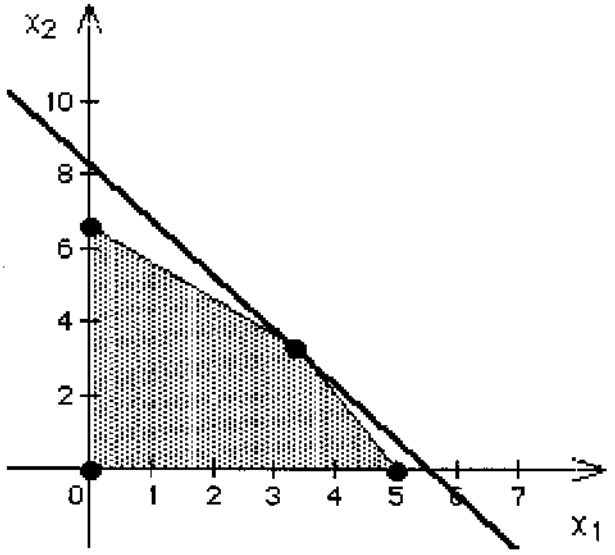
C_W: \$200 → \$175. % of allowable decrease $= 100\left(\frac{200-175}{50}\right) = 50\%$

Sum = 125%.

g) When the profit for grandfather clocks increases to \$375, the optimal solution is still: $(G, W) = (x_1, x_2) = (3.333, 3.333)$ and $P = 1666.67$.



If both profit estimates change then the optimal solution changes to: $(G, W) = (x_1, x_2) = (5,0)$ and $P = 1875$. (5,0) with a profit of \$1875.



h) Lydia should increase her hours slightly since she has the highest shadow price connected to her.

i) The shadow price for David is zero because all of his available hours are not being used and consequently an increase in his hours would not impact total profit.

j) Yes, this increase is within the allowable increase. The increase in total profit will be \$33.33 x 5 = \$166.65.

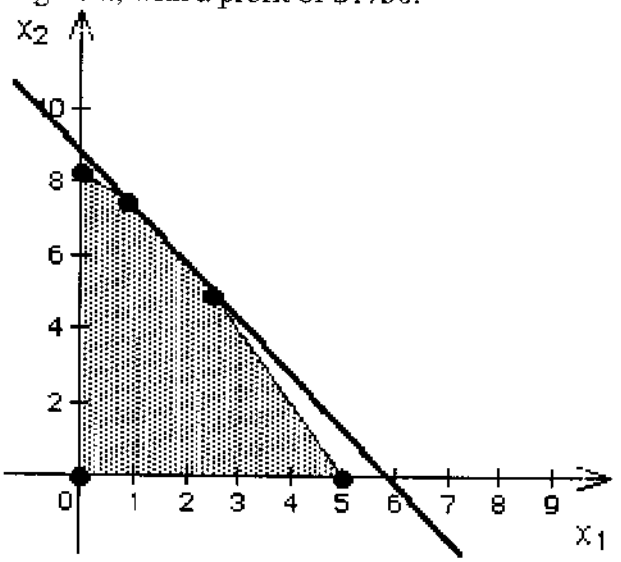
k) By the 100% rule for simultaneous changes in right-hand sides, the optimal solution may or may not change. The shadow prices may not be used to determine the effect on profit.

C_L: $20 \rightarrow 25$. % of allowable increase $= 100\left(\frac{25-20}{10}\right) = 50\%$

C_D: $40 \rightarrow 35$. % of allowable decrease $= 100\left(\frac{40-35}{6.667}\right) = 75\%$

Sum = 125%.

l) The revised graph is shown below. The optimal solution changes from (3.333,3.333) with a profit of \$1666.70 to (2.5,5), (.833,7.5), and all points on the connecting line segment, with a profit of \$1750.



6.7-20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$G$4	Serve 6-8am Totals	48	10	48	6	48
$G$5	Serve 8-10am Totals	79	160	79	1E+30	6
$G$6	Serve 10-12am Totals	79	0	65	14	1E+30
$G$7	Serve 12-2pm Totals	118	0	87	31	1E+30
$G$8	Serve 2-4pm Totals	70	0	64	6	1E+30
$GS9	Serve 4-6pm Totals	82	0	73	9	1E+30
$GS10	Serve 6-8pm Totals	82	175	82	1E+30	6
$GS11	Serve 8-10pm Totals	43	5	43	6	6
$GS12	Serve 10-12pm Totals	58	0	52	6	1E+30
$G$13	Serve 12-6am Totals	15	195	15	1E+30	6

a) The following shift can be increased by the indicated amounts:

Serve 10-12 am	14
Serve 12-2 pm	31
Serve 2-4 pm	6
Serve 4-6 pm	9
Serve 10-12 pm	6

b) For each of the following shifts, total cost will increase by the amount indicated per unit increase. These costs hold for the indicated increases

Serve 6-8 am	\$10	6
Serve 8-10 am	\$160	∞
Serve 6-8 pm	\$175	∞
Serve 8-10 pm	\$5	6
Serve 12-6 am	\$195	∞

c) By the 100% rule for simultaneous changes in right-hand sides, the shadow prices are still valid.

C_{6-8am}: $48 \rightarrow 49$. % of allowable increase $= 100\left(\frac{49-48}{6}\right) = 17\%$

C_{8-10am}: $79 \rightarrow 80$ % of allowable increase $= 100\left(\frac{80-79}{\infty}\right) = 0\%$

C_{6-8pm}: $82 \rightarrow 83$ % of allowable increase $= 100\left(\frac{83-82}{\infty}\right) = 0\%$

C_{8-10pm}: $43 \rightarrow 44$ % of allowable increase $= 100\left(\frac{44-43}{6}\right) = 17\%$

C_{12-6am}: $15 \rightarrow 16$. % of allowable increase $= 100\left(\frac{16-15}{\infty}\right) = 0\%$

Sum = 34%.

d) By the 100% rule for simultaneous changes in right-hand sides, the shadow prices are still valid.

C_{6-8am}: $48 \rightarrow 49$. % of allowable increase $= 100\left(\frac{49-48}{6}\right) = 17\%$

C_{8-10am}: $79 \rightarrow 80$ % of allowable increase $= 100\left(\frac{80-79}{\infty}\right) = 0\%$

$C_{10-12am}$: $65 \rightarrow 66$. % of allowable increase $= 100\left(\frac{66-65}{14}\right) = 7\%$

C_{12-2pm}: $87 \rightarrow 88$. % of allowable increase $= 100\left(\frac{88-87}{31}\right) = 3\%$

C_{2-4pm}: $64 \rightarrow 65$. % of allowable increase $= 100\left(\frac{65-64}{6}\right) = 17\%$

C_{4-6pm}: $73 \rightarrow 74$. % of allowable increase $= 100\left(\frac{74-73}{9}\right) = 11\%$

C_{6-8pm}: $82 \rightarrow 83$ % of allowable increase $= 100\left(\frac{83-82}{\infty}\right) = 0\%$

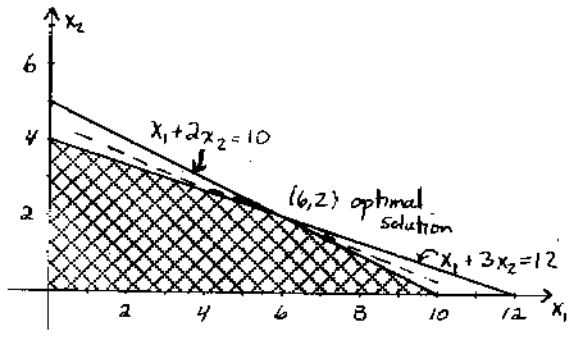
C_{8-10pm}: $43 \rightarrow 44$ % of allowable increase $= 100\left(\frac{44-43}{6}\right) = 17\%$

$C_{10-12pm}$: $52 \rightarrow 53$. % of allowable increase $= 100\left(\frac{53-52}{6}\right) = 17\%$

C_{12-6am}: $15 \rightarrow 16$. % of allowable increase $= 100\left(\frac{16-15}{\infty}\right) = 0\%$ SUM=89%

e) All numbers can increase by $\frac{100}{89}$ (or 1.124) before it is no longer definite that the shadow prices remain valid.

6.7-21 (a)



Parameter	Allowable Range	Likely, but not Allowable Range(s)
b_1	[8, 12]	[5, 8) and (12, 15]
b_2	[10, 15]	[6, 10) and (15, 18]
c_1	$[1\frac{2}{3}, 2\frac{1}{2}]$	$[1, 1\frac{2}{3})$ and $(2\frac{1}{2}, 3]$
c_2	[4, 6]	$[2\frac{1}{2}, 4)$ and $(6, 7\frac{1}{2}]$

(b) For b_1 the current basis is optimal if

$$b^* + B^{-1}\begin{pmatrix}\Delta b_1\\ 0\end{pmatrix} \geq 0$$

$$\text{-or-} \quad \begin{pmatrix}6\\2\end{pmatrix} + \begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}\begin{pmatrix}\Delta b_1\\ 0\end{pmatrix} \geq 0$$

$$\text{-or-} \quad \left.\begin{aligned} 6 + 3\Delta b_1 &\geq 0\\ 2 - \Delta b_1 &\geq 0\end{aligned}\right\} -2 \leq \Delta b_1 \leq 2$$

The allowable range for b_1 is

$$8 \leq b_1 \leq 12.$$

For c_1 the current basis is optimal if Row 0 + Δc_1 Row 1 ≥ 0

$$\text{-or-} \quad \left.\begin{aligned} 1 + 3\Delta c_1 &\geq 0\\ 1 - 2\Delta c_1 &\geq 0\end{aligned}\right\} -\tfrac{1}{3} \leq \Delta c_1 \leq \tfrac{1}{2}$$

The allowable range for c_1 is:

$$1\tfrac{2}{3} \leq c_1 \leq 2\tfrac{1}{2}.$$

(c) For b_2 the current basis is optimal if $b^* + B^{-1}\begin{pmatrix}0\\ \Delta b_2\end{pmatrix} \geq 0$

$$\text{-or-} \quad \begin{pmatrix}6\\2\end{pmatrix} + \begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}\begin{pmatrix}0\\ \Delta b_2\end{pmatrix} \geq 0$$

$$\text{-or-} \quad \left.\begin{aligned} 6 - 2\Delta b_2 &\geq 0\\ 2 + \Delta b_2 &\geq 0\end{aligned}\right\} -2 \leq \Delta b_2 \leq 3$$

The allowable range for b_2 is

$$10 \leq b_2 \leq 15$$

(d) For c_2 the current basis remains optimal if Row 0 + Δc_2 Row 2 ≥ 0

$$\text{-or-} \quad \left.\begin{aligned} 1 - \Delta c_2 &\geq 0\\ 1 + \Delta c_2 &\geq 0\end{aligned}\right\} -1 \leq \Delta c_2 \leq 1$$

The allowable range for c_2 is:

$$4 \leq c_2 \leq 6$$

6.7-22

a) The allowable range for b_1 is [8, 12] and for b_2 is [10, 15]
originally $b_1 = 10$ and $b_2 = 12$

(1) We have to solve the equations:

$$\frac{10 - x_1}{2} = \frac{12 - x_2}{2} \quad \text{and} \quad \frac{10 - x_1}{2} + \frac{12 - x_2}{2} = 1$$

$x_1 = 8$ and $x_2 = 11$ satisfy these equations (50% and)
(CONT'D)

6.7-22 (cont'd)

(2) We need to solve the equations:

$$\frac{x_1-10}{2} = \frac{x_2-12}{3} \quad \text{and} \quad \frac{x_1-10}{2} + \frac{x_2-12}{3} = 1$$

$x_1 = 11$ and $x_2 = 13.5$ satisfy these equations (50% error)

(3) Same analysis as before. We get $x_1 = 9$ & $x_2 = 13.5$ (50% error)

(4) Same analysis as before. We get $x_1 = 11$ & $x_2 = 11$ (50% error)

b) We have to ensure that $b + B^{-1}\Delta b \geq 0$; where $b = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$; $B^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

and $\Delta b = \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix}$ and where b_1 & b_2 change by the same %.

100% → (1) $b_1 = 8$ and $b_2 = 10$ satisfy the equation above, and stay within range.

100% → (2) $b_1 = 12$ and $b_2 = 15$ satisfy the equation above, and stay within range.

50% → (3) $b_1 = 9$ and $b_2 = 13.5$ satisfy the equation above, and stay within range.

50% → (4) $b_1 = 11$ and $b_2 = 11$ satisfy the equations above, and stay within range.

c) The allowable range for c_1 is $[1\frac{2}{3}, 2\frac{1}{2}]$ and for c_2 is $[4, 6]$

originally $c_1 = 2$ and $c_2 = 5$.

(1) we have to solve the equations: $\frac{2-x_1}{1/3} = \frac{5-x_2}{1}$ and $\frac{2-x_1}{1/3} + \frac{5-x_2}{1} = 1$

Sol^n: $x_1 = 1\frac{5}{6}$ and $x_2 = 4\frac{1}{2}$ (50%)

(2) We have to solve the equations: $\frac{x_1-2}{1/2} = \frac{x_2-5}{1}$ and $\frac{x_1-2}{1/2} + \frac{x_2-5}{1} = 1$

Sol^n: $x_1 = 2\frac{1}{4}$ and $x_2 = 5\frac{1}{2}$ (50%)

(3) We have the equations $\frac{2-x_1}{1/3} = \frac{x_2-5}{1}$ and $\frac{2-x_1}{1/3} + \frac{x_2-5}{1} = 1$

Sol^n: $x_1 = 1\frac{5}{6}$; $x_2 = 5\frac{1}{2}$ (50% error)

(4) The equations are $\frac{x_1-2}{1/2} = \frac{5-x_2}{1}$ and $\frac{x_1-2}{1/2} + \frac{5-x_2}{1} = 1$

Sol^n: $x_1 = 2\frac{1}{4}$ and $x_2 = 4.5$. (50% error)

d) We have to ensure that $\text{Row } 0 + \Delta c_1 \text{Row } 1 + \Delta c_2 \text{Row } 2 \geq 0$

where c_1 and c_2 change by the same percentage.

where Row $0 = [1, 1]$; Row $1 = [3, -2]$ and Row $2 = [-1, 1]$.

Solving we get in each case:

(1): 100% error (2) 100% error (3): 50% error (4): 50% error

6.7-23(a) For b_2 the current basis remains optimal if $b^* + B^{-1}\begin{pmatrix}0\\ \Delta b_2\end{pmatrix} \geq 0$

-or- $\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}\begin{pmatrix}0\\ \Delta b_2\end{pmatrix} \geq 0$

-or- $\left.\begin{array}{l} 2 - 5\Delta b_2 \geq 0 \\ 1 + 2\Delta b_2 \geq 0 \end{array}\right\} -\frac{1}{2} \leq \Delta b_2 \leq \frac{2}{5}$

The allowable range for b_2 is

$$4\tfrac{1}{2} \leq b_2 \leq 5\tfrac{2}{5}$$

(b) For c_2 the current basis is optimal if

$\bar{c}_{2new} = \bar{c}_2 - \Delta c_2 = 1 - \Delta c_2 \geq 0$

-or- $\Delta c_2 \leq 1$

The allowable range for c_2 is

$$c_2 \leq 5$$

(c) For a_{22} the current basis remains optimal if $\bar{c}_{2new} = \bar{c}_2 + (1,1)\begin{pmatrix}0\\ \Delta a_{22}\end{pmatrix} \geq 0$

-or- $1 + \Delta a_{22} \geq 0 \Rightarrow \Delta a_{22} \geq -1$

The allowable range for a_{22} is

$$a_{22} \geq 1$$

(d) For c_3 the current basis is optimal if Row 0 + Δc_3 Row 2 ≥ 0

-or- $\left.\begin{array}{l} 1 + \Delta c_3 \geq 0 \\ 1 - \Delta c_3 \geq 0 \\ 1 + 2\Delta c_3 \geq 0 \end{array}\right\} -\frac{1}{2} \leq \Delta c_3 \leq 1$

The allowable range for c_3 is

$$7\tfrac{1}{2} \leq c_3 \leq 9$$

(e) For a_{12} the current basis is optimal if $\bar{c}_{2new} = \bar{c}_2 + (1,1)\begin{pmatrix}\Delta a_{12}\\ 0\end{pmatrix} \geq 0$

-or- $1 + \Delta a_{12} \geq 0 \Rightarrow \Delta a_{12} \geq -1$

The allowable range for a_{12} is

$$a_{12} \geq 2$$

(f) For b_1 the current basis remains optimal if $b^* + B^{-1}\begin{pmatrix}\Delta b_1\\ 0\end{pmatrix} \geq 0$

-or- $\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}\begin{pmatrix}\Delta b_1\\ 0\end{pmatrix} \geq 0$

-or- $\left.\begin{array}{l} 2 + 3\Delta b_1 \geq 0 \\ 1 - \Delta b_1 \geq 0 \end{array}\right\} -\frac{2}{3} \leq \Delta b_1 \leq 1$

The allowable range for b_1 is

$$8\tfrac{1}{3} \leq b_1 \leq 10$$

The results of (a)-(f) are summarized below:

Parameter	Allowable Range	Likely, but not Allowable Range(s)
b_2	$[4\frac{1}{2}, 5\frac{2}{5}]$	$[2\frac{1}{2}, 4\frac{1}{2})$ and $(5\frac{2}{5}, 7\frac{1}{2}]$
c_2	$(-\infty, 5]$	$(5, 6]$
a_{22}	$[1, \infty)$	none
c_3	$[7\frac{1}{2}, 9]$	$[4, 7\frac{1}{2})$ and $(9, 12]$
a_{12}	$[2, \infty)$	$[1\frac{1}{2}, 2)$
b_1	$[8\frac{1}{3}, 10]$	$[4\frac{1}{2}, 8\frac{1}{3})$ and $(10, 13\frac{1}{2}]$

6.7-24

$\Delta c_1 = \theta \Rightarrow \Delta(z_1^* - c_1) = -\theta$

$\Delta c_2 = 2\theta \Rightarrow \Delta(z_2^* - c_2) = -2\theta$

Revised Final Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z(θ)	0	1	$-\theta$	-2θ	3/4	0	3/4	33/2
X1	1	0	1	0	1	0	0	4
X2	2	0	0	1	-3/4	0	1/4	3/2
X4	3	0	0	0	9/4	1	-3/4	39/2

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3			Right side
Z(θ)	0	1	0	0	$3/4 - \theta/2$	0	$3/4 + \theta/2$	$33/2 + 7\theta$
X1	1	0	1	0	1	0	0	4
X2	2	0	0	1	-3/4	0	1/4	3/2
X4	3	0	0	0	9/4	1	-3/4	39/2

This basic solution is optimal if;

$\left.\begin{array}{l} 3/4 - \theta/2 \geq 0 \\ 3/4 + \theta/2 \geq 0 \end{array}\right\} -\frac{3}{2} \leq \theta \leq \frac{3}{2}$

6.7-25 $\Delta c_1 = \theta \Rightarrow \Delta(z_1^* - c_1) = -\theta$

$\Delta c_2 = -\theta \Rightarrow \Delta(z_2^* - c_2) = \theta$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	$-\theta$	θ	2	1	1	0	18
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

The current basic solution is optimal if,

$2-\theta \geq 0$ and $1-\theta \geq 0$ $\Rightarrow \theta \leq 1$

Clearly $Z(\theta) = 18 - 3\theta$ is maximized when θ is as small as possible. θ is restricted to be nonnegative so $\theta = 0$ is optimal.

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	Right side
Z	0	1	0	0	$2-\theta$	1	$1-\theta$	0	$18-3\theta$
X2	1	0	0	1	5	1	3	0	24
X6	2	0	0	0	2	0	1	1	7
X1	3	0	1	0	4	1	2	0	21

6.7-26 a) Row 0 of the final tableau will look like this: $[4\theta, \theta, 3-\theta, 2, 2 \mid 24]$

Eliminating the X_1 and X_2 coefficients using row 1 and 2, resp. we get:

Final row 0 $= [4\theta, \theta, 3-\theta, 2, 2 \mid 24] - 4\theta[1, 0, -1, 1, -1 \mid 2] - \theta[0, 1, 5, -2, 3 \mid 1]$

$= [0, 0, 3-2\theta, 2-2\theta, 2+\theta \mid 24-9\theta]$

To remain optimal, therefore, we need:

$3-2\theta \geq 0 \Rightarrow \theta \leq 3/2$

$2-2\theta \geq 0 \Rightarrow \theta \leq 1$

$2+\theta \geq 0 \Rightarrow \theta \geq -2$

So the range of values over which this BF stays optimal is

$-2 \leq \theta \leq 1$

Since $Z - 9\theta$ is decreasing in θ, the best choice of θ is -2.

$(\theta = -2 \Rightarrow Z = 42)$

b) Adding Δb_1 (Δb_2) to b_1 (b_2), we get

b_1: $b^* = S^* \bar{b} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7+\Delta b_1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Delta b_1$

b_2: $b^* = S^* \bar{b} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 5+\Delta b_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \Delta b_2$

So the allowable range to stay feasible is:

b_1: $2+\Delta b_1 \geq 0$, $1-2\Delta b_1 \geq 0$ $\Rightarrow -2 \leq \Delta b_1 \leq \frac{1}{2} \Rightarrow 5 \leq b_1 \leq \frac{15}{2}$

b_2: $2-\Delta b_2 \geq 0$, $1+3\Delta b_2 \geq 0$ $\Rightarrow -\frac{1}{3} \leq \Delta b_2 \leq 2 \Rightarrow \frac{14}{3} \leq b_2 \leq 7$

c) From the final row 0 in part (a), we get $y_1^* = 2-2\theta$, $y_2^* = 2+\theta$

Decreasing resource 1 (b_1) by 1 and increasing resource 2 (b_2) by 1 gives us a new objective value $\tilde{Z} = Z - (2-2\theta) + (2+\theta) = Z + 3\theta$

Z would increase by θ

6.7-26 d) Dual: Minimize $V(\theta) = 7Y_1 + 5Y_2$

subject to
$$3Y_1 + 2Y_2 \geq 10 - 4\theta$$
$$Y_1 + Y_2 \geq 4 - \theta$$
$$2Y_1 + 3Y_2 \geq 7 + \theta$$
$$Y_1 \geq 0,\ Y_2 \geq 0$$

Starting with tableau:

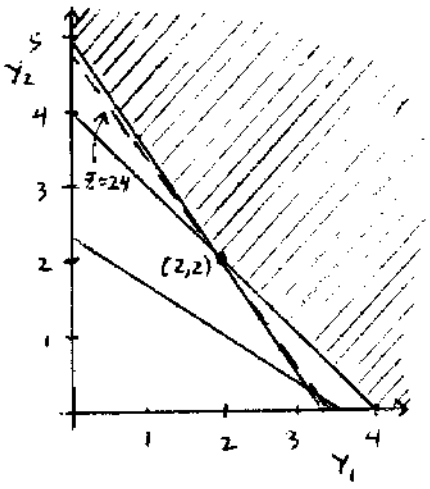
-7	-5	0	0	0	0
-3	-2	1	0	0	$-10+4\theta$
-1	-1	0	1	0	$-4+\theta$
-2	-3	0	0	1	$-7-\theta$

We force Y_1, Y_2 into basis and Y_3, Y_4 out of basis, we get:

0	0	-2	-1	0	$24-9\theta$
0	1	1	-3	0	$2+\theta$
1	0	-1	2	0	$2-2\theta$
0	0	1	-5	1	$3-2\theta$

With shadow prices $(Y_1^*, Y_2^*) = (2-2\theta, 2+\theta)$ as in (c).

At $\theta = 0$, graphically:



Optimal at $(Y_1^*, Y_2^*) = (2, 2)$

6.7-27. a) Using $b^* = S^* \bar{b}$, we get

$$b^* = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5+\theta \\ 6+2\theta \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}\theta \Rightarrow \text{BF is } (3-\theta, 0, 1+\theta, 0, 0)$$

Thus, to be feasible, we need $3-\theta \geq 0$, $1+\theta \geq 0$ or $-1 \leq \theta \leq 3$.

The new Z becomes $(1\ 1)\begin{pmatrix} 5+\theta \\ 6+2\theta \end{pmatrix} = 11+3\theta$ (increasing in θ)

So the best choice of θ would be $\theta = 3$ ($Z = 20$)

b) Incrementing C_1 by ΔC_1, then adding ΔC_1 times row 1 to row 0, we get:

Final row 0 $= [-\Delta C_1, 1, 0, 1, 1 : 11+3\theta] + \Delta C_1[1, 5, 0, 3, -2, 3-\theta]$

$$= [0, 1+5\Delta C_1, 0, 1+3\Delta C_1, 1-2\Delta C_1, 11+3\Delta C_1 + (3-\Delta C_1)\theta]$$

In order to stay optimal, we need

$$\left.\begin{array}{lll} 1+5\Delta C_1 \geq 0 & \Rightarrow & \Delta C_1 \geq -\frac{1}{5} \\ 1+3\Delta C_1 \geq 0 & \Rightarrow & \Delta C_1 \geq -\frac{1}{3} \\ 1-2\Delta C_1 \geq 0 & \Rightarrow & \Delta C_1 \leq \frac{1}{2} \end{array}\right\} -\frac{1}{5} \leq \Delta C_1 \leq \frac{1}{2}$$

Thus, the allowable range to stay optimal for C_1 is

$$\frac{9}{5} \leq C_1 \leq \frac{5}{2}$$

6.7-28.a)-c) $B^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. So new right hand side is

$$\text{new RHS} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5+5\theta \\ 10-10\theta \end{pmatrix} = \begin{pmatrix} 10-10\theta \\ 5-15\theta \end{pmatrix}$$

$$\text{new } Z = (2-\theta)(10-10\theta) = 20-30\theta+10\theta^2$$

$$\text{new row 0} = \text{old row 0} - \theta(\text{row 1})$$

$$= [1-2\theta, \theta, 1-\theta, 0, M, M+2] - \theta[3, 1, 2, 0, 0, 1]$$

$$= [1-5\theta, 0, 1-3\theta, 0, M, M+2-\theta]$$

The complete tableau is:

Basic Var	Eq. No.	Coefficient of Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
Z	0	1	$1-5\theta$	0	$1-3\theta$	0	M	$M+2-\theta$	$20-30\theta+10\theta^2$
x_2	1	0	3	1	2	0	0	1	$10-10\theta$
x_4	2	0	-1	0	2	1	-1	1	$5-15\theta$

(b) To be feasible, $10-10\theta \geq 0$, $5-15\theta \geq 0$ $\Rightarrow 0 \leq \theta \leq \frac{1}{3}$

To be optimal, $1-5\theta \geq 0$, $1-3\theta \geq 0$ $\Rightarrow 0 \leq \theta \leq \frac{1}{5}$

(c) Or $0 \leq \theta \leq \frac{1}{5}$ to be feasible and optimal.

So the solution is:

$(x_1^*, x_2^*) = (0, 10-10\theta)$, $Z^* = 20-30\theta+10\theta^2$

for $0 \leq \theta \leq \frac{1}{5}$

The best choice for θ is the one that maximizes $Z^*(\theta) = 20-30\theta+10\theta^2$ over $0 \leq \theta \leq \frac{1}{5}$.

Since $Z^*(\theta)$ is convex in θ, we need only look at the endpoints.

$Z^*(0) = 20$

$Z^*(\frac{1}{5}) = 20 - 30(\frac{1}{5}) + 10(\frac{1}{25}) = 14\frac{2}{5}$

So $\theta = 0$ is the best choice of θ.

6.7-29.

(a) $\Delta c_1 = -2\theta$, $\Delta a_{11} = 1$

$\Delta(z_1^* - c_1) = 2\theta + (2,2)\begin{pmatrix}1\\0\end{pmatrix} = 2\theta + 2$

$\Delta a_{11}^* = (-2,3)\begin{pmatrix}1\\0\end{pmatrix} = -2$

$\Delta a_{21}^* = (1,-1)\begin{pmatrix}1\\0\end{pmatrix} = 1$

$\Delta c_2 = \theta \Rightarrow \Delta(z_2^* - c_2) = -\theta$

$\Delta b_1 = 10 \Rightarrow \Delta z^* = (2,2)\begin{pmatrix}10\\0\end{pmatrix} = 20$

$\Delta b_1^* = (-2,3)\begin{pmatrix}10\\0\end{pmatrix} = -20$

$\Delta b_2^* = (1,-1)\begin{pmatrix}10\\0\end{pmatrix} = 10$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	Right side
Z	0	1	$2\theta+2$	$-\theta$	2	2	130
X2	1	0	-2	1	-2	3	-5
X1	2	0	2	0	1	-1	15

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	Right side
Z	0	1	0	0	$-3\theta+1$	$4\theta+3$	$-20\theta+115$
X2	1	0	0	1	-1	2	10
X1	2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{15}{2}$

For θ near 0, the optimal solution is $(x_1, x_2, x_3, x_4) = (\frac{15}{2}, 10, 0, 0)$ with $z = -20\theta + 115$.

(b) The solution above remains optimal, if

$$\left.\begin{aligned} -3\theta+1 &\geq 0 \\ 4\theta+3 &\geq 0 \end{aligned}\right\} \Rightarrow -\tfrac{3}{4} \leq \theta \leq \tfrac{1}{3}$$

(c) $z(\theta) = -20\theta + 115$ is largest when θ is smallest. $\theta = 0$ gives the largest objective function value.

6.7-30.

(a) $\Delta c_1 = 9$, $\Delta a_{11} = 1$, $\Delta a_{21} = 1$

$\Delta(z_1^* - c_1) = -9 + (2,1)\begin{pmatrix}1\\1\end{pmatrix} = -6$

$\Delta a_{11}^* = (3,-1)\begin{pmatrix}1\\1\end{pmatrix} = 2$

$\Delta a_{21}^* = (-5,2)\begin{pmatrix}1\\1\end{pmatrix} = -3$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	-6	2	0	2	1	19
X1	1	0	3	5	0	3	-1	1
X3	2	0	-3	-7	1	-5	2	2

6.7-30. (a) (cont')

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	12	0	8	-1	21
X1	1	0	1	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	$\frac{1}{3}$
X3	2	0	0	-2	1	-2	1*	3

Optimal Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	10	1	6	0	24
X1	1	0	1	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{4}{3}$
X5	2	0	0	-2	1	2	1	3

With the new technology, $(x_1, x_2, x_3, x_4, x_5) = (\frac{4}{3}, 0, 0, 0, 3)$ is optimal with $z = 24$.

(b) Changes to $(z_j^* - c_j)$, a_{ij}^*, a_{2j}^* will be θ times the results in (a)

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	-6θ	2	0	2	1	19
X1	1	0	$1+2\theta$	5	0	3	-1	1
X3	2	0	-3θ	7	1	-5	2	2

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	$\frac{34\theta+2}{2\theta+1}$	0	$\frac{22\theta+2}{2\theta+1}$	$\frac{-4\theta+1}{2\theta+1}$	$\frac{44\theta+19}{2\theta+1}$
X1	1	0	1	$\frac{5}{2\theta+1}$	0	$\frac{3}{2\theta+1}$	$\frac{-1}{2\theta+1}$	$\frac{1}{2\theta+1}$
X3	2	0	0	$\frac{29\theta+7}{2\theta+1}$	1	$\frac{19\theta+5}{2\theta+1}$	$\frac{\theta+2}{2\theta+1}$	$\frac{7\theta+2}{2\theta+1}$

Since $2\theta+1 > 0$ for all choices of θ between 0 and 1, the right-hand side will always be positive. The current basic solution is feasible for all choices of $\theta \in [0,1]$.

For optimality we require:

$$\left.\begin{aligned} 34\theta+2 &\geq 0 \\ 22\theta+2 &\geq 0 \\ -4\theta+1 &\geq 0 \end{aligned}\right\} \quad \theta \leq \tfrac{1}{4}$$

The current basis is optimal for $\theta \in [0, \frac{1}{4}]$.

$z(\theta)$ is maximized when $\theta = \frac{1}{4}$ on the range $0 \leq \theta \leq \frac{1}{4}$.

6.7-31. $\Delta c_1 = 2\theta \Rightarrow \Delta(z_1^* - c_1) = -2\theta$

$\Delta c_2 = \theta \Rightarrow \Delta(z_2^* - c_2) = -\theta$

$\Delta c_3 = -\theta \Rightarrow \Delta(z_3^* - c_3) = \theta$

$\left.\begin{matrix}\Delta b_1 = 6\theta \\ \Delta b_2 = -8\theta\end{matrix}\right\}$ $\Delta z^* = (9,7)\begin{pmatrix}6\theta \\ -8\theta\end{pmatrix} = -2\theta$

$\Delta b_1^* = (1,1)\begin{pmatrix}6\theta \\ -8\theta\end{pmatrix} = -2\theta$

$\Delta b_2^* = (3,2)\begin{pmatrix}6\theta \\ -8\theta\end{pmatrix} = 2\theta$

New Tableau:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	-2θ	$20-\theta$	θ	9	7	$115-2\theta$
X1	1	0	1	3	0	1	1	$15-2\theta$
X3	2	0	0	8	1	3	2	$35+2\theta$

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	$20-3\theta$	0	$9-\theta$	7	$115-7\theta-8\theta^2$
X1	1	0	1	3	0	1	1	$15-2\theta$
3	2	0	0	8	1	3	2	$35+2\theta$

For $\theta \geq 0$ the current basic solution is feasible if $\left.\begin{matrix}15-2\theta \geq 0 \\ 35+2\theta \geq 0\end{matrix}\right\}$ $\theta \leq 15/2$

For $\theta \geq 0$ the current basic solution is optimal if $\left.\begin{matrix}20-3\theta \geq 0 \\ 9-\theta \geq 0\end{matrix}\right\}$ $\theta \leq 20/3$

The current basic solution is both feasible and optimal if $\theta \leq 20/3$.

For $0 \leq \theta \leq 20/3$, $z(\theta) = 115-7\theta-8\theta^2$ is maximized with $\theta = 0$.

6.7-32. a) $B^{-1} = \begin{pmatrix}1 & 1/3 & 1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3\end{pmatrix}$, New $b = \begin{pmatrix}1 & 1/3 & 1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3\end{pmatrix}\begin{pmatrix}4-\theta \\ 12-4\theta \\ 18-3\theta\end{pmatrix} = \begin{pmatrix}2 \\ 6 \\ 2\end{pmatrix} - \begin{pmatrix}4/3 \\ 2 \\ -1/3\end{pmatrix}\theta$

$Z = 3\left(2+\frac{1}{3}\theta\right) + 5(6-2\theta) = 36 - 9\theta$

To remain feasible, we need $2-\frac{4}{3}\theta \geq 0$, $6-2\theta \geq 0$, $2+\frac{1}{3}\theta \geq 0$

or $-6 \leq \theta \leq \frac{3}{2}$

So for $-6 \leq \theta \leq \frac{3}{2}$,

$(x_1^*, x_2^*) = (2+\frac{1}{3}\theta, 6-2\theta)$, $Z^*(\theta) = 36-9\theta$

b) Since $Z^*(\theta) = 36-9\theta$, every unit of change (increase) in the production of the old product results in a change (decrease) in the profit (of the optimal product mix of the two new products) of 9 (which represents ₦9000 per batch). Thus, if unit profit for the old product is more than this, θ should be > 0. If less, then θ should be < 0.

Break even point is ₦9000 per batch of old product.

c) As shown in (a), to remain feasible, $\theta \leq \frac{3}{2}$, so old product's production rate cannot be increased more than $\frac{3}{2}$ units without changing BF soln.

d) Similarly, the old product's production rate cannot decreased more than 6 units without changing the BF solution.

6.7-33 $\Delta c_2 = 4 \Rightarrow \Delta(z_2^* - c_2) = -4$

$\Delta c_3 = 1 \Rightarrow \Delta(z_3^* - c_3) = -1$

$\Delta b_3 = -1 \Rightarrow \Delta z^* = (2, 0, 1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -1$

$\Delta b_1^* = (1, 0, -1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 1$

$\Delta b_2^* = (1, -1, 0)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$

$\Delta b_3^* = (0, 0, 1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -1$

New Tableau

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	$\bar{X}4$	X5	$\bar{X}6$	X7	Right side
Z	0	1	0	1	-1	M+2	0	M	1	7
X1	1	0	1	-1	0	1	0	0	-1	2
X5	2	0	0	3	0	1	1	-1	0	2
X3	3	0	0	2	1	0	0	0	1	1

Proper Form:

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	$\bar{X}4$	X5	$\bar{X}6$	X7	Right side
Z	0	1	0	3	0	M+2	0	M	2	8
X1	1	0	1	-1	0	1	0	0	-1	2
X5	2	0	0	3	0	1	1	-1	0	2
X3	3	0	0	2	1	0	0	0	1	1

The current basic solution is feasible and optimal.

CP6-1: a) The formulation is given in Sec. 3.4 (The order of the constraints and variables were kept the same). The results were:

Optimal Solution

Value of the
Objective Function: Z = 32.1546313

Variable	Value
X_1	1
X_2	0.6227
X_3	0.34348
X_4	1
X_5	0.04757
X_6	1

Sensitivity Analysis

Objective Function Coefficient

Current Value	Allowable Range To Stay Optimal Minimum	Maximum
8	-∞	8.33621
10	9.33304	10.4294
7	4.98854	7.38163
6	-∞	7.81609
11	10.9554	13.9752
9	-∞	9.04416

Constraint	Slack or Surplus	Shadow Price
1	0	-0.111
2	0	-0.1268
3	0	-0.0693
4	0	0.33621
5	0.3773	0
6	0.65652	0
7	0	1.81609
8	0.95243	0
9	0	0.04416

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
60	52.52	74.2971
150	148.31	170.453
125	103.308	127.042
1	0.25152	1.24623
1	0.6227	+∞
1	0.34348	+∞
1	0	1.11061
1	0.04757	+∞
1	0.03729	1.04809

(Pt. b) Since the shadow prices for the first 3 constraints are non-zero, all the b_i's (required reductions in emission rates) are sensitive (as one would expect...). Similarly, since all 3 constraints are tight (binding), all the coefficients (a_{ij}) of these constraints are also sensitive. Finally, since each C_i has an allowable range to stay optimal around it, so that the optimal solution is insensitive to small changes in C_i, all the C_i's are insensitive, but note that most C_i's don't have much leeway to stay optimal, so that a modest change in C_5, for example, from 11 to 10.9 results in a different optimum.

Since the shadow price of constraint 2 is largest in absolute value, and the fact that b_2, being a larger number, is probably more uncertain (so a 10% change in b_2 has a larger effect than a 10% change in b_1) b_2 should be estimated more closely. (So, also, should the coefficients of that constraints, though these are not as critical) Even though C_i's are not sensitive, it seems that these should also be estimated more closely since many are close to the end of their allowable ranges to stay optimal. C_5 and C_6 are most critical (meaning closest to "non-optimality") here.

c) If cost parameters are changed, would the optimal solution change? Here is a table:

True value is:	C_1	C_2	C_3	C_4	C_5	C_6
10% less	N	Y	N	N	Y	N
10% more	Y	Y	Y	N	N	Y

than estimated

This suggests that C_2 should estimated more closely, since it affects the optimal solution in both directions.

CP6-1.d) Dual of the OR Courseware converted (to maximization) problem:

Minimize $V = 60Y_1 + 150Y_2 + 125Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 + Y_9$

subject to

$12Y_1 + 35Y_2 + 37Y_3 + Y_4 \geq -8$

$9Y_1 + 42Y_2 + 53Y_3 + Y_5 \geq -10$

$25Y_1 + 18Y_2 + 28Y_3 + Y_6 \geq -7$

$20Y_1 + 31Y_2 + 24Y_3 + Y_7 \geq -6$

$17Y_1 + 56Y_2 + 29Y_3 + Y_8 \geq -11$

$13Y_1 + 49Y_2 + 20Y_3 + Y_9 \geq -9$

$Y_1 \leq 0,\ Y_2 \leq 0,\ Y_3 \leq 0;\ Y_4 \geq 0,\ Y_5 \geq 0,\ Y_6 \geq 0,\ Y_7 \geq 0,\ Y_8 \geq 0,\ Y_9 \geq 0$

Looking at the shadow prices of the constraints to get dual solution:

$Y_1^* = -0.111$ $\quad Y_4^* = 0.33621$ $\quad Y_7^* = 1.8161$

$Y_2^* = -0.1268$ $\quad Y_5^* = 0$ $\quad Y_8^* = 0$

$Y_3^* = -0.0693$ $\quad Y_6^* = 0$ $\quad Y_9^* = 0.04416$

If the primal had been left in minimization form, the dual would change as follows: ① the right side would be positive (not negative)
② objective would be maximized (not minimized)
③ the signs of all the variable will be reversed.
Of course, the optimal dual solution would be exactly the same, except that the signs are reversed (+ → − and − → +)

e) The numbers of interest here are Y_1^*, Y_2^* and Y_3^* which represent the rate at which the objective value changes as the values of b_1, b_2 and b_3 (respectively) change. So, for example, an increase of b_1 by 1 unit increases the optimal cost by 0.111 units (for small changes).
The amount that b_1, b_2, b_3 can be changed without affecting the rate of change in the total cost is simply the "allowable range to stay feasible."
In summary:

pollutant	Current reg. reduction	Rate that Z changes with change in b_i	Lowest value before rate changes	Highest value before rate changes
Particulates	60	+0.111	52.52	74.297
Sulfur oxides	150	+0.1268	148.31	170.453
Hydrocarbons	125	+0.0693	103.308	127.042

CP6-1. f) To keep optimal solution total cost unchanged, the rates of change must "cancel out." That is, for particulates, a unit change in the policy standard results in +0.111 units of change in Z. To counteract this, we need -0.111 units of change in Z due to changes in sulfur oxides. Thus, we would change the policy standard for sulfur oxides by $\frac{-0.111}{0.1268} = -0.875$ units. Similarly for every unit change for particulates policy standard, we must change that of hydrocarbons by $\frac{-0.111}{0.0693} = -1.602$ units. If both sulfur oxides and hydrocarbons are simultaneously used, the change required is $\frac{-0.111}{0.1268+0.0693} = -0.566$ units. (Here negative signs means "change in the opposite direction").

g) Formulation of simultaneous proportional increases:

Let θ = % increase (so $\theta = 1$ means 1% increase, not 100%)

The problem remains exactly the same as before, except that the right side changes from

$$\begin{bmatrix} 60 \\ 150 \\ 125 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} 60 + .6\theta \\ 150 + 1.5\theta \\ 125 + 1.25\theta \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The costs, other coefficients, inequalities, etc. stay the same.

The rate at which the optimal increases with increase in θ would be $.6(0.111) + 1.5(0.1268) + 1.25(0.0693) = 0.3434$ units per unit change in θ.

h) The optimal solutions and objective value are:

θ	x_1	x_2	x_3	x_4	x_5	x_6	Z	$Z - .35\theta$ (tax incentives)
0	1	0.6227	0.3435	1	0.0476	1	32.155	32.155
10	1	0.7188	0.4360	1	0.2136	1	35.590	32.090
20	1	0.8150	0.5285	1	0.3796	1	39.025	32.025
30	1	0.9111	0.6210	1	0.5456	1	42.460	31.960
40	1	1	0.7053	1	0.7816	0.9293	45.898	31.898 *
50	1	1	0.9491	1	1	0.8963	49.710	32.210

Subtracting 3.5 for each 10% (due to tax incentives) for optimal Z, we see that, to minimize total costs of pollution abatement and taxes, $\theta = 40$ should be chosen.

C6-4, i) Looking at the output for the case $\epsilon = 40$:

Optimal Solution

Value of the
Objective Function: Z = 45.8981883

Variable	Value
X_1	1
X_2	1
X_3	0.70528
X_4	1
X_5	0.78158
X_6	0.92932

Constraint	Slack or Surplus	Shadow Price
1	0	-0.0993
2	0	-0.124
3	0	-0.0817
4	0	0.55269
5	0	0.42945
6	0.29472	0
7	0	1.78923
8	0.21842	0
9	0.07068	0

Sensitivity Analysis

Objective Function Coefficient

Current Value	Allowable Range To Stay Optimal Minimum	Maximum
8	$-\infty$	8.55269
10	$-\infty$	10.4294
7	5.70764	7.38163
6	$-\infty$	7.78923
11	10.9554	11.3844
9	8.62811	9.04416

Right Hand Sides

Current Value	Allowable Range To Stay Feasible Minimum	Maximum
84	83.1542	84.2648
210	203.706	211.112
175	174.747	175.864
1	0.95669	1.01442
1	0.9773	1.00727
1	0.70528	$+\infty$
1	0.88376	2.17467
1	0.78158	$+\infty$
1	0.92932	$+\infty$

From this, we follow the same steps as in e) and f) to get the desired rates and values:

Pollutant	Current rel. reduction	Rate that Z changes with change in b.	Lowest value until rate changes	Highest value until rate changes
Particulates	84	0.0993	83.154	84.265
Sulfur oxides	210	0.124	203.706	211.112
Hydrocarbons	175	0.0817	174.747	175.864

To keep Z unchanged, for every unit of change of particulate policy standard we need to change the policy standard(s) for:

sulfur oxides by $\frac{-0.0993}{0.124} = -0.801$ units

hydrocarbons by $\frac{-0.0993}{0.0817} = -1.215$ units

both simultaneously by $\frac{-0.0993}{0.124 + 0.0817} = -0.483$ units each.

Cases

6-1 a) The decisions to be made are how which types of abatement methods will be used and at what fractions of their abatement capacities for the blast furnaces and the open-hearth furnaces. The constraints on these decisions are the technological limits on how heavily each method can be used and the required reductions in the annual emission rate. The overall measure of performance is cost, which is to be minimized.

Benefit Contribution Per Unit of Abatement Method

	Taller Smokestacks		Filters		Better Fuels				
Constraints	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth	Totals		Acceptable Level
reduce particulates	12	9	25	20	17	13	60	≥	60
reduce sulfer oxides	35	42	18	31	58	49	150	≥	150
reduce hydrocarbons	37	53	28	24	29	20	125	≥	125
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.6226975	≤	1
filters - blast	0	0	1	0	0	0	0.3434794	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.0475728	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	$ 32.155		
Solution	1	0.6227	0.3435	1	0.0476	1			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Solution Blast	1	0	8	0.336210968	1E+30
$C$14	Solution Open-hearth	0.6227	0.0000	10	0.429446289	0.666961637
$D$14	Solution Blast	0.3435	0.0000	7	0.381632655	2.011459969
$E$14	Solution Open-hearth	1	0	6	1.816085017	1E+30
$F$14	Solution Blast	0.0476	0.0000	11	2.975225225	0.044638358
$G$14	Solution Open-hearth	1	0	9	0.044161638	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$4	reduce particulates Totals	60	0.111046969	60	14.29714286	7.48
$H$5	reduce sulfer oxides Totals	150	0.126817108	150	20.453125	1.689655172
$H$6	reduce hydrocarbons Totals	125	0.069325636	125	2.041666667	21.69195612
$H$7	smokestacks - blast Totals	1	-0.336210968	1	0.246231156	0.748477435
$H$8	smokestacks - open-hearth Totals	0.622697455	0	1	1E+30	0.377302545
$H$9	filters - blast Totals	0.343479402	0	1	1E+30	0.656520598
$H$10	filters - open-hearth Totals	1	-1.816085017	1	0.110609481	1
$H$11	fuels - blast Totals	0.047572816	0	1	1E+30	0.952427184
$H$12	fuels - open-hearth Totals	1	-0.044161638	1	0.048086359	0.962708538

b) The right-hand-side of each constraint with a non-zero shadow price is sensitive, since changing its value will impact the total cost. All three required reductions in emission rates are sensitive parameters. All of the objective coefficients have an allowable range to stay optimal around them, and thus are not as sensitive. However, for some, the allowable change is small—in particular, the cost of the two better fuel options (with an allowable increase of only 0.045 and an allowable decrease of 0.044, respectively) are fairly sensitive. Thus, all five of these parameters should be estimated more closely, if possible.

c) The following table shows in which cases the optimal solution will change:

Current Value	10% Less Value	Solution Changes?	10% More Value	Solution Changes?
8	7.2	No	8.8	Yes
10	9	Yes	11	Yes
7	6.3	No	7.7	Yes
6	5.4	No	6.6	No
11	9.9	Yes	12.1	No
9	8.1	No	9.9	Yes

This suggests that focus should be put on estimating all of the costs except the one that is currently \$6 million since it's optimal solution will not change with a 10% increase or decrease. Special consideration should be given to the estimate of the current \$10 million cost since it affects the optimal solution for both an increase and a decrease.

d) Here is the ~~corresponding~~ dual problem.

Benefit Contribution Per Unit of Abatement Method

Constraints	Taller Smokestacks Blast	Taller Smokestacks Open-earth	Filters Blast	Filters Open-earth	Better Fuels Blast	Better Fuels Open-earth	Dual Variables		Accepatabl Level
reduce particulates	12	9	25	20	17	13	y1	<= 0	60
reduce sulfer-oxides	35	42	18	31	56	49	y2	<= 0	150
reduce hydrocarbons	37	53	28	24	29	20	y3	<= 0	125
smokestacks - blast	1	0	0	0	0	0	y4	>= 0	1
smokestacks - open-earth	0	1	0	0	0	0	y5	>= 0	1
filters - blast	0	0	1	0	0	0	y6	>= 0	1
filters - open-earth	0	0	0	1	0	0	y7	>= 0	1
fuels - blast	0	0	0	0	1	0	y8	>= 0	1
fuels - open-earth	0	0	0	0	0	1	y9	>= 0	1
Totals									
Unit Cost	-8	-10	-7	-6	-11	-9			

This is the sensitivity report of primal (maximization problem)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$16	Solution Blast	1.00	0.00	-8	1E+30	0.336210968
$D$16	Solution Open-earth	0.62	0.00	-10	0.668961637	0.429446294
$E$16	Solution Blast	0.34	0.00	-7	2.01145997	0.381632659
$F$16	Solution Open-earth	1.00	0.00	-6	1E+30	1.816085017
$G$16	Solution Blast	0.05	0.00	-11	0.044638358	2.975225225
$H$16	Solution Open-earth	1.00	0.00	-9	1E+30	0.044161638

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$I$6	reduce particulates Totals	60	-0.111046969	60	14.29714286	7.48
$I$7	reduce sulfer-oxides Totals	150	-0.126817108	150	20.453125	1.689655172
$I$8	reduce hydrocarbons Totals	125	-0.069325636	125	2.041666667	21.69195612
$I$9	smokestacks - blast Totals	1	0.336210968	1	0.246231156	0.748477435
$I$10	smokestacks - open-earth Totals	0.623	0.000	1	1E+30	0.377302545
$I$11	filters - blast Totals	0.343	0.000	1	1E+30	0.656520598
$I$12	filters - open-earth Totals	1	1.816085017	1	0.110609481	1
$I$13	fuels - blast Totals	0.0476	0.0000	1	1E+30	0.952427184
$I$14	fuels - open-earth Totals	1	0.044161638	1	0.048086359	0.962708538

The dual variables are the shadow prices of the constraints.
If the primal had been left in minimization form, the sign of the dual variables would be the opposite.
The dual would be the same except that the "sign" constraints on the dual variables changes from ≥ to ≤ and visceversa, and the dual functional constraints all change from ≥ to ≤. (CONT'D)

d) (CONT'D) It would also be a maximization problem, instead of minimization.

e)

Pollutant	Rate that cost changes	Maximum increase before rate changes	Maximum decrease before rate changes
Particulates	0.111	14.297	7.48
Sulfur oxides	0.127	20.453	1.69
Hydrocarbons	0.069	2.042	21.692

b) Particulates and sulfur oxides:
For each unit increase in particulate reduction, cost will increase by \$0.111 million.
For each unit decrease in sulfur oxide reduction, cost will decrease by \$0.127 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide reduction is reduced by \$0.111 / \$0.127 = 0.874 units.

Particulates and hydrocarbons:
For each unit increase in particulate reduction, cost will increase by \$0.111 million.
For each unit decrease in hydrocarbon reduction, cost will decrease by \$0.069 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the hydrocarbon reduction is reduced by \$0.111 / \$0.069 = 1.609 units.

Particulates and both sulfur oxides and hydrocarbons:
For each unit increase in particulate reduction, cost will increase by \$0.111 million.
For each simultaneous unit decrease in sulfur oxide and hydrocarbon reduction, cost will decrease by \$0.127 + \$0.069 = \$0.196.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide and hydrocarbon reduction are each reduced by \$0.111 / \$0.196 = 0.566 units.

g) The formulation is the same except that the right hand side of the constraints corresponding to Table 3.12 become $60 + \theta\frac{60}{100}$; $150 + \theta\frac{150}{100}$; and $125 + \theta\frac{125}{100}$.

The rate at which the optimal cost of an optimal solution would increase with a small increase in θ from zero is given by:

$$y^* \Delta b = [0.111;\ 0.127;\ 0.0693] \begin{bmatrix} 0.6\theta \\ 1.5\theta \\ 1.25\theta \end{bmatrix} = 0.344\theta$$

These are the shadow prices for the first 3 constraints.

So the rate of increase is 0.344θ.

h) 10% increase

Benefit Contribution Per Unit of Abatement Method

Constraints	Taller Smokestacks		Filters		Better Fuels		Totals		Acceptable Level
	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth			
reduce particulates	12	9	25	20	17	13	66	≥	66
reduce sulfer oxides	35	42	18	31	56	49	165	≥	165
reduce hydrocarbons	37	53	28	24	29	20	137.5	≥	137.5
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.7188402	≤	1
filters - blast	0	0	1	0	0	0	0.4359748	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.2135922	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	$ 35.590		
Solution	1	0.7188	0.4360	1	0.2136	1			

20% increase

Benefit Contribution Per Unit of Abatement Method

Constraints	Taller Smokestacks		Filters		Better Fuels		Totals		Acceptable Level
	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth			
reduce particulates	12	9	25	20	17	13	72	≥	72
reduce sulfer oxides	35	42	18	31	56	49	180	≥	180
reduce hydrocarbons	37	53	28	24	29	20	150	≥	150
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.8149829	≤	1
filters - blast	0	0	1	0	0	0	0.5284702	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.3796117	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	$ 39.025		
Solution	1	0.8150	0.5285	1	0.3796	1			

30% increase

Benefit Contribution Per Unit of Abatement Method

Constraints	Taller Smokestacks		Filters		Better Fuels		Totals		Acceptable Level
	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth			
reduce particulates	12	9	25	20	17	13	78	≥	78
reduce sulfer oxides	35	42	18	31	56	49	195	≥	195
reduce hydrocarbons	37	53	28	24	29	20	162.5	≥	162.5
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	0.9111257	≤	1
filters - blast	0	0	1	0	0	0	0.6209656	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.5456311	≤	1
fuels - open-hearth	0	0	0	0	0	1	1	≤	1
Unit Cost	8	10	7	6	11	9	$ 42.460		
Solution	1	0.9111	0.6210	1	0.5456	1			

40% increase

	Benefit Contribution Per Unit of Abatement Method								
	Taller Smokestacks		Filters		Better Fuels				Acceptable
Constraints	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth	Totals		Level
reduce particulates	12	9	25	20	17	13	84	≥	84
reduce sulfer oxides	35	42	18	31	56	49	210	≥	210
reduce hydrocarbons	37	53	28	24	29	20	175	≥	175
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	1	≤	1
filters - blast	0	0	1	0	0	0	0.705282	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	0.7815769	≤	1
fuels - open-hearth	0	0	0	0	0	1	0.9293187	≤	1
Unit Cost	8	10	7	6	11	9	$ 45.898		
Solution	1	1.0000	.7053	1	0.7816	0.9293187			

50% increase

	Benefit Contribution Per Unit of Abatement Method								
	Taller Smokestacks		Filters		Better Fuels				Acceptable
Constraints	Blast	Open-hearth	Blast	Open-hearth	Blast	Open-hearth	Totals		Level
reduce particulates	12	9	25	20	17	13	93.378953	≥	90
reduce sulfer oxides	35	42	18	31	56	49	225	≥	225
reduce hydrocarbons	37	53	28	24	29	20	187.5	≥	187.5
smokestacks - blast	1	0	0	0	0	0	1	≤	1
smokestacks-open-hearth	0	1	0	0	0	0	1	≤	1
filters - blast	0	0	1	0	0	0	0.9491107	≤	1
filters - open-hearth	0	0	0	1	0	0	1	≤	1
fuels - blast	0	0	0	0	1	0	1	≤	1
fuels - open-hearth	0	0	0	0	0	1	0.8962451	≤	1
Unit Cost	8	10	7	6	11	9	$ 49.710		
Solution	1	1.0000	0.9491	1	1.0000	0.8962451			

Subtracting $3.5 for each 10% reduction gives the following costs:

10% - $32.090
20% - $32.025
30% - $31.960
40% - $31.898
50% - $32.210

To minimize the total cost of both pollution abatement and taxes, a 40% reduction should be chosen.

i) The sensitivity report for a 40% reduction is shown below.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Solution Blast	1	0	8	0.552692013	1E+30
$C$14	Solution Open-hearth	1.0000	0.0000	10	0.429446287	1E+30
$D$14	Solution Blast	0.7053	0.0000	7	0.381632653	1.292358804
$E$14	Solution Open-hearth	1	0	6	1.789231947	1E+30
$F$14	Solution Blast	0.7816	0.0000	11	0.384387352	0.044638358
$G$14	Solution Open-hearth	0.929318704	0	9	0.044161637	0.371892925

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$4	reduce particulates Totals	84	0.099260015	84	0.264818356	0.845849802
$H$5	reduce sulfer oxides Totals	210	0.124011227	210	1.112449799	6.294117647
$H$6	reduce hydrocarbons Totals	175	0.081653483	175	0.863773968	0.253199269
$H$7	smokestacks - blast Totals	1	-0.552692013	1	0.014418823	0.043313262
$H$8	smokestacks-open-hearth Totals	1	-0.429446287	1	0.007268433	0.022703764
$H$9	filters - blast Totals	0.70528196	0	1	1E+30	0.29471804
$H$10	filters - open-hearth Totals	1	-1.789231947	1	1.174670633	0.116240034
$H$11	fuels - blast Totals	0.781576933	0	1	1E+30	0.218423067
$H$12	fuels - open-hearth Totals	0.929318704	0	1	1E+30	0.070681296

Pollutant	Rate that cost changes	Maximum increase before rate changes	Maximum decrease before rate changes
Particulates	0.099	0.265	0.846
Sulfur oxides	0.124	1.112	6.294
Hydrocarbons	0.082	0.864	0.253

Particulates and sulfur oxides:
For each unit increase in particulate reduction, cost will increase by $0.099 million.
For each unit decrease in sulfur oxide reduction, cost will decrease by $0.124 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide reduction is reduced by $0.099 / $0.124 = 0.798 units.

Particulates and hydrocarbons:
For each unit increase in particulate reduction, cost will increase by $0.099 million.
For each unit decrease in hydrocarbon reduction, cost will decrease by $0.082 million.
Thus, cost will remain equal if for each unit increase in particulate reduction, the hydrocarbon reduction is reduced by $0.099 / $0.082 = 1.207 units.

Particulates and both sulfur oxides and hydrocarbons:
For each unit increase in particulate reduction, cost will increase by $0.099 million.
For each simultaneous unit decrease in sulfur oxide and hydrocarbon reduction, cost will decrease by $0.124 + $0.082 = $0.206.
Thus, cost will remain equal if for each unit increase in particulate reduction, the sulfur oxide and hydrocarbon reduction are each reduced by $0.099 / $0.206 = 0.481 units.

6-2 a) The decisions to be made are how much acreage should be planted in each of the crops and how many cows and hens to have for the coming year. The constraints on these decisions are amount of labor hours available, the investment funds available, the number of acres available, the space available in the barn and chicken house, the minimum requirements for feed to be planted. The overall measure of performance is monetary worth, which is to be maximized.

b & c)

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
	Resource Usage Per Unit of Activity											
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	30	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	0	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	0	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	0	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	**$ 46,817**		
Net Value	$70	$60	$40	$1,060	$1,350	$1.88	$2.25	$0	$0	**$ 72,550**		
Remaining Investment										**$ 20,000**		
Living Expenses										**$ (40,000)**		
Solution	**450**	**30**	**100**	30	**0**	2000	**0**	**1063**	**1364**	**$ 99,367**		Total Monetary Worth

Note that the cells for current cows and current hens are not changing cells but fixed amounts.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$17	Solution soybean	450	0	70	1E+30	8.400000002
$C$17	Solution corn	30	0	60	8.400000002	1E+30
$D$17	Solution wheat	100	0	40	17.15005129	1E+30
$F$17	Solution cows	0	-53.00000097	699.9999983	53.00000097	1E+30
$H$17	Solution hens	0	-0.857502564	3.499997547	0.857502564	1E+30
$I$17	Solution W/S Labor	1063	0	5	57.3	0.915371347
$J$17	Solution S/F Labor	1364	0	5.5	34.5	0.929824579

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$K$4	acreage Totals	640	57.3	640	974.2857143	450
$K$5	barn space Totals	30	0	42	1E+30	12
$K$6	chicken house space Totals	2000	0	5000	1E+30	3000
$K$7	winter/spring hours Totals	4000	5	4000	1E+30	1063
$K$8	summer/fall hours Totals	4500	5.5	4500	1E+30	1364
$K$9	investment fund Totals	0	0	20000	1E+30	20000
$K$10	feed for cows Totals	0	8.400000002	0	30	450
$K$11	feed for chickens Totals	0	24.15	0	100	450

This model predicts that the family's monetary worth at the end of the coming year will be $99, 367.

d) Range of optimality - soybeans: $61.6 \leq$ soybeans value $\leq \infty$
corn: $-\infty \leq$ corn value ≤ 68.4
wheat: $-\infty \leq$ wheat value ≤ 57.15

e) Drought

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	259.33333	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2666.6667	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	20000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	4.547E-13	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	1.705E-11	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 55,544		
Net Value	-$10	-$15	$0	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 52,320		
Remaining Investment Living Expenses										$ (40,000)		
Solution	0	42	133	30	12	2000	667	562	1036	$ 67,864		Total Monetary Worth

Flood

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2666.6667	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	20000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	-380.6667	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	-7.11E-15	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 51,318		
Net Value	$15	$20	$10	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 62,737		
Remaining Investment Living Expenses										$ (40,000)		
Solution	0	423	133	30	12	2000	667	220	579	$ 74,055		Total Monetary Worth

Early Frost

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	30	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	0	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	6.673E-11	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	7.104E-11	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 46,817		
Net Value	$50	$40	$30	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 61,950		
Remaining Investment Living Expenses										$ 20,000		
										$ (40,000)		
Solution	450	30	100	30	0	2000	0	1063	1364	$ 88,767		Total Monetary Worth

Drought and Early Frost

Resource Usage Per Unit of Activity

Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	226	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	18000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	-1.28E-12	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	0	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	$ 55,039		
Net Value	-$15	-$20	-$10	$1,050	$1,350	$1.88	$2.25	$0	$0	$ 49,610		
Remaining Investment Living Expenses										$ 2,000		
										$ (40,000)		
Solution	0	42	100	30	12	2000	0	782	1260			

Flood and Early Frost

Resources	Resource Usage Per Unit of Activity: acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	362	≤	640
barn space	0	0	0	1	1	0	0	0	0	37.333333	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	5000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	20000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	-2.69E-12	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	7.671E-11	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	**$ 56,336**		
Net Value	$10	$10	$5	$1,050	$1,350	$1.88	$2.25	$0	$0	**$ 53,523**		
Remaining Investment										**(0)**		
Living Expenses										**(40,000)**		
Solution	**0**	**37.3**	**250**	30	**7.333**	2000	**3000**	**76**	**540**	**$ 69,860**		Total Monetary Worth

f)

Opt. Sol. Used	Family's monetary worth at year's end if the scenario is actually: Good Weather	Drought	Flood	Early Frost	Drought & Early Frost	Flood & Early Frost
Good Weather	99,367	57,117	70,417	88,767	53,717	67,367
Drought	76,348	67,864	70,668	74,174	66,321	69,581
Flood	94,962	57,929	74,055	85,175	54,482	69,162
Early Frost	99,367	57,117	70,417	88,767	53,717	67,367
Drought & Early Frost	75,009	67,859	70,329	73,169	66,649	69,409
Flood & Early Frost	80,476	67,676	71,483	77,230	64,990	69,860

The "Flood & Early Frost" solution looks like the best conservative option. The "Flood" option looks good for those who would like more risk.

g and h)

The expected net value for each of the crops is calculated as follows:

Soybeans: ($70)(0.4) + (-$10)(0.2) + ($15)(0.1) + ($50)(0.15) + (-$15)(0.1) + ($10)(0.05) = $34 million,

Corn: ($60)(0.4) + (-$15)(0.2) + ($20)(0.1) + ($40)(0.15) + (-$20)(0.1) + ($10)(0.05) = $27.5 million,

Wheat: ($40)(0.4) + ($0)(0.2) + ($10)(0.1) + ($30)(0.15) + (-$10)(0.1) + ($5)(0.05) = $20.75 million.

The resulting spreadsheet solution is shown below:

	Resource Usage Per Unit of Activity											
Resources	acres soybean	acres corn	acres wheat	current cows	new cows	current hens	new hens	leftover W/S Labor	leftover S/F Labor	Totals		Resource Available
acreage	1	1	1	2	2	0	0	0	0	640	≤	640
barn space	0	0	0	1	1	0	0	0	0	42	≤	42
chicken house space	0	0	0	0	0	1	1	0	0	2000	≤	5000
winter/spring hours	1	0.9	0.6	60	60	0.3	0.3	1	0	4000	≤	4000
summer/fall hours	1.4	1.2	0.7	60	60	0.3	0.3	0	1	4500	≤	4500
investment fund	0	0	0	0	1500	0	3	0	0	18000	≤	20000
feed for cows	0	-1	0	1	1	0	0	0	0	0	≤	0
feed for chickens	0	0	-1	0	0	0.05	0.05	0	0	0	≤	0
Net Income	$0	$0	$0	$850	$850	$4.25	$4.25	$5	$5.50	49,781		
Net Value	$34.0	$27.5	$20.8	$1,050	$1,350	$1.88	$2.25	$0	$0	68,756		
Remaining Investment										2,000		
Living Expenses										(40,000)		
										80,537		Total Monetary Worth
Solution	414	42	100	30	12	2000	0	368	680			

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$17	Solution soybean	414	0	34	7.499999997	0.400002814
$C$17	Solution corn	42	0	27.5	4.899999997	22.49999999
$D$17	Solution wheat	100	0	20.75	0.400002814	1E+30
$F$17	Solution cows	12	0	700	1E+30	22.49999999
$H$17	Solution hens	0	-0.020000141	3.499999875	0.020000141	1E+30
$I$17	Solution W/S Labor	368	0	5	0.388601036	0.071429073
$J$17	Solution S/F Labor	680	0	5.5	0.394736842	0.075472229

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$K$4	acreage Totals	640	21.3	640	368.2	414
$K$5	barn space Totals	42	22.49999999	42	1.333333333	12
$K$6	chicken house space Totals	2000	0	5000	1E+30	3000
$K$7	winter/spring hours Totals	4000	5	4000	1E+30	368.2
$K$8	summer/fall hours Totals	4500	5.5	4500	1E+30	680
$K$9	investment fund Totals	18000	0	20000	1E+30	2000
$K$10	feed for cows Totals	0	4.899999997	0	42	414
$K$11	feed for chickens Totals	0	7.399999998	0	100	414

This model predicts that the family's monetary worth at the end of the coming year will be $80,537.

i) The shadow price for the investment constraint is zero, indicating that additional investment funds will not increase their total monetary worth at all. Thus, it is not worthwhile to obtain a bank loan. The shadow price would need to be at least $1.10 before a loan at 10% interest would be worthwhile.

j) The expected net value for soybeans can increase up to \$7.50 or decrease up to \$0.40; for corn can increase up to \$4.90 or decrease up to \$22.50; for wheat can increase up to \$0.40 or decrease any amount without changing the optimal solution. The expected net value for soybeans and wheat should be estimated most carefully.

The solution is sensitive to decreases in the expected value of soybeans and increases in the expected value of wheat. If the *cumulative* decrease in the expected value of soybeans *and* increase in the expected value of wheat exceeds \$0.40, then the 100% rule will be violated, and the solution might change.

k) Answers will vary.

6-3 a)

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost (\$/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	422.222222	177.777778	600	=	600
Area 3	0	227.777778	322.222222	550	=	550
Area 4	350	0	0	350	=	350
Area 5	366.666667	0	133.333333	500	=	500
Area 6	83.3333333	0	366.666667	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = \$ 555,555.56

Grade Constraints:

	School 1	School 2	School 3
6th Graders	269.333333	368.555556	339.111111
7th Graders	288	362.111111	300.888889
8th Graders	242.666667	369.333333	360
30% of Total	240	330	300
36% of Total	288	396	360

b)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Area 1 School 1	0	177.77778	300	1E+30	177.77778
$C$14	Area 1 School 2	450	0	0	177.77778	1.554E+17
$D$14	Area 1 School 3	0	266.66667	700	1E+30	266.66667
$B$15	Area 2 School 1	0	-800	0	1E+30	800
$C$15	Area 2 School 2	422.22222	0	400	34.210526	4.5454555
$D$15	Area 2 School 3	177.77778	0	500	4.5454555	34.210526
$B$16	Area 3 School 1	0	11.111114	600	1E+30	11.111114
$C$16	Area 3 School 2	227.77778	0	300	4.5454555	34.210526
$D$16	Area 3 School 3	322.22222	0	200	34.210526	7.6923092
$B$17	Area 4 School 1	350	0	200	366.66667	2.339E+16
$C$17	Area 4 School 2	0	366.66667	500	1E+30	366.66667
$D$17	Area 4 School 3	0	-433.3333	0	1E+30	433.33333
$B$18	Area 5 School 1	366.66667	0	0	16.66667	108.33333
$C$18	Area 5 School 2	0	233.33333	0	1E+30	233.33333
$D$18	Area 5 School 3	133.33333	0	400	108.33333	16.66667
$B$19	Area 6 School 1	83.333333	0	500	33.333342	166.66667
$C$19	Area 6 School 2	0	200	300	1E+30	200
$D$19	Area 6 School 3	366.66667	0	0	166.66667	33.333342

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$30	8th Graders School 1	242.66667	0	0	1E+30	45.333333
$C$30	8th Graders School 2	369.33333	0	0	1E+30	26.666667
$D$30	8th Graders School 3	360	-6666.667	0	5.3333333	0.6666667
$B$20	Total School 1	800	0	900	1E+30	100
$C$20	Total School 2	1100	-177.7778	1100	36.363636	3.7735849
$D$20	Total School 3	1000	-144.4444	1000	42.105263	3.8834951
$B$28	6th Graders School 1	269.33333	0	0	29.333333	1E+30
$C$28	6th Graders School 2	368.55556	0	0	38.555556	1E+30
$D$28	6th Graders School 3	339.11111	0	0	39.111111	1E+30
$B$28	6th Graders School 1	269.33333	0	0	1E+30	18.666667
$C$28	6th Graders School 2	368.55556	0	0	1E+30	27.444444
$D$28	6th Graders School 3	339.11111	0	0	1E+30	20.888889
$B$29	7th Graders School 1	288	0	0	48	1E+30
$C$29	7th Graders School 2	362.11111	0	0	32.111111	1E+30
$D$29	7th Graders School 3	300.88889	0	0	0.8888889	1E+30
$B$29	7th Graders School 1	288	-2777.778	0	0.2580645	2.9090909
$C$29	7th Graders School 2	362.11111	0	0	1E+30	33.888889
$D$29	7th Graders School 3	300.88889	0	0	1E+30	59.111111
$B$30	8th Graders School 1	242.66667	0	0	2.6666667	1E+30
$C$30	8th Graders School 2	369.33333	0	0	39.333333	1E+30
$D$30	8th Graders School 3	360	0	0	60	1E+30
$E$14	Area 1 Total	450	177.77778	450	3.7735849	36.363636
$E$15	Area 2 Total	600	577.77778	600	3.7735849	36.363636
$E$16	Area 3 Total	550	477.77778	550	3.7735849	36.363636
$E$17	Area 4 Total	350	311.11111	350	72.727273	6.4516129
$E$18	Area 5 Total	500	-55.55556	500	12.903226	145.45455
$E$19	Area 6 Total	450	277.77778	450	3.2258065	36.363636

c) The bussing cost from area 6 to school 1 can increase $33.33 before the current optimal solution would no longer be optimal. The new solution with a 10% increase ($50) is shown below.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	550	300	0
				Capacity	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	72.7272726	50	427.272727	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.181818	0	181.818182	500	=	500
Area 6	59.0909093	0	390.909091	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 559,318.18

Grade Constraints:

	School 1	School 2	School 3
6th Graders	264	381	332
7th Graders	288	355	308
8th Graders	248	364	360
30% of Total	240	330	300
36% of Total	288	396	360

d) The bussing cost from area 6 to school 2 can increase any amount and the current optimal solution will still be optimal.

e) According to the 100% rule, the bussing cost from area 6 can increase uniformly up to 6.67% ($33 for school 1, and $20 for school 2) without changing the solution. Beyond that the solution might change. This calculation is shown below.

School 1: S500 → $533.33. % of allowable increase $= 100\left(\frac{533.33-500}{33.33}\right) = 100\%$

School 2: $300 → $320. % of allowable increase $= 100\left(\frac{320-300}{\infty}\right) = \underline{0\%}$

Sum = 100%.

The new spreadsheet solution is shown below.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	550	330	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	72.7272727	50	427.272727	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.181818	0	181.818182	500	=	500
Area 6	59.0909091	0	390.909091	450	=	450
Total	800	1100	1000			
	≤	≤	≤			
Capacity	900	1100	1000			

Total Bussing Cost = $ 559,318.18

Grade Constraints:

	School 1	School 2	School 3
6th Graders	264	381	332
7th Graders	288	355	308
8th Graders	248	364	360
30% of Total	240	330	300
36% of Total	288	396	360

f) The shadow price for school 1 is zero. Thus, adding a temporary classroom at school 1 would not save any money, and thus would not be worthwhile.

The shadow price for school 2 is -$177.78. Thus, adding a temporary classroom at school 2 would save ($177.78)(20) = $3555.60 in bussing cost. This is worthwhile, since it exceeds the $2500 leasing cost.

The shadow price for school 3 is -$144.44. Thus, adding a temporary classroom at school 3 would save ($144.44)(20) = $2888.80 in bussing cost. This is also worthwhile, since it exceeds the $2500 leasing cost.

g) For school 2, the allowable increase for school capacity is 36. This means the shadow price is only valid for a single additional portable classroom.

For school 3, the allowable increase for school capacity is 42. This means the shadow price is valid for up to two additional portable classrooms.

h) The following combinations do not violate the 100% rule:

Portables to add to school 2	Portables to add to school 3	100%-rule calculation	Bussing Cost Savings
1	0	(20/36) + (0/42) = 55.6%	($177.78)(20)=$2888.80
0	1	(0/36) + (20/42) = 47.6%	
0	2	(0/36) + (40/42) = 95.23%	

Each combination yields the following total savings

Portables to add to school 2	Portables to add to school 3	Bussing Cost Savings	Lease Cost	Total Savings
1	0	($177.78)(20) = $3555.60	$2500	$1055.60
0	1	($144.44)(20) = $2888.80	$2500	$388.80
0	2	($144.44)(40) = $5777.60	$5000	$777.60

Of these combinations, adding one portable to school 2 is best in terms of minimizing total cost. The spreadsheet solution is shown below.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0 3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	520	80	600	=	600
Area 3	0	150	400	550	=	550
Area 4	350	0	0	350	=	350
Area 5	340	0	160	500	=	500
Area 6	90	0	360	450	=	450
Total	780	1120	1000			
	≤	≤	≤			
Capacity	900	1120	1000			

Total Bussing Cost = $ 552,000.00
Leasing Cost = $ 2,500.00
Total Cost = $ 554,500.00

Grade Constraints:

	School 1	School 2	School 3
6th Graders	261.2	381.4	334.4
7th Graders	280.8	364.6	305.6
8th Graders	238	374	360
30% of Total	234	336	300
36% of Total	280.8	403.2	360

i) Adding two portables to school 2 yields the following solution. This is the best plan.

Data:

Area	Number of Students	Percentage in 6th Grade	Percentage in 7th Grade	Percentage in 8th Grade	Bussing Cost ($/Student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	-	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	-
5	500	0.39	0.34	0.27	0	-	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	900	1100	1000

Solution:

	Number of Students Assigned School 1	School 2	School 3	Total		
Area 1	0	450	0	450	=	450
Area 2	0	600	0	600	=	600
Area 3	0	90	460	550	=	550
Area 4	350	0	0	350	=	350
Area 5	318.947368	0	181.052632	500	=	500
Area 6	95.2631579	0	354.736842	450	=	450
Total	764.210526	1140	995.789474			
	≤	≤	≤			
Capacity	900	1140	1000			

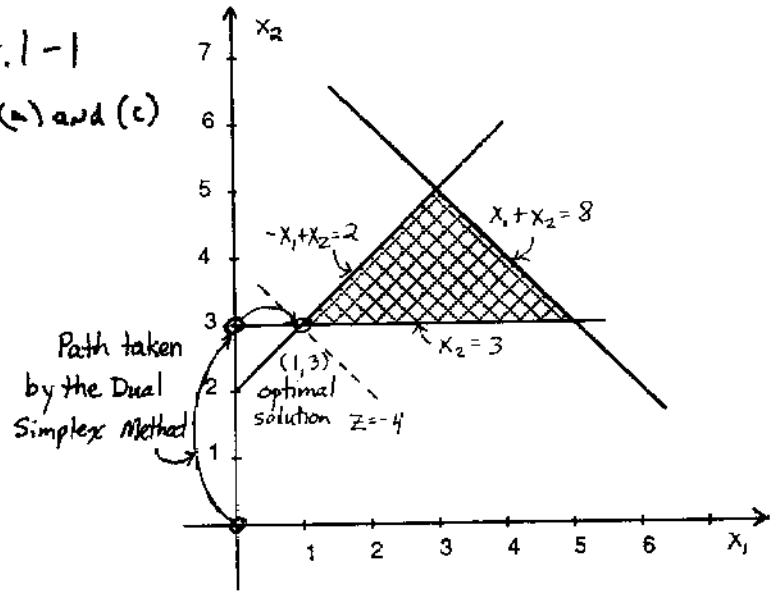
Total Bussing Cost = $ 549,052.63
Leasing Cost = $ 5,000.00
Total Cost = $ 554,052.63

Grade Constraints:

	School 1	School 2	School 3
6th Graders	254.778947	393	329.221053
7th Graders	275.115789	367.8	308.084211
8th Graders	234.315789	379.2	358.484211
30% of Total	229.263158	342	298.736842
36% of Total	275.115789	410.4	358.484211

7.1-1

(a) and (c)



(b)

	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RHS
	Z	0	1	1	1	0	0	0	0
(0)	x_3	1	0	1	1	1	0	0	8
	x_4	2	0	0	-1*	0	1	0	-3
	x_5	3	0	-1	1	0	0	1	2
	Z	0	1	1	0	0	1	0	-3
(1)	x_3	1	0	1	0	1	1	0	5
	x_2	2	0	0	1	0	-1	0	3
	x_5	3	0	-1*	0	0	1	1	-1
	Z	0	1	0	0	0	2	1	-4
(2)	x_3	1	0	0	0	1	2	1	4
	x_2	2	0	0	1	0	-1	0	3
	x_1	3	0	1	0	0	-1	-1	1

So $(x_1, x_2) = (1, 3)$ with $Z = -4$ is optimal

7.1-2

	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RHS
	Z	0	-1	5	2	4	0	0	0
(0)	x_4	1	0	-3	-1	-2	1	0	-4
	x_5	2	0	-6	-3*	-5	0	1	-10
	Z	0	-1	1	0	2/3	0	2/3	-20/3
(1)	x_4	1	0	-1*	0	-1/3	1	-1/3	-2/3
	x_2	2	0	2	1	5/3	0	-1/3	10/3
	Z	0	-1	0	0	1/3	1	1/3	-22/3
(2)	x_1	1	0	1	0	1/3	-1	1/3	2/3
	x_2	2	0	0	1	1	2	-1	2

So $(x_1, x_2, x_3) = (2/3, 2, 0)$ with $Z = 7\ 1/3$ is optimal.

7.1-3

	BV	Eq #	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
	z	0	-1	7	2	5	4	0	0	0	0
(0)	x_5	1	0	-2	-4	-7	-1	1	0	0	-5
	x_6	2	0	8	-4*	-6	-4	0	1	0	-8
	x_7	3	0	-3	-8	-1	-4	0	0	1	-4
	z	0	-1	3	0	2	2	0	1/2	0	-4
(1)	x_5	1	0	6	0	-1	3	1	-1	0	3
	x_2	2	0	2	1	3/2	1	0	-1/4	0	2
	x_7	3	0	13	0	11	4	0	-2	1	12

So $(x_1, x_2, x_3, x_4) = (0, 2, 0, 0)$ with $z = 4$ is optimal.

7.1-4(a)

	BV	Eq #	z	x_1	x_2	x_3	x_4	x_5	RHS	Primal Solution	Dual Solution
	z	0	1	-3	-2	0	0	0	0		
(0)	x_3	1	0	3*	1	1	0	0	12		
	x_4	2	0	1	1	0	1	0	6	(0, 0, 12, 6, 27)	(0, 0, 0, -3, 2)
	x_5	3	0	5	3	0	0	1	27		
	z	0	1	0	-1	1	0	0	12		
(1)	x_1	1	0	1	1/3	1/3	0	0	4		
	x_4	2	0	0	2/3*	-1/3	1	0	2	(4, 0, 0, 2, 7)	(1, 0, 0, 0, -1)
	x_5	3	0	0	4/3	-5/3	0	1	7		
	z	0	1	0	0	1/2	3/2	0	15		
(2)	x_1	1	0	1	0	1/2	-1/2	0	3		
	x_2	2	0	0	1	-1/2	3/2	0	3	(3, 3, 0, 0, 3)	(1/2, 3/2, 0, 0, 0)
	x_5	3	0	0	0	-1	-2	1	3		

So $(x_1, x_2) = (3, 3)$ with $z = 15$ is optimal

(b) The dual is minimize $12y_1 + 6y_2 + 27y_3$

subject to $3y_1 + y_2 + 5y_3 \geq 3$

$y_1 + y_2 + 3y_3 \geq 2$

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

	BV	Eq. #	w	y_1	y_2	y_3	y_4	y_5	RHS	Primal Solution	Dual Solution
	z	0	-1	12	6	27	0	0	0		
(0)	y_4	1	0	-3*	-1	-5	1	0	-3		
	y_5	2	0	-1	-1	-3	0	1	-2	(0, 0, 12, 6, 27)	(0, 0, 0, -2, -2)
	z	0	-1	0	2	7	4	0	12		
(1)	y_1	1	0	1	1/3	5/3	-1/3	0	1		
	y_5	2	0	0	-4/3*	-4/3	-1/3	1	-1	(4, 0, 0, 2, 7)	(1, 0, 0, 0, -1)
	z	0	-1	0	0	3	3	3	15		
(2)	y_1	1	0	1	0	1	-1/2	1/2	1/2		
	y_2	2	0	0	1	2	-2	-3/2	3/2	(3, 3, 0, 0, 3)	(1/2, 3/2, 0, 0, 0)

The dual optimal solution is $(y_1, y_2, y_3) = (1/2, 3/2, 0)$ with $z = 15$.

The sequence of basic solutions and complementary basic solutions is identical to that in part (a).

7.1-5

	BV	Eq.#	z	x_1	x_2	x_3	x_4	x_5	RHS
	z	0	1	0	0	0	3/2	1	54
		1	0	0	0	1	1/3	-1/3	6
(0)	x_3	2	0	0	1	0	1/2	0	12
	x_2	3	0	1	0	0	-1/3 *	1/3	-2
	x_1								
	z	0	1	3/2	0	0	0	5/2	45
	x_3	1	0	1	0	1	0	0	4
(1)	x_2	2	0	3/2	1	0	0	1/2	9
	x_4	3	0	-3	0	0	1	-1	6

So the new optimal solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 9, 4, 6, 0)$ with $z = 45$.

7.1-6 (a)

	BV	Eq.#	z	x_1	x_2	x_3	x_4	x_5	RHS
	z	0	1	0	0	2	5	0	150
(0)	x_2	1	0	-1	1	3	1	0	30
	x_5	2	0	16	0	-2 *	-4	1	-30
	z	0	1	16	0	0	1	1	120
(1)	x_2	1	0	23	1	0	-5 *	3/2	-15
	x_3	2	0	-8	0	1	2	-1/2	15
	z	0	1	103/5	1/5	0	0	13/10	117
(2)	x_4	1	0	23/5	-1/5	0	1	-3/10	3
	x_3	2	0	6/5	2/5	1	0	1/10	9

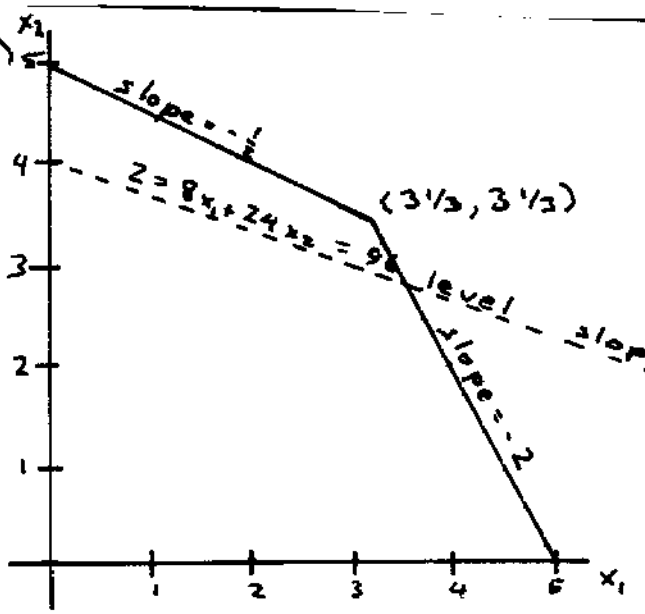
So $(x_1, x_2, x_3, x_4, x_5) = (-, -, 9, 3, 0)$ with $z = 117$ is optimal.

(b)

	BV	Eq.#	z	x_1	x_2	x_3	x_4	x_5	RHS
	z	0	1	0	0	2	5	0	100
(0)	x_2	1	0	-1	1	3	1	0	20
	x_5	2	0	16	0	-2 *	-4	1	-10
	z	0	1	16	0	0	1	1	90
(1)	x_2	1	0	23	1	0	-5	3/2	5
	x_3	2	0	-8	0	1	2	-1/2	5

so $(x_1, x_2, x_3, x_4, x_5) = (0, 5, 5, 0, 0)$ with $z = 90$ is optimal.

7.2-1 (a)



(CONT.)

72-1 (CONT) $\Rightarrow$ (0,5) with $z = 120$ is optimum.

(0,5) will remain optimal until

$$-\frac{(8+\theta)}{(24-2\theta)} = -\frac{1}{2} \quad \text{or} \quad \theta = 2$$

at which point $(10/3, 10/3)$ will be optimal. In turn, $(10/3, 10/3)$ will remain optimal until

$$-\frac{(8+\theta)}{(24-2\theta)} = -2 \quad \text{or} \quad \theta = 8$$

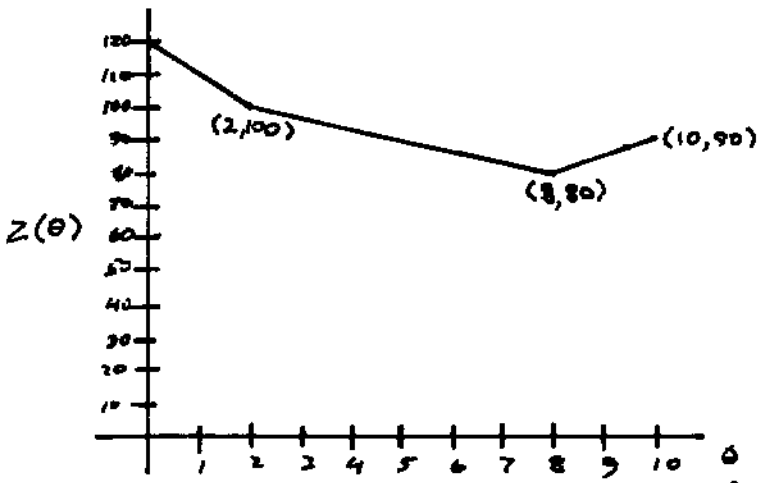
at which point (5,0) will be optimal

Summarizing:

θ	(x_1^*, x_2^*)	$z^*(\theta)$
$0 \le \theta \le 2$	(0, 5)	$120 - 10\theta$
$2 \le \theta \le 8$	$(10/3, 10/3)$	$(320 - 10\theta)/3$
$8 \le \theta \le 10$	(5, 0)	$40 + 5\theta$

(b)

BV	Eq.#	z	x_1	x_2	x_3	x_4	RHS	optimal for:
z	0	1	$-8-\theta$	$-24+2\theta$	0	0	0	
x_3	1	0	1	2*	1	0	10	
x_4	2	0	2	1	0	1	10	
z	0	1	$4-2\theta$	0	$12-\theta$	0	$120-10\theta$	$0 \le \theta \le 2$
x_2	1	0	1/2	1	1/2	0	5	
x_4	2	0	3/2 *	0	-1/2	1	5	
z	0	1	0	0	$(40-5\theta)/3$	$(8+\theta)/3$	$(320-10\theta)/3$	$2 \le \theta \le 8$
x_2	1	0	0	1	2/3 *	-1/3	10/3	
x_1	2	0	1	0	-1/3	2/3	10/3	
z	0	1	0	$(-40+5\theta)/2$	0	$(8+\theta)/2$	$40+5\theta$	$8 \le \theta \le 10$
x_3	1	0	0	3/2	1	-1/2	5	
x_1	2	0	1	1/2	0	1/2	5	



(c) From the above graph $\theta = 0$ is optimal. Since $z(\theta)$ is a convex function of θ, its maximum must occur at $\theta = 0$ or $\theta = 10$. Thus, only the linear programs corresponding to $\theta = 10$ and $\theta = 0$ need be solved.

7.2-2

BV	Eq.#	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	optimal solution
z	0	1	$-20-4\theta$	$-30+3\theta$	-5	0	0	0	0	
x_4	1	0	3	3*	1	1	0	0	30	
x_5	2	0	8	6	4	0	1	0	75	
x_6	3	0	6	1	1	0	0	1	45	
z	0	1	$10-7\theta$	0	$5-\theta$	$10-\theta$	0	0	$300-30\theta$	
x_2	1	0	1	1	1/3	1/3	0	0	10	For $0 \le \theta \le 10/7$
x_5	2	0	2	0	2	-2	1	0	15	$x^* = (0, 10, 0)$
x_6	3	0	5*	0	2/3	-1/3	0	1	35	$z = 300-30\theta$
z	0	1	0	0	$2/3 - \theta/15$	$33/3 - \frac{22}{15}\theta$	0	$-2+\frac{7}{5}\theta$	$230+19\theta$	
x_2	1	0	0	1	1/5	2/5*	0	-1/5	3	For $10/7 \le \theta \le 80/11$
x_5	2	0	0	0	26/5	-28/15	1	-2/5	1	$x^* = (7, 3, 0)$
x_1	3	0	1	0	2/15	-1/15	0	1/5	7	$z = 230+19\theta$
z	0	1	0	$-80/3+11/3\theta$	$-14/3+2/3\theta$	0	0	$10/3+\frac{2}{3}\theta$	$150+30\theta$	
x_4	1	0	0	5/2	1/2	1	0	-1/2	15/2	For $80/11 \le \theta$
x_5	2	0	0	14/3	8/3	0	1	-4/3	15	$x^* = (15/2, 0, 0)$
x_1	3	0	1	1/6	1/6	0	0	1/6	15/2	$z = 150+30\theta$

7.2-3

(a) Starting with the optimal tableau for $\theta = 0$ obtained after two iterations:

BV	Eq.#	z	x_1	x_2	x_3	x_4	x_5	RHS
z	0	1	0	0	$5-\theta$	$2+2\theta$	$8-3\theta$	220
x_2	1	0	0	1	1	1	-1	10
x_1	2	0	1	0	0	-1	2*	10
z	0	1	$\frac{-8+3\theta}{2}$	0	$5-\theta$	$\frac{12+\theta}{2}$	0	$180+15\theta$
x_2	1	0	1/2	1	1*	1/2	0	15
x_5	2	0	1/2	0	0	-1/2	1	5
z	0	1	$\frac{-13+4\theta}{2}$	$-5+\theta$	0	$\frac{7+2\theta}{2}$	0	$105+30\theta$
x_3	1	0	1/2	1	1	1/2	0	15
x_5	2	0	1/2	0	0	-1/2	1	5

optimal solution

For $0 \le \theta \le 8/3$
$x^* = (10, 10, 0)$
$z = 220$

For $8/3 \le \theta \le 5$
$x^* = (0, 15, 0)$
$z = 180 + 15\theta$

For $\theta \ge 5$
$x^* = (0, 0, 15)$
$z = 105 + 30\theta$

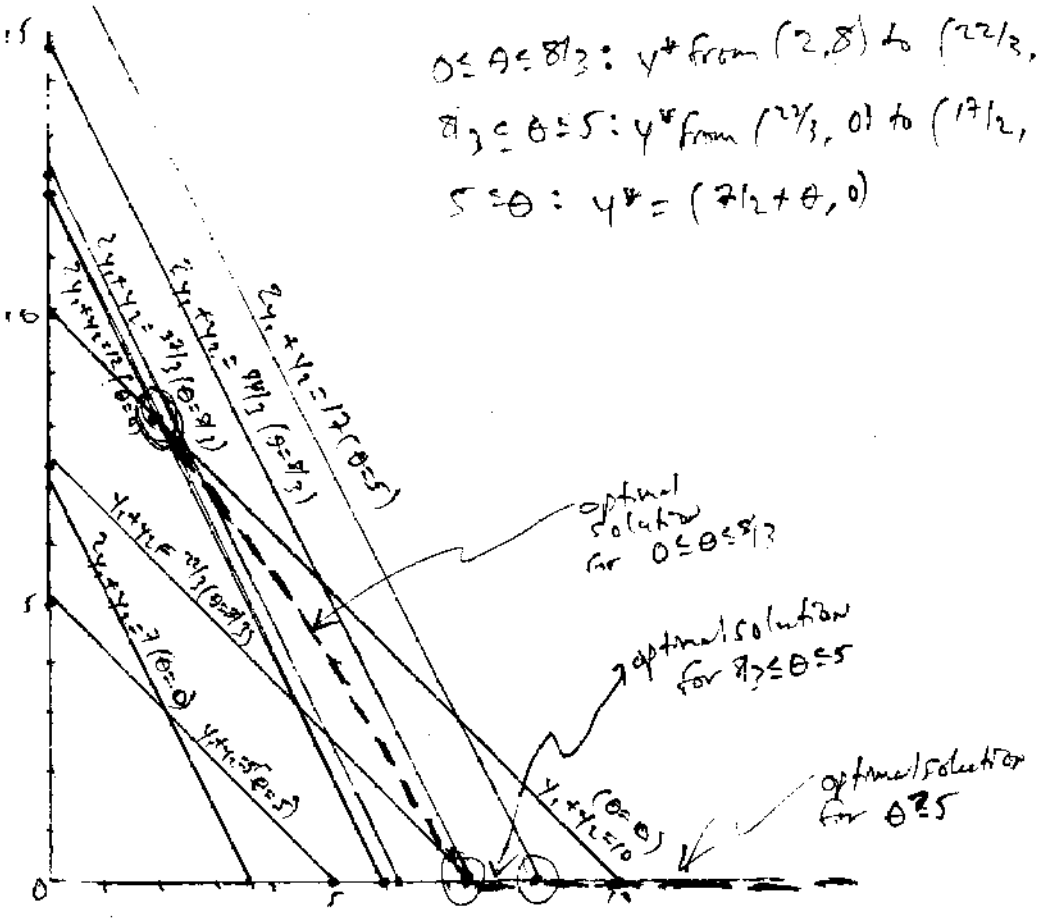
(b) The dual is minimize $30y_1 + 20y_2$
subject to
$y_1 + y_2 \ge 10 - \theta$
$2y_1 + y_2 \ge 12 + \theta$
$2y_1 + y_2 \ge 7 + 2\theta$
$y_1 \ge 0, y_2 \ge 0$

(CONT.)

7.2-3 b) (cont.) Starting with the optimal tableau for $\theta = 0$ obtained after two iterations:

BV	Eq #	ω	y_1	y_2	y_3	y_4	y_5	RHS	optimal solution
ω	0	-1	0	0	10	10	0	-220	For $0 \le \theta \le 8/3$
y_2	1	0	0	1	-2^*	1	0	$8-3\theta$	$y^* = (2+2\theta, 8-3\theta)$
y_1	2	0	1	0	1	-1	0	$2+2\theta$	and $\omega^* = 220$
y_5	3	0	0	0	0	-1	1	$5-\theta$	
ω	0	-1	0	5	0	15	0	$-180-15\theta$	For $8/3 \le \theta \le 5$
y_3	1	0	0	$-1/2$	1	$-1/2$	0	$-4+3\theta/2$	$y^* = (6+\theta/2, 0)$
y_1	2	0	1	$1/2$	0	$-1/2$	0	$6+\theta/2$	and $\omega^* = 180+15\theta$
y_5	3	0	0	0	0	-1^*	1	$5-\theta$	
ω	0	-1	0	5	0	0	15	$-105-30\theta$	For $5 \le \theta$
y_3	1	0	0	$-1/2$	1	0	$-1/2$	$-17/2+2\theta$	$y^* = (7/2+\theta, 0)$
y_1	2	0	1	$1/2$	0	0	$-1/2$	$7/2+\theta$	and $\omega^* = 105+30\theta$
y_4	3	0	0	0	0	1	-1	$-5+\theta$	

The basic solutions are the same as those obtained in part (a).



$0 \le \theta \le 8/3$: y^* from $(2, 8)$ to $(22/3, 0)$

$8/3 \le \theta \le 5$: y^* from $(22/3, 0)$ to $(17/2, 0)$

$5 \le \theta$: $y^* = (7/2+\theta, 0)$

7.2-4

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	-2	-1	0	0	0	0
X3	1	0	1	0	1	0	0	10+2θ
X4	2	0	1*	1	0	1	0	25-θ
X5	3	0	0	1	0	0	1	10+2θ

optimal Solution

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	1	0	2	0	50-2θ
X3	1	0	0	-1*	1	-1	0	15+3θ
X1	2	0	1	1	0	1	0	25-θ
X5	3	0	0	1	0	0	1	10+2θ

For $5 \le \theta \le 25$

$X^* = (25-\theta, 0)$

$Z^* = 50-2\theta$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	1	1	0	35+θ
X2	1	0	0	1	-1	1	0	15-3θ
X1	2	0	1	0	1	0	0	10+2θ
X5	3	0	0	0	1	-1	1	-5+5θ

For $1 \le \theta \le 5$

$X^* = (10+2\theta, 15-3\theta)$

$Z^* = 35+\theta$

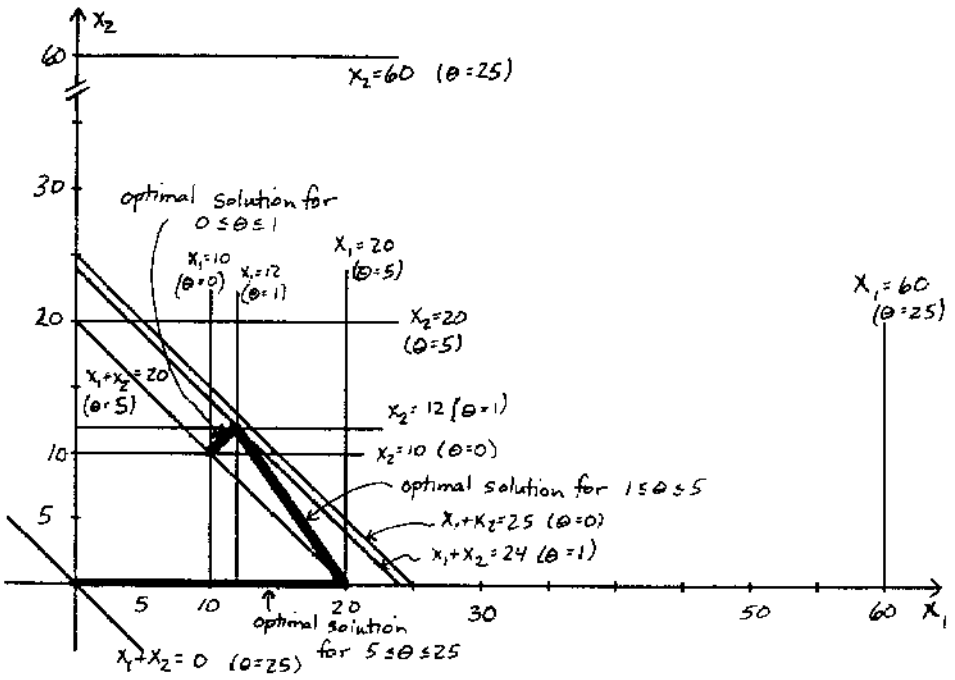
Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	0	1	30+6θ
X2	1	0	0	1	0	0	1	10+2θ
X1	2	0	1	0	1	0	0	10+2θ
X4	3	0	0	0	-1	1	-1	5-5θ

For $0 \le \theta \le 1$

$X^* = (10+2\theta, 10+2\theta)$

$Z^* = 30+6\theta$

Graphically we get:



7.2-5 Starting with the optimal tableau for $\theta = 0$ obtained after 2 iterations,

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	4	1	3	0	0	5	$150+5\theta$
X5	1	0	0	-8	-2	-3	1	0	-3	$45-5\theta$
X6	2	0	0	0	-3*	-2	0	1	-2	$18-3\theta$
X1	3	0	1	2	1	2	0	0	1	$30+\theta$

optimal solution

For $0 \le \theta \le 6$

$X^* = (30+\theta, 0, 0, 0)$

$Z^* = 150+5\theta$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	4	0	$7/3$	0	$1/3$	$13/3$	$144+4\theta$
X5	1	0	0	-8	0	$-5/3$	1	$-2/3$*	$-5/3$	$33-3\theta$
X3	2	0	0	0	1	$2/3$	0	$-1/3$	$2/3$	$-6+\theta$
X1	3	0	1	2	0	$4/3$	0	$1/3$	$1/3$	36

For $6 \le \theta \le 11$

$X^* = (36, 0, -6+\theta, 0)$

$Z^* = 144+4\theta$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	Right side
Z	0	1	0	0	0	$3/2$	$1/2$	0	$7/2$	$160\frac{1}{2}+\frac{5}{2}\theta$
X6	1	0	0	12	0	$5/2$	$-3/2$	1	$5/2$	$-49\frac{1}{2}+\frac{9}{2}\theta$
X3	2	0	0	4	1	$3/2$	$-1/2$	0	$3/2$	$-22\frac{1}{2}+\frac{5}{2}\theta$
X1	3	0	1	-2	0	$1/2$	$1/2$	0	$-1/2$	$52\frac{1}{2}-\frac{3}{2}\theta$

For $11 \le \theta \le 35$

$X^* = (52\frac{1}{2}-\frac{3}{2}\theta, 0, -22\frac{1}{2}+\frac{5}{2}\theta, 0)$

$Z^* = 160\frac{1}{2}+\frac{5}{2}\theta$

$\theta = 30$ provides the largest value of the objective

$X^*(30) = (7\frac{1}{2}, 0, 52\frac{1}{2}, 0)$, $Z^*(30) = 235\frac{1}{2}$

7.2-6

(a)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	$100+10\theta$
X2	1	0	-1	1	3	1	0	$20+2\theta$
X5	2	0	16	0	-2*	-4	1	$10-9\theta$

For $0 \le \theta \le 10/9$

$X^* = (0, 20+2\theta, 0)$

$Z^* = 100+10\theta$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	16	0	0	1	1	$110+\theta$
X2	1	0	23	1	0	-5*	$3/2$	$35-\frac{23}{2}\theta$
X3	2	0	-8	0	1	2	$-1/2$	$-5+\frac{9}{2}\theta$

For $10/9 \le \theta \le \frac{70}{23}$

$X^* = (0, 35-\frac{23}{2}\theta, -5+\frac{9}{2}\theta)$

$Z^* = 110+\theta$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	$103/5$	$1/5$	0	0	$13/10$	$117-\frac{13}{10}\theta$
X4	1	0	$-23/5$	$-1/5$	0	1	$-3/10$	$-7+\frac{23}{10}\theta$
X3	2	0	$6/5$	$3/5$	1	0	$1/10$	$9-\frac{\theta}{10}$

For $\frac{70}{23} \le \theta \le 90$

$X^* = (0, 0, 9-\frac{\theta}{10})$

$Z^* = 117-\frac{13}{10}\theta$

(b)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	$100-10\theta'$
X2	1	0	-1*	1	3	1	0	$20-2\theta'$
X5	2	0	16	0	-2	-4	1	$10+9\theta'$

For $-10 \le \theta \le 0$

$X^* = (0, 20+2\theta, 0)$

$Z^* = 100-10\theta$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
Z	0	1	0	0	2	5	0	$100-10\theta'$
X2	1	0	1	-1	-3	-1	0	$-20+2\theta'$
X3	2	0	0	16	46	12	1	$330-23\theta'$

For $-\frac{330}{23} \le \theta \le -10$

$X^* = (0, -20-2\theta, 330+23\theta)$

$Z^* = 100-10\theta$

For $\theta < -330/23$ the problem is infeasible.

7.2-7 (a) Let $x^{(k)}$ be the kth optimal solution obtained as θ is increased from 0. Each $x^{(k)}$ is optimal for some θ-interval, say $\theta_k \le \theta \le \theta_{k+1}$, and the objective function $z(\theta) = \alpha_k + \beta_k \theta$ for some α_k and β_k. So $z(\theta)$ is linear in this interval. As the interval changes, so do α_k and β_k so that a different linear function is obtained for each interval.

(b) The problem is maximize $z(\theta) = \sum_{j=1}^{n} (c_j + \alpha_j \theta) x_j$
subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1, 2, \ldots, m$
$x_j \ge 0 \quad j = 1, 2, \ldots, n$

Note that the feasible region does not change with θ. Consider $\theta_1 < \theta_2$ and let $\theta_3 = \lambda\theta_1 + (1-\lambda)\theta_2$ for $0 \le \lambda \le 1$. Let $x_j^{(1)}$, $x_j^{(2)}$ and $x_j^{(3)}$ be the optimal values of x_j $(j = 1, 2, \ldots, n)$ for θ_1, θ_2 and θ_3 respectively.
Let $z(\theta, x) = \sum_{j=1}^{n} (c_j + \alpha_j \theta) x_j$.

Now, $z^*(\theta_1) = z(\theta_1, x^{(1)}) \ge z(\theta_1, x^{(3)}) \Rightarrow \lambda z^*(\theta_1) \ge \lambda z(\theta_1, x^{(3)})$
and $z^*(\theta_2) = z(\theta_2, x^{(2)}) \ge z(\theta_2, x^{(3)}) \Rightarrow (1-\lambda) z^*(\theta_2) \ge (1-\lambda) z(\theta_2, x^{(3)})$
Adding gives $\lambda z^*(\theta_1) + (1-\lambda) z^*(\theta_2) \ge \lambda z(\theta_1, x^{(3)}) + (1-\lambda) z(\theta_2, x^{(3)})$
But, $\lambda z(\theta_1, x^{(3)}) + (1-\lambda) z(\theta_2, x^{(3)}) = \lambda \sum_{j=1}^{n} (c_j + \alpha_j \theta_1) x_j^{(3)} + (1-\lambda) \sum_{j=1}^{n} (c_j + \alpha_j \theta_2) x_j^{(3)}$

$$= \sum_{j=1}^{n} (c_j + (\lambda\theta_1 + (1-\lambda)\theta_2)\alpha_j) x_j^{(3)}$$
$$= \sum_{j=1}^{n} (c_j + \alpha_j \theta_3) x_j^{(3)}$$
$$= z(\theta_3, x^{(3)})$$
$$= z^*(\theta_3)$$

Hence, $\lambda z^*(\theta_1) + (1-\lambda) z^*(\theta_2) \ge z^*(\lambda\theta_1 + (1-\lambda)\theta_2)$
Hence, $z^*(\theta)$ is convex.

7.2-8 (a) Same argument as in problem 7.2-7 (a) above.
(b) The problem is maximize $z(\theta) = \sum_{j=1}^{n} c_j x_j$
subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i + \alpha_i \theta \quad i = 1, 2, \ldots, m$
$x_j \ge 0 \quad j = 1, 2, \ldots, n$

Consider $\theta_1 < \theta_2$ and let $\theta_3 = \lambda\theta_1 + (1-\lambda)\theta_2$, $0 \le \lambda \le 1$.
Let $x_j^{(1)}$, $x_j^{(2)}$, and $x_j^{(3)}$ be the optimal values of x_j for θ_1, θ_2 and θ_3 respectively.
Thus, $\lambda z^*(\theta_1) + (1-\lambda) z^*(\theta_2) = \lambda \sum_{j=1}^{n} c_j x_j^{(1)} + (1-\lambda) \sum_{j=1}^{n} c_j x_j^{(2)}$

$$= \sum_{j=1}^{n} c_j (\lambda x_j^{(1)} + (1-\lambda) x_j^{(2)}) \quad (*)$$

Note that if $x_j' = \lambda x_j^{(1)} + (1-\lambda) x_j^{(2)}$ for $j = 1, 2, \ldots, n$, then x' is feasible for $\theta = \theta_3$ since

$$\sum_{j=1}^{n} a_{ij} x_j' = \lambda \sum_{j=1}^{n} a_{ij} x_j^{(1)} + (1-\lambda) \sum_{j=1}^{n} a_{ij} x_j^{(2)}$$
$$\le \lambda (b_i + \alpha_i \theta_1) + (1-\lambda)(b_i + \alpha_i \theta_2)$$
$$= b_i + \alpha_i \theta_3 \quad i = 1, 2, \ldots, m \quad \text{(CONT.)}$$

7.2-8 (CONT) Since $x^{(3)}$ is optimal for θ_3

$$\sum_{j=1}^{n} c_j(\lambda x_j^{(1)} + (1-\lambda)x_j^{(2)}) \le \sum_{j=1}^{n} c_j x_j^{(3)} = z^*(\theta_3)$$

So from (*) above, $\lambda z^*(\theta_1) + (1-\lambda) z^*(\theta_2) \le z^*(\theta_3)$

Hence, $z^*(\theta)$ is concave.

7.2-9

From Duality Theory, $z^{**} =$ minimum $\sum_{i=1}^{m} (b_i + k_i) y_i$

subject to: $\sum_{i=1}^{m} a_{ij} y_i \ge c_j$

$y_i \ge 0$.

$(y_1^*, y_2^*, \ldots, y_m^*)$ is feasible for this problem, therefore,

$$z^{**} \le \sum_{i=1}^{m} (b_i + k_i) y_i^* = z^* + \sum_{i=1}^{m} k_i y_i^*.$$

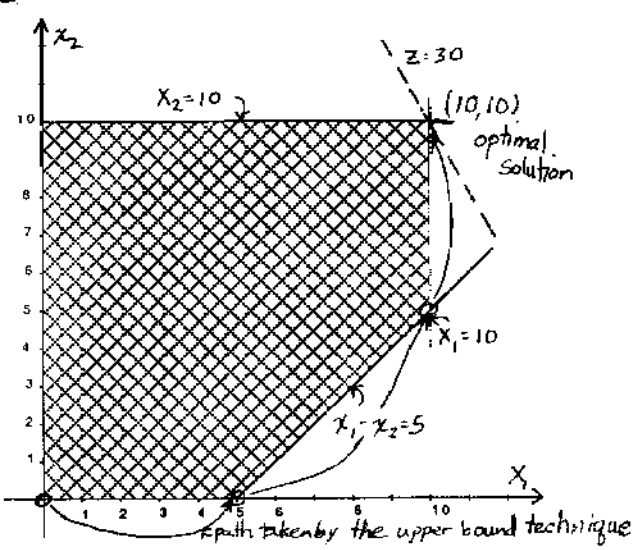
7.3-1

	BV	Eq #	Z	x_1	x_2	x_3	RHS
(0)	Z	0	1	-3	-5	0	0
	x_3	1	0	3	2*	1	18
(1)	Z	0	1	-3	5	0	30
	x_3	1	0	3*	-2	1	6
(2)	Z	0	1	0	3	1	36
	x_1	1	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	2

$x_2 \le 6$ [from $2x_2 \le 12$]
$x_2 \le 9$ from p.vot
So x_2 moves to upper bound.
$x_1 \le 4$ from constraint
$x_1 \le 2$ from p.vot

So $(x_1, x_2) = (2, 6)$ with $z = 36$ is optimal.

7.3-2 (a) and (c)



The optimal solution is $(x_1, x_2) = (10, 10)$ with $z = 30$

7.3-2

(b)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	Right side	
Z	0	1	-2	-1	0	0	$X1 \le 10$
X3	1	0	1*	-1	1	5	$X1 \le 5$

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	Right side	
Z	0	1	0	-3	2	10	$X2 \le 10$
X1	1	0	1	-1	1	5	$X2 \le 5$

Bas Var	Eq No	Z	Coefficient of Y1	X2	X3	Right side	
Z	0	1	0	-3	2	10	$X2 \le 10$
Y1	1	0	1	1*	-1	5	$X2 \le 5$

Bas Var	Eq No	Z	Coefficient of Y1	X2	X3	Right side	
Z	0	1	3	0	-1	25	$X3 \le 5$
X2	1	0	1	1	-1	5	

BV	Eq	Z	Y1	Y2	X3	RS	
Z	0	1	3	0	-1	25	$X_3 \le 5$
Y2	1	0	-1	1	1*	5	

BV	Eq	Z	Y1	Y2	X3	RS
Z	0	1	2	1	0	30
X3	1	0	-1	1	1	5

(X_1, X_2) is optimal with $Z = 30$

7.3-3

	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RHS	
	Z	0	1	-1	-3	2	0	0	0	$x_2 \le 3$
(0)	x_4	1	0	0	1*	-2	1	0	1	$x_2 \le 1$
	x_5	2	0	2	1	2	0	1	8	$x_2 \le 8$
	Z	0	1	-1	0	-4	3	0	3	$x_3 \le 2$
(1)	x_2	1	0	0	1	-2	1	0	1	$x_3 \le 1$
	x_5	2	0	2	0	4	-1	1	7	$x_3 \le 1\frac{3}{4}$

	BV	Eq. #	Z	x_1	x_2	x_3	x_4	x_5	RHS	
	Z	0	1	-1	0	-4	3	0	3	$x_3 \le 2$
(2)	x_2	1	0	0	1	2*	-1	0	2	$x_3 \le 1$
	x_5	2	0	2	0	4	-1	1	7	$x_3 \le 1\frac{3}{4}$
	Z	0	1	-1	2	0	1	0	7	$x_1 \le 1$
(3)	x_3	1	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1	
	x_5	2	0	2	-2	0	1	1	3	$x_1 \le 1\frac{1}{2}$

(CONT)

7.3-3 (CONT)

	BV	Eq. #	Z	X_1	X_2	X_3	X_4	X_5	RHS
	Z	0	1	-1	2	0	1	0	8
(4)	X_3	1	0	0	1/2	1	-1/2	0	1
	X_5	2	0	2	-2	0	1	1	1

So $(x_1, x_2, x_3) = (1, 3, 1)$ with $z = 8$ is optimal.

7.3-4 Initial Tableau:

BV	Eq #	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
Z	0	1	-2	-3	2	-5	0	0	0
X_5	1	0	2	2	1	2	1	0	5
X_6	2	0	1	2	-3	4	0	1	5

Final Tableau (After five iterations):

BV	Eq #	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS
Z	0	1	0	1/7	0	3/7	4/7	4/7	54/7
X_1	1	0	1	-8/7	0	-10/7	3/7	1/7	2/7
X_3	2	0	0	4/7	1	4/7	1/7	-2/7	3/7

So $(x_1, x_2, x_3, x_4) = (2/7, 1, 3/7, 1)$ with $z = 75/7$ is optimal.

7.3-5 Initial Tableau:

BV	Eq. #	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
Z	0	1	-2	-5	-3	-4	-1	0	0	0
X_6	1	0	1	3	2	3	1	1	0	6
X_7	2	0	4	6	5	7	1	0	1	15

Final Tableau (after seven iterations):

BV	Eq. #	Z	Y_1	Y_2	Y_3	Y_4	X_5	X_6	X_7	RHS
Z	0	1	2/3	1	1/3	0	1/3	4/3	0	10
Y_4	1	0	1/3	1	2/3	1	-1/3	-1/3	0	1
X_7	2	0	-5/3	1	-1/3	0	-4/3	-7/3	1	0

So $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 0, 0)$ with $z = 10$ is optimal.

7.3-6

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	3	4	2	0	0	0	$X_1 \le 25$
X4	1	0	-1*	-1	0	1	0	-15	$X_1 \le 15$
X5	2	0	0	-1	-1	0	1	-10	

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	0	1	2	3	0	-45	$X_2 \le 5$
X1	1	0	1	1	0	1	0	15	$X_2 \le 15$
X5	2	0	0	-1	1	0	1	-10	$X_2 \le 10$

(CONT.)

7.3-6 (CONT.)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side	
-Z	0	1	0	-1	2	3	0	-50	X3≤15
X1	1	0	1	-1	0	1	0	10	
X5	2	0	0	1	-1*	0	1	-5	X3≤5

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	Right side
-Z	0	1	0	1	0	3	2	-60
X1	1	0	1	-1	0	1	0	10
X3	2	0	0	-1	1	0	-1	5

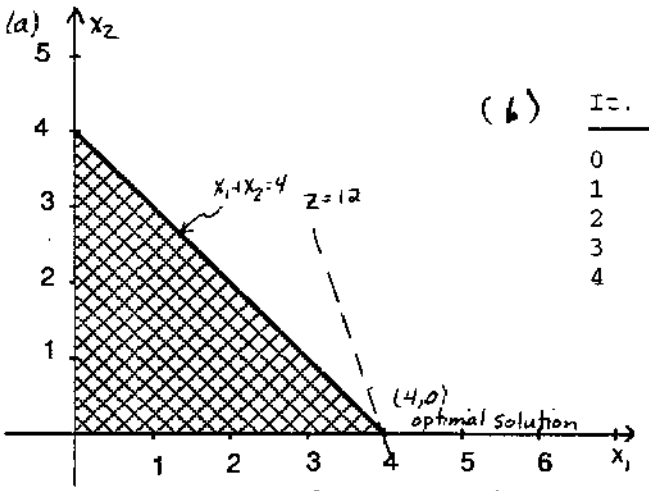
(X1, X2, X3) = (10, 5, 5) is optimal with Z = 60

7.4-1

It.	x_1	x_2	Z
0	1	3	7
1	1.04605	4.95395	10.9539
2	0.93406	6.06594	13.0659

7.4-2

(a)



The corner point feasible solutions are (0,0), (0,4) and (4,0). The last one is optimal with Z=12.

(b)

It.	x_1	x_2	Z
0	1	1	4
1	1.875	1.125	6.75
2	2.6981	0.8019	8.89621
3	3.34396	0.40095	10.4328
4	3.6671	0.20047	11.2018

(cont.)

7.4-2
c)

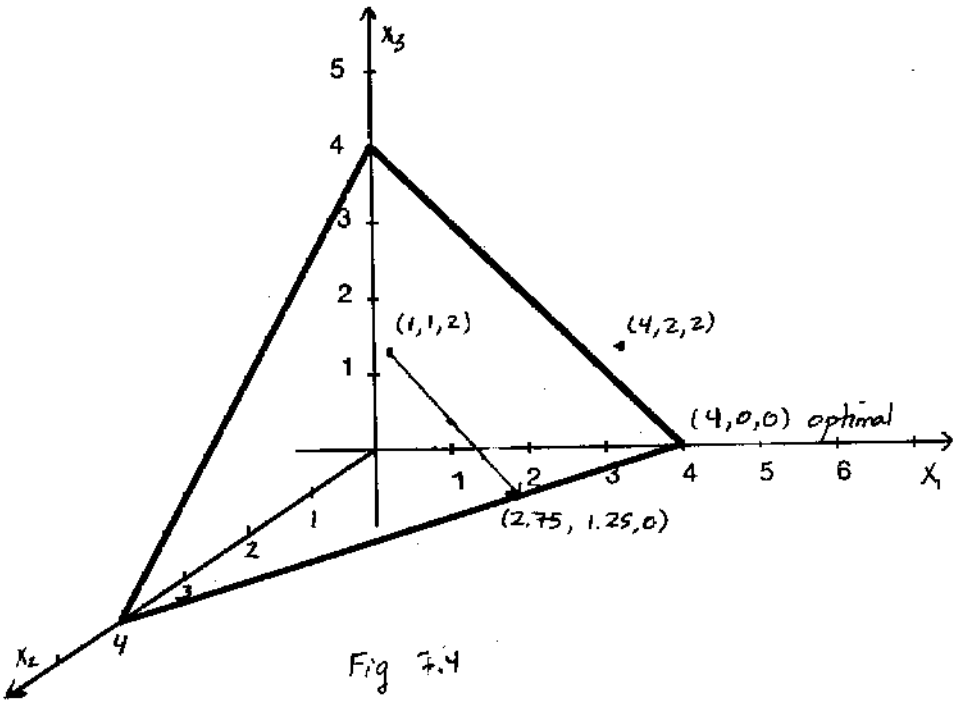


Fig 7.4

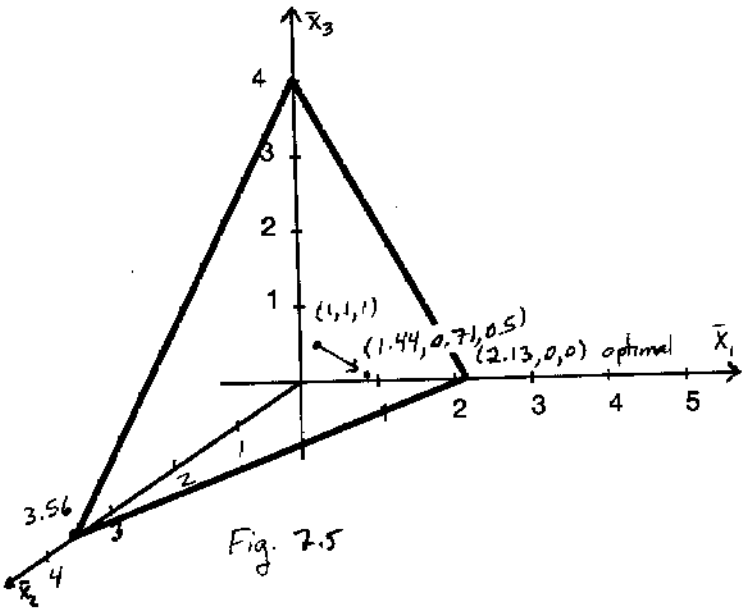


Fig. 7.5

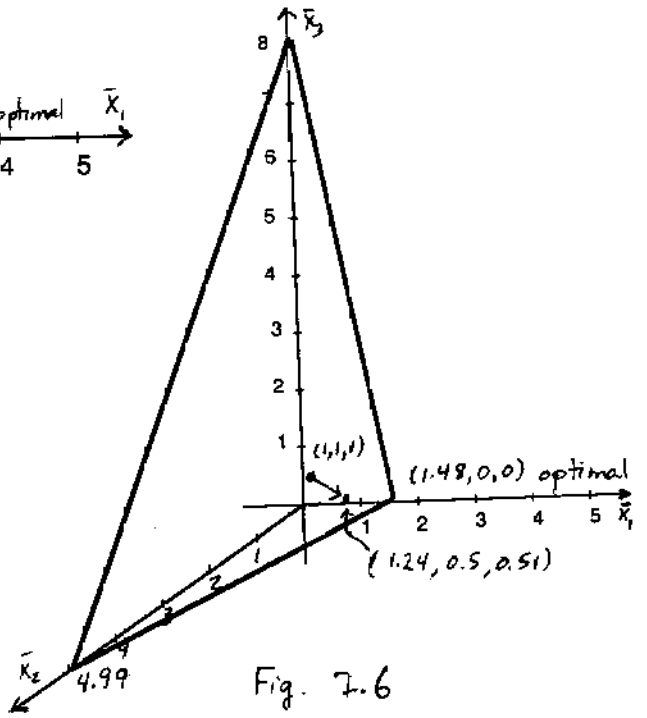


Fig. 7.6

7.4-2
c)
(CONT.)

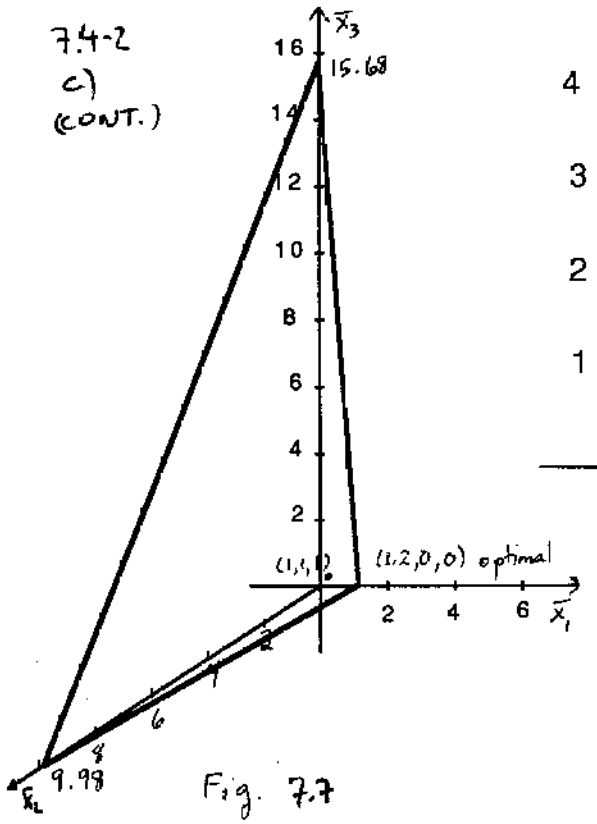


Fig. 7.7

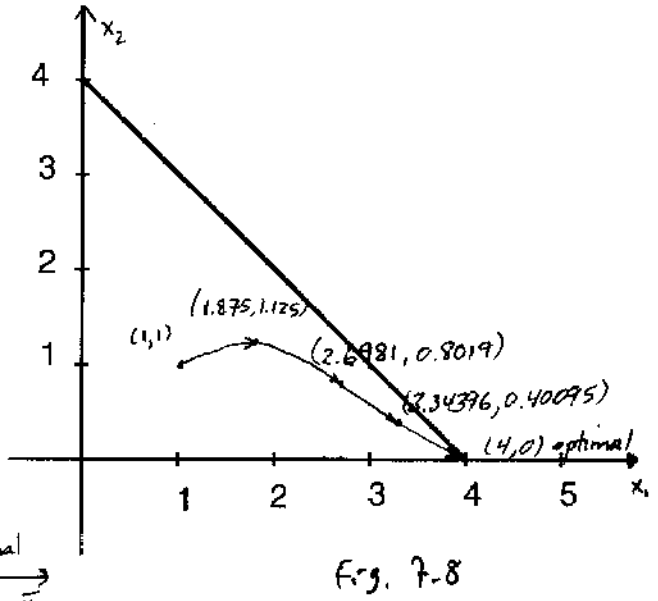


Fig. 7.8

7.4-3

(a) and (c)

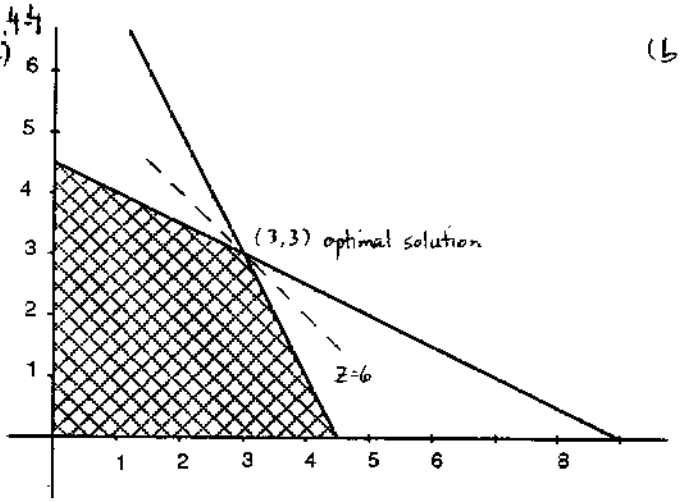
It.	X_1	X_2	Z
0	4	4	12
1	2	6	14
2	1	7	15
3	0.5	7.5	15.5
4	0.25	7.75	15.75
5	0.125	7.875	15.875
6	0.0625	7.9375	15.9375
7	0.03125	7.96875	15.9688
8	0.01562	7.98438	15.9844
9	0.00781	7.99219	15.9922

(b) The value of X_1 is halved at each step so subsequent trial solutions should be of the form $(X_1, X_2) = (2^{-i}, 8 - 2^{-i})$ for $i = 1, 2, \ldots$

(c) We need to determine when $2^{-i} - 2^{-(i+1)} \leq 0.01$. This is true for $i = 6$ with $(X_1, X_2) = (2^{-7}, 8 - 2^{-7}) = (0.0078, 7.9922)$ on iteration 9.

7.4-4

a)



The optimal solution is $(x_1, x_2) = (3, 3)$ with $z = 6$

(b) The gradient is $(1,1)$. Moving from the origin in the direction $(1,1)$, the first boundary point encountered is the optimal solution $(3,3)$.

c)

It.	x_1	x_2	z
0	1	1	2
1	2	2	4
2	2.5	2.5	5
3	2.75	2.75	5.5
4	2.875	2.875	5.75
5	2.9375	2.9375	5.875
6	2.96875	2.96875	5.9375
7	2.98437	2.98438	5.96875
8	2.99219	2.99219	5.98438
9	2.99609	2.99609	5.99219
10	2.99805	2.99805	5.99609

alpha = 0.5

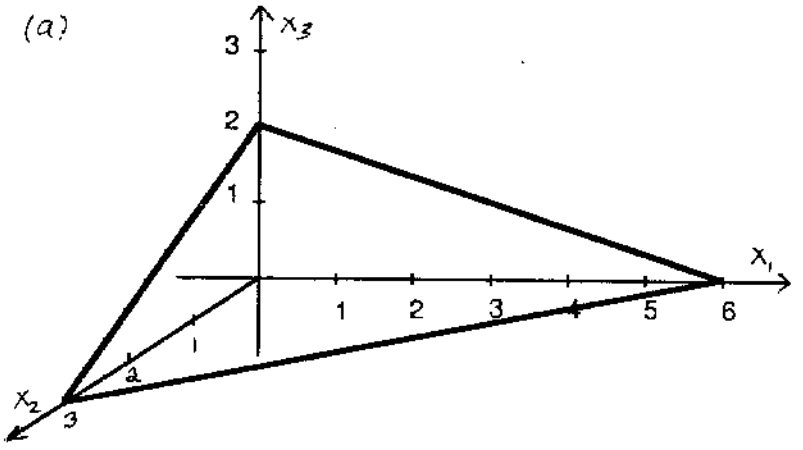
(d)

It.	x_1	x_2	z
0	1	1	2
1	2.8	2.8	5.6
2	2.98	2.98	5.96
3	2.998	2.998	5.996
4	2.9998	2.9998	5.9996
5	2.99998	2.99998	5.99995
6	3	3	6
7	3	3	6
8	3	3	6
9	3	3	6
10	3	3	6

alpha = 0.9

7.4-5

(a)



7.4-5 (b) gradient = $(2,5,7)$

$$\text{projected gradient} = P\begin{pmatrix}2\\5\\7\end{pmatrix} = \left[I - \begin{pmatrix}1\\2\\3\end{pmatrix}\left[(1,2,3)\begin{pmatrix}1\\2\\3\end{pmatrix}\right]^{-1}(1,2,3)\right]\begin{pmatrix}2\\5\\7\end{pmatrix}$$

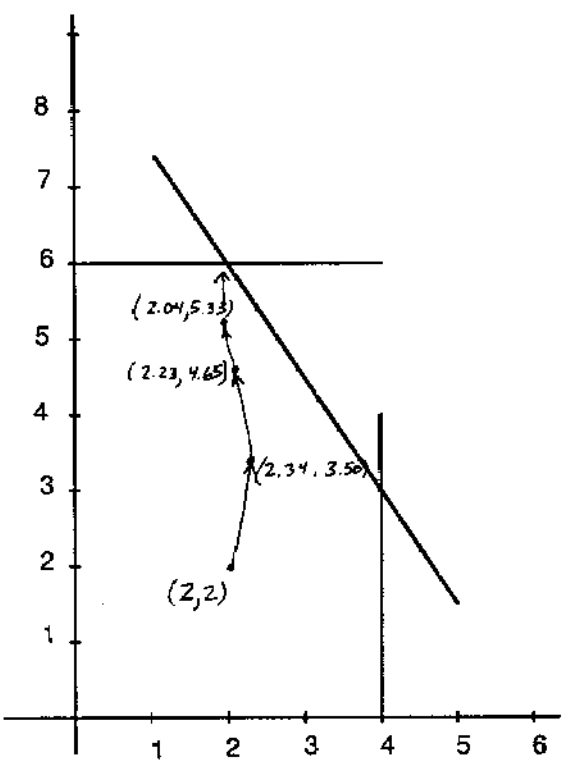
$$= \begin{pmatrix}2\\5\\7\end{pmatrix} - \frac{1}{14}\begin{pmatrix}33\\66\\99\end{pmatrix} = \frac{1}{14}\begin{pmatrix}-5\\4\\-1\end{pmatrix}$$

(c) and (d)

It.	x_1	x_2	x_3	z
0	1	1	1	14
1	0.5	1.4	0.9	14.3
2	0.25969	2.19516	0.45	14.6452
3	0.17947	2.57276	0.225	14.7978
4	0.1069	2.7778	0.1125	14.8903
5	0.05595	2.88765	0.05625	14.9439
6	0.0281	2.94376	0.02812	14.9719
7	0.01406	2.97188	0.01406	14.9859
8	0.00703	2.98594	0.00703	14.993
9	0.00352	2.99297	0.00352	14.9965
10	0.00176	2.99648	0.00176	14.9982

7.4-6

It.	x_1	x_2	z
0	2	2	16
1	2.336	3.496	24.488
2	2.23067	4.65399	29.962
3	2.03597	5.32699	32.7429
4	1.95211	5.6635	34.1738
5	1.95054	5.83175	35.0104
6	1.97169	5.91587	35.4944
7	1.98588	5.95788	35.7471
8	1.99296	5.97891	35.8734
9	1.99648	5.98945	35.9367
10	1.99824	5.99473	35.9684
11	1.99912	5.99736	35.9842
12	1.99956	5.99868	35.9921
13	1.99978	5.99934	35.996
14	1.99989	5.99967	35.998
15	1.99995	5.99984	35.999



7.5-1

a) $3x_1+4x_2+2x_3-y^++y^-=60$

b) let c^+ be the coefficient of y^+ and c^- the one for y^-.

So $c^+=2c^-$

7.5-2

a). minimize (amount under market share for product 1) + (amount under market share for product 2)

subject to:

$x_1+x_2+x_3 \le 55$

$x_3 \ge 10$

$x_1 \ge 0$; $x_2 \ge 0$

b) let

$y_1 = 0.5x_1+0.2x_3-15$; $y_1=y_1^+-y_1^-$

$y_2 = 0.3x_2+0.2x_3-10$; $y_2=y_2^+-y_2^-$

The problem is:

minimize $y_1^-+y_2^-$

subject to:

$0.5x_1 + 0.2x_3 - y_1^+ + y_1^- = 15$

$0.3x_2 + 0.2x_3 - y_2^+ + y_2^- = 10$

$x_1 + x_2 + x_3 \le 55$

$x_3 \ge 10$

$x_1 \ge 0$; $x_2 \ge 0$

$y_1^+, y_1^-, y_2^+, y_2^- \ge 0$

c)

Goals	Unit Contribution Per Unit of Each Activity: Campaign 1	Campaign 2	Campaign 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Market Share 1	0.5	0	0.2	15	≥	15	0	0	15	=	15
Market Share 2	0	0.3	0.2	8.33333	≥	10	0	1.667	10	=	10
Budget	1	1	1	55	≤	55			55	≤	55
Campaign 3 budget	0	0	1	41.6667	≥	10			41.667	≥	10
Solution	13.333	0	41.667								

Weighted sum of deviations = 1.667

7.5-3

a) $6x_1 + 4x_2 + 5x_3 - y_1^+ + y_1^- = 50$

$8x_1 + 7x_2 + 5x_3 - y_2^+ + y_2^- = 75$

$P = 20x_1 + 15x_2 + 25x_3$

b) $Z = 20x_1 + 15x_2 + 25x_3 - 6y_1^+ - 6y_1^- - 3y_2^-$

c) Maximize $20x_1 + 15x_2 + 25x_3 - 6y_1^+ - 6y_1^- - 3y_2^-$

subject to:

$6x_1 + 4x_2 + 5x_3 - y_1^+ + y_1^- = 50$

$8x_1 + 7x_2 + 5x_3 - y_2^+ + y_2^- = 75$

$y_1^+, y_1^-, y_2^+, y_2^-, x_1, x_2, x_3$ all ≥ 0

d)

Goals	Unit Contribution Per Unit of Each Activity: Product 1	Product 2	Product 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Profit	20	15	25	375							
Employment	8	4	5	75	=	50	**25**	**0**	50	=	50
Earnings	8	7	5	75	≥	75	**0**	**0**	75	=	75
Solution	**0**	**0**	**15**								

Weighted sum of deviations = 225

7.5-4

a) No, we would not expect the optimal solution to change. Goal 1 is already met, so increasing the weight on that goal would not change anything. We already exceed goal 2, so decreasing the penalty weight for goal 2 would only decrease our desire to avoid exceeding this goal.

b)

Goals	Product 1	Product 2	Product 3	Achieved		Goal	Over	Under	Totals		Side
Profit	12	9	15	140	≥	140	**0**	**0**	140	=	140
Employment	5	3	4	58.3333	=	40	**18.3333**	**0**	40	=	40
Investment	5	7	8	58.3333	≤	55	**3.33333**	**0**	55	=	55
Solution	**11.667**	**0**	**0**								

Weighted sum of deviations = 225

c)

Goals	Unit Contribution Per Unit of Each Act: Product 1	Product 2	Product 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Profit	12	9	15	140	≥	140	**0**	**0**	140	=	140
Employment	5	3	4	58.3333	=	40	**18.3333**	**0**	40	=	40
Investment	5	7	8	58.3333	≤	55	**3.33333**	**0**	55	=	55
Solution	**11.667**	**0**	**0**								

Weighted sum of deviations = 225

7.5-5

a) Minimize (amount under foreign capital goal)/100 +
(amount under citizens fed goal) +
(amount under goal for citizens employed) +
(amount over goal for citizens employed)

b)

Minimize $z = \frac{1}{100}y_1^- + y_2^- + y_3^+ + y_3^-$

Subject to $1000x_1 + 1000x_2 + 1000x_3 + x_4 = 15M$

$3000x_1 + 5000x_2 + 4000x_3 - (y_1^+ - y_1^-) = 70M$

$150x_1 + 75x_2 + 100x_3 - (y_2^+ - y_2^-) = 1.75M$

$10x_1 + 15x_2 + 12x_3 - (y_3^+ - y_3^-) = .2M$

$x_j \geq 0$ $j = 1,2,3,4$ $y_i^+ \geq 0, y_i^- \geq 0$ $i = 1,2,3$

c)

Goals	Unit Contribution Per Unit of Each Activity: Product 1	Product 2	Product 3	Level Achieved		Goal	Amount Over	Amount Under	Totals		Right-Hand Side
Foreign Capital	3000	5000	4000	58333333	≥	70000000	0	11666666.7	70000000	=	70000000
Citizens Fed	150	75	100	1750000	≥	1750000	0	0	1750000	=	1750000
Citizens Employed	10	15	12	183333.3	=	200000	0	16666.7	200000	=	200000
Acres	1000	1000	1000	15000000	≤	15000000					
Solution	8333.333	6666.667	0								

Weighted sum of deviations = 133333.3

d)

Minimize $z = M_2 y_1^- + M_1 y_2^- + y_3^+ + y_3^-$

Subject to $1000x_1 + 1000x_2 + 1000x_3 + x_4 = 15M$

$3000x_1 + 5000x_2 + 4000x_3 - (y_1^+ - y_1^-) = 70M$

$150x_1 + 75x_2 + 100x_3 - (y_2^+ - y_2^-) = 1.75M$

$10x_1 + 15x_2 + 12x_3 - (y_3^+ - y_3^-) = .2M$

$x_j \geq 0$ $j = 1,2,3,4$ $y_i^+ \geq 0, y_i^- \geq 0$ $i = 1,2,3$

e)

BV	Eq#	z	x_1	x_2	x_3	x_4	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	RHS
z	0	-1	$-1500M_1$ $-3000M_2$ -10	$-7500M_1$ $-5000M_2$ -15	$-1000M_1$ $-4000M_2$ -12	0	M_2	0	M_1	0	2	0	$-175M_1$ $-70000M_2$ -20
x_4	1	0	1000	1000	1000	1	0	0	0	0	0	0	1500
y_1^-	2	0	2000	5000	4000	0	-1	1		0	0	0	70000
y_2^-	3	0	150*	75	100	0	0	0	-1	1	0	0	175
y_3^-	4	0	10	15	12	0	0	0	0	0	-1	1	20
z	0	-1	0	$-3500M_2$ -10	$-2000M_2$ -12	0	M_2	0	$-20M_2$ $-1/15$	$M_1 + 20M_2$ $1/15$	2	0	$3500M_2$ $-2/3$
x_4	1	0	0	500*	1000/3	1	0	0	20/3	-20/3	0	0	1000/3
y_1^-	2	0	0	3500	2000	0	-1	1	20	-20	0	0	3500
x_1	3	0	1	1/2	2/3	0	0	0	-1/150	1/150	0	0	7/6
y_3^-	4	0	0	10	12	0	0	0	1/15	-1/15	-1	1	2/3
z	0	-1	0	0	$100M_2 + 4/3$	$7M_2 + 1/100$	M_2	0	$8M_2 - 1/15$	$M_1 - 8M_2 - 1/15$	2	0	$350M_2$ $-2/3$
x_2	1	0	0	1	2/3	1/500	0	0	1/75	-1/75	0	0	2/3
y_1^-	2	0	0	0	-1000/3	-7	-1	1	-80/3	80/3	0	0	3500/3
x_1	3	0	1	0	1/3	-1/1000	0	0	0	0	0	0	5/6
y_3^-	4	0	0	0	-4/3	-1/50	0	0	-1/15	1/15	-1	1	6/3

(CONT.)

7.5-5 e) (CONT'D)

Solution is $(x_1, x_2, x_3) = \left(\frac{50000}{6}, \frac{20000}{3}, 0\right)$, in units of 1000 acres, with $z = \frac{35,000,000}{3} M_2 + \frac{50,000}{3}$.

(F) With only $M_1 Y_2^-$ in the objective function, we set $Y_2^- = 0$, $z = 0$. So fix $Y_2^- = 0$ and bring $M_1 Y_1^-$ into the objective function; now $Y_1^- = 11,666,666\ 2/3$. Fix Y_1^- at this value (remembering to subtract from RHS) and optimize for third priority: Now the solution in (c) is reached.

$x_1 = 8333\ 1/3$, $x_2 = 6666\ 2/3$, $Y_1^- = 11,666,666\ 2/3$, $Y_3^- = 16,666\ 2/3$

7.5-6

(a) Minimize $M_1 Y_1^+ + M_2 Y_2^+ + M_2 Y_2^- + Y_3^-$

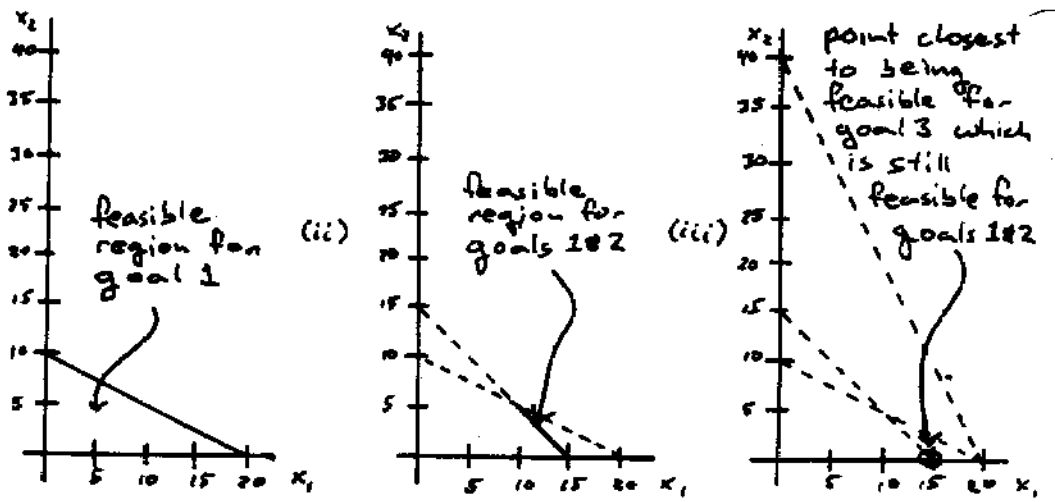
subject to
$$x_1 + 2x_2 - (Y_1^+ - Y_1^-) = 20$$
$$x_1 + x_2 - (Y_2^+ - Y_2^-) = 15$$
$$2x_1 + x_2 - (Y_3^+ - Y_3^-) = 40$$
$$x_1 \ge 0, x_2 \ge 0, Y_1^+ \ge 0, Y_1^- \ge 0, Y_2^+ \ge 0, Y_3^- \ge 0, Y_3^+ \ge 0, Y_3^- \ge 0$$

(b) & (c)

	BV	Eq.#	z	x_1	x_2	Y_1^+	Y_1^-	Y_2^+	Y_2^-	Y_3^+	Y_3^-	RHS
	z	0	-1	$-M_2-2$	$-M_2-1$	M_1	0	$2M_2$	0	1	0	$-15M_2-40$
(0)	Y_1^-	1	0	1	2	-1	1	0	0	0	0	20
	Y_2^-	2	0	1*	1	0	0	-1	1	0	0	15
	Y_3^-	3	0	2	1	0	0	0	0	-1	1	40
	z	0	-1	0	1	M_1	0	M_2-2	M_2+2	1	0	-10
(1)	Y_1^-	1	0	0	1	-1	1	1	-1	0	0	5
	x_1	2	0	1	1	0	0	-1	1	0	0	15
	Y_3^-	3	0	0	-1	0	0	2	-2	-1	1	10

solution is $(x_1, x_2) = (15, 0)$ with $z = 10$.

(d)



7.5-6

(e) The first program: minimize $z_1 = M_1 y_1^+$

subject to

$$x_1 + 2x_2 - (y_1^+ - y_1^-) = 20$$
$$x_1 + x_2 - (y_2^+ - y_2^-) = 15$$
$$2x_1 + x_2 - (y_3^+ - y_3^-) = 40$$
$$x_1 \geq 0,\ x_2 \geq 0$$

from (iii) the solution is the feasible region in (i) above

Similarly, we let $y_1^+ = 0$ and take $z_2 = M_2 y_3^+ + M_2 y_2^-$ and the resulting program yields the feasible region in (ii).

Finally, letting $y_1^+ = y_2^+ = y_2^- = 0$ and $z_3 = y_3^-$ we get the solution $(15, 0)$, $z_3 = 10$.

7.5-7

(a) Minimize $M_1 y_1^+ + M_2 y_2^- + y_3^-$

subject to

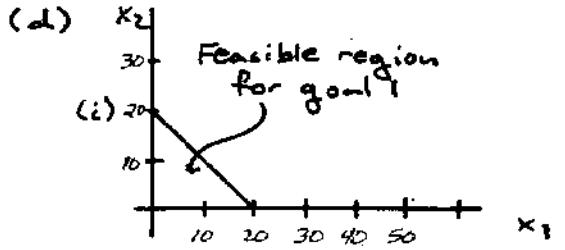
$$x_1 + x_2 - (y_1^+ - y_1^-) = 20$$
$$x_1 + x_2 - (y_2^+ - y_2^-) = 30$$
$$x_1 + 2x_2 - (y_3^+ - y_3^-) = 50$$
$$x_1 \geq 0,\ x_2 \geq 0,\ y_i^+ \geq 0,\ y_i^- \geq 0 \quad i = 1, 2, 3$$

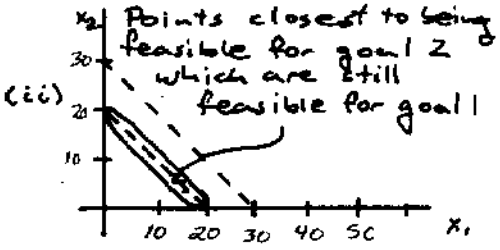
(b) & (c)

BV	Eq.#	z	x_1	x_2	y_1^+	y_1^-	y_2^+	y_2^-	y_3^+	y_3^-	RHS
z	0	-1	$-M_1-1$	$-M_1-2$	M_1	0	M_2	0	1	0	$-30M_2-50$
y_1^-	1	0	1	1 ✱	-1	1	0	0	0	0	20
y_2^-	2	0	1	1	0	0	-1	1	0	0	30
y_3^-	3	0	1	2	0	0	0	0	-1	1	50
z	0	-1	1	0	M_1-M_2-2	M_2+2	M_2	0	1	0	$-10M_2-10$
x_2	1	0	1	1	-1	1	0	0	0	0	20
y_2^-	2	0	0	0	1	-1	-1	1	0	0	10
y_3^-	3	0	-1	0	2	-2	0	0	-1	1	10

So solution is $(x_1, x_2) = (0, 20)$ with $z = 10M_2 + 10$

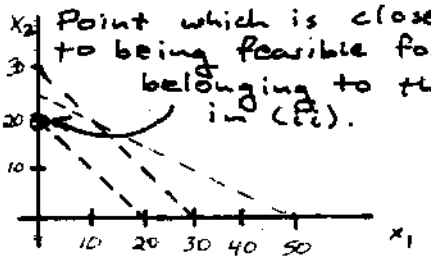
(d)





(CONT.)

7.5-7 (CONT.)

(iii) 

Point which is closest to being feasible for goal 3 belonging to the set in (ii).

(e) Again, the sequential programming method yields the above regions as well.

7.5-8

Note that if $z_i = z_i^+ - z_i^-$ where $z_i^+ \geq 0, z_i^- \geq 0$ then $|z_i| = z_i^+ + z_i^-$

(a) Minimize $\sum_{i=1}^{n} (z_i^+ + z_i^-)$
subject to $z_i^+ - z_i^- = y_i - (a + bx_i)$ for $i = 1, 2, \ldots, n$
$z_i^+ \geq 0, z_i^- \geq 0$ for $i = 1, 2, \ldots, n$

(b) Minimize z
subject to $z_i^+ - z_i^- = y_i - (a + bx_i)$ for $i = 1, 2, \ldots, n$
$0 \leq z_i^+ \leq z$ for $i = 1, 2, \ldots, n$
$0 \leq z_i^- \leq z$ for $i = 1, 2, \ldots, n$

7-1 a) We can apply goal programming to Mr. Baker's problem

b) We need to develop a goal programming problem whose solution characterizes Mr. Baker's shipping policy. We define the following variables:

Basic = Number (in 1000's) of basic packages sent
Advanced = Number (in 1000's) of advanced packages sent
Supreme = Number (in 1000's) of supreme packages sent
Doctors = Number of doctors sent

For Mr. Baker's goals we define the following variables:

V-cost = Amount (in 1000's of dollars) over the cost goal
U-cost = Amount (in 1000's of dollars) under the cost goal
V-people = Number (in 1000's) of people over the people goal
U-people = Number (in 1000's) of people under the people goal
V-sup = Number (in 1000's) of supreme packages over the supreme goal
U-sup = Number (in 1000's) of supreme packages under the supreme goal

Note: Measuring most variables in 1000's greatly improves the reliability of the Excel Solver.

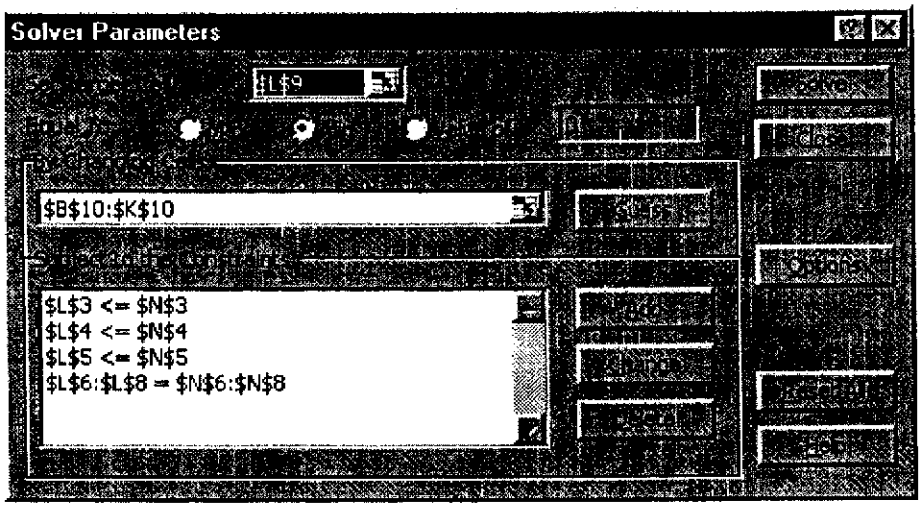
Mr. Baker faces three hard constraints. First, the sum of the package variables (Basic + Advanced + Supreme) must not exceed 40 (Note, packages are measured in 1000's). Second, the weight of all packages combined (120*Basic + 180*Advanced + 220*Supreme) must be less than or equal to 6000 (since packages are measured in 1000's, we divide also the total weight by 1000). Third, we need to incorporate the safety restriction Supreme <= 0.1*Doctors.

In addition, we need to include three constraints for Mr. Baker's goals. For example, for his cost goal the constraint equals

300*Basic + 350*Advanced + 720*Supreme + U-cost – V-cost = 20000.

Three deviations from Mr. Baker's goals are penalized resulting in an objective function of 0.001*V-cost + 0.07*U-people + U-sup

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2	Server	Basic	Advanced	Supreme	Doctors	U-cost	V-cost	U-people	V-people	U-sup	V-sup	Totals		
3	packets	1	1	1	0	0	0	0	0	0	0	40	<=	40
4	weight	120	180	220	0	0	0	0	0	0	0	6000	<=	6000
5	safety	0	0	1	-0.1	0	0	0	0	0	0	0	<=	0
6	cost	300	350	720	33	1	-1	0	0	0	0	20000	=	20000
7	people	30	35	54	0	0	0	1	-1	0	0	2200	=	2200
8	supreme	0	0	1	0	0	0	0	0	1	-1	3	=	3
9	Points	0	0	0	0	0	0.001	0.07	0	1	0	**50.84**		
10	Solution	**28**	**0**	**12**	**120**	**0**	**1000**	**712**	**0**	**0**	**9**			
11														
12														
13			Formula in cell L3:			"=SUMPRODUCT(B3:K3,B10:K10)"								
14			Formula in cell L4:			"=SUMPRODUCT(B4:K4,B10:K10)"								
15			Formula in cell L5:			"=SUMPRODUCT(B5:K5,B10:K10)"								
16			Formula in cell L6:			"=SUMPRODUCT(B6:K6,B10:K10)"								
17			Formula in cell L7:			"=SUMPRODUCT(B7:K7,B10:K10)"								
18			Formula in cell L8:			"=SUMPRODUCT(B8:K8,B10:K10)"								
19			Formula in cell L9:			"=SUMPRODUCT(B9:K9,B10:K10)"								



Mr. Baker should send 28,000 basic packages and 12,000 supreme packages along with 120 doctors to Cuba.

c) The coefficient of U-people in the objective function changes. The new penalty weight for being below the people goal equals 10/55.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2	Server	Basic	Advanced	Supreme	Doctors	U-cost	V-cost	U-people	V-people	U-sup	V-sup	Totals		
3	packets	1	1	1	0	0	0	0	0	0	0	40	<=	40
4	weight	120	180	220	0	0	0	0	0	0	0	6000	<=	6000
5	safety	0	0	1	-0.1	0	0	0	0	0	0	0	<=	0
6	cost	300	350	720	33	1	-1	0	0	0	0	20000	=	20000
7	people	30	35	54	0	0	0	1	-1	0	0	2200	=	2200
8	supreme	0	0	1	0	0	0	0	0	1	-1	3	=	3
9	Points	0	0	0	0	0	0.001	0.18182	0	1	0	130.4545		
10	Solution	**28**	**0**	**12**	**120**	**0**	**1000**	**712**	**0**	**0**	**9**			
11														
12														
13			Formula in cell L3:			"=SUMPRODUCT(B3:K3,B10:K10)"								
14			Formula in cell L4:			"=SUMPRODUCT(B4:K4,B10:K10)"								
15			Formula in cell L5:			"=SUMPRODUCT(B5:K5,B10:K10)"								
16			Formula in cell L6:			"=SUMPRODUCT(B6:K6,B10:K10)"								
17			Formula in cell L7:			"=SUMPRODUCT(B7:K7,B10:K10)"								
18			Formula in cell L8:			"=SUMPRODUCT(B8:K8,B10:K10)"								
19			Formula in cell L9:			"=SUMPRODUCT(B9:K9,B10:K10)"								

The optimal shipping policy did not change. The plan appears to be very insensitive to drastic increases in the penalty weight for violating the goal to reach at least 20% of the Cuban population.

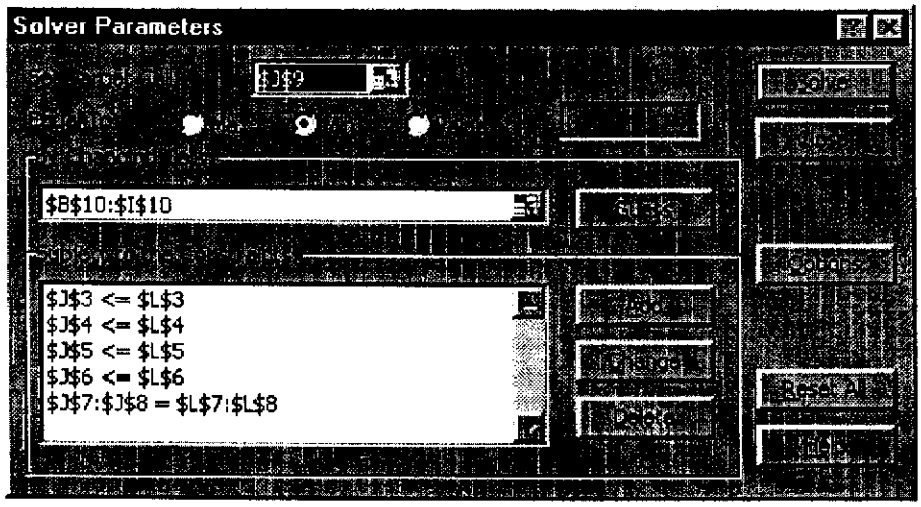
d) The coefficient of Doctors in the safety constraint changes from –0.1 to –0.075.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2	Server	Basic	Advanced	Supreme	Doctors	U-cost	V-cost	U-people	V-people	U-sup	V-sup	Totals		
3	packets	1	1	1	0	0	0	0	0	0	0	40	<=	40
4	weight	120	180	220	0	0	0	0	0	0	0	6000	<=	6000
5	safety	0	0	1	-0.075	0	0	0	0	0	0	0	<=	0
6	cost	300	350	720	33	1	-1	0	0	0	0	20000	=	20000
7	people	30	35	54	0	0	0	1	-1	0	0	2200	=	2200
8	supreme	0	0	1	0	0	0	0	0	1	-1	3	=	3
9	Points	0	0	0	0	0	0.001	0.18182	0	1	0	131.7745		
10	Solution	**28**	**0**	**12**	**160**	**0**	**2320**	**712**	**0**	**0**	**9**			
11														
12														
13			Formula in cell L3:			"=SUMPRODUCT(B3:K3,B10:K10)"								
14			Formula in cell L4:			"=SUMPRODUCT(B4:K4,B10:K10)"								
15			Formula in cell L5:			"=SUMPRODUCT(B5:K5,B10:K10)"								
16			Formula in cell L6:			"=SUMPRODUCT(B6:K6,B10:K10)"								
17			Formula in cell L7:			"=SUMPRODUCT(B7:K7,B10:K10)"								
18			Formula in cell L8:			"=SUMPRODUCT(B8:K8,B10:K10)"								
19			Formula in cell L9:			"=SUMPRODUCT(B9:K9,B10:K10)"								

While the number of packages Mr. Baker should ship has not changed, the number of doctors is now 160.

e) The budget restriction is now a hard constraint and the penalty variables for the cost goal can be eliminated.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	Basic	Advanced	Supreme	Doctors	U-people	V-people	U-sup	V-sup	Totals		
3	packets	1	1	1	0	0	0	0	0	40	<=	40
4	weight	120	180	220	0	0	0	0	0	6000	<=	6000
5	safety	0	0	1	-0.1	0	0	0	0	0	<=	0
6	cost	300	350	720	33	0	0	0	0	20000	<=	20000
7	people	30	35	54	0	1	-1	0	0	2200	=	2200
8	supreme	0	0	1	0	0	0	1	-1	3	=	3
9	Points	0	0	0	0	0.07	0	1	0	**51.485**		
10	Solution	**27**	**2.5**	**10.5**	**105**	**735.5**	**0**	**0**	**7.5**			
11												
12												
13			Formula in cell L3:			"=SUMPRODUCT(B3:I3,B10:I10)"						
14			Formula in cell L4:			"=SUMPRODUCT(B4:I4,B10:I10)"						
15			Formula in cell L5:			"=SUMPRODUCT(B5:I5,B10:I10)"						
16			Formula in cell L6:			"=SUMPRODUCT(B6:I6,B10:I10)"						
17			Formula in cell L7:			"=SUMPRODUCT(B7:I7,B10:I10)"						
18			Formula in cell L8:			"=SUMPRODUCT(B8:I8,B10:I10)"						
19			Formula in cell L9:			"=SUMPRODUCT(B9:I9,B10:I10)"						



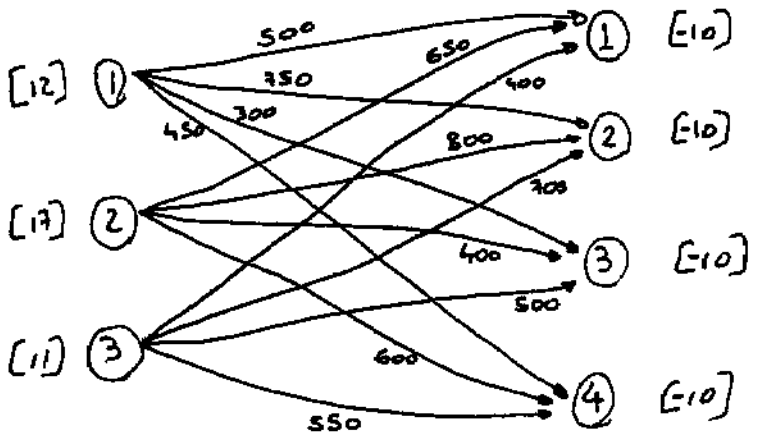
Mr. Baker should send 27,000 basic packages, 2,500 advanced packages, and 10,500 supreme packages along with 105 doctors to Cuba.

8.1-1

a)

		Unit Cost ($) Destination (Distribution Center) 1	2	3	4	Supply
Source (Plant)	1	500	750	300	450	12
	2	650	800	400	600	17
	3	400	700	500	550	11
Demand		10	10	10	10	

b)



c)

Shipment Quantities
Destination (Distribution Center)

		1	2	3	4	Totals		Supply
Source (Plant)	1	**0**	**0**	**2**	**10**	12	=	12
	2	**0**	**9**	**8**	**0**	17	=	17
	3	**10**	**1**	**0**	**0**	11	=	11
Totals		10	10	10	10			
		=	=	=	=	Total Cost =		**$ 20,200**
Demand		10	10	10	10			

8.1-2(a) Let X1 be the number of pints purchased from Dick today,
X2 be the number of pints purchased from Dick tomorrow,
X3 be the number of pints purchased from Harry today and
X4 be the number of pints purchased from Harry tomorrow.

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8.1-2 (cont'd)

The linear program is:

Min Z = 3 X1 + 2.7 X2 + 2.9 X3 + 2.8 X4

subject to

$$
\begin{aligned}
1\,X1 + 1\,X2 + 0\,X3 + 0\,X4 &\le 5\\
0\,X1 + 0\,X2 + 1\,X3 + 1\,X4 &\le 4\\
1\,X1 + 0\,X2 + 1\,X3 + 0\,X4 &= 3\\
0\,X1 + 1\,X2 + 0\,X3 + 1\,X4 &\ge 4
\end{aligned}
$$

and

X1 >= 0, X2 >= 0, X3 >= 0, X4 >= 0.

INITIAL TABLEAU

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	Right side
			-1M	-1M	-1M	-1M	1M					-7M
Z	0	-1	3	2.7	2.9	2.8	0	0	0	0	0	0
X6	1	0	1	1	0	0	0	1	0	0	0	5
X7	2	0	0	0	1	1	0	0	1	0	0	4
X8	3	0	1	0	1	0	0	0	0	1	0	3
X9	4	0	0	1	0	1	-1	0	0	0	1	4

(b)

	Destination 1 (Today)	2 (Tomorrow)	3 (Dummy)	Supply
Dick 1	3	2.7	0	5
Harry 2	2.9	2.8	0	4
Demand	3	4	2	

c)

	Destination 1	2	3	Supply
Source 1		4	1	5
Source 2	3		1	4
Demand	3	4	2	Cost is 19.5

8.1-3

a)

	Cost Per Unit Distributed, Destination (product) 1	2	3	dummy	Supply
Source (plant) 1	31	45	38	0	400
Source (plant) 2	29	41	35	0	600
Source (plant) 3	32	46	40	0	400
Source (plant) 4	28	42	1M	0	600
Source (plant) 5	29	43	1M	0	1000
Demand	600	1000	800	600	

8.1-3 (CONT'D)

b)

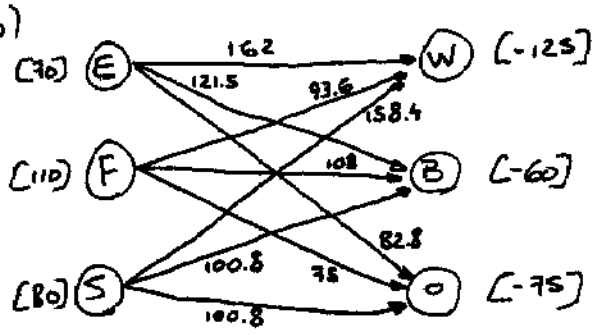
		Destination				
		1	2	3	4	Supply
Source	1			200	200	400
	2			600		600
	3				400	400
	4	600				600
	5		1000			1000
Demand		600	1000	800	600	Cost is 88400

8.1-4

a)

		Unit Cost ($) Destination			
		Wheat	Barley	Oats	Supply
Source	England	162	121.5	82.8	70
	France	93.6	108	75	110
	Spain	158.4	100.8	100.8	80
Demand		125	60	75	

b)



c)

		Allocation Quantities Destination					
		Wheat	Barley	Oats	Totals		Supply
Source	England	**0**	**0**	**70**	70	=	70
	France	**110**	**0**	**0**	110	=	110
	Spain	**15**	**60**	**5**	80	=	80
Totals		125	60	75			
		=	=	=	Total Cost	=	**$ 25,020**
Demand		125	60	75			

8.1-5

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$D$15	Bellingham Sacramento	0	14.9999999	464	1E+30	15
$E$15	Bellingham Sa t Lake City	20	0	513	15	21
$F$15	Bellingham Rapid City	0	83.9999996	654	1E+30	84
$G$15	Bellingham Albuquerque	55	0	867	21	351
$D$16	Eugene Sacramento	80	0	352	15	1E+30
$E$16	Eugene Salt Lake City	45	0	416	21	15
$F$16	Eugene Rapid City	0	217	690	1E-30	217
$G$16	Eugene Albuquerque	0	20.9999997	791	1E-30	21
$D$17	Albert Lea Sacramento	0	728	995	1E-30	728
$E$17	Albert Lea Salt Lake City	0	351	682	1E-30	351
$F$17	Albert Lea Rapid City	70	0	388	84	1E+30
$G$17	Albert Lea Albuquerque	30	0	685	351	84

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$18	Totals Sacramento	80	267	80	0	20
$E$18	Totals Salt Lake City	65	331	65	0	20
$F$18	Totals Rapid City	70	388	70	0	70
$G$18	Totals Albuquerque	85	685	85	0	30
$H$15	Bellingham Totals	75	182	75	30	0
$H$16	Eugene Totals	125	85	125	20	0
$H$17	Albert Lea Totals	100	0	100	0	1E+30

		Range of Optimality			
		Destination			
		Sacramento	Salt Lake City	Rapid City	Albuquerque
Source	Bellingham	449 to ∞	492 to 528	570 to ∞	516 to 888
	Eugene	-∞ to 367	401 to 437	473 to ∞	770 to ∞
	Albert Lea	267 to ∞	331 to ∞	-∞ to 472	601 to 1036

These ranges tell management how much each individual cost can change without changing the optimal solution.

8.1-6.

a) Customer 5 represents the amount sent to customer 3 in excess of 20 units. Plant 4 is a "dummy" plant which "sells" units to customers 5 and 4 that they don't really receive.

(CONT'D)

8.1-6 (cont'd)

a), c) & d)

Profit per unit distributed

		Customer 1	2	3	4	5	Supply
	1	800	700	500	200	500	60
Plant	2	500	200	100	300	100	80
	3	600	400	300	500	300	40
	4	1.00E+06	1.00E+06	1.00E+06	0	0	60
	Demand	40	60	20	60	60	

Shipments

		Customer 1	2	3	4	5	Supply
	1		60				60
Plant	2	40			40		80
	3			20	20		40
	4					60	60
	Demand	40	60	20	60	60	Profit is 90000

b) & e)

Cost per unit distributed

		Customer 1	2	3	4	5	Supply
	1	-800	-700	-500	-200	-500	60
Plant	2	-500	-200	-100	-300	-100	80
	3	-600	-400	-300	-500	-300	40
	4	1.00E+06	1.00E+06	1.00E+06	0	0	60
	Demand	40	60	20	60	60	

Shipments

		Customer 1	2	3	4	5	Supply
	1		60				60
Plant	2	40			40		80
	3			20	20		40
	4					60	60
	Demand	40	60	20	60	60	Cost is -90000

The profit is equal to minus the cost

8.1-7

a) & b)

		Distribution center 1	2	3	4 Dummy	Supply
Plant	1	800	700	400	0	50
	2	600	800	500	0	50
	Demand	20	20	20	40	

c)

		Distribution center 1	2	3	4 Dummy	Supply
Plant	1		20	20	10	50
	2	20			30	50
	Demand	20	20	20	40	Cost is 34000

8.1-8 a) and b)

Let destination 2i-1 represent the demand of 10 at center i, and destination 2i is the extra demand up to 20 shipped to center i. For i = 1, 2, 3.

Cost per unit distributed

		Destination 1	2	3	4	5	6	7 Dummy	Supply
plant	1	800	800	700	700	400	400	0	50
plant	2	600	600	800	800	500	500	0	50
dummy	3	1.00E+06	0	1.00E+06	0	1.00E+06	0	1.00E+06	30
	Demand	10	20	10	20	10	20	40	

8.1-8 c)

Cost per unit distributed

		Destination 1	2	3	4	5	6	7 Dummy	Supply
plant	1			10		10	20	10	50
plant	2	10	10					30	50
dummy	3		10		20				30
	Demand	10	20	10	20	10	20	40	Cost is 31000

8.1-9 a) & b)

source 1 is initial inventory
souce 2 is week 1 regular hours
source 3 is week 1 overtime
source 4 is week 2 regular hours
source 5 week 2 overtime
source 6 is week 3 regular hours
source 7 is week 3 overtime

		week 1	2	3	4 dummy	supply
	1	0	50	100	0	2
	2	300	350	400	0	2
	3	400	450	500	0	2
source	4	1.00E-06	500	550	0	3
	5	1.00E+06	600	650	0	2
	6	1.00E+06	1.00E+06	400	0	1
	7	1.00E+06	1.00E+06	500	0	2
	demand	3	3	3	5	

		1	2	3	4 dummy	supply
	1	2	0	0	0	2
	2	1	1	0	0	2
	3	0	2	0	0	2
source	4	0	0	0	3	3
	5	0	0	0	2	2
	6	0	0	1	0	1
	7	0	0	2	0	2
	demand	3	3	3	5	Total cost is 2950

8.1-10

a)

Sources:
1 week 1 RT
2 week 1 OT
3 week 2 RT
4 week 2 OT
5 week 3 RT
6 week 3 OT

destinations:
1 product 1, week 1
2 product 2, week 1
3 product 1, week 2
4 product 2, week 2
5 product 1, week 3
6 product 2, week 3
7 dummy

Cost Per Unit Distributed (in $1,000's)

		Destination							
		1	2	3	4	5	6	7	Supply
Source	1	15	15	16	18	18	19	0	10
	2	18	20	19	22	21	23	0	3
	3	1M	1M	17	15	19	16	0	8
	4	1M	1M	20	18	22	19	0	2
	5	1M	1M	1M	1M	19	17	0	10
	6	1M	1M	1M	1M	22	22	0	3
Demand		5	3	3	5	4	4	12	

b)

		Destination							
		1	2	3	4	5	6	7	Supply
Source	1	5	3	2					10
	2							3	3
	3			1	5		2		8
	4							2	2
	5					4	2	4	10
	6							3	3
Demand		5	3	3	5	4	4	12	Cost is 389,000

No overtime is necessary.

8.2-1

a)

	Destination 1	2	3	Supply	Row Difference
1	6	3	5	4	2
Source 2	4	M	7	3	3 ←
3	3	4	3	2	0
Demand	4	2	3		
Column Difference	1	1	2		

Vogel's method would choose x_{21} as the first basic variable.

b)

	Destination 1	2	3	Supply	Row Maximum
1	6 / -6	3 / -M-3	5 / -8	4	6
2	4 / -M-2	M / -M	7 / -M	3	M
3	3 / -7	4 / -M	3 / -8	2	4
Demand	4	2	3		
Column Maximum	6	M	7		

Russell's Method would choose x_{12} as the first basic variable.

c) NW corner rule initial BF solution:

4				4
0	2	1		3
		2		2
4	2	3		

(after the first step, we eliminate the row, as step 3 says)

8.2-2

(a) NW corner rule

	Destination 1	2	3	4	5	Supply
1	2 / 4 B	4 / 0 B	6 / 0	5 / 0	7 / 0	4
2	7 / 0	6 / 4 B	3 / 2 B	M / 0 B	4 / 0	6
3	8 / 0	7 / 0	5 / 0	2 / 5 B	5 / 1 B	6
4	0 / 0	0 / 0	0 / 0	0 / 0	0 / 4 B	4
Demand	4	4	2	5	5	

Cost = 53

(b) Vogel's

	Destination 1	2	3	4	5	Supply
1	2 / 4 B	4 / 0	6 / 0	5 / 0	7 / 0	4
2	7 / 0	6 / 0	3 / 2 B	M / 0	4 / 4 B	6
3	8 / 0 B	7 / 0 B	5 / 0	2 / 5 B	5 / 1 B	6
4	0 / 0	0 / 4 B	0 / 0	0 / 0	0 / 0	4
Demand	4	4	2	5	5	

Cost = 45

8.2.2 (c)

	Destination 1	2	3	4	5	Supply	u[i]
1	2	4	6	5	7		
	B						
	4	2	7	8	7	4	-5
2	7	6	3	M	4		
			B		B		
	1	0	2	1M- 1	4	6	-1
3	8	7	5	2	5		
		B		B	B		
	1	0	1	5	1	6	0
4	0	0	0	0	0		
	B	B					
	0	4	3	5	2	4	-7
Demand	4	4	2	5	5		
v[j]	7	7	4	2	5		

Cost = 45

Note that the initial BF solutions found by Vogel's and by Russell's methods are optimal.

8.2-3

a) Method used to construct initial basic feasible solution: Northwest Corner Rule

Initial Basic Feasible Solution:

Source	Destination 1	2	3	4	5	6	Supply
1	13 (3)	10 (2)	22	29	18	0	5
2	14	13 (3)	16 (3)	21	1M	0	6
3	3	0	1M (1)	11 (5)	6 (1)	0	7
4	18	9	19	23	11 (4)	0	4
5	30	24	34	36	28 (1)	0 (2)	3
Demand	3	5	4	5	6	2	

Cost = M+279

b) Method used to construct initial basic feasible solution: Vogel's Method

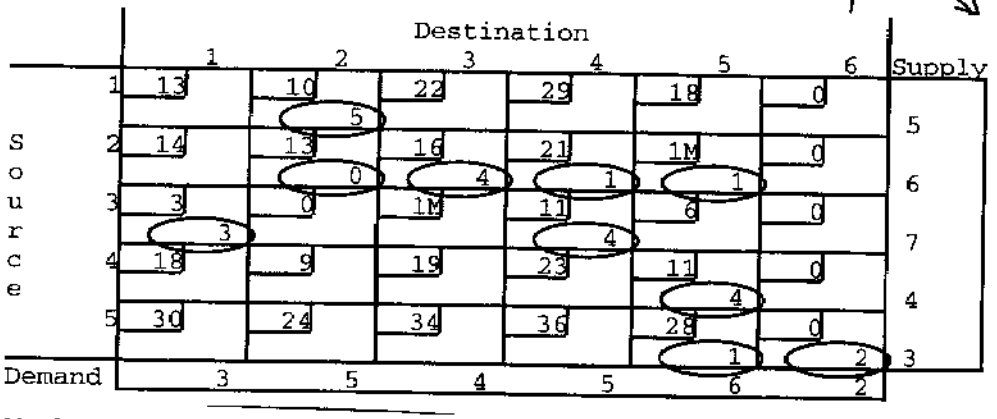
Initial Basic Feasible Solution:

Source	Destination 1	2	3	4	5	6	Supply
1	13	10 (3)	22	29	18 (2)	0	5
2	14 (1)	13 (1)	16 (4)	21	1M	0	6
3	3 (2)	0	1M	11 (5)	6	0	7
4	18	9	19	23	11 (4)	0	4
5	30	24 (1)	34	36	28	0 (2)	3
Demand	3	5	4	5	6	2	

Cost = 296

8.2-3 b) Method used to construct initial basic feasible solution: Vogel's Method

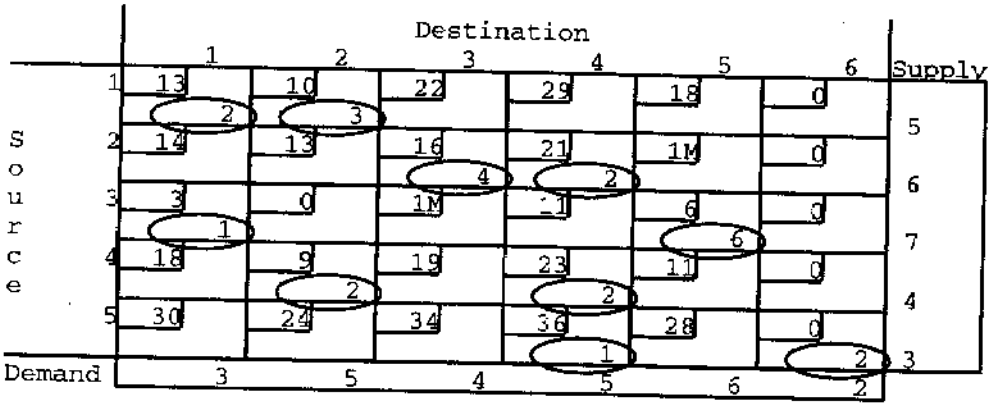
Initial Basic Feasible Solution: (arbitrarily breaking ties differently) ↘ !



Cost = M+260

c) Method used to construct initial basic feasible solution: Russell's Method

Initial Basic Feasible Solution:



Cost = 301

(8.2-4 is on Next Page)

8.2-5

				u_i
464 15	513 (20)	654 84	867 (55)	182
352 (80)	416 (45)	690 217	791 21	85
995 728	682 351	388 (70)	685 (30)	0
v_j	267	331	388	685

Cost = 152,535.

$c_{ij} - u_i - v_j \geq 0$ for all i and j
so the solution is optimal.

8.2-4

(a) The Integer Solutions Property guarantees that, since supplies and demands are integers, the resulting basic feasible solutions will be integral. Supplies and demands are 1, so the only possible values of variables in a basic feasible solution are 0 or 1. The 1's indicate the assignment of a source to a destination.

(b) There are 7 basic variables in every basic feasible solution and 3 of them are degenerate.

(d)

	Destination 1	2	3	4	Supply	u[i]
1	7 | ---- | 0	4 | ---- | 0	1 | ---- B | 1	4 | ---- | 0	1	0
2	4 | ---- | 0	6 | ---- | 0	7 | ---- | 0	2 | ---- B | 1	1	0
3	8 | ---- | 0	5 | ---- B | 1	4 | ---- | 0	6 | ---- | 0	1	0
4	6 | ---- B | 1	7 | ---- B | 0	6 | ---- B | 0	3 | ---- B | 0	1	0
Demand	1	1	1	1		
v[j]	0	0	0	0		

The variables are chosen in the order,
x_{13}, x_{24}, x_{44}, x_{32}, x_{41}, x_{43}, x_{42}

(c) and (e)

(c)

	Destination 1	2	3	4	Supply	u[i]
1	7 | ---- L | 1	4 | ---- | -5	1 | ---- E | -7	4 | ---- | -1	1	2
2	4 | ---- P | 0	6 | ---- P | 1	7 | ---- | 2	2 | ---- | 0	1	-1
3	8 | ---- | 5	5 | ---- P | 0	4 | ---- P | 1	6 | ---- | 5	1	-2
4	6 | ---- | 1	7 | ---- | 0	6 | ---- B | 0	3 | ---- B | 1	1	0
Demand	1	1	1	1		
v[j]	5	7	6	3		

(e)

	Destination 1	2	3	4	Supply	u[i]
1	7 | ---- | 7	4 | ---- | 2	1 | ---- B | 1	4 | ---- | 6	1	1
2	4 | ---- B | 1	6 | ---- B | 0	7 | ---- | 2	2 | ---- | 0	1	5
3	8 | ---- | 5	5 | ---- B | 1	4 | ---- B | 0	6 | ---- | 5	1	4
4	6 | ---- | 1	7 | ---- | 0	6 | ---- B | 0	3 | ---- B | 1	1	6
Demand	1	1	1	1		
v[j]	-1	1	0	-3		

The optimal assignment is (source, destination):
(1,3), (2,1), (3,2), (4,4)

Cost = 13

8.2-6

	Destination 1	2	3	4	5	Supply	u[i]
1	41	27	28	24	0		
	----	---- B	---- B	----	---- B		
	2	30	30	1	15	75	0
2	40	29	M	23	0		
	----	----	----	---- B	---- B		
	1	2	1M-28	15	60	75	0
3	37	30	27	21	0		
	---- B	----	----	---- B	----		
	20	5	1	25	2	45	-2
Demand	20	30	30	40	75		
v[j]	39	27	28	23	0		

In the tableau representing the solution from Sect. 8.3 shown above $c_{ij}-u_i-v_j \geq 0$ for all i and j so the solution is optimal.

8.2-7

(0)

	Destination 1	2	3	4	5	Supply	u[i]
1	8	6	3	7	5		
	----	----	---- B	----	----		
	1	3	20	1	2	20	2
2	5	M	8	4	7		
	---- P	----	----	---- P	----		
	25	1M- 1	7	5	6	30	0
3	6	3	9	6	8		
	----	---- P	----	---- P	----		
	-1	25	6	5	5	30	2
4	0	0	0	0	0		
	---- E	---- L	---- B	----	---- B		
	-4	0	0	-3	20	20	-1
Demand	25	25	20	10	20		
v[j]	5	1	1	4	1		

(1)

	Destination 1	2	3	4	5	Supply	u[i]
1	8	6	3	7	5		
	----	----	---- B	----	----		
	5	7	20	5	2	20	3
2	5	M	8	4	7		
	---- P	----	----	---- P	----		
	25	1M- 1	3	5	2	30	5
3	6	3	9	6	8		
	---- E	---- B	----	---- L	----		
	-1	25	2	5	1	30	7
4	0	0	0	0	0		
	---- B	----	---- B	----	---- B		
	0	4	0	1	20	20	0
Demand	25	25	20	10	20		
v[j]	0	-4	0	-1	0		

(2)

	Destination 1	2	3	4	5	Supply	u[i]
1	8	6	3	7	5		
	----	----	---- B	----	----		
	5	6	20	5	2	20	3
2	5	M	8	4	7		
	---- B	----	----	---- B	----		
	20	1M- 2	3	10	2	30	5
3	6	3	9	6	8		
	---- B	---- B	----	----	----		
	5	25	3	1	2	30	6
4	0	0	0	0	0		
	---- B	----	---- B	----	---- B		
	0	3	0	1	20	20	0
Demand	25	25	20	10	20		
v[j]	0	-3	0	-1	0		

The current solution is optimal:

$x_{13}=20$, $x_{21}=20$, $x_{24}=10$,

$x_{31}=5$, $x_{32}=25$ and $x_{45}=20$

Cost = 305.

The optimality condition is $c_{ij}-u_i-v_j \geq 0$ which has been met.

8.2-8

a) Method used to construct initial basic feasible solution: Northwest Corner Rule

Interactive Transportation Simplex Method:

(0)

Source	1	2	3	4	Supply	u_i
1	3	7	6	4		
	3	2	0	1	5	0
2	2	4	3	2		
	2	1	1	2	2	-3
3	4	3	8	5		
	-1	+	1	2	3	2
Demand	3	3	2	2		
v_j	3	7	6	3	Z = 48	

(1)

Source	1	2	3	4	Supply	u_i
1	3	7	6	4		
	3	2	0	+	5	7
2	2	4	3	2		
	2	0	2	-4	2	4
3	4	3	8	5		
	5	+	6	-	3	3
Demand	3	3	2	2		
v_j	-4	0	-1	2	Z = 42	

(2)

Source	1	2	3	4	Supply	u_i
1	3	7	6	4		
	3	5	5	2	5	0
2	2	4	3	2		
	-3	0	2	+	2	2
3	4	3	8	5		
	0	+	6	-	3	1
Demand	3	3	2	2		
v_j	3	2	1	4	Z = 32	

(3)

Source	1	2	3	4	Supply	u_i
1	3	7	6	4		
	3	1	1	2	5	2
2	2	4	3	2		
	1	0	2	0	2	0
3	4	3	8	5		
	4	3	6	4	3	-1
Demand	3	3	2	2		
v_j	1	4	3	2	Z = 32	

3 iterations are required to reach optimality.

8.2-8 cont.

b) Vogel's

Source	Destination 1	2	3	4	Supply	u_i
1	3	7	6	4	5	0
2	2	4	3	2	2	-3
3	4	3	8	5	3	-4
Demand	3	3	2	2		
v_j	3	7	6	4	Z = 32	

c) Russell's

Source	Destination 1	2	3	4	Supply	u_i
1	3	7	6	4	5	0
2	2	4	3	2	2	-3
3	4	3	8	5	3	-4
Demand	3	3	2	2		
v_j	3	7	6	4	Z = 32	

Both of these initial BF solutions are optimal, so zero iterations of network simplex are required.

8.2-9 a)

		Unit Cost ($) Destination (Retail Outlet) 1	2	3	4	Supply
Source (Plant)	1	500	600	400	200	10
	2	200	900	100	300	20
	3	300	400	200	100	20
	4	200	100	300	200	10
Demand		20	10	10	20	

b & c)

(0)

	Destination 1	2	3	4	Supply	u[i]
1	5 / ---- P / 10	6 / ---- / -1M+ 3	4 / ---- / -1N+ 3	2 / ---- E / -1N+ 2	10	1M + 3
2	2 / ---- P / 10	M / ---- L / 10	1 / ---- / -1N+ 3	3 / ---- / -1M+ 6	20	1M
3	3 / ---- / 1N- 3	4 / ---- P / 0	2 / ---- B / 10	1 / ---- P / 10	20	4
4	2 / ---- / 1N- 5	1 / ---- / -4	3 / ---- / 0	2 / ---- B / 10	10	5
Demand	20	10	10	20		
v[j]	-1M + 2	0	-2	-3		

(1)

	Destination 1	2	3	4	Supply	u[i]
1	5 / ---- B / 0	6 / ---- / 1	4 / ---- / 1	2 / ---- B / 10	10	1
2	2 / ---- B / 20	M / ---- / 1M- 2	1 / ---- / 1	3 / ---- / 4	20	-2
3	3 / ---- / -1	4 / ---- L / 10	2 / ---- B / 10	1 / ---- P / 0	20	0
4	2 / ---- / -3	1 / ---- E / -4	3 / ---- / 0	2 / ---- P / 10	10	1
Demand	20	10	10	20		
v[j]	4	4	2	1		

cont.

345

8.2-9
b) and c) (cont.)

(2)

	Destination 1	2	3	4	Supply	u[i]
1	5 | ---- P | 0	6 | ---- | 5	4 | ---- | 1	2 | ---- P | 10	10	2
2	2 | ---- B | 20	M | 1M+ 2	1 | ---- | 1	3 | ---- | 4	20	-1
3	3 | ---- | -1	4 | ---- | 4	2 | ---- B | 10	1 | ---- B | 10	20	1
4	2 | ---- E | -3	1 | ---- B | 10	3 | ---- | 0	2 | ---- L | 0	10	2
Demand	20	10	10	20		
v[j]	3	-1	1	0		

(3)

	Destination 1	2	3	4	Supply	u[i]
1	5 | ---- L | 0	6 | ---- | 2	4 | ---- | 1	2 | ---- P | 10	10	5
2	2 | ---- B | 20	M | 1M- 1	1 | ---- | 1	3 | ---- | 4	20	2
3	3 | ---- E | -1	4 | ---- | 1	2 | ---- B | 10	1 | ---- P | 10	20	4
4	2 | ---- B | 0	1 | ---- B | 10	3 | ---- | 3	2 | ---- | 3	10	2
Demand	20	10	10	20		
v[j]	0	-1	-2	-3		

(4)

	Destination 1	2	3	4	Supply	u[i]
1	5 | ---- | 1	6 | ---- | 3	4 | ---- | 1	2 | ---- B | 10	10	1
2	2 | ---- B | 20	M | 1M- 1	1 | ---- | 0	3 | ---- | 3	20	-1
3	3 | ---- B | 0	4 | ---- | 2	2 | ---- B | 10	1 | ---- B | 10	20	0
4	2 | ---- B | 0	1 | ---- B | 10	3 | ---- | 2	2 | ---- | 2	10	-1
Demand	20	10	10	20		
v[j]	3	2	2	1		

The optimal solution is:

$x_{14} = 10$, $x_{21} = 20$, $x_{33} = 10$,

$x_{34} = 10$, $x_{42} = 10$

Cost = 100.

8.2-10

(a)

	1 Electricity	2 Water	3 Space	4 Dummy	Supply
Electricity 1	50	90	80	0	60
Nat. Gas 2	M	60	50	0	40
Solar 3	M	30	40	0	30
Demand	20	10	30	70	

(b) and (c) 8.2-10

(0)

	Destination 1	2	3	4	Supply	u[i]
	50	90	80	0		
1	---- B	---- P	---- P	----		
	20	10	30	-30	60	0
	M	60	50	0		
2	----	----	---- L	---- P		
	1M-20	0	0	40	40	-30
	M	30	40	0		
3	----	---- E	----	---- P		
	1M-20	-30	-10	30	30	-30
Demand	20	10	30	70		
v[j]	50	90	80	30		

(1)

	Destination 1	2	3	4	Supply	u[i]
	50	90	80	0		
1	---- B	---- L	---- B	---- E		
	20	10	30	-60	60	0
	M	60	50	0		
2	----	----	----	---- B		
	1M+10	30	30	40	40	-60
	M	30	40	0		
3	----	---- P	----	---- P		
	1M+10	0	20	30	30	-60
Demand	20	10	30	70		
v[j]	50	90	80	60		

(2)

	Destination 1	2	3	4	Supply	u[i]
	50	90	80	0		
1	---- B	----	---- P	---- P		
	20	60	30	10	60	0
	M	60	50	0		
2	----	----	----	---- B		
	1M-50	30	-30	40	40	0
	M	30	40	0		
3	----	---- B	---- E	---- L		
	1M-50	10	-40	20	30	0
Demand	20	10	30	70		
v[j]	50	30	80	0		

(3)

	Destination 1	2	3	4	Supply	u[i]
	50	90	80	0		
1	---- B	----	---- L	---- P		
	20	20	10	30	60	0
	M	60	50	0		
2	----	----	---- E	---- P		
	1M-50	-10	-30	40	40	0
	M	30	40	0		
3	----	---- B	---- B	----		
	1M-10	10	20	40	30	-40
Demand	20	10	30	70		
v[j]	50	70	80	0		

(4)

	Destination 1	2	3	4	Supply	u[i]
	50	90	80	0		
1	---- B	----	----	---- B		
	20	50	30	40	60	0
	M	60	50	0		
2	----	----	---- B	---- B		
	1M-50	20	10	30	40	0
	M	30	40	0		
3	----	---- B	---- B	----		
	1M-40	10	20	10	30	-10
Demand	20	10	30	70		
v[j]	50	40	50	0		

The optimal solution is to meet

20 units of electricity demand with electricity

10 units of space heating demand with Natural Gas

10 units of water heating demand with Solar heating

20 units of space heating with solar heating.

Cost = $2600

8.2-10

The initial BF solution from both Vogel's and Russell's method provides the same optimal solution given in (c).

(f) and (g) Russell's

		Destination 1	2	3	4	Supply	u_i
Source	1	50	90	80	0		
		20	50	30	40	60	0
	2	1M	60	50	0		
		1M-50	20	10	30	40	0
	3	1M	30	40	0		
		1M-40	10	20	10	30	-10
Demand		20	10	30	70		
v_j		50	40	50	0	Z = 2600	

(d) and (e) Vogel's

(c)

	Destination 1	2	3	4	Supply	u[i]
	50	90	80	0		
1	---- B	----	----	---- B		
	20	50	30	40	60	0
	M	60	50	0		
2	----	----	---- B	---- B		
	1M-50	20	10	30	40	0
	M	30	40	0		
3	----	---- B	---- B	----		
	1M-40	10	20	10	30	-10
Demand	20	10	30	70		
v[j]	50	40	50	0		

(1)

	Destination 1	2	3	4	5	Supply	u[i]
	1.08	1.09	1.11	1.13	0		
1	---- B	---- B	---- B	----	----		
	10	15	0	0.01	0.02	25	0
	M	1.11	1.13	1.14	0		
2	----	----	----	---- B	---- B		
	1M- 1	0	0	5	30	35	0.02
	M	M	1.1	1.11	0		
3	----	----	---- B	---- B	----		
	1M- 1	1M- 1	25	5	0.03	30	-0.01
	M	M	M	1.13	0		
4	----	----	----	---- B	----		
	1M- 1	1M- 1	1M- 1	10	0.01	10	0.01
Demand	10	15	25	20	30		
v[j]	1.08	1.09	1.11	1.12	-0.02		

8.2-11 Using Vogel's Method.

(0)

	Destination 1	2	3	4	5	Supply	u[i]
	1.08	1.09	1.11	1.13	0		
1	---- B	---- B	---- B	----	----		
	10	15	0	0.03	0.02	25	-0.02
	M	1.11	1.13	1.14	0		
2	----	----	----	---- P	---- P		
	1M- 1	0	0	15	20	35	0
	M	M	1.1	1.11	0		
3	----	----	---- B	---- B	----		
	1M- 1	1M- 1	25	5	0.03	30	-0.03
	M	M	M	1.13	0		
4	----	----	----	---- E	---- L		
	1M- 1	1M- 1	1M- 1	-0.01	10	10	0
Demand	10	15	25	20	30		
v[j]	1.1	1.11	1.13	1.14	0		

The optimal solution is to produce

10 engines in month 1 for installation in month 1

15 engines in month 1 for installation in month 2

5 engines in month 2 for installation in month 4

25 engines in month 3 for installation in month 3

5 engines in month 3 for installation in month 4

10 engines in month 4 for installation in month 4

Cost = 77.3

8.2-12

a) (0)

	Destination 1	2	3	4	Supply	u[i]
	500	750	300	450		
1	---- P	---- P	----	----		
	10	2	-50	50	12	0
	650	800	400	600		
2	----	---- P	---- P	----		
	100	8	9	150	17	50
	400	700	500	550		
3	---- E	----	---- L	---- B		
	-250	-200	1	10	11	150
Demand	10	10	10	10		
v[j]	500	750	350	400		

(1)

	Destination 1	2	3	4	Supply	u[i]
	500	750	300	450		
1	---- L	---- B	----	---- E		
	9	3	-50	-200	12	0
	650	800	400	600		
2	----	---- B	---- B	----		
	100	7	10	-100	17	50
	400	700	500	550		
3	---- P	----	----	---- P		
	1	50	250	10	11	-100
Demand	10	10	10	10		
v[j]	500	750	350	650		

(2)

	Destination 1	2	3	4	Supply	u[i]
	500	750	300	450		
1	----	---- L	---- E	---- B		
	200	3	-50	9	12	0
	650	800	400	600		
2	----	---- P	---- P	----		
	300	7	10	100	17	50
	400	700	500	550		
3	---- B	----	----	---- B		
	10	-150	50	1	11	100
Demand	10	10	10	10		
v[j]	300	750	350	450		

(3)

	Destination 1	2	3	4	Supply	u[i]
	500	750	300	450		
1	----	----	---- P	---- P		
	200	50	3	9	12	0
	650	800	400	600		
2	----	---- P	---- P	----		
	250	10	7	50	17	100
	400	700	500	550		
3	---- B	---- E	----	---- L		
	10	-100	100	1	11	100
Demand	10	10	10	10		
v[j]	300	700	300	450		

(4)

	Destination 1	2	3	4	Supply	u[i]
	500	750	300	450		
1	----	----	---- B	---- B		
	100	50	2	10	12	0
	650	800	400	600		
2	----	---- B	---- B	----		
	150	9	8	50	17	100
	400	700	500	550		
3	---- B	---- B	----	----		
	10	1	200	100	11	0
Demand	10	10	10	10		
v[j]	400	700	300	450		

The optimal solution is to send

2 shipments from plant 1 to center 3
10 shipments from plant 1 to center 4
9 shipments from plant 2 to center 2
8 shipments from plant 2 to center 3
10 shipments from plant 3 to center 1
1 shipment from plant 3 to center 2

Total cost: # 30,200.

8.2-13 (0)

	Destination 1	2	3	Supply	u[i]
1	3 | ---- P | 3	2.7 | ---- P | 2	0 | ---- | 0.1	5	0
2	2.9 | ---- E | -0.2	2.8 | ---- L | 2	0 | ---- B | 2	4	0.1
Demand	3	4	2		
v[j]	3	2.7	-0.1		

(1)

	Destination 1	2	3	Supply	u[i]
1	3 | ---- L | 1	2.7 | ---- B | 4	0 | ---- E | -0.1	5	0
2	2.9 | ---- P | 2	2.8 | ---- | 0.2	0 | ---- P | 2	4	-0.1
Demand	3	4	2		
v[j]	3	2.7	0.1		

	Destination 1	2	3	Supply	u[i]
1	3 | ---- | 0.1	2.7 | ---- B | 4	0 | ---- B | 1	5	0
2	2.9 | ---- B | 3	2.8 | ---- | 0.1	0 | ---- B | 1	4	0
Demand	3	4	2		
v[j]	2.9	2.7	0		

The optimal solution is:
purchase 4 pints from Dick tomorrow
and 3 pints from Harry today.

Cost: #19.50

8.2-14

	Destination 1	2	3	4	Supply	u[i]
1	31 | ---- B | 400	45 | ---- | 2	38 | ---- | -1M+37	0 | ---- | -1	400	31
2	29 | ---- B | 200	41 | ---- B | 400	35 | ---- | -1M+36	0 | ---- | 1	600	29
3	32 | ---- | -2	46 | ---- B | 400	40 | ---- | -1M+36	0 | ---- | -4	400	34
4	28 | ---- | -2	42 | ---- B | 200	M | ---- B | 400	0 | ---- | 0	600	30
5	29 | ---- | -1	43 | ---- | 1	M | ---- B | 400	0 | ---- B | 600	1000	30
Demand	600	1000	800	600		
v[j]	0	12	1M - 30	-30		

Z = 800M+61400

(CONT'D)

(CONT'D)

	Destination 1	2	3	4	Supply	u[i]
1	31 | B 400	45 | 2	38 | 1	0 | 1M−37	400	2
2	29 | L 200	41 | B 0	35 | P 400	0 | 1M−35	600	0
3	32 | −2	46 | B 400	40 | 0	0 | 1M−40	400	5
4	28 | −2	42 | B 600	M | 1M−36	0 | 1M−36	600	1
5	29 | E −1M+35	43 | −1M+37	M | P 400	0 | B 600	1000	1M − 35
Demand	600	1000	800	600		
v[j]	29	41	35	−1M + 35	Z = 400M+75800	

	Destination 1	2	3	4	Supply	u[i]
1	31 | B 400	45 | −1M+37	38 | −1M−36	0 | −2	400	2
2	29 | 1M−35	41 | P 0	35 | P 600	0 | 1M−35	600	−1M + 35
3	32 | 1M−37	46 | B 400	40 | 0	0 | 1M−40	400	−1M + 40
4	28 | 1M−37	42 | B 600	M | 1M−36	0 | 1M−36	600	−1M + 36
5	29 | B 200	43 | E −1M+37	M | L 200	0 | B 600	1000	0
Demand	600	1000	800	600		
v[j]	29	1M + 6	1M	0	Z = 200M+82800	

(CONT'D)

(CONT'D)

	Destination 1	2	3	4	Supply	u[i]
1	31 | B 400	45 | 0	38 | −1	0 | −2	400	45
2	29 | 2	41 | B −200	35 | B 800	0 | 2	600	41
3	32 | 0	46 | L 400	40 | 0	0 | E −3	400	46
4	28 | 0	42 | B 600	M | 1M−36	0 | 1	600	42
5	29 | B 200	43 | P 200	M | 1M−37	0 | P 600	1000	43
Demand	600	1000	800	600		
v[j]	−14	0	−6	−43		

Z = 90200

	Destination 1	2	3	4	Supply	u[i]
1	31 | P 400	45 | 0	38 | −1	0 | E −2	400	2
2	29 | 2	41 | B −200	35 | B 800	0 | 2	600	−2
3	32 | 3	46 | 3	40 | 3	0 | B 400	400	0
4	28 | 0	42 | B 600	M | 1M−36	0 | 1	600	−1
5	29 | P 200	43 | B 600	M | 1M−37	0 | L 200	1000	0
Demand	600	1000	800	600		
v[j]	29	43	37	0		

Z = 89000 (CONT'D)

(cont'd)

	Destination 1	2	3	4	Supply	u[i]
1	31 — L — 200	45 — 0	38 — E — −1	0 — B — 200	400	45
2	29 — 2	41 — P — −200	35 — P — 800	0 — 4	600	41
3	32 — 1	46 — 1	40 — 1	0 — B — 400	400	45
4	28 — 0	42 — B — 600	M — 1M−36	0 — 3	600	42
5	29 — P — 400	43 — P — 600	M — 1M−37	0 — 2	1000	43
Demand	600	1000	800	600		
v[j]	−14	0	−6	−45		

Z = 88600

	Destination 1	2	3	4	Supply	u[i]
1	31 — 1	45 — 1	38 — B — 200	0 — B — 200	400	44
2	29 — 2	41 — B — 0	35 — B — 600	0 — 3	600	41
3	32 — 2	46 — 2	40 — 2	0 — B — 400	400	44
4	28 — 0	42 — B — 600	M — 1M−36	0 — 2	600	42
5	29 — B — 600	43 — B — 400	M — 1M−37	0 — 1	1000	43
Demand	600	1000	800	600		
v[j]	−14	0	−6	−44		

Z = 88400 ← minimum cost.

The optimal soln is: $x_{13}=200$ $x_{14}=200$ $x_{25}=600$ $x_{34}=400$ $x_{42}=600$ $x_{51}=600$ $x_{52}=400$.

8.2-15

(0)

	Destination 1	2	3	Supply	u[i]
	54	40.5	27.6		
1	---- P	----	---- E		
	70	-18.3	-31.2	70	22.8
	31.2	36	25		
2	---- P	---- L	----		
	55	55	-11	110	0
	52.8	33.6	33.6		
3	----	---- P	---- P		
	24	5	75	80	-2.4
Demand	125	60	75		
v[j]	31.2	36	36		

(1)

	Destination 1	2	3	Supply	u[i]
	54	40.5	27.6		
1	---- L	----	---- P		
	15	12.9	55	70	0
	31.2	36	25		
2	---- B	----	----		
	110	31.2	20.2	110	-22.8
	52.8	33.6	33.6		
3	---- E	---- B	---- P		
	-7.2	60	20	80	6
Demand	125	60	75		
v[j]	54	27.6	27.6		

(2)

	Destination 1	2	3	Supply	u[i]
	54	40.5	27.6		
1	----	----	---- B		
	7.2	12.9	70	70	-6
	31.2	36	25		
2	---- B	----	----		
	110	24	13	110	-21.6
	52.8	33.6	33.6		
3	---- B	---- B	---- B		
	15	60	5	80	0
Demand	125	60	75		
v[j]	52.8	33.6	33.6		

The optimal allocation of land is
70 million acres in England for Oats
110 million acres in France for Wheat
15 million acres in Spain for Wheat
60 million acres in Spain for Barley
5 million acres in Spain for Oats.

Cost = $8,340,000,000.

8.2-16

(0)

	Destination 1	2	3	4	5	Supply	u[i]
	-800	-700	-500	-200	-500		
1	L	P					
	40	20	100	600	300	60	-500
	-500	-200	-100	-300	-100		
2	E	B	B	P			
	-200	0	20	60	200	80	0
	-600	-400	-300	-500	-300		
3		P					
	-100	40	0	0	200	40	-200
	M	M	M	0	0		
4				B	B		
	1M+ 0	1M-1e	1M-2e	0	60	60	300
Demand	40	60	20	60	60		
v[j]	-300	-200	-100	-300	-300		

Z = -82000

Initial BF solution obtained with Russell's method.

(cont'd)

(CONT'D)

(1)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 | ——— P 40	−700 | ——— P 20	−500 | ——— −100	−200 | ——— 400	−500 | ——— 100	60	−300
2	−500 | ——— P 0	−200 | ——— 200	−100 | ——— L 20	−300 | ——— B 60	−100 | ——— 200	80	0
3	−600 | ——— −100	−400 | ——— P 40	−300 | ——— E −200	−500 | ——— −200	−300 | ——— 0	40	0
4	M | ——— 1M+2e	M | ——— 1M+1e	M | ——— 1M−2e	0 | ——— B 0	0 | ——— B 60	60	300
Demand	40	60	20	60	60		
v[j]	−500	−400	−100	−300	−300		

Z = −82000

(2)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 | ——— P 20	−700 | ——— P 40	−500 | ——— 100	−200 | ——— 400	−500 | ——— 100	60	0
2	−500 | ——— P 20	−200 | ——— 200	−100 | ——— 200	−300 | ——— P 60	−100 | ——— 200	80	300
3	−600 | ——— −100	−400 | ——— L 20	−300 | ——— B 20	−500 | ——— E −200	−300 | ——— 0	40	300
4	M | ——— 1M+2e	M | ——— 1M+1e	M | ——— 1M+ 0	0 | ——— B 0	0 | ——— B 60	60	600
Demand	40	60	20	60	60		
v[j]	−800	−700	−600	−600	−600		

Z = −86000

(CONT'D)

8.2-16 (CONT'D)

(3)

	Destination 1	2	3	4	5	Supply	u[i]
1	−800 | B 0	−700 | B 60	−500 | −100	−200 | 400	−500 | 100	60	−600
2	−500 | B 40	−200 | 200	−100 | 0	−300 | B 40	−100 | 200	80	−300
3	−600 | 100	−400 | 200	−300 | B 20	−500 | B 20	−300 | 200	40	−500
4	M | 1M+2e	M | 1M+1e	M | 1M−2e	0 | B 0	0 | B 60	60	0
Demand	40	60	20	60	60		
v[j]	−200	−100	200	0	0	Z = −90000	

optimal solution:

ship counits from plant 1 to cust 2, 40 from 2 to 1, 40 from 2 to 4, 20 from 3 to 3, 20 from 3 to 4. Profit = 90,000

8.2-17 a) & c) [For b) see 8.1-7 b)]

Northwest Corner Rule:

(a)

	Destination 1	2	3	4	Supply	u[i]
1	800 | P 20	700 | B 20	400 | P 10	0 | 100	50	−100
2	600 | E −300	800 | 0	500 | L 10	0 | B 40	50	0
Demand	20	20	20	40		
v[j]	900	800	500	0	Z = 39000	

initial BF solution
22 seconds
15% above optimal

(d)

	Destination 1	2	3	4	Supply	u[i]
1	800 | L 10	700 | B 20	400 | B 20	0 | E −200	50	0
2	600 | P 10	800 | 300	500 | 300	0 | P 40	50	−200
Demand	20	20	20	40		
v[j]	800	700	400	200	Z = 36000	

(CONT'D)

8.2-17 (cont'd)

(2)

	Destination 1	2	3	4	Supply	u[i]
1	800	700	400	0		
	200	20	20	10	50	0
2	600 B 20	800	500	0 B 30	50	0
Demand	20	20	20	40		
v[j]	600	700	400	0	Z = 34000	

2 iterations
48 sec.

Vogel's Method: initial BF solution 44 sec, 6% above optimal. 1 iteration: 28 sec.

(0)

	Destination 1	2	3	4	Supply	u[i]
1	800 L	700 B	400 B	0 E		
	10	20	20	−200	50	200
2	600 P 10	800 300	500 300	0 P 40	50	0
Demand	20	20	20	40		
v[j]	600	500	200	0	Z = 36000	

(1)

	Destination 1	2	3	4	Supply	u[i]
1	800	700 B	400 B	0 B		
	200	20	20	10	50	0
2	600 B 20	800 100	500 100	0 B 30	50	0
Demand	20	20	20	40		
v[j]	600	700	400	0	Z = 34000	

Russell's Method: initial BF solution: 25 sec, 9% above optimal

(0)

	Destination 1	2	3	4	Supply	u[i]
1	800	700	400 P	0 P		
	300	0	10	40	50	0
2	600 B 20	800 B 20	500 L 10	0 E −100	50	100
Demand	20	20	20	40		
v[j]	500	700	400	0	Z = 37000	

(1)

	Destination 1	2	3	4	Supply	u[i]
1	800	700 E	400 B	0 P		
	200	−100	20	30	50	0
2	600 B 20	800 L 20	500 100	0 P 10	50	0
Demand	20	20	20	40		
v[j]	600	800	400	0	Z = 36000	

(2)

	Destination 1	2	3	4	Supply	u[i]
1	800	700 B	400 B	0 B		
	200	20	20	10	50	0
2	600 B 20	800 100	500 100	0 B 30	50	0
Demand	20	20	20	40		
v[j]	600	700	400	0	Z = 34000	

2 iterations: 45 sec.

8.2-18

a) NW Corner

Source	Destination 1	2	3	4	5	6	7	Supply	u_i
1	8 (10)	8 (20)	7 (10)	7 (10)	4 (0)	4 (0)	3 (1)	50	0
2	6 (−3)	6 (−3)	8 (0)	8 (10)	5 (10)	5 (20)	3 (10)	50	1
3	1M (−9)	0 (+)	1M (−8)	0 (−1M− 8)	1M (−5)	0 (−1M− 5)	1M (30)	30	1M+1
Demand	10	20	10	20	10	20	40		
v_j	8	8	7	7	4	4	−1		30M

Z = +610

a) 40 sec., b) M% (6 of 5), c) 7 iterations, 4 minutes

Vogel's

Source	Destination 1	2	3	4	5	6	7	Supply	u_i
1	8 (0)	8 (10)	7 (10)	7 (−1)	4 (10)	4 (20)	0 (+)	50	0
2	6 (10)	6 (0)	8 (3)	8 (2)	5 (3)	5 (3)	0 (40)	50	−2
3	1M (1M)	0 (10)	1M (1M+ 1)	0 (20)	1M (1M+ 4)	0 (4)	1M (1M+ 6)	30	−8
Demand	10	20	10	20	10	20	40		
v_j	8	8	7	8	4	4	2		Z = 330

a) 55 sec., b) 6%, c) 2 iterations, 1 minute

Russell's

Source	Destination 1	2	3	4	5	6	7	Supply	u_i
1	8 (3)	8 (4)	7 (0)	7 (0)	4 (10)	4 (0)	0 (40)	50	−1
2	6 (10)	6 (1)	8 (10)	8 (20)	5 (0)	5 (10)	0 (−1)	50	0
3	1M (1M− 1)	0 (20)	1M (1M− 3)	0 (+)	1M (10)	0 (−)	1M (1M+ 4)	30	−5
Demand	10	20	10	20	10	20	40		
v_j	6	5	8	8	5	5	1		Z = 390

a) 1:03, b) 26%, c) 5 iterations, 2 minutes

optimal solution:

Source	Destination 1	2	3	4	5	6	7	Supply	u_i
1	8 (2)	8 (2)	7 (10)	7 (1)	4 (10)	4 (20)	0 (10)	50	0
2	6 (10)	6 (10)	8 (1)	8 (2)	5 (1)	5 (1)	0 (30)	50	0
3	1M (1M)	0 (10)	1M (1M− 1)	0 (20)	1M (1M+ 2)	0 (2)	1M (1M+ 6)	30	−6
Demand	10	20	10	20	10	20	40		
v_j	6	6	7	6	4	4	0		Z = 310 ($31,000)

8.2-19

(a) Initial Solution using the Northwest Corner Rule is:

		u_i
8 ③	5 ①	0
6 -1	4 ②	-1
v_j 8	5	

Final Tableau:

		u_i
8 ①	5 ③	0
6 ②	4 1	-2
v_j 8	5	

cost = 35

(b) Minimize $8x_{11} + 5x_{12} + 6x_{21} + 4x_{22}$

subject to
$$x_{11} + x_{12} = 4$$
$$x_{21} + x_{22} = 2$$
$$x_{11} + x_{21} = 3$$
$$x_{12} + x_{22} = 3$$
$$x_{ij} \geq 0 \quad i = 1,2 \;,\; j = 1,2$$

Iter.	B.V.	Eq.#	Z	x_{11}	x_{12}	x_{21}	x_{22}	w_1	w_2	w_3	w_4	RHS
0	Z	0	-1	8-2M	5-2M	6-2M	4-2M	0	0	0	0	-12M
	w_1	1	0	1	1	0	0	1	0	0	0	4
	w_2	2	0	0	0	1	1	0	1	0	0	2
	w_3	3	0	1	0	1	0	0	0	1	0	3
	w_4	4	0	0	1	0	1	0	0	0	1	3
4	Z	0	-1	0	0	0	1	2M-8	2M-6	0	3	-35
	x_{11}	1	0	1	0	0	-1	1	0	0	-1	1
	x_{21}	2	0	0	0	1	1	0	1	0	0	2
	w_3	3	0	0	0	0	0	-1	-1	1	1	0
	x_{12}	4	0	0	1	0	1	0	0	0	1	3

Hence, the transportation simplex method takes 1 iteration while the general simplex method takes four iterations.

Fill in your own computation times — do they agree with the 4:1 ratio? Should they?

8.2-20 Let $z_1 = x_1 - 10$, $z_2 = x_1 + x_2 - 25$, $z_3 = x_1 + x_2 + x_3 - 50$ and $z_4 = x_1 + x_2 + x_3 + x_4 - 70$

Then Minimize $1.080x_1 + 1.110x_2 + 1.100x_3 + 1.130x_4 + .015(z_1 + z_2 + z_3 + z_4)$

subject to

$$\begin{aligned} x_1 \qquad\qquad\qquad\qquad - z_1 &= 10 \\ x_1 + x_2 \qquad\qquad\qquad - z_2 &= 25 \\ x_1 + x_2 + x_3 \qquad\qquad - z_3 &= 50 \\ x_1 + x_2 + x_3 + x_4 \qquad - z_4 &= 70 \end{aligned}$$

$0 \le x_1 \le 25$

$0 \le x_2 \le 35$

$0 \le x_3 \le 30$

$0 \le x_4 \le 10$

$z_i \ge 0 \quad i = 1,2,3,4.$

Initial Simplex Tableau

B.V.	Eq#	Z	x_1	x_2	x_3	x_4	z_1	z_2	z_3	z_4	w_1	w_2	w_3	w_4	Y_1	Y_2	Y_3	Y_4	RHS
Z	0	-1	-4M+1.08	-3M+1.11	-2M+1.1	-M+1.13	M+.015	M+.015	M+.015	M+.015	0	0	0	0	0	0	0	0	-155M
w_1	1		1				-1				1								10
w_2	2		1	1				-1				1							25
w_3	3		1	1	1				-1				1						50
w_4	4		1	1	1	1				-1				1					70
Y_1	5		1												1				25
Y_2	6			1												1			35
Y_3	7				1												1		30
Y_4	8					1												1	10

Simplex tableau: 16 variables and 8 constraints.
Transportation tableau: 20 variables and 9 constraints
But even though the transportation tableau is "larger," it is less work to use than the simplex tableau.

8.2-21 If we multiply the demand constraints by -1, each constraint column will have a +1 entry and a -1 entry with remaining entries 0. If we then sum all constraints, we get

$$0x = \Sigma \text{supplies} - \Sigma \text{demands} = 0 \quad \text{since} \quad \Sigma \text{supplies} = \Sigma \text{demands}$$

Therefore, we have a redundant constraint.

8.2-22 In the initialization step, after selecting the next basic variable, the allocation made is equal to either the (remaining) supply or demand for that row or column. Since these quantities are known to be integer, the allocation will be integer.

Given a current BF solution that is integer, step 3 of an iteration adds and subtracts, around the chain-reaction cycle, the current value of the leaving basic variable. Since we know this is an integer, and all the other basic variables in the cycle began with integer values, the new BF solution must be all integer.

During the initialization step, we can select the next basic variable for allocation arbitrarily from among the rows and columns not already eliminated. Thus by altering our selections we can construct any BF solution as our initial one. And because we've shown that the initialization step gives integer solutions, this means all BF solutions must be integer.

8.2-23.

(a) Let x_{ij} be the number of tons hauled from pit i (North, South) to site j (1, 2, 3).

The linear program is:

Minimize $130x_{11} + 160x_{12} + 150x_{13} + 180x_{21} + 150x_{22} + 160x_{23}$

subject to:

$$x_{11} + x_{12} + x_{13} \le 18$$
$$x_{21} + x_{22} + x_{23} \le 14$$
$$x_{11} + x_{21} = 10$$
$$x_{12} + x_{22} = 5$$
$$x_{13} + x_{23} = 10$$
$$x_{ij} \ge 0 \quad \text{for } i = 1, 2 \quad j = 1, 2, 3.$$

Initial Tableau:

Bas Var	Eq No	Z	Coefficient of X1,1	X1,2	X1,3	X2,1	X2,2	X2,3	X7	X8	X9	X10	X11	Right side
Z	0	1	-M+130	-M+160	-M+150	-M+180	-M+150	-M+160	0	0	0	0	0	-25M
X7	1	0	1	1	1	0	0	0	1	0	0	0	0	18
X8	2	0	0	0	0	1	1	1	0	1	0	0	0	14
X9	3	0	1	0	0	1	0	0	0	0	1	0	0	10
X10	4	0	0	1	0	0	1	0	0	0	0	1	0	5
X11	5	0	0	0	1	0	0	1	0	0	0	0	1	10

(b)

	Destination 1	2	3	4 Dummy	Supply
North 1	130	160	150	0	18
South 2	180	150	160	0	14
Demand	10	5	10	7	

This table is much smaller than the simplex tableau necessary to store the same information

(c)

	Destination 1	2	3	4	Supply	u[i]
	130	160	150	0		
1	---- B	---- B	----	---- B		
	10	5	-10	3	18	0
	180	150	160	0		
2	----	----	---- B	---- B		
	50	-10	10	4	14	0
Demand	10	5	10	7		
v[j]	130	160	160	0		

The solution is not optimal since

$c_{22} - u_2 - v_2 = -10$

and $c_{13} - u_1 - v_3 = -10$

(d)

(0)	Destination 1	2	3	4	Supply	u[i]
	130	160	150	0		
1	---- B	---- L	---- P	----		
	10	5	3	10	18	0
	180	150	160	0		
2	----	---- E	---- P	---- B		
	40	-20	7	7	14	10
Demand	10	5	10	7		
v[j]	130	160	150	-10		

(cont.)

d)

(1)	Destination 1	2	3	4	Supply	u[i]
	130	160	150	0		
1	---- B	----	---- B	----		
	10	20	8	10	18	-10
	180	150	160	0		
2	----	---- B	---- B	---- B		
	40	5	2	7	14	0
Demand	10	5	10	7		
v[j]	140	150	160	0		

The optimal solution is to haul

10 tons from the north pit to site 1

5 tons from the south pit to site 2

8 tons from the north pit to site 3

2 tons from the south pit to site 3

Cost = #3,570

(e) From the reduced costs $(c_{ij} - u_i - v_j)$ in the final tableau we see

$\Delta c_{12} \geq -20 \Rightarrow c_{12} \geq 140$

$\Delta c_{21} \geq -40 \Rightarrow c_{21} \geq 140$

If the contractor can negotiate a hauling cost per ton of 40 or less from the North pit to site 2, or of 20 or less from the South pit to site 1, a new solution, using these options, would give a cost of at least as small as $3570, currently optimal.

8.2-24

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$D$15	Colombo Berdoo	2.22045E-16	0	160	0	20
$E$15	Colombo Los Devils	5	0	130	20	1E+30
$F$15	Colombo San Go	0	10	220	1E+30	10
$G$15	Colombo Hollyglass	0	0	170	1E+30	0
$D$16	Sacron Berdoo	2	0	140	10	0
$E$16	Sacron Los Devils	0	20	130	1E+30	20
$F$16	Sacron San Go	2.5	0	190	10	10
$G$16	Sacron Hollyglass	1.5	0	150	0	1E+30
$D$17	Calorie Berdoo	0	10	190	1E+30	10
$E$17	Calorie Los Devils	0	50	200	1E+30	50
$F$17	Calorie San Go	1.5	0	230	10	20
$G$17	Calorie Hollyglass	0	-190	0	1E+30	190

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$15	Colombo Totals	5	-20	5	1.5	0
$H$16	Sacron Totals	6	-40	6	1.5	2.5
$H$17	Calorie Totals	1.5	0.0	5	1E+30	3.5
$D$18	Totals Berdoo	2	180	2	2.5	1.5
$E$18	Totals Los Devils	5	150	5	0	1.5
$F$18	Totals San Go	4	230	4	3.5	1.5
$G$18	Totals Hollyglass	1.5	190	1.5	2.5	1.5

(CONT'D)

(CONT'D) 8.2-24

a) The optimal solution would change because the decrease of \$30 million is outside the allowable decrease of \$20 million.

b) The optimal solution would remain the same since the allowable increase is ∞.

c) By the 100% rule for simultaneous changes, the optimal solution will not change.

C_{CS}: \$230 → \$215 % of allowable decrease $= 100\left(\frac{230-215}{20}\right) = 75\%$

C_{SL}: \$130 → \$145 % of allowable increase $= 100\left(\frac{145-130}{\infty}\right) = 0\%$

Sum = 75%.

d) By the 100% rule for simultaneous changes, the shadow prices may or may not remain valid.

C_{Sacron}: \$6 → \$5.5 % of allowable decrease $= 100\left(\frac{6-5.5}{2.5}\right) = 20\%$

$C_{Hollyglass}$: \$1.5 → \$1 % of allowable decrease $= 100\left(\frac{1.5-1}{0}\right) = \infty\%$

Sum = ∞%.

8.2-25.

(a) $\Delta c_{34} = -3 \Rightarrow \Delta(c_{34} - u_3 - v_4)^* = -3$

$(c_{34} - u_3 - v_4)^* = -2$

New Tableau

Iteration 3	Destination 1	2	3	4	5	Supply	u_i
1	16 +4	16 +4	13 (30)	22 +7	17 +2	50	-7
2	14 +2	14 +2	13 (20)	19 +4	15 (40)	60	-7
Source 3	19 (30)	19 (20)	20 (0)	20 -2	M M-22	50	0
4(D)	M M+3	0 +3	M M+2	0 (30)	0 (20)	50	-22
Demand	30	20	70	30	60	$\Sigma = 2460$	
v_j	19	19	20	22	22		

The current basic solution is feasible, but not optimal.

(b) $\Delta c_{23} = 3 \Rightarrow \Delta(c_{23} - u_2 - v_3) = 3$

we can revise the tableau by changing u_2 from -7 to -7+3 = -4

This causes v_5 to change to 22-3 = 19

u_4 to change to -22+3 = -19

v_4 to change to 22-3 = 19

Δ(Reduced cost x_{41}) $= -\Delta u_4 = -3$

Δ(Reduced cost x_{42}) $= -\Delta u_4 = -3$

Δ(Reduced cost x_{43}) $= -\Delta u_4 = -3$

Δ(Reduced cost x_{34}) $= -\Delta v_4 = 3$

Δ(Reduced cost x_{35}) $= -\Delta v_5 = 3$

Δ(Reduced cost x_{14}) $= -\Delta v_4 = 3$

Δ(Reduced cost x_{15}) $= -\Delta v_5 = 3$

The New Tableau is:

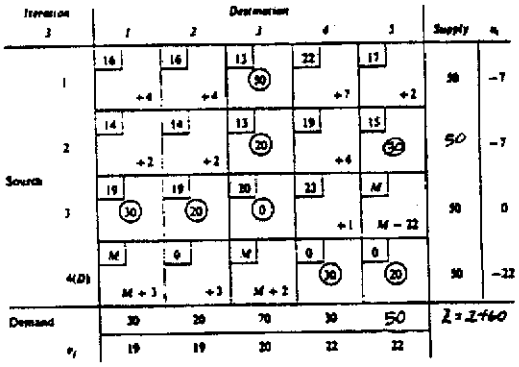
Iteration 3	Destination 1	2	3	4	5	Supply	u_i
1	16 +4	16 +4	13 (30)	22 10	17 5	50	-7
2	14 +2	14 +2	16 (20)	19 +4	15 (40)	60	-4
Source 3	19 (30)	19 (20)	20 (0)	23 4	M M-19	50	0
4(D)	M M	0 0	M M-1	0 (30)	0 (20)	50	-19
Demand	30	20	70	30	60	$\Sigma = 2460$	
v_j	19	19	20	19	19		

The basic solution remains feasible and optimal.

(c) $\Delta s_2 = -10 \Rightarrow \Delta x_{25} = 10$

$\Delta d_5 = 10$

New Tableau:



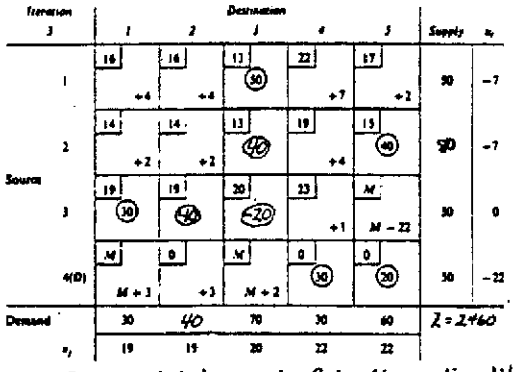
The basic solution remains feasible and optimal

(d) $\Delta s_2 = 20 \Rightarrow \Delta x_{23} = 20$

$\Delta d_2 = 20 \qquad \Delta x_{33} = -20$

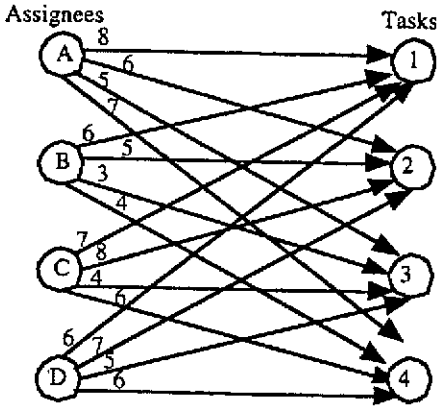
$\Delta x_{32} = 20$

New Tableau:



This solution satisfies the optimality criterion, but is infeasible.

8.3-1 a)



b & c)

		Unit Cost ($) Task 1	2	3	4	Supply
	A	8	6	5	7	1
Assignee	B	6	5	3	4	1
	C	7	8	4	6	1
	D	6	7	5	6	1
Demand		1	1	1	1	

d)

		Assignments Task 1	2	3	4	Totals		Supply
	A	**0**	**1**	**0**	**0**	1	=	1
Assignee	B	**0**	**0**	**0**	**1**	1	=	1
	C	**0**	**0**	**1**	**0**	1	=	1
	D	**1**	**0**	**0**	**0**	1	=	1
Totals		1	1	1	1			
		=	=	=	=	Total Cost	=	**$ 20**
Demand		1	1	1	1			

9.3-2 (a) Ships are assignees and Ports are assignments.

b)

Cost Table

Assignee \ Task	1	2	3	4
1	500	400	600	700
2	600	600	700	500
3	700	500	700	600
4	500	400	600	600

Optimal Solution: The X's in the table indicate an optimal assignment of assignees to tasks.

Assignee \ Task	1	2	3	4
1	X			
2				X
3		X		
4			X	

Cost is 2,100

c)

	Destination 1	2	3	4	Supply
1	500	400	600	700	1
2	600	600	700	500	1
3	700	500	700	600	1
4	500	400	600	600	1
Demand	1	1	1	1	

(d) and (e)

(d)

	Destination 1	2	3	4	Supply	u[i]
1	500 P 1	400 P 0	600 100	700 300	1	0
2	600 -100	600 1	700 0	500 -100	1	200
3	700 0	500 -100	700 1	600 0	1	200
4	500 E -200	400 -200	600 -100	600 P 1	1	200
Demand	1	1	1	1		
v[j]	500	400	500	400		

(f)

	Destination 1	2	3	4	Supply	u[i]
1	500 P 0	400 P 1	600 100	700 100	1	0
2	600 -100	600 L 0	700 B 1	500 E -300	1	200
3	700 200	500 100	700 200	600 1	1	0
4	500 P 1	400 0	600 100	600 P 0	1	0
Demand	1	1	1	1		
v[j]	500	400	500	600		

(cont.)

8.3-2(e) cont.

(2)

	Destination 1	2	3	4	Supply	u[i]
1	500 ---- B 0	400 ---- B 1	600 ---- -200	700 ---- 100	1	600
2	600 ---- 200	600 ---- 300	700 ---- P 1	500 ---- P 0	1	500
3	700 ---- 200	500 ---- 100	700 ---- -100	600 ---- B 1	1	600
4	500 ---- B 1	400 ---- 0	600 ---- E -200	600 ---- L 0	1	600
Demand	1	1	1	1		
v[j]	-100	-200	200	0		

(3)

	Destination 1	2	3	4	Supply	u[i]
1	500 ---- P 0	400 ---- P 1	600 ---- 0	700 ---- 300	1	0
2	600 ---- 0	600 ---- 100	700 ---- 1	500 ---- P 0	1	100
3	700 ---- 0	500 ---- E -100	700 ---- -100	600 ---- P 1	1	200
4	500 ---- P 1	400 ---- 0	600 ---- P 0	600 ---- 200	1	0
Demand	1	1	1	1		
v[j]	500	400	600	400		

(4)

	Destination 1	2	3	4	Supply	u[i]
1	500 ---- B 1	400 ---- B 0	600 ---- 0	700 ---- 200	1	0
2	600 ---- 100	600 ---- 200	700 ---- 100	500 ---- B 1	1	0
3	700 ---- 100	500 ---- B 1	700 ---- 0	600 ---- B 0	1	100
4	500 ---- B 0	400 ---- 0	600 ---- B 1	600 ---- 100	1	0
Demand	1	1	1	1		
v[j]	500	400	600	500		

One optimal assignment is (ship, port) =
(1,1), (2,4), (3,2), (4,3)

Cost = 21

f) Continuing to pivot where reduced cost are 0, we get

(5)

	Destination 1	2	3	4	Supply	u[i]
1	500 ---- B 1	400 ---- B 0	600 ---- 0	700 ---- 200	1	400
2	600 ---- 100	600 ---- 200	700 ---- 100	500 ---- B 1	1	400
3	700 ---- 100	500 ---- P 1	700 ---- E 0	600 ---- B 0	1	500
4	500 ---- 0	400 ---- P 0	600 ---- L 1	600 ---- 100	1	400
Demand	1	1	1	1		
v[j]	100	0	200	200		

Alternative optimal matching (ship, port)
(1,1) (2,4) (3,3) (4,2)

8.3-2 (CONT'D)

(F) (6)

	Destination 1	2	3	4	Supply	u[i]
	500	400	600	700		
1	---- L	---- P	----	----		
	1	0	0	200	1	-100
	600	600	700	500		
2	----	----	----	---- B		
	000	200	100	1	1	-100
	700	500	700	600		
3	----	---- B	---- B	---- B		
	100	0	1	0	1	0
	500	400	600	600		
4	---- E	---- P	----	----		
	0	1	0	100	1	-100
Demand	1	1	1	1		
v[j]	600	500	700	600		

Alternative optimal matching (ship, port)
(1,2), (2,4), (3,3), (4,1)

(7)

	Destination 1	2	3	4	Supply	u[i]
	500	400	600	700		
1	----	---- P	---- E	----		
	0	1	0	200	1	-100
	600	600	700	500		
2	----	----	----	---- B		
	100	200	100	1	1	-100
	700	500	700	600		
3	----	---- P	---- L	---- B		
	100	0	1	0	1	0
	500	400	600	600		
4	---- B	---- B	----	----		
	1	0	0	100	1	-100
Demand	1	1	1	1		
v[j]	600	500	700	600		

Alternative optimal matching (ship, port)
(1,3), (2,4), (3,2), (4,1)

8.3-3 a)

Cost Table

	Task 1	2	3	4	5
1	7440	18000	12160	0	0
2	6960	16400	11200	0	0
Assignee 3	7680	18400	12800	0	0
4	6720	16800	1M	0	0
5	6960	17200	1M	0	0

b)

	Task 1	2	3	4	5
1				X	
2			X		
Assignee 3					X
4		X			
5	X				

Cost is
34960

c)

	Product 1	2	3	4	5	Supply
1	74.4	180	121.6	0	0	1
2	69.6	164	112	0	0	1
Plant 3	76.8	184	128	0	0	1
4	67.2	168	M	0	0	1
5	69.6	172	M	0	0	1
Demand	1	1	1	1	1	

8.3-3(d)

(o)	1	2	3	4	5	Supply	u[i]
1	74.4	180	122	0	0 B		
	4.8	9.6	3.2	0	1	1	6.4
2	69.6	164	112	0	0		
	6.4	0 B	0 B	6.4	6.4	1	0
3	76.8	184	128	0	0		
	7.2	13.6	9.6	1 B	0 B	1	6.4
4	67.2	168	M	0	0		
	0 B	1 B	1M-e3	2.4	2.4	1	4
5	69.6	172	M	0	0		
	1	1.6	1M-e3	0 B	0 B	1	6.4
Demand	1	1	1	1	1		
v[j]	63.2	164	112	-6.4	-6.4		

The initial solution from Vogel's Method is optimal.

Plant 2 produces Product 3
Plant 4 produces Product 2
Plant 5 produces Product 1
Cost = 349.6

8.3-4

(a) After adding a "dummy" stroke which everyone can swim in 0 seconds, the problem becomes that of assigning 5 swimmers to 5 strokes

see cost table below

The optimal solution is
David swims the backstroke
Tony swims the breaststroke
Chris swims the butterfly
Carl swims Freestyle.

Time = 126.2 sec.

8.3-4 a)

Cost Table

Assignee \ Task	Carl 1	Chris 2	David 3	Tony 4	Ken 5
back 1	37.7	32.9	33.8	37	35.4
breast 2	43.4	33.1	42.2	34.7	41.8
fly 3	33.3	28.5	38.9	30.4	33.6
free 4	29.2	26.4	29.6	28.5	31.1
dummy 5	0	0	0	0	0

Optimal Solution: The X's in the table indicate an optimal assignment of assignees to tasks.

b)

Assignee \ Task	1	2	3	4	5
1			X		
2				X	
3		X			
4	X				
5					X

Cost is 126.2

8.3-5

(a) The assignments are the loads needed at sites 1, 1', 2 and 3 and the assignees are the three trucks from North and 2 from South

	Sites				
	1	1'	2	3	3'
North 1	650	650	800	750	750
North 2	650	650	800	750	750
North 3	650	650	800	750	750
South 4	900	900	750	800	800
South 5	900	900	750	800	800

b)

	Task				
	1	2	3	4	5
1	X				
2		X			
Assignee 3				X	
4			X		
5					X

Cost 3,600

c)

	Site			
	1	2	3	Supply
North 1	650	800	750	3
South 2	900	750	800	2
Demand	2	1	2	

d)

	Destination			
	1	2	3	Supply
Source 1	2		1	3
2		1	1	2
Demand	2	1	2	Cost is 3,600

8.3-6 (a)

	Product					
	1	2	3	4	5	Supply
Plant 1	820	810	840	960	0	2
2	800	870	M	920	0	2
3	740	900	810	840	0	1
Demand	1	1	1	1	1	

(b) and (c)

	Destination						
	1	2	3	4	5	Supply	u[i]
	820	810	840	960	0		
1	----	---- B	---- B	----	---- B		
	20	1	1	40	0	2	0
	800	870	M	920	0		
2	---- B	----	----	---- B	---- B		
	1	60	1M-e3	0	1	2	0
	740	900	810	840	0		
3	----	----	----	---- B	----		
	20	170	50	1	80	1	-80
Demand	1	1	1	1	1		
v[j]	800	810	840	920	0		

Since all reduced costs are nonnegative, this solution is optimal.

d)

	Product					
	1	2	3	4	5	Supply
1	820	810	840	960	0	1
2	820	810	840	960	0	1
Plant 3	800	870	M	920	0	1
4	800	870	M	920	0	1
5	740	900	810	840	M	1
Demand	1	1	1	1	1	

This is identical to the table in part (a) except Plants 1 and 2 have been split into 2 plants each.

9.3-8(e)

	Destination 1	2	3	4	5	Supply	u[i]
1	820	810	840	960	0		
	----	---- B	----	----	----		
	20	1	0	40	0	1	0
2	820	810	840	960	0		
	----	---- B	---- B	----	---- B		
	20	0	1	40	0	1	0
3	800	870	M	920	0		
	---- B	----	----	----	----		
	1	60	1M-e3	0	0	1	0
4	800	870	M	920	0		
	---- B	----	----	---- B	---- B		
	0	60	1M-e3	0	1	1	0
5	740	900	810	840	M		
	----	----	----	---- B	----		
	20	170	50	1	1M+80	1	-80
Demand	1	1	1	1	1		
v[j]	800	810	840	920	0		

The basic feasible solution for the transformed problem above corresponds to that given in part (e).

9.3-7

o	Destination 1	2	3	4	Supply	u[i]
1	13	16	12	11		
	----	----	----	---- B		
	7	8	8	1	1	-9
2	15	M	13	20		
	---- B	----	---- P	---- L		
	1	1M-17	0	0	1	0
3	5	7	10	6		
	---- B	---- B	----	----		
	0	1	7	-4	1	-10
4	0	0	0	0		
	----	----	---- P	---- E		
	-2	-4	1	-7	1	-13
Demand	1	1	1	1		
v[j]	15	17	13	20		

(1)

	Destination 1	2	3	4	Supply	u[i]
1	13	16	12	11		
	----	----	----	---- B		
	0	1	1	1	1	-2
2	15	M	13	20		
	---- L	----	---- P	----		
	1	1M-17	0	7	1	0
3	5	7	10	6		
	---- P	---- P	----	----		
	0	1	7	3	1	-10
4	0	0	0	0		
	----	---- E	---- P	---- B		
	-2	-4	1	0	1	-13
Demand	1	1	1	1		
v[j]	15	17	13	13		

(2)

	Destination 1	2	3	4	Supply	u[i]
1	13	16	12	11		
	----	----	----	---- B		
	4	5	1	1	1	11
2	15	M	13	20		
	----	----	---- B	----		
	4	1M-13	1	7	1	13
3	5	7	10	6		
	---- B	---- L	----	---- E		
	1	0	3	-1	1	7
4	0	0	0	0		
	----	---- P	---- B	---- P		
	2	1	0	0	1	0
Demand	1	1	1	1		
v[j]	-2	0	0	0		

(cont.)

8.3-7 (cont.)

(3)	Destination 1	2	3	4	Supply	u[i]
1	13 / 3	16 / 5	12 / 1	11 / 1 B	1	11
2	15 / 3	M / 1M-13	13 / 1 B	20 / 7	1	13
3	5 B / 1	7 / 1	10 / 4	6 B / 0	1	6
4	0 / 1	0 B / 1	0 B / 0	0 B / 0	1	0
Demand	1	1	1	1		
v[j]	-1	0	0	0		

This solution corresponds to that given in Sec. 8.3, though the set of basic variables is different, variable values are the same.

8.3-8 a) let assignees 1 and 2 represent Plant A; assignees 3 and 4 represent Plant B; and tasks be distribution centers.

Cost Table

	Task 1	2	3	dummy 4
1	8000	14000	12000	0
2	8000	14000	12000	0
Assignee 3	6000	16000	15000	0
4	6000	16000	15000	0

b)

	Task 1	2	3	4
1		X		
2			X	
Assignee 3	X			
4				X

Cost is 32,000

8.3-8 c)

Cost Per Unit Distributed

	Destination 1	2	3	4	Supply
1	8000	14000	12000	0	1
2	8000	14000	12000	0	1
Source 3	6000	16000	15000	0	1
4	6000	16000	15000	0	1
Demand	1	1	1	1	

d)

	Destination 1	2	3	4	Supply
1		1			1
2			1		1
Source 3	1				1
4				1	1
Demand	1	1	1	1	Cost is 32000

e)

Cost Per Unit Distributed

	Destination 1	2	3	4	Supply
1	8000	14000	12000	0	2
Source 2	6000	16000	15000	0	2
Demand	1	1	1	1	

f)

	Destination 1	2	3	4	Supply
1		1	1		2
Source 2	1			1	2
Demand	1	1	1	1	Cost is 32000

8-3-9

a)

Assignee	Task 1	2	3
1			X
2	X		
3		X	

Cost 10

b)

Cost Per Unit Distributed

Source	Destination 1	2	3	Supply
1	5	7	4	1
2	3	6	5	1
3	2	3	4	1
Demand	1	1	1	

c)

Source	Destination 1	2	3	Supply
1			1	1
2	1			1
3		1		1
Demand	1	1	1	Cost is 10

8-3-10 Minimize $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

subject to $\sum_{j=1}^{n} x_{ij} = 1$ for $i = 1,2,\ldots,n$

$\sum_{i=1}^{n} x_{ij} = 1$ for $j = 1,2,\ldots,n$

$x_{ij} \geq 0$ for $i = 1,2,\ldots,n$

$j = 1,2,\ldots,n.$

The table of constraint coefficients is identical to that for the transportation problem (Table 7.6). The assignment problem has a more special structure because $m=n$ and $s_i - d_i = 1$ for all i.

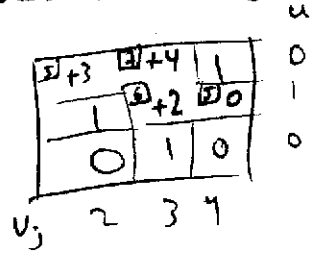
d) A transportation problem of size $m \times n$ has $m+n-1$ basic variables; since $m=n$ for the assignment problem, there are $2(3)-1 = 5$ basic variables, but only 3 assignments. Thus, 2 basic variables are degenerate (equal to zero). Assignment problems will always be highly degenerate. If you use the interactive routine in the OR Courseware you'll see this.

e) x_{A1}, x_{A2}, x_{B2} are nonbasic; and one of (x_{B3}, x_{C3}) is.

x_{C1} and one of (x_{B3}, x_{C3}) are basic and $= 0$.

(from network simplex - try working it out yourself).

dual variables:



looking at $c_{ij} - u_i - v_j$, we see that allowable ranges to stay optimal for c_{ij} are: $c_{A1} \geq 2$, $c_{A2} \geq 3$, $c_{B2} \geq 4$, $c_{B3} \geq 5$

Cases

8-1 Option 1:

Unit Cost (1,000's)

		Destination (Market) 1	2	3	4	5	Supply
	1	61	72	45	55	66	15
Source 1	2	69	78	60	49	56	20
	3	59	66	63	61	47	15
Demand		11	12	9	10	8	

Unit Cost (1,000's)

		Destination (Market) 1	2	3	4	5	Totals		Supply
	1	**6**	**0**	**9**	**0**	**0**	15	=	15
Source 1	2	**2**	**0**	**0**	**10**	**8**	20	=	20
	3	**3**	**12**	**0**	**0**	**0**	15	=	15
Totals		11	12	9	10	8			
		=	=	=	=	=	Total Cost	=	**$ 2,816.00**
Demand		11	12	9	10	8			

Option 2:

Unit Cost (1,000's)

		Destination (Market) 1	2	3	4	5	Supply
	1	58.5	68.3	47.8	55	63.5	15
Source 1	2	65.3	74.8	55	49	57.5	20
	3	59	61.3	63.5	58.8	50	15
Demand		11	12	9	10	8	

Unit Cost (1,000's)

		Destination (Market) 1	2	3	4	5	Totals		Supply
	1	**6**	**0**	**9**	**0**	**0**	15	=	15
Source 1	2	**5**	**0**	**0**	**10**	**5**	20	=	20
	3	**0**	**12**	**0**	**0**	**3**	15	=	15
Totals		11	12	9	10	8			
		=	=	=	=	=	Total Cost	=	**$ 2,770.80**
Demand		11	12	9	10	8			

Option 3:

Unit Cost (1,000's)

		Destination (Market) 1	2	3	4	5	Supply
	1	58.5	68.3	45	55	63.5	15
Source 1	2	65.3	74.8	55	49	56	20
	3	59	61.3	63	58.8	47	15
Demand		11	12	9	10	8	

Unit Cost (1,000's)

		Destination (Market) 1	2	3	4	5	Totals		Supply
	1	**6**	**0**	**9**	**0**	**0**	15	=	15
Source 1	2	**5**	**0**	**0**	**10**	**5**	20	=	20
	3	**0**	**12**	**0**	**0**	**3**	15	=	15
Totals		11	12	9	10	8			
		=	=	=	=	=	Total Cost	=	**$ 2,729.10**
Demand		11	12	9	10	8			

When comparing the three options, it is best to use the combination plan, while shipping entirely by rail leads to the highest costs.

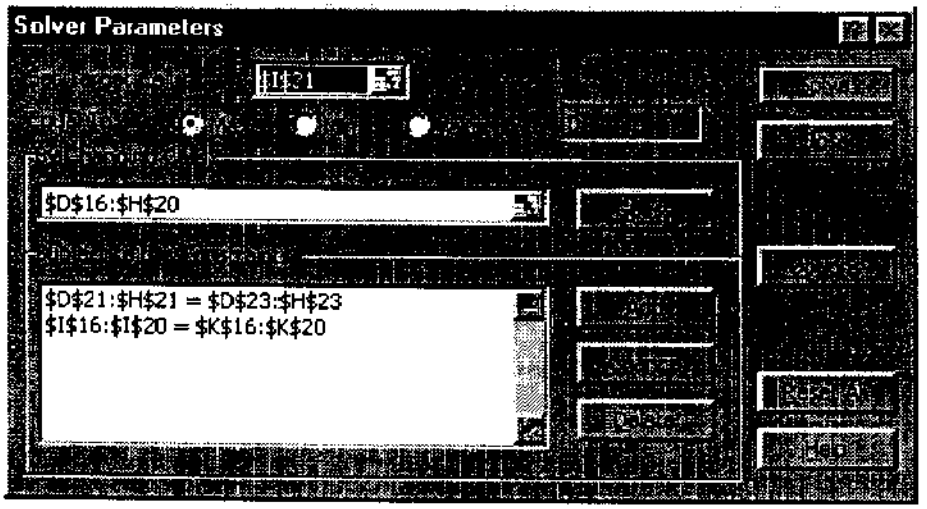
If costs of shipping by water are expected to rise considerably more than for shipping by rail, stay with rail and use Option 1. If the reverse is true, then use Option 2. If the cost comparisons will remain roughly the same, use Option 3. Option 3 is clearly the most feasible but may not be chosen if it is too logistically cumbersome. More knowledge of the situation is necessary to determine this.

8-2 The problem in this case can be solved using the assignment problem. Throughout this case we use the template for the assignment problem.

a) The projects are the tasks, and the scientists are the assignees in this assignment problem.

	A	B	C	D	E	F	G	H	I	J	K
1											
2				Points							
3				Task							
4				Up	Stable	Choice	Hope	Release	Supply		
5			Kvaal	100	400	200	200	100	1		
6		Assignee	Zuner	0	200	800	0	0	1		
7			Tsai	100	100	100	100	600	1		
8			Mickey	267	153	99	451	30	1		
9			Rollins	100	33	33	34	800	1		
10		Demand		1	1	1	1	1			
11											
12											
13				Assignments							
14				Task							
15				Up	Stable	Choice	Hope	Release	Totals		Supply
16			Kvaal	**0**	**1**	**0**	**0**	**0**	1	=	1
17		Assignee	Zuner	**0**	**0**	**1**	**0**	**0**	1	=	1
18			Tsai	**1**	**0**	**0**	**0**	**0**	1	=	1
19			Mickey	**0**	**0**	**0**	**1**	**0**	1	=	1
20			Rollins	**0**	**0**	**0**	**0**	**1**	1	=	1
21		Totals		1	1	1	1	1	**2551**	=	Total Points
22				=	=	=	=	=			
23		Demand		1	1	1	1	1			

The solver dialogue box appears as follows:



The solver options throughout this case are:

To maximize the scientists preferences you want to assign Dr. Tsai to lead project Up, Dr. Kvaal to lead project Stable, Dr. Zuner to lead project Choice, Dr. Mickey to lead project Hope, and Dr. Rollins to lead project Release.

b) Since there are only four assignees we introduce a dummy assignee with preferences of –1. The task that gets assigned the dummy assignee will not be done.

	A	B	C	D	E	F	G	H	I	J	K
1											
2				Points							
3				Task							
4				Up	Stable	Choice	Hope	Release	Supply		
5			Kvaal	100	400	200	200	100	1		
6		Assignee	Zuner	0	200	800	0	0	1		
7			Tsai	100	100	100	100	600	1		
8			Mickey	267	153	99	451	30	1		
9			dummy	-1	-1	-1	-1	-1	1		
10		Demand		1	1	1	1	1			
11											
12											
13				Assignments							
14				Task							
15				Up	Stable	Choice	Hope	Release	Totals		Supply
16			Kvaal	**0**	**1**	**0**	**0**	**0**	1	=	1
17		Assignee	Zuner	**0**	**0**	**1**	**0**	**0**	1	=	1
18			Tsai	**0**	**0**	**0**	**0**	**1**	1	=	1
19			Mickey	**0**	**0**	**0**	**1**	**0**	1	=	1
20			dummy	**1**	**0**	**0**	**0**	**0**	1	=	1
21		Totals		1	1	1	1	1	**2250**	=	Total Points
22				=	=	=	=	=			
23		Demand		1	1	1	1	1			

The solver dialogue box remains the same.

We give up on project Up.

c) Since two of the assignees can do two tasks we need to double them. We include assignees Zuner-1, Zuner-2, Mickey-1, and Mickey-2 into the problem. In order to have an equal number of assignees and tasks we also need to include one dummy task. In order to ensure that neither Dr. Kvaal nor Dr. Tsai can get assigned the dummy task and thus no project, we insert a large negative number as their point bid for the dummy project.

					Points							
					Task							
				Up	Stable	Choice	Hope	Release	dummy	Supply		
			Kvaal	100	400	200	200	100	-10000	1		
			Zuner-1	0	200	800	0	0	-1	1		
		Assignee	Zuner-2	0	200	800	0	0	-1	1		
			Tsai	100	100	100	100	600	-10000	1		
			Mickey-1	267	153	99	451	30	-1	1		
			Mickey-2	267	153	99	451	30	-1	1		
		Demand		1	1	1	1	1	1			
					Assignments							
					Task							
				Up	Stable	Choice	Hope	Release	dummy	Totals		Supply
			Kvaal	0	1	0	0	0	0	1	=	1
			Zuner-1	0	0	0	0	0	1	1	=	1
		Assignee	Zuner-2	0	0	1	0	0	0	1	=	1
			Tsai	0	0	0	0	1	0	1	=	1
			Mickey-1	0	0	0	1	0	0	1	=	1
			Mickey-2	1	0	0	0	0	0	1	=	1
		Totals		1	1	1	1	1	1	2517	=	Total Points
				=	=	=	=	=	=			
		Demand		1	1	1	1	1	1			

Dr. Kvaal leads project Stable, Dr. Zuner leads project Choice, Dr. Tsai leads project Release, and Dr. Mickey leads the projects Hope and Up.

d) Under the new bids of Dr. Zuner the assignment does not change:

					Points							
					Task							
				Up	Stable	Choice	Hope	Release	dummy	Supply		
			Kvaal	100	400	200	200	100	-10000	1		
			Zuner-1	20	450	451	39	40	-1	1		
		Assignee	Zuner-2	20	450	451	39	40	-1	1		
			Tsai	100	100	100	100	600	-10000	1		
			Mickey-1	267	153	99	451	30	-1	1		
			Mickey-2	267	153	99	451	30	-1	1		
		Demand		1	1	1	1	1	1			
					Assignments							
					Task							
				Up	Stable	Choice	Hope	Release	dummy	Totals		Supply
			Kvaal	0	1	0	0	0	0	1	=	1
			Zuner-1	0	0	1	0	0	0	1	=	1
		Assignee	Zuner-2	0	0	0	0	0	1	1	=	1
			Tsai	0	0	0	0	1	0	1	=	1
			Mickey-1	0	0	0	1	0	0	1	=	1
			Mickey-2	1	0	0	0	0	0	1	=	1
		Totals		1	1	1	1	1	1	2168	=	Total Points
				=	=	=	=	=	=			
		Demand		1	1	1	1	1	1			

e) Certainly Dr. Zuner could be disappointed that she is not assigned to project Stable, especially when she expressed a higher preference for that project than the scientist assigned. The optimal solution maximizes the preferences overall, but individual scientists may be disappointed. We should therefore make sure to communicate the reasoning behind the assignments to the scientists.

f) Whenever a scientist cannot lead a particular project we use a large negative number as the point bid.

	A	B	C	D	E	F	G	H	I	J	K
1											
2				Points							
3				Task							
4				Up	Stable	Choice	Hope	Release	Supply		
5			Kvaal	86	343	171	-10000	-10000	1		
6		Assignee	Zuner	0	200	800	0	0	1		
7			Tsai	100	100	100	100	600	1		
8			Mickey	300	-10000	125	-10000	175	1		
9			Rollins	-10000	50	50	100	600	1		
10		Demand		1	1	1	1	1			
11											
12											
13				Assignments							
14				Task							
15				Up	Stable	Choice	Hope	Release	Totals		Supply
16			Kvaal	**0**	**1**	**0**	**0**	**0**	1	=	1
17		Assignee	Zuner	**0**	**0**	**1**	**0**	**0**	1	=	1
18			Tsai	**0**	**0**	**0**	**1**	**0**	1	=	1
19			Mickey	**1**	**0**	**0**	**0**	**0**	1	=	1
20			Rollins	**0**	**0**	**0**	**0**	**1**	1	=	1
21		Totals		1	1	1	1	1	**2143**	=	Total Points
22				=	=	=	=	=			
23		Demand		1	1	1	1	1			

Dr. Kvaal leads project Stable, Dr. Zuner leads project Choice, Dr. Tsai leads project Hope, Dr. Mickey leads project Up, and Dr. Rollins leads project Release.

g) When we want to assign two assignees to the same task we need to duplicate that task.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2				Points									
3				Task									
4				Up	Stable	Choice	Hope-A	Hope-B	Release-A	Release-B	Supply		
5			Kvaal	86	343	171	-10000	-10000	-10000	-10000	1		
6		Assignee	Zuner	0	200	800	0	0	0	0	1		
7			Tsai	100	100	100	100	100	600	600	1		
8			Mickey	300	-10000	125	-10000	-10000	175	175	1		
9			Rollins	-10000	50	50	100	100	600	600	1		
10			Arriaga	250	250	-10000	250	250	250	250	1		
11			Santos	111	1	-10000	333	333	555	555	1		
12		Demand		1	1	1	1	1	1	1			
13													
14													
15				Assignments									
16				Task									
17				Up	Stable	Choice	Hope-A	Hope-B	Release-A	Release-B	Totals		Supply
18			Kvaal	0	1	0	0	0	0	0	1	=	1
19		Assignee	Zuner	0	0	1	0	0	0	0	1	=	1
20			Tsai	0	0	0	0	0	1	0	1	=	1
21			Mickey	1	0	0	0	0	0	0	1	=	1
22			Rollins	0	0	0	0	0	0	1	1	=	1
23			Arriaga	0	0	0	0	1	0	0	1	=	1
24			Santos	0	0	0	1	0	0	0	1	=	1
25		Totals		1	1	1	1	1	1	1	3226	=	Total Points
26				=	=	=	=	=	=	=			
27		Demand		1	1	1	1	1	1	1			

Project Up is led by Dr. Mickey, Stable by Dr. Kvaal, Choice by Dr. Zuner, Hope by Dr. Arriaga and Dr. Santos, and Release by Dr. Tsai and Dr. Rollins.

h) No. Maximizing overall preferences does not maximize individual preferences. Scientists who do not get their first choice may become resentful and therefore lack the motivation to lead their assigned project. For example, in the optimal solution of part (g), Dr. Santos clearly elected project Release as his first choice, but he was assigned to lead project Hope.

In addition, maximizing preferences ignores other considerations that should be factored into the assignment decision. For example, the scientist with the highest preference for a project may not be the scientist most qualified to lead the project.

Supplement to Chapter 8

(Appears on the book's Web site, www.mhhe.com/hillier)

8S-1: Starting with

5	4	6	7
6	6	7	5
7	5	7	6
5	4	6	6

If we subtract the minimum element in each column from each element in the column, and then continue the algorithm, we have:

0	0	0	2
1	2	1	0
2	1	1	1
0	0	0	1

0	0	[0]	3
0	1	0	[0]
1	[0]	0	1
[0]	0	0	2

One optimal solution is to assign ships (1, 2, 3, 4) to Ports (3, 4, 2, 1), cost = 21.

8S-2. Subtracting the minimum element in each row from each element of the row, and then continuing the algorithm, we get:

4.8	0	0.9	4.1	25
10.3	0	9.1	1.6	8.7
4.8	0	10.4	1.9	5.1
2.8	0	3.2	2.1	4.7
0	0	0	0	0

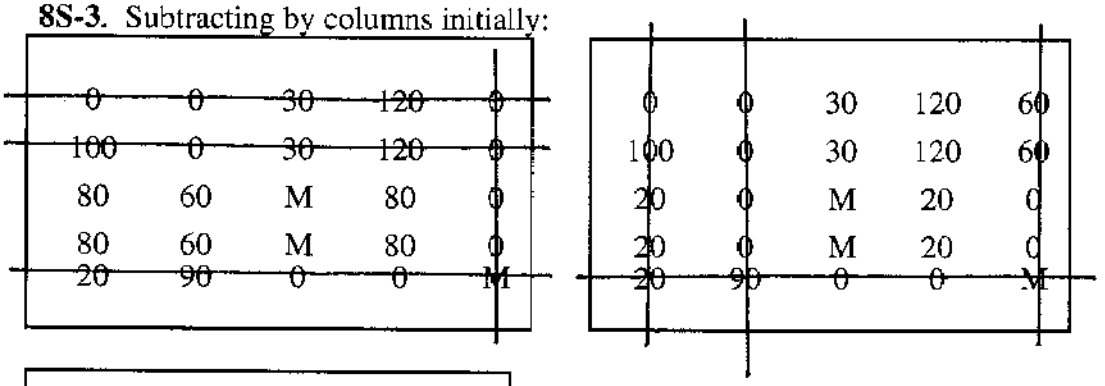
3.9	0	0	3.2	1.6
9.4	0	8.2	0.7	7.8
3.9	0	9.5	1	4.2
1.9	0	2.3	1.2	3.6
0	0.9	0	0	0

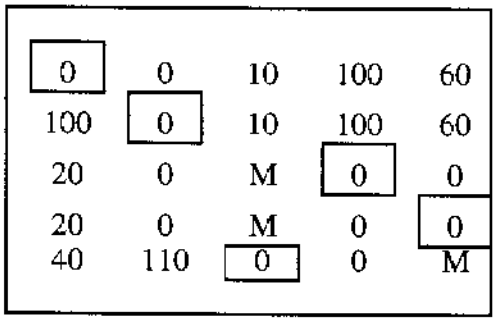
3.9	0.7	0	3.2	1.6
8.7	0	7.5	0	7.1
3.2	0	8.8	0.3	3.5
1.2	0	1.6	0.5	2.9
0	1.6	0	0	0

2.7	0.7	[0]	3.2	0.4
7.5	0	7.5	[0]	5.9
2.0	[0]	8.8	0.3	2.3
[0]	0	1.6	0.5	1.7
0	2.8	1.2	1.2	[0]

One optimal solution is David does the backstroke, Tony swims the Breaststroke, Chris swims the Butterfly, Carl does the Freestyle. Total time = 126.2.

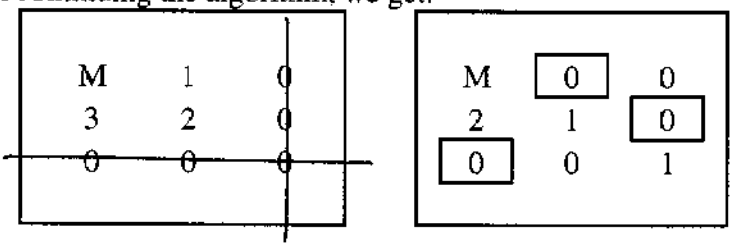
8S-3. Subtracting by columns initially:





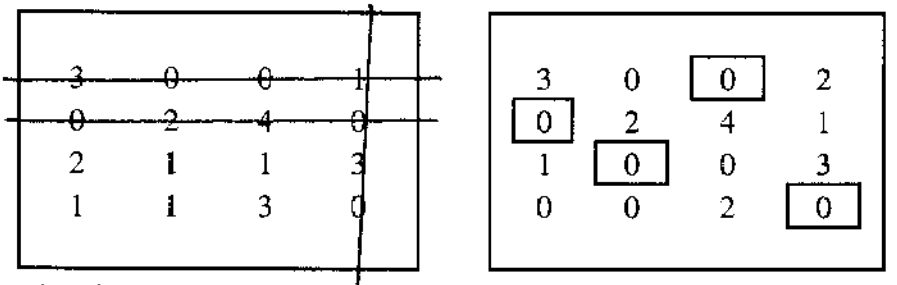
Cost = 3260.

8S-4. Subtracting the minimum element in each row from every element in the row, and then continuing the algorithm, we get:



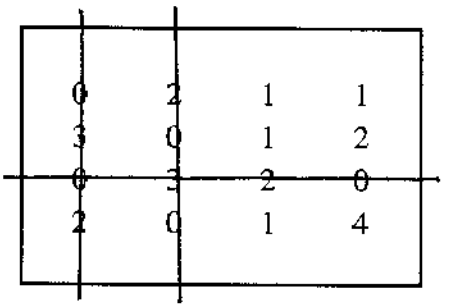
Cost = 12.

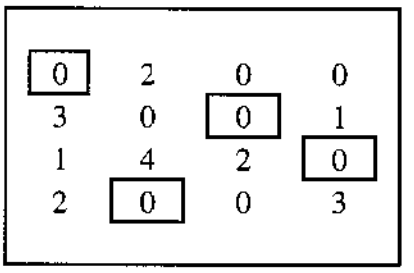
8S-5. Subtracting the minimum element in each column from each entry in the column, and then continuing the algorithm, we get:



An optimal assignment is (A, 3), (B, 1), (C, 2), (D, 4). Cost = 3.

8S-6. Subtracting the minimum element in each row from each element in the row, and then continuing the algorithm, we get:





An optimal assignment is (A, 1), (B, 3), (C, 4), (D, 2). Cost = 16.

9.2-1

a) AD - DC - CE - EF (A → D → C → E → F) is a directed path from A to F

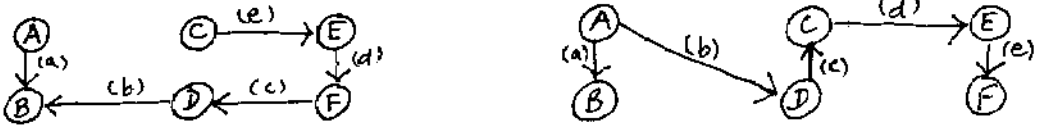
AD - FD (A → D → F)
CA - CE - EF (A → C → E → F)
AD - ED - EF (A → D - E → F)
} are undirected paths from A to F

b) AD - DC - CA
DC - CE - ED
DC - CE - EF - FD
} are directed cycles

CA - CE - EF - FD - DB - AB is an undirected cycle which includes every node.

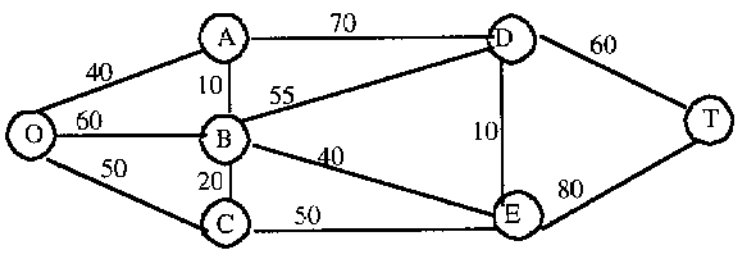
c) {CA, CE, DC, FD, DB} is a spanning tree.

d)



9.3-1

a)



b)

n	Solved nodes connected to unsolved nodes	its closest connected unsolved node	total distance involved	nth nearest node	its minimum distance	its last connection
1	O	A	40	A	40	OA
		C	50	C	50	OC
2,3	O	B	40+10=50	B	50	AB
	A	D	40+70=110			
4	A	E	50+40=90	E	90	BE
	B	E	50+50=100			
	C					
	A	D	40+70=110			
5	B	D	50+55=105	D	100	ED
	E	D	90+10=100			
6	D	T	100+60=160	T	160	DT
	E	T	90+80=170			

shortest route: O-A-B-E-D-T Total distance = 160

c) Yes d) Yes

e)

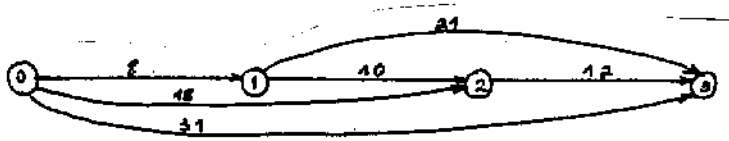
From	To	On Route	Distance
Origin	A	1	40
Origin	B	0	60
Origin	C	0	50
A	B	1	10
A	D	0	70
B	C	0	20
B	D	0	55
B	E	1	40
C	E	0	50
D	E	0	10
D	Destination	1	60
E	Destination	0	80
E	D	1	10
	Total Distance =	160	

Nodes	Net Flow		Supply/Demand
Origin	1	=	1
A	0	=	0
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
Destination	-1	=	-1

9.3-2

a) Nodes are the years

d_{ij} = cost (in $ thousands) of using same tractor from end of year i to end of year j.



(b)

n	Solved nodes connected to unsolved node	its closest connected unsolved node	total distance involved	nth nearest node	its minimum distance	its last connection
1	0	1	8	1	8	01
2	0	2	18	2	18	02
	1	2	8+10 = 18	2		12
3	0	3	31			
	1	3	8+21 = 29	3	29	13
	2	3	18+12 = 30			

After buying new tractor, replace it at end of year 1 and then keep the new till the end of year 3, for total cost of 29,000

c)

From	To	On Route	Cost
Node 0	Node 1	1	$8,000
Node 0	Node 2	0	$18,000
Node 0	Node 3	0	$31,000
Node 1	Node 2	0	$10,000
Node 1	Node 3	1	$21,000
Node 2	Node 3	0	$12,000

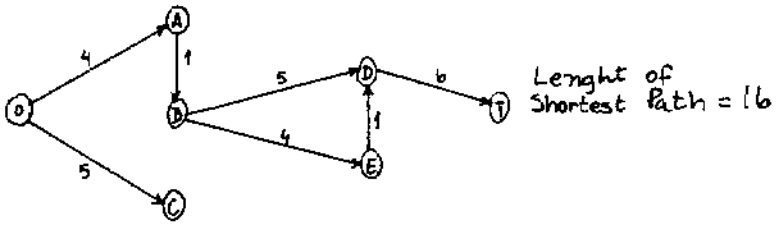
Total Cost = **$ 29,000**

Nodes	Net Flow		Supply/Demand
0	1	=	1
1	0	=	0
2	0	=	0
3	-1	=	-1

9.3-3

(a)

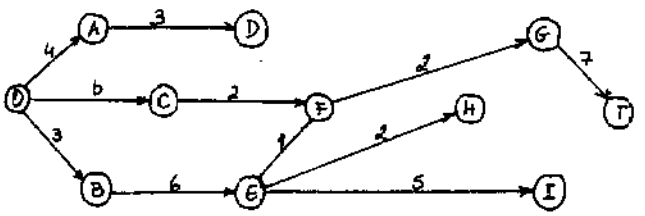
0	4	5	5	10	9	16
ⓞ	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ	Ⓣ
(OA-4)	(AB-1)	DA~~<1~~	CB~~<2~~	DB~~<1~~	(ED-1)	
(OC-5)	AD~~<7~~	(BE-4)	CO~~<5~~	(DT-6)	EB~~<4~~	
OD~~<6~~		(BD-3)		DA~~>7~~	EO~~<5~~	
					E~~>8~~	



Lenght of Shortest Path = 16

(b)

0	4	3	6	7	9	8	10	11	14	17
ⓞ	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ	Ⓕ	Ⓖ	Ⓗ	Ⓘ	Ⓣ
(OB-3)	(AD-3)	BA~~<4~~	CB~~<3~~	DA~~<2~~	EA~~<1~~	(EE-1)	GA~~<2~~	HE~~<2~~	IA~~<3~~	
(OA-4)	AO~~<5~~	(BE-6)	(CF-2)	DB~~<2~~	(EH-2)	FE~~<2~~	GH~~<2~~	HB~~<3~~	IB~~<4~~	
OC~~<6~~			CB~~<4~~	DA~~<3~~	(EI-5)	FB~~<3~~	GB~~<4~~	HI~~<3~~	IE~~<5~~	
			CO~~<5~~	DB~~<4~~	EE~~<6~~	(EG-2)	(GT-7)	HF~~<5~~		
			CA~~<5~~		EB~~<6~~	FA~~<5~~		HT~~<8~~		



Length of Shortest Path = 17.

9.3-4.

This is just the minimum cost flow problem with a unit source at the origin and a unit sink at the destination.

Assume without loss of generality that the origin is node 1 and the destination is node n. The LP formulation is

$$\text{MIN.} \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{1j} - \sum_{j=1}^{n} x_{j1} = 1$$

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0 \qquad 2 \le i \le n-1$$

$$\sum_{j=1}^{n} x_{nj} - \sum_{j=1}^{n} x_{jn} = -1$$

$$0 \le x_{ij} \le 1 \;, \quad 1 \le i, j \le n$$

9.3-5

a) Times play the role of distances.

b)

n	solved nodes connected to unsolved nodes	its closest connected unsolved node	total distance involved	nth nearest node	its minimum distance	its last connection
1	SE	C	4.2	C	4.2	SE-C
2	SE C	A F	4.6 4.2+3.4=7.6	A	4.6	SE-A
3	SE C A	B F E	4.7 4.2+3.4=7.6 4.6+3.4=8	B	4.7	SE-B
4	A B C	E E F	4.6+3.4=8 4.7+3.2=7.9 4.2+3.4=7.6	F	7.6	C-F
5	A B C F	E E E LN	4.6+3.4=8 4.7+3.2=7.9 4.2+3.5=7.7 7.6+3.8=11.4	E	7.7	C-E
6	A B F E	D D LN LN	4.6+3.5=8.1 4.7+3.6=8.3 7.6+3.8=11.4 7.7+3.6=11.3	D	8.1	A-D
7	D E F	LN LN LN	8.1+3.4=11.5 7.7+3.6=11.3 7.6+3.8=11.4	LN	11.3	E-LN

optimal path: SE-C-E-LN

minimum distance = 11.3

c)

From	To	On Route	Time
SE	A	0	4.6
SE	B	0	4.7
SE	C	1	4.2
A	D	0	3.5
A	E	0	3.4
B	D	0	3.6
B	E	0	3.2
B	F	0	3.3
C	E	1	3.5
C	F	0	3.4
D	LN	0	3.4
E	LN	1	3.6
F	LN	0	3.8

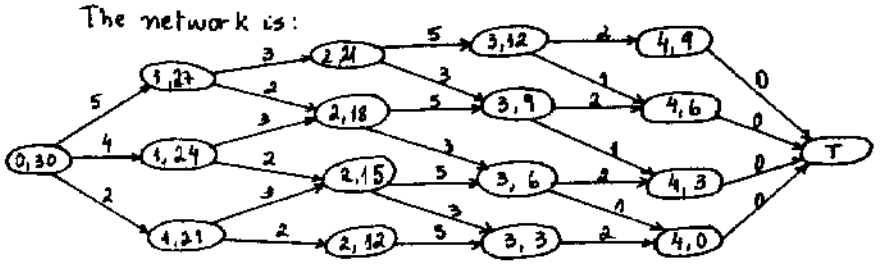
Total Time = **11.3**

Nodes	Net Flow		Supply/Demand
SE	1	=	1
A	0	=	0
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
F	0	=	0
LN	-1	=	-1

9.3-6

(a) Let node (i,j) denote phase i being completed with j left to spend. $t_{(i,j),(i+1,k)}$ = the time taken to complete phase $i+1$ if $(j-k)$ million is spent.

The network is:



(b)

n	Solved nodes connected to unsolved nodes	Its closest connected unsolved node	total distance involved	Its nearest node	Its minimum distance	Its last connection
1	(0,30)	(1,21)	2	(1,24)	2	(0,30)-(1,21)
2	(0,30)	(1,24)	4	(1,24)	4	(0,30)-(1,24)
	(1,21)	(2,12)	2+2=4	(2,12)	4	(1,21)-(2,12)
4	(0,30)	(1,27)	5	(1,27)	5	(0,30)-(1,27)
	(1,21)	(2,15)	2+3=5	(2,15)	5	(1,21)-(2,15)
	(1,24)	(2,15)	4+2=6			
	(2,12)	(3,3)	4+5=9			
6	(1,24)	(2,18)	4+3=7	(2,18)	7	(1,24)-(2,18)
	(1,27)	(2,18)	5+2=7	(2,18)	7	(1,27)-(2,18)
	(2,12)	(3,3)	4+5=9			
	(2,15)	(3,3)	5+3=8			
7	(1,27)	(2,21)	5+3=8	(2,21)	8	(1,27)-(2,21)
	(2,12)	(3,3)	4+5=9			
	(2,15)	(3,3)	5+3=8	(3,3)	8	(2,15)-(3,3)
	(2,18)	(3,6)	7+3=10			
9	(2,15)	(3,6)	5+5=10	(3,6)	10	(2,15)-(3,6)
	(2,18)	(3,6)	7+3=10	(3,6)	10	(2,18)-(3,6)
	(2,21)	(3,9)	8+3=11			
	(3,3)	(4,0)	8+2=10	(4,0)	10	(3,3)-(4,0)
11	(2,18)	(3,9)	7+5=12			
	(2,21)	(3,9)	8+3=11			
	(3,6)	(4,3)	10+2=12			
	(4,0)	T	10+0=10	T	10	(4,0)-T

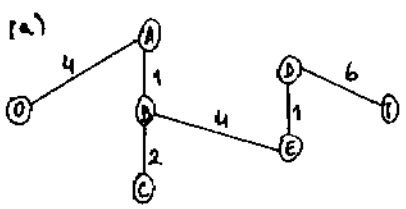
Shortest Route: 0,30 —2→ 1,21 —3→ 2,15 —3→ 3,3 —2→ 4,0 —0→ T

Phase	Level	Cost	Time
Research	crash	9	2
Development	priority	6	3
Design	crash	12	3
Production	priority	3	2

Total time = 10

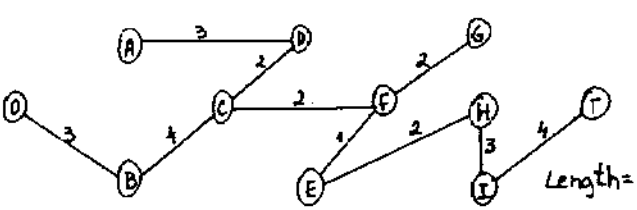
9.4-1

(a)



Length = 18

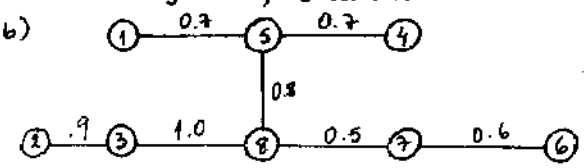
(b)



Length = 26

9.4-2 (a) nodes ~ groves, branches ~ roads

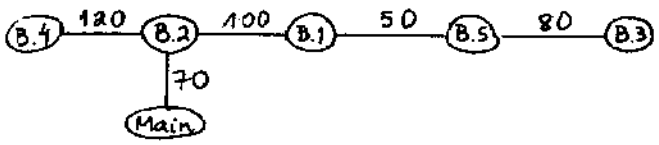
(b)



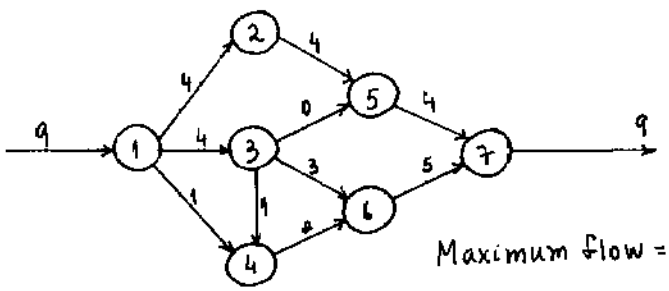
Length = 5.2

9.4-3 (a) nodes ~ {Main office, Branch 1, ..., Branch 5}, branches ~ telephone lines

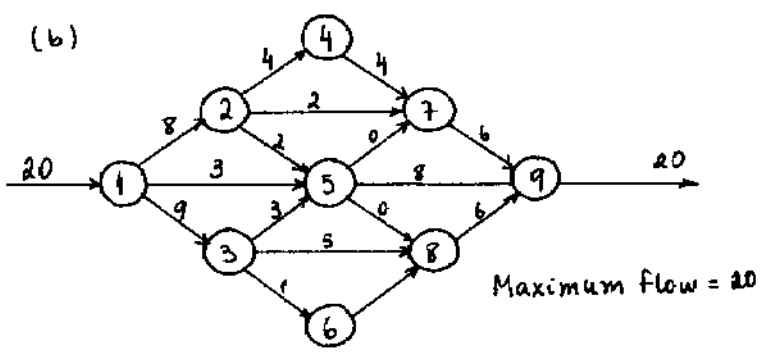
(b)



9.5-1 (a)



(b)



9.5-2 Let node 1 be the source and node N be the sink. Then:

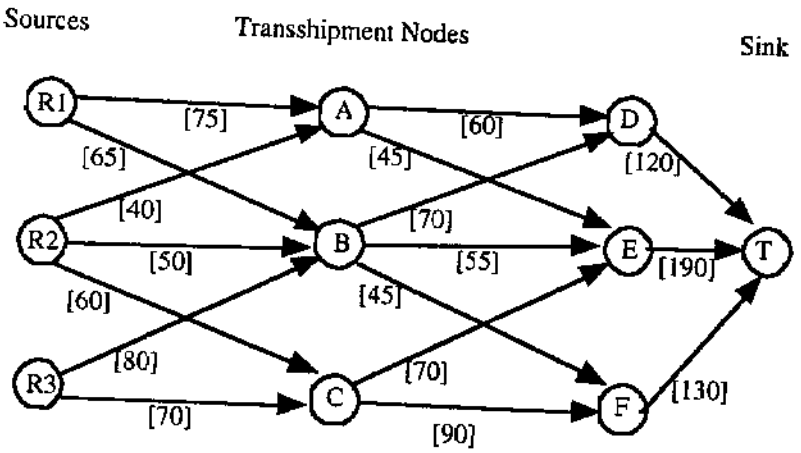
Maximize $\sum_{j=2}^{N} x_{1j}$

subject to $\sum_{\substack{j=1 \\ j \neq i}}^{N} x_{ij} - \sum_{\substack{j=1 \\ j \neq i}}^{N} x_{ji} = 0$ for $i = 2, 3, \ldots, N-1$

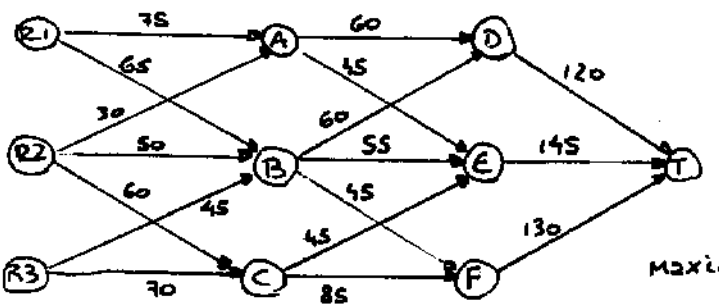
$0 \leq x_{ij} \leq c_{ij}$ where $c_{ij} = 0$ if (i,j) is not a branch

9.5-3

a)



b)



maximum flow = 395

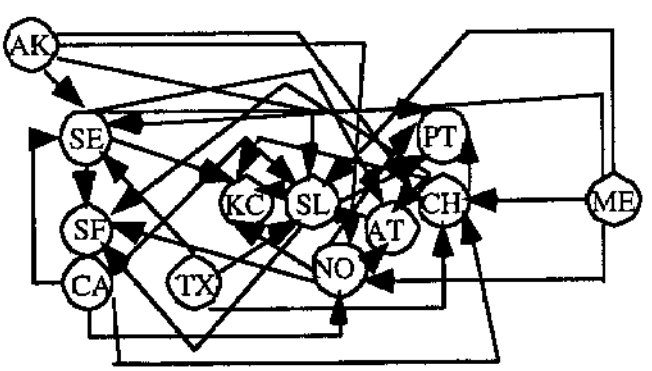
c)

From	To	Ship	Capacity
R1	A	75	75
R1	B	65	65
R2	A	30	40
R2	B	50	80
R3	B	45	80
R3	C	70	70
A	D	60	60
A	E	45	45
B	D	60	70
C	E	45	70
C	F	85	90
D	T	120	120
E	T	145	190
F	T	130	130

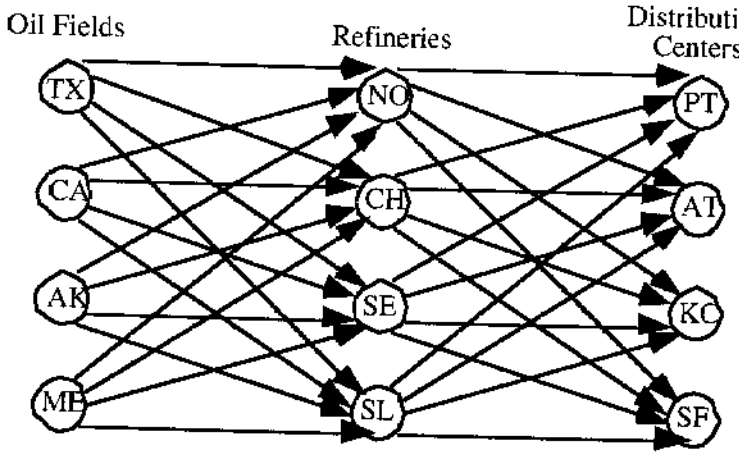
Maximum Flow = 395

Nodes	Net Flow		Supply/Demand
R1	140		
R2	140		
R3	115		
C	0	=	0
D	0	=	0
E	0	=	0
F	0	=	0
T	-395		

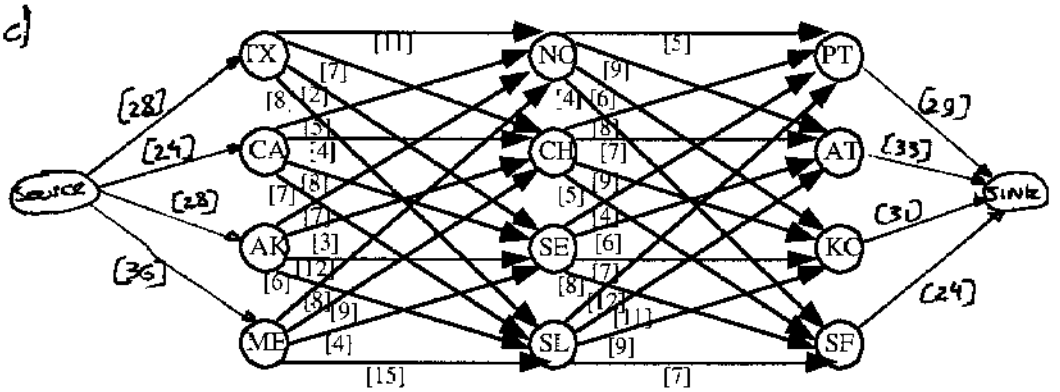
9.5-4 a)



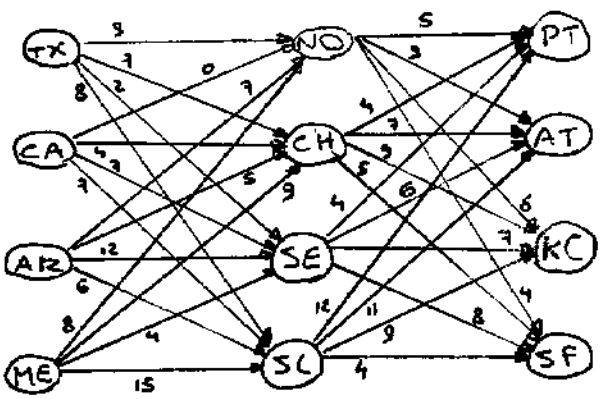
9.3-4 b)


Oil Fields
Refineries
Distribution Centers
TX
CA
AK
ME
NO
CH
SE
SL
PT
AT
KC
SF

c)


Source
TX
CA
AK
ME
NO
CH
SE
SL
PT
AT
KC
SF
Sink
[28]
[24]
[28]
[36]
[29]
[33]
[31]
[24]

d)


TX
CA
AK
ME
NO
CH
SE
SL
PT
AT
KC
SF

e)

From	To	Ship	Capacity
TX	NO	9	11
TX	CH	7	7
TX	SE	2	2
TX	SL	8	8
CA	NO	0	5
CA	CH	4	4
CA	SE	7	8
CA	SL	7	7
AK	NO	7	7
AK	CH	5	5
AK	SE	12	12
AK	SL	6	6
ME	NO	8	8
ME	CH	9	9
ME	SE	4	4
ME	SL	15	15
NO	PT	5	5
NO	AT	9	9
NO	KC	6	6
NO	SF	4	4
CH	PT	4	8
CH	AT	7	7
CH	KC	9	9
CH	SF	5	5
SE	PT	4	4
SE	AT	6	6
SE	KC	7	7
SE	SF	8	8
SL	PT	12	12
SL	AT	11	11
SL	KC	9	9
SL	SF	4	7

Maximum Flow = 110

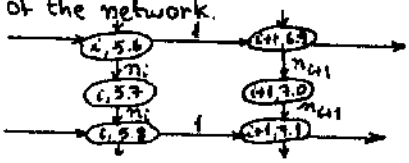
Nodes	Net Flow		Supply/Demand
TX	26		
CA	18		
AK	30		
ME	36		
NO	0	=	0
CH	0	=	0
SE	0	=	0
SL	0	=	0
PT	-25		
AT	-33		
KC	-31		
SF	-21		

9.5-5 For convenience, call Faireparc station siding 0 and the Portstown station siding s+1. Let node (i,j) represent siding i at time j for $i = 0, 1, \ldots, s, s+1$ and $j = 0, .1, .2, \ldots, 23.9$. Node $(0,0)$ is the source and node $(s+1, 23.9)$ the sink.

Arcs with capacity 1 will exist between nodes (i,j) and $(i+1, j+t_i)$ if and only if a freight train leaving siding i at time j could not be overtaken by a scheduled passenger train before it reached siding $i+1$. Arcs with capacity n_i will exist between nodes (i,j) and $(i,j+1)$ for $j = 0, .1, .2, \ldots, 23.8$

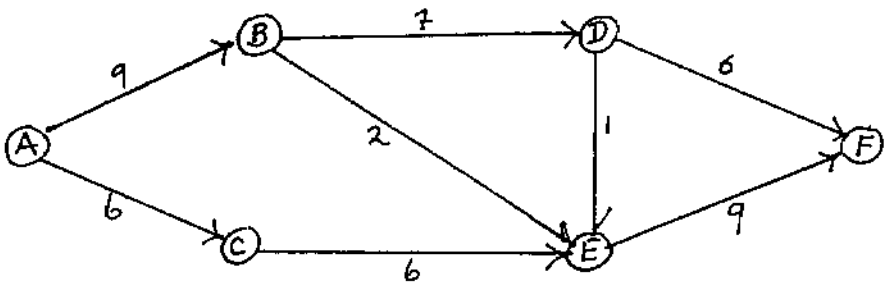
No other arcs exist.

For example, if $t_i = 1.3$ and a scheduled passenger train could overtake a freight train leaving siding i at time 5.7 before it reached siding $i+1$, the following would be a portion of the network.



Solving the maximal flow problem will maximize the number of freight trains that are sent.

9.5-6

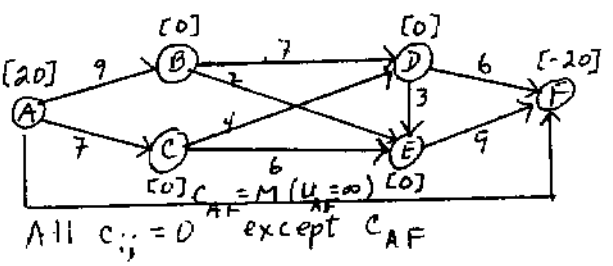


From	To	Ship	Capacity
A	B	8	9
A	C	7	7
B	D	7	7
B	E	1	2
C	D	2	4
C	E	5	6
D	E	3	3
D	F	6	6
E	F	9	9

Nodes	Net Flow		Supply/Demand
A	15		
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
F	-15		

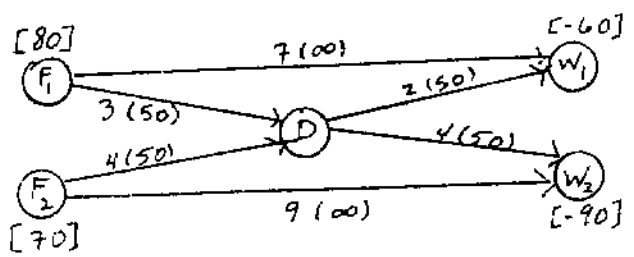
Maximum Flow = **15**

9.6-1



All $c_{ij} = 0$ except c_{AF}

9.6-2 a)



9.6-2 b) Min $7X_{F_1W_1} + 3X_{F_1D} + 2X_{DW_1} + 4X_{F_2D} + 4X_{DW_2} + 9X_{F_2W_2}$

subject to

$X_{F_1W_1} + X_{F_1D} + X_{F_2D} + X_{F_2W_2} = 80$

$X_{F_1W_1} + X_{DW_1} = 70$

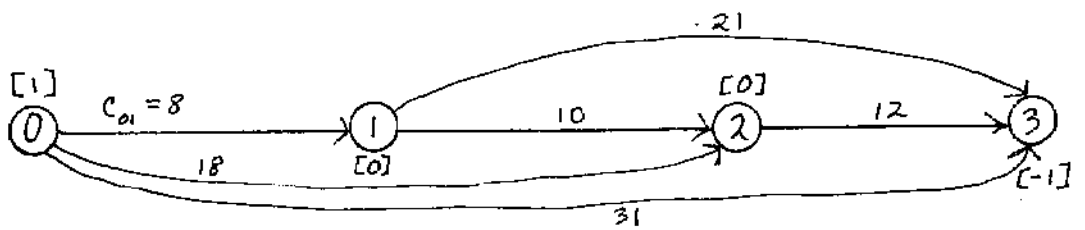
$X_{DW_2} + X_{F_2W_2} = 60$

$X_{F_1D} - X_{DW_1} + X_{F_2D} - X_{DW_2} = 90$

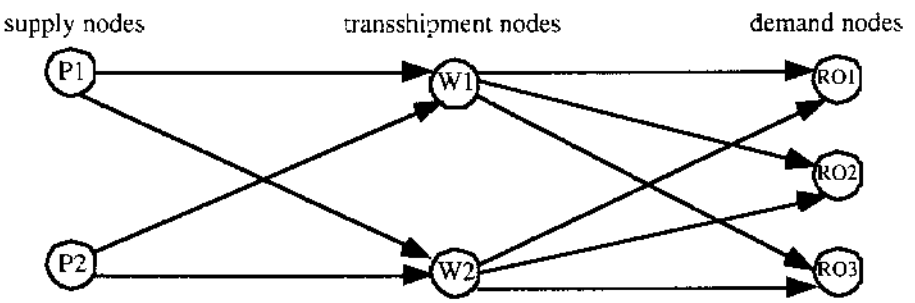
$0 = 0$

$0 \le X_{F_1D}, X_{F_2D}, X_{DW_1}, X_{DW_2} \le 50$

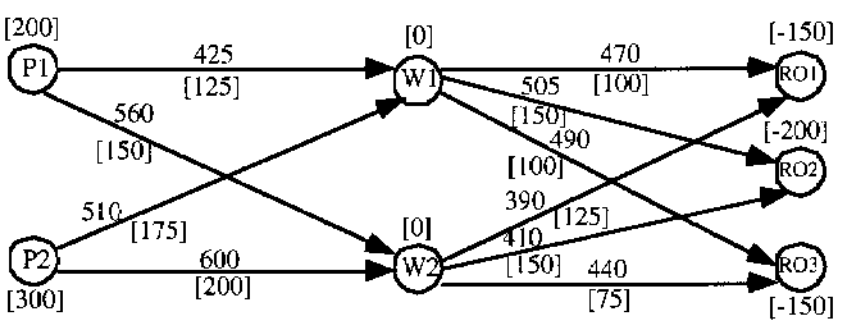
9.6-3



9.6-4 a)



b)



9.6-4

c)

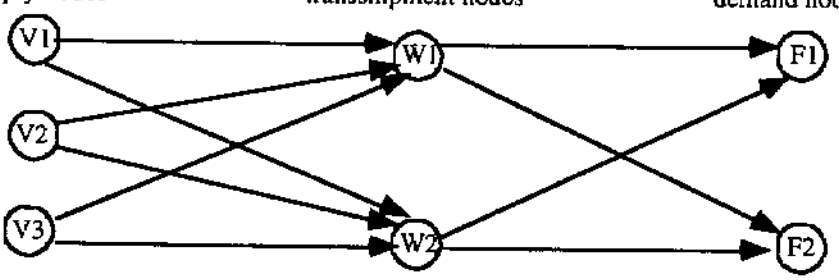
From	To	Ship		Capacity	Unit Cost
P1	W1	**125**	≤	125	$425
P1	W2	**75**	≤	150	$560
P2	W1	**125**	≤	175	$510
P2	W2	**175**	≤	200	$600
W1	RO1	**100**	≤	100	$470
W1	RO2	**50**	≤	150	$505
W1	RO3	**100**	≤	100	$490
W2	RO1	**50**	≤	125	$390
W2	RO2	**150**	≤	150	$410
W2	RO3	**50**	≤	75	$440

Nodes	Net Flow		Output/Demand
P1	200	=	200
P2	300	=	300
W1	0	=	0
W2	0	=	0
RO1	-150	=	-150
RO2	-200	=	-200
RO3	-150	=	-150

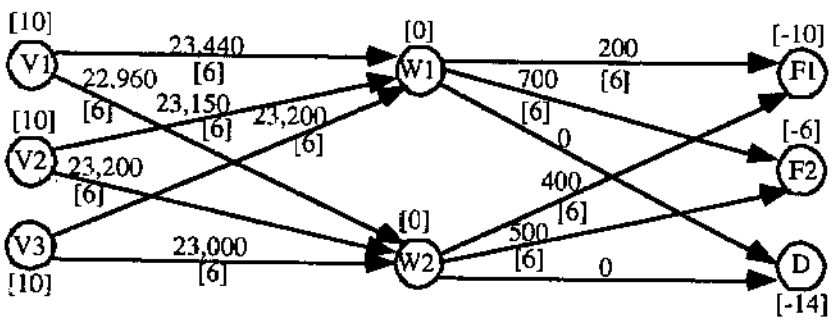
Total Cost = **$ 488,125**

9.6-5

a) supply nodes transshipment nodes demand nodes



b)



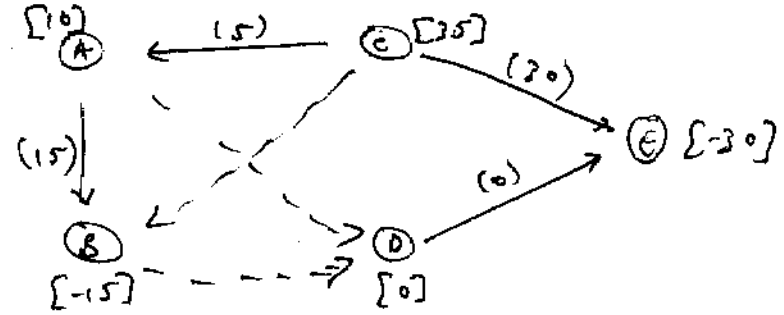
c)

From	To	Ship		Capacity	Unit Cost
V1	W1	**4**	≤	6	$23,440
V1	W2	**6**	≤	6	$22,960
V2	W1	**6**	≤	6	$23,150
V2	W2	**4**	≤	6	$23,200
V3	W1	**4**	≤	6	$23,200
V3	W2	**6**	≤	6	$23,000
W1	F1	**6**	≤	6	$200
W1	F2	**0**	≤	6	$700
W1	D	**8**		-	$0
W2	F1	**4**	≤	6	$400
W2	F2	**6**	≤	6	$500
W2	D	**6**		-	$0

Nodes	Net Flow		Output/Demand
V1	10	=	10
V2	10	=	10
V3	10	=	10
W1	0	=	0
W2	0	=	0
F1	-10	=	-10
F2	-6	=	-6
D	-14	=	-14

Total Cost = **$ 699,820**

9.7-1 a)



b) calculate Δ for nonbasic arcs:

$\Delta_{BD} = 5+4-3+(-6)+2 = 2$

$\Delta_{AD} = 5+4-3+(-6) = 0$

$\Delta_{CB} = (-3)-2-(-6) = 1$

} all ≥ 0 $\Rightarrow$ optimal

$\Delta_{AD} = 0 \Rightarrow$ multiple optima exist

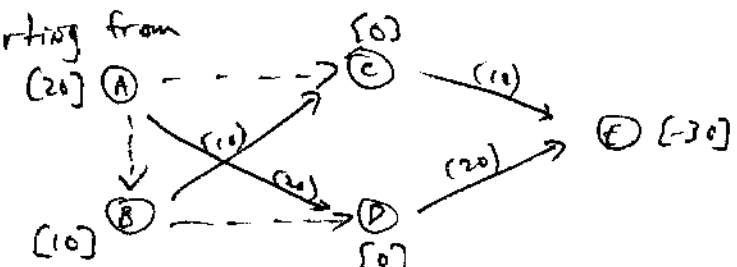
Network simplex:

$\theta = 5$, CA is leaving basic arc (reverses)

From this and part (a) we see that optimal nonbasic solutions have $x_{AB} = 15$, $x_{AC} = \theta$, $x_{AD} = 5-\theta$, $x_{CE} = 25+\theta$, $x_{DE} = 5-\theta$, where $0 \leq \theta \leq 5$ and $C \to B$, $B \to D$ are 'nonbasic' arcs.

---

c) starting from



Network simplex gives

$\Delta_{AC} = 6+3-4-5 = 0$

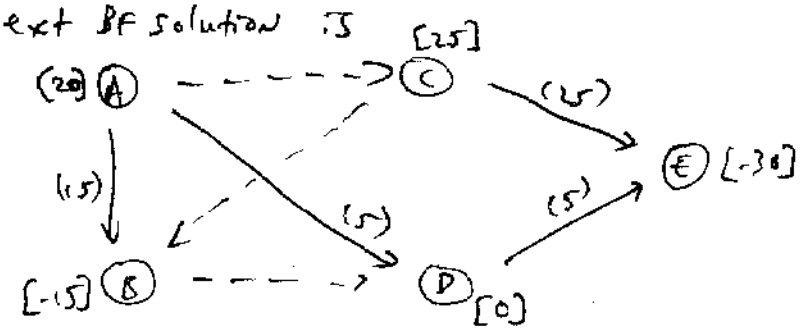
$\Delta_{AB} = 2+3+3-4-5 = -1 < 0 \leftarrow$ entering arc

$\Delta_{BD} = 5+4-3-3 = 3$

$\theta = 15$, BC is leaving arc (reverses)

9.7-1 c) (CONT'D)

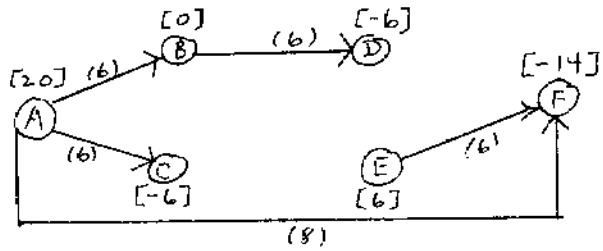
Next BF solution :



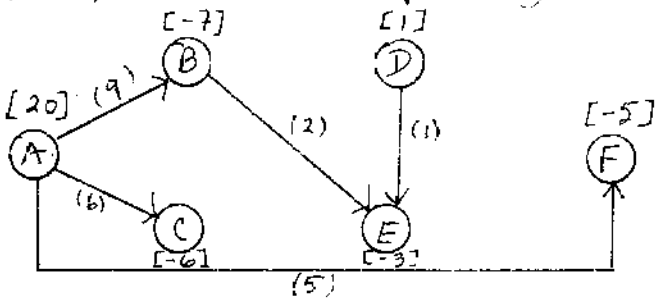
From (6) we recognize this as optimal.

9.7-2

a)



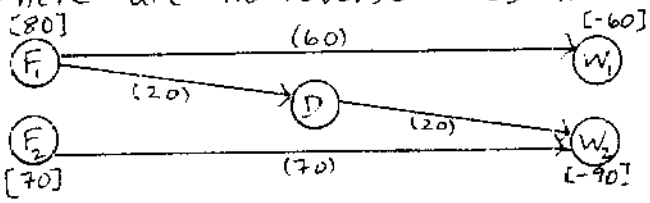
b) The final feasible spanning tree is



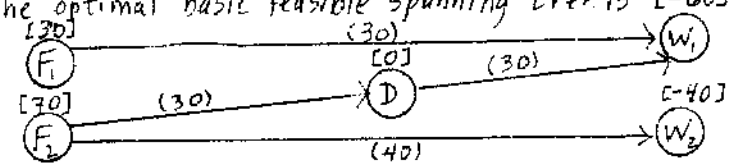
And the flow to which it corresponds is the same as in 9.5-6

9.7-3 There are no reverse arcs in this solution

a)

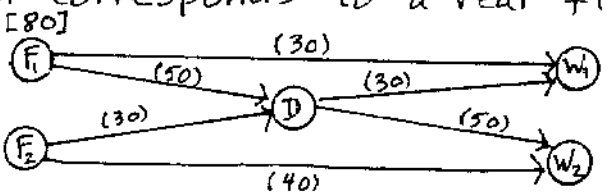


b) The optimal basic feasible spanning tree is [-60]



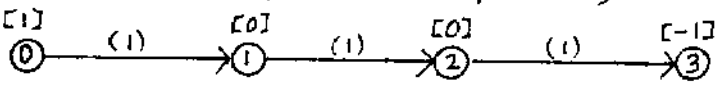
9.7-3 (CONT'D)

Which corresponds to a real flow of

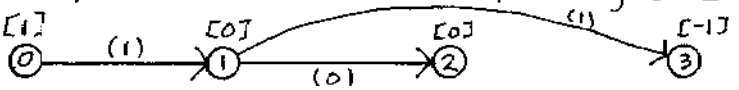


and a cost of 1,100

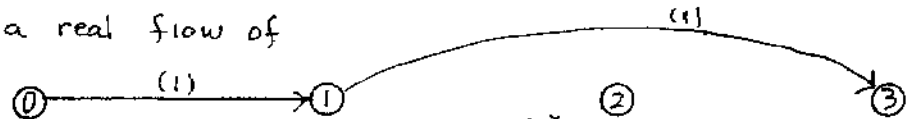
9.7-4 Initial basic feasible spanning tree is



The optimal basic feasible spanning tree is



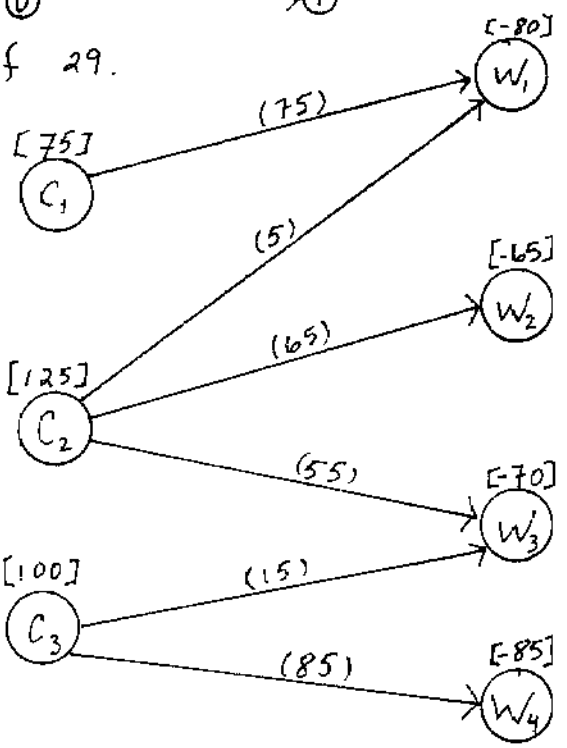
which has a real flow of



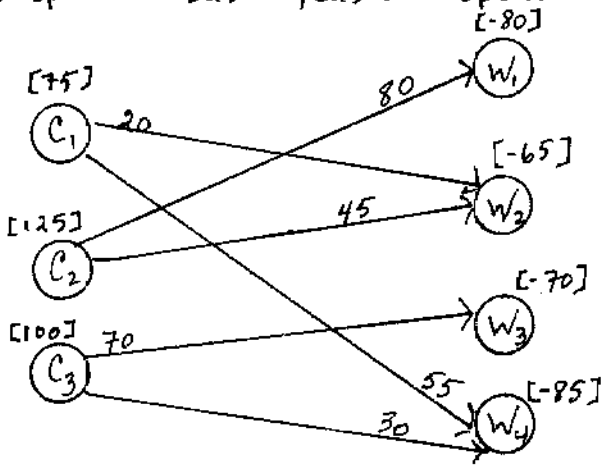
and a cost of 29.

9.7-5

Initial basic spanning tree:

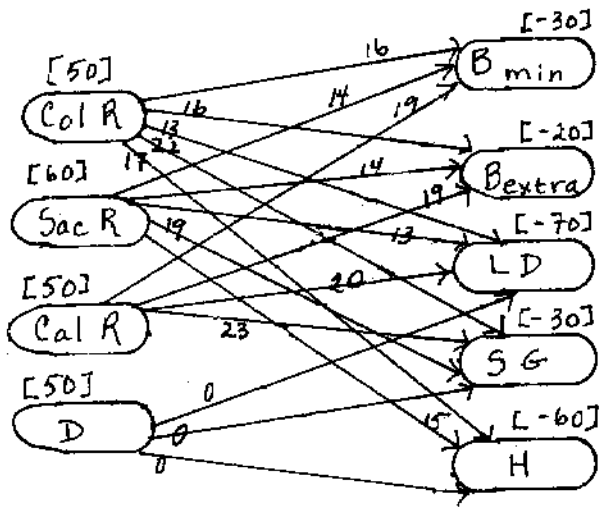


9.7-5 (continued)
The optimal basic feasible spanning tree is:

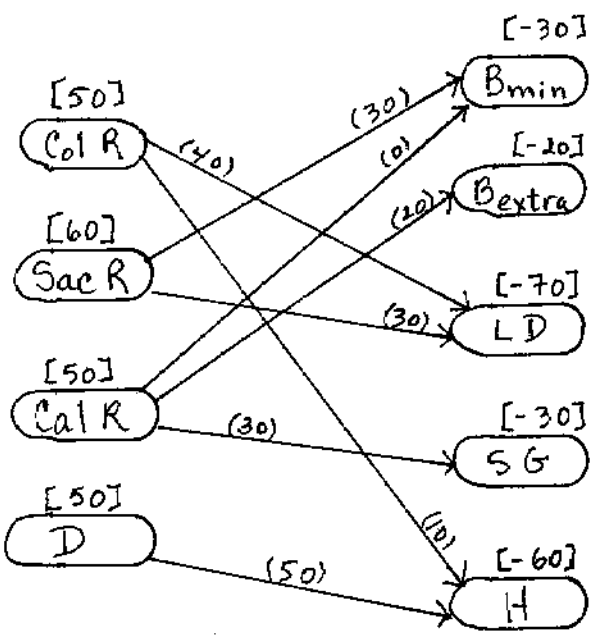


which corresponds to the optimal solution given in Sec. 8.1.

9.7-6 a)

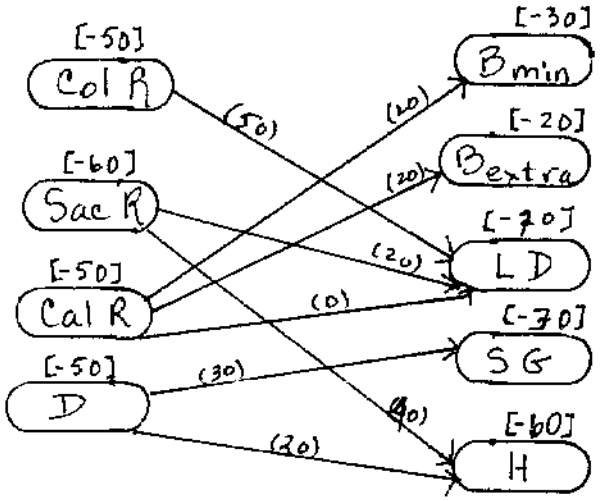


b) Initial basic feasible spanning tree



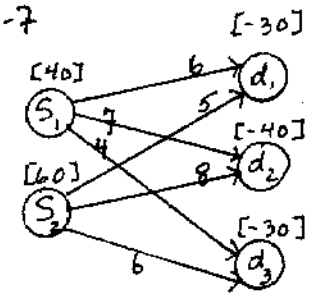
9.7-6 b) (continued)

The optimal basic feasible spanning tree is:

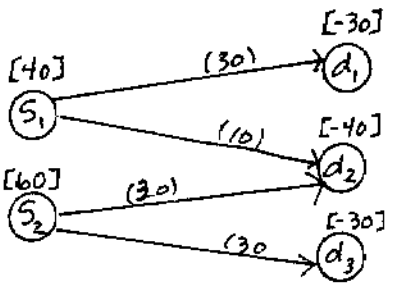


The sequence of basic feasible solutions is identical with the transportation simplex method.

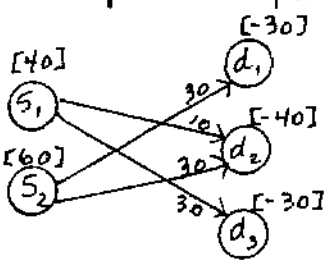
9.7-7



Initial Basic Feasible Spanning Tree →

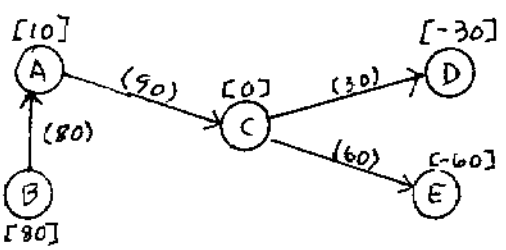


The optimal feasible spanning tree is:

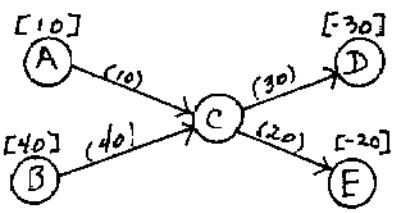


With a cost of 580

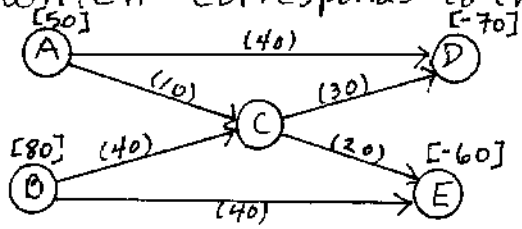
9.7-8



The optimal basic feasible Spanning tree is →



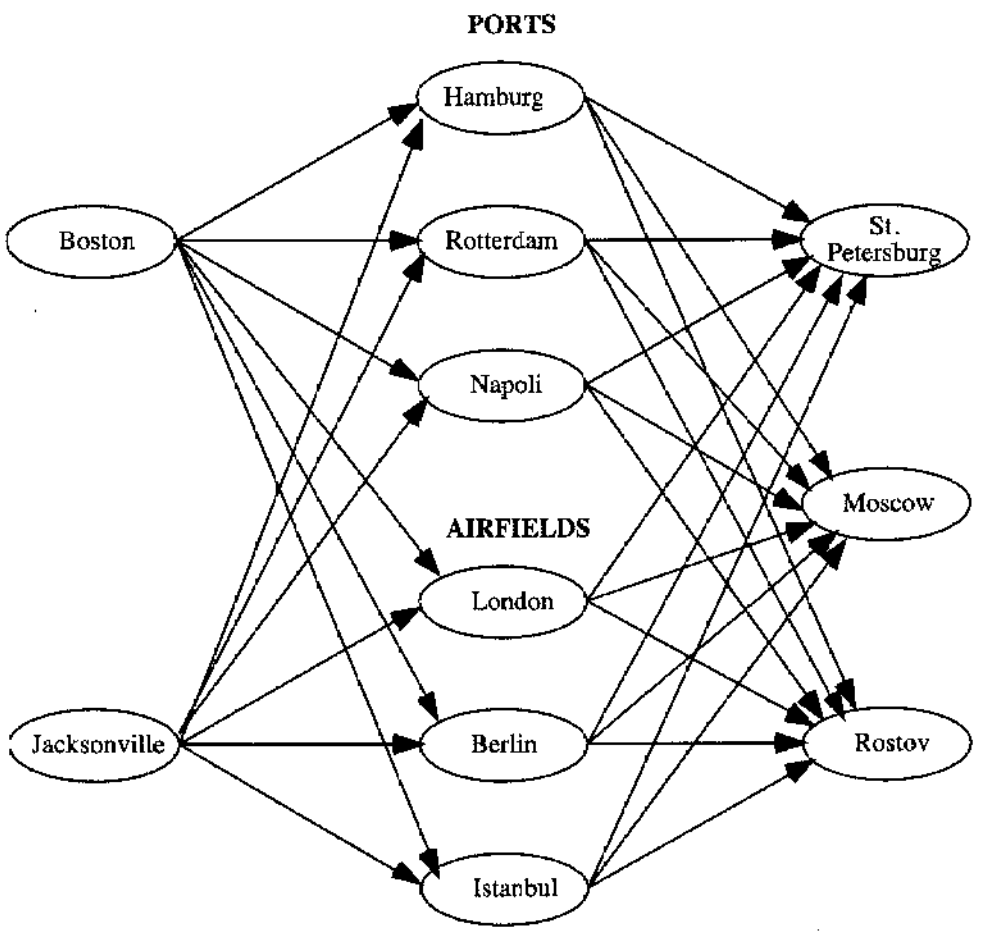
which corresponds to the real flows



and a cost of 750.

Cases

9-1 a) The network showing the different routes troops and supplies may follow to reach the Russian Federation appears below.



2400

b) The President is only concerned about how to most quickly move troops and supplies from the United States to the three strategic Russian cities. Obviously, the best way to achieve this goal is to find the fastest connection between the US and the three cities. We therefore need to find the shortest path between the US and each of the three cities.

The President only cares about the time it takes to get the troops and supplies to Russia. It does not matter how great a distance the troops and supplies cover. Therefore we define the arc length between two nodes in the network to be the time it takes to travel between the respective cities. For example, the distance between Boston and London equals 6,200 km. The mode of transportation between the cities is a Starfighter traveling at a speed of 400 miles per hour * 1.609 km per mile = 643.6 km per hour. The time is takes to bring troops and supplies from Boston to London equals 6,200 km / 643.6 km per hour = 9.6333 hours. Using this approach we can compute the time of travel along all arcs in the network.

By simple inspection and common sense it is apparent that the fastest transportation involves using only airplanes. We therefore can restrict ourselves to only those arcs in the network where the mode of transportation is air travel. We can omit the three port cities and all arcs entering and leaving these nodes.

Finally, we define a new node ("dummy" node) in the network called "US," and we introduce two new arcs: one going from the US to Boston and the other going from the US to Jacksonville. The arc length on both new arcs equals 0. The objective is now to find the shortest path from the US to each of the three Russian cities. We define the US node to be a supply node with supply 3, and we define each of the three nodes representing Russian cities as demand nodes with a demand of -1. The nodes representing the three European airfields – London, Berlin, and Istanbul – are all transshipment nodes.

The following spreadsheet shows the entire linear programming model, which identifies the three shortest paths.

	A	B	C	D	E	F	G	H	I	J	K
1		From	To	On Route	Distance	Time (hr)		Nodes	Net Flow		Supply/Demand
2		US	Boston	3	0	0		US	3	=	3
3		US	Jacksonville	0	0	0		Boston	0	=	0
4		Boston	London	2	6200	9.63331		Jacksonville	0	=	0
5		Boston	Berlin	1	7250	11.2648		London	0	=	0
6		Boston	Istanbul	0	8300	12.8962		Berlin	0	=	0
7		Jacksonville	London	0	7900	12.2747		Istanbul	0	=	0
8		Jacksonville	Berlin	0	9200	14.2946		St. Petersburg	-1	=	-1
9		Jacksonville	Istanbul	0	10100	15.693		Moscow	-1	=	-1
10		London	St. Petersburg	1	1980	3.07644		Rostov	-1	=	-1
11		London	Moscow	1	2300	3.57365					
12		London	Rostov	0	2860	4.44375					
13		Berlin	St. Petersburg	0	1280	1.98881					
14		Berlin	Moscow	0	1600	2.48602					
15		Berlin	Rostov	1	1730	2.688					
16		Istanbul	St. Petersburg	0	2040	3.16967					
17		Istanbul	Moscow	0	1700	2.64139					
18		Istanbul	Rostov	0	990	1.53822					
19											
20			Total Time =	39.86948415							

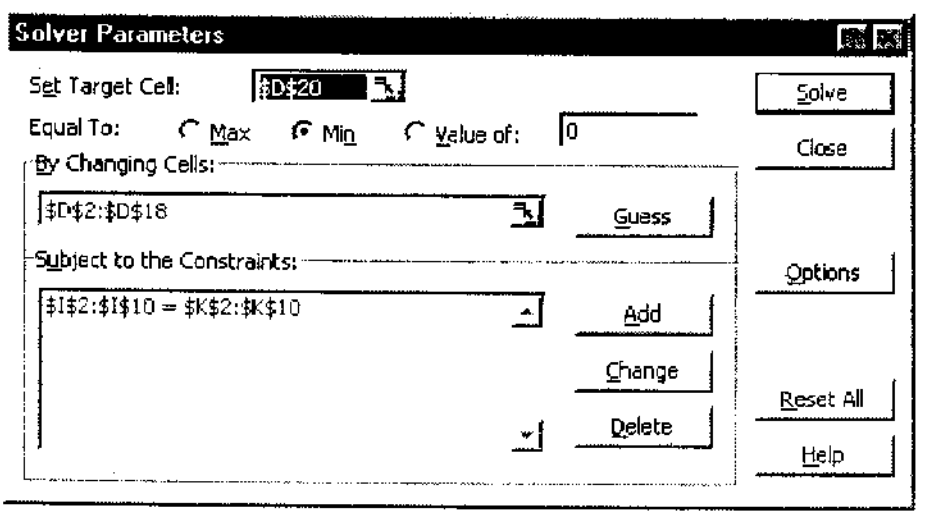
The spreadsheet contains the following formulas:

	F
1	Time (hr)
2	0
3	0
4	=E4/(400*1.609)
5	=E5/(400*1.609)
6	=E6/(400*1.609)
7	=E7/(400*1.609)
8	=E8/(400*1.609)
9	=E9/(400*1.609)
10	=E10/(400*1.609)
11	=E11/(400*1.609)
12	=E12/(400*1.609)
13	=E13/(400*1.609)
14	=E14/(400*1.609)
15	=E15/(400*1.609)
16	=E16/(400*1.609)
17	=E17/(400*1.609)
18	=E18/(400*1.609)
19	
20	

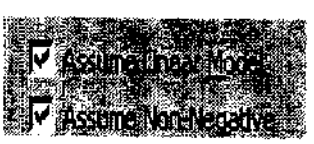
	I
1	Net Flow
2	=SUM(D2:D3)
3	=-D2+SUM(D4:D6)
4	=-D3+SUM(D7:D9)
5	=-D4-D7+D10+D11+D12
6	=-D5-D8+D13+D14+D15
7	=-D6-D9+D16+D17+D18
8	=-D10-D13-D16
9	=-D11-D14-D17
10	=-D12-D15-D18
11	

	C	D
20	Total Time =	=SUMPRODUCT(D2:D18,F2:F18)

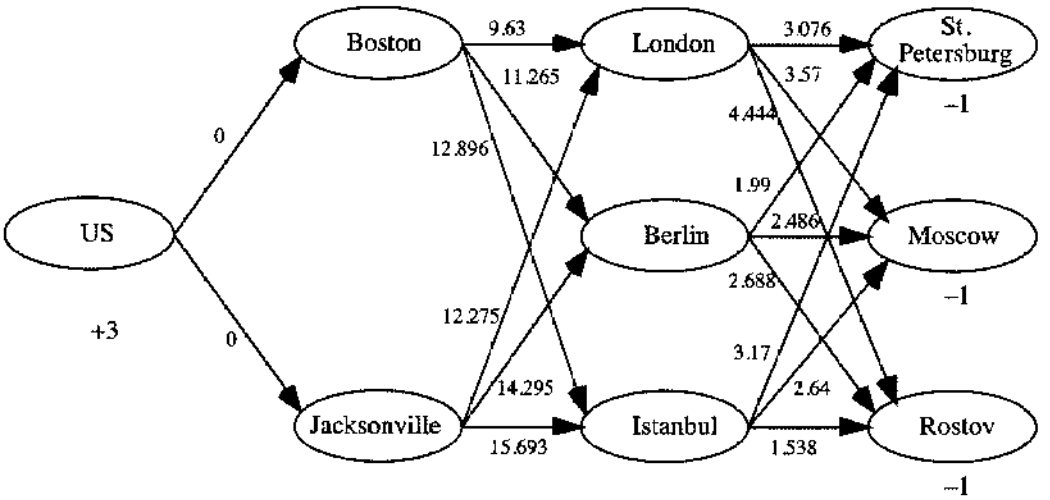
The solver dialogue box appears as follows.



Throughout the analysis of this case we use the following solver options.



From the optimal solution to the linear programming model we see that the shortest path from the US to Saint Petersburg is Boston → London → Saint Petersburg with a total travel time of 12.710 hours. The shortest path from the US to Moscow is Boston → London → Moscow with a total travel time of 13.207 hours. The shortest path from the US to Rostov is Boston → Berlin → Rostov with a total travel time of 13.953 hours. The following network diagram highlights these shortest paths.



c) The President must satisfy each Russian city's military requirements at minimum cost. Therefore, this problem can be solved as a minimum-cost network flow problem. The two nodes representing US cities are supply nodes with a supply of 500 each (we measure all weights in 1000 tons). The three nodes representing Saint Petersburg, Moscow, and Rostov are demand nodes with demands of –320, -440, and –240, respectively. All nodes representing European airfields and ports are transshipment nodes. We measure the flow along the arcs in 1000 tons. For some arcs, capacity constraints are given. All arcs from the European ports into Saint Petersburg have zero capacity. All truck routes from the European ports into Rostov have a transportation limit of 2,500*16 = 40,000 tons. Since we measure the arc flows in 1000 tons, the corresponding arc capacities equal 40. An analogous computation yields arc capacities of 30 for both the arcs connecting the nodes London and Berlin to Rostov. For all other nodes we determine natural arc capacities based on the supplies and demands at the nodes. We define the unit costs along the arcs in the network in $1000 per 1000 tons. For example, the cost of transporting 1 ton of material from Boston to Hamburg equals $30,000 / 240 = $125, so the costs of transporting 1000 tons from Boston to Hamburg equals $125,000.

The objective is to satisfy all demands in the network at minimum cost. The following spreadsheet shows the entire linear programming model.

	A	B	C	D	E	F	G	H	I	J	K
1	From	To	Ship	Capacity (in 1000 tons)	Cost of Transport	Unit Cost (in $1000 per 1000 tons)		Nodes	Net Flow		Supply/Demand
2	Boston	Hamburg	440	500	30000	125		Boston	500	=	500
3	Boston	Rotterdam	0	500	30000	125		Jacksonville	500	=	500
4	Boston	Napoli	0	500	32000	133.3333333		Hamburg	0	=	0
5	Boston	London	0	500	45000	300		Rotterdam	0	=	0
6	Boston	Berlin	0	500	50000	333.3333333		Napoli	0	=	0
7	Boston	Istanbul	60	500	55000	366.6666667		London	0	=	0
8	Jacksonville	Hamburg	0	500	49000	200		Berlin	0	=	0
9	Jacksonville	Rotterdam	0	500	44000	183.3333333		Istanbul	0	=	0
10	Jacksonville	Napoli	0	500	56000	233.3333333		St. Petersburg	-320	=	-320
11	Jacksonville	London	350	500	49000	326.6666667		Moscow	-440	=	-440
12	Jacksonville	Berlin	0	500	57000	380		Rostov	-240	=	-240
13	Jacksonville	Istanbul	150	500	61000	406.6666667					
14	Hamburg	St. Petersburg	0	0	3000	187.5					
15	Rotterdam	St. Petersburg	0	0	3000	187.5					
16	Napoli	St. Petersburg	0	0	5000	312.5					
17	London	St. Petersburg	320	320	22000	146.6666667					
18	Berlin	St. Petersburg	0	320	24000	160					
19	Istanbul	St. Petersburg	0	320	28000	186.6666667					
20	Hamburg	Moscow	440	440	4000	250					
21	Rotterdam	Moscow	0	440	5000	312.5					
22	Napoli	Moscow	0	440	5000	312.5					
23	London	Moscow	0	440	19000	126.6666667					
24	Berlin	Moscow	0	440	22000	146.6666667					
25	Istanbul	Moscow	0	440	25000	166.6666667					
26	Hamburg	Rostov	0	40	7000	437.5					
27	Rotterdam	Rostov	0	40	8000	500					
28	Napoli	Rostov	0	40	9000	562.5					
29	London	Rostov	30	30	4000	26.66666667					
30	Berlin	Rostov	0	30	23000	153.3333333					
31	Istanbul	Rostov	210	240	2000	13.33333333					
32											
33		Total Cost =	412866.6667								

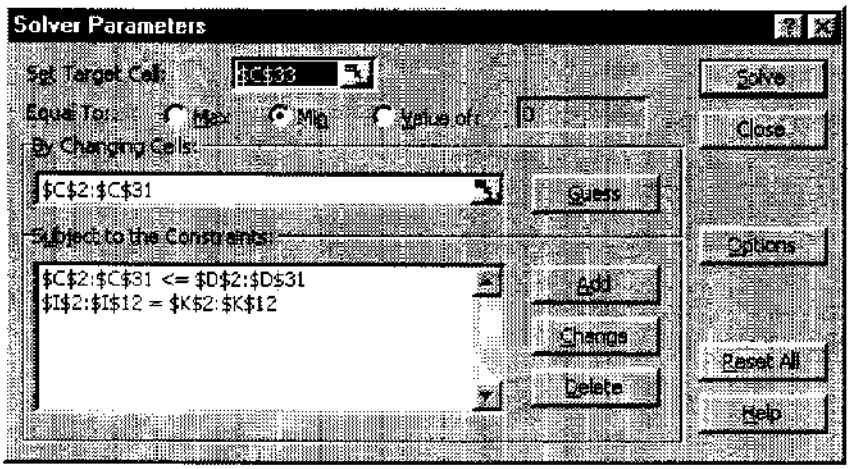
The following formulas appear in the spreadsheet.

	I
1	Net Flow
2	=SUM(C2:C7)
3	=SUM(C8:C13)
4	=-C2-C8+C14+C20+C26
5	=-C3-C9+C15+C21+C27
6	=-C4-C10+C16+C22+C28
7	=-C5-C11+C17+C23+C29
8	=-C6-C12+C18+C24+C30
9	=-C7-C13+C19+C25+C31
10	=-SUM(C14:C19)
11	=-SUM(C20:C25)
12	=-SUM(C26:C31)
13	

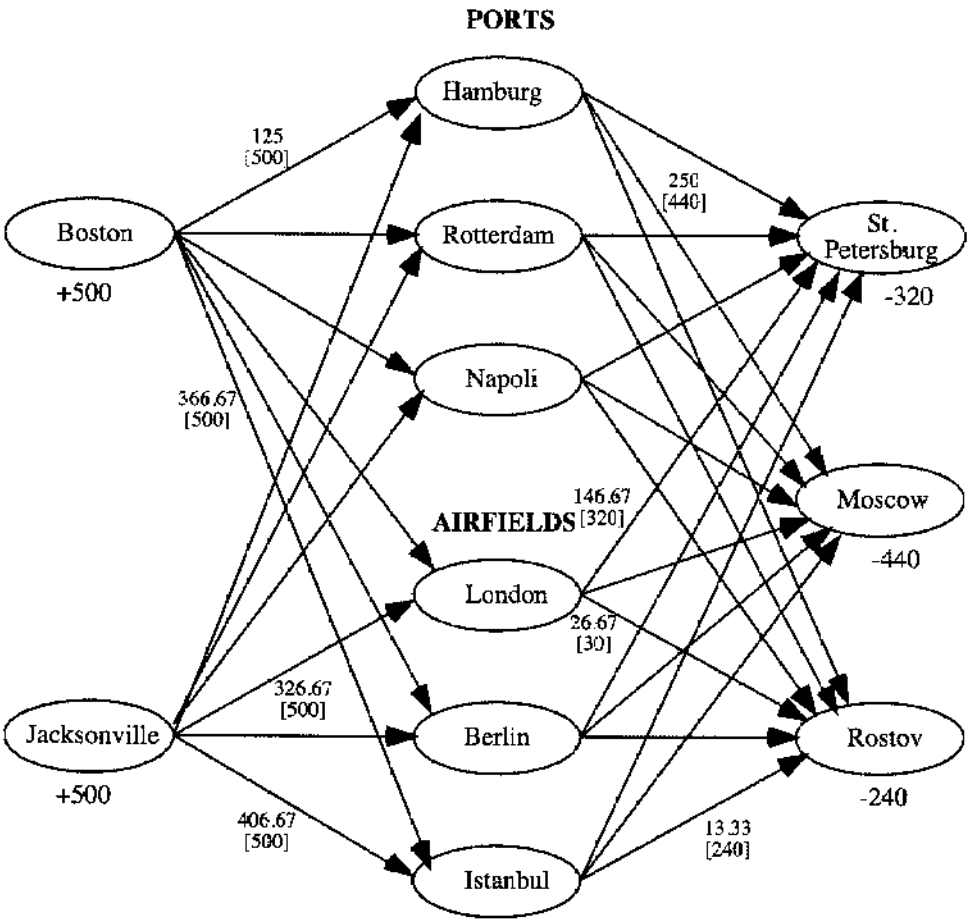
	F
1	Unit Cost (in $1000 per 1000 tons)
2	=E2/240
3	=E3/240
4	=E4/240
5	=E5/150
6	=E6/150
7	=E7/150
8	=E8/240
9	=E9/240
10	=E10/240
11	=E11/150
12	=E12/150
13	=E13/150
14	=E14/16
15	=E15/16
16	=E16/16
17	=E17/150
18	=E18/150
19	=E19/150
20	=E20/16
21	=E21/16
22	=E22/16
23	=E23/150
24	=E24/150
25	=E25/150
26	=E26/16
27	=E27/16
28	=E28/16
29	=E29/150
30	=E30/150
31	=E31/150
32	

	B	C
33	Total Cost =	**=SUMPRODUCT(C2:C31,F2:F31)**

We use the following solver dialogue box for this model.



The total cost of the operation equals $412,866,666.67. The entire supply for Saint Petersburg is supplied from Jacksonville via London. The entire supply for Moscow is supplied from Boston via Hamburg. Of the 240 (= 240,000 tons) demanded by Rostov, 60 are shipped from Boston via Istanbul, 150 are shipped from Jacksonville via Istanbul, and 30 are shipped from Jacksonville via London. The paths used to ship supplies to Saint Petersburg, Moscow, and Rostov are highlighted on the following network diagram.


PORTS
Hamburg
Rotterdam
Napoli
London
Berlin
Istanbul
Boston
+500
Jacksonville
+500
St. Petersburg
-320
Moscow
-440
Rostov
-240
125
[500]
366.67
[500]
326.67
[500]
406.67
[500]
250
[440]
146.67
[320]
26.67
[30]
13.33
[240]
AIRFIELDS

d) Now the President wants to maximize the amount of cargo transported from the US to the Russian cities. In other words, the President wants to maximize the flow from the two US cities to the three Russian cities. All the nodes representing the European ports and airfields are once again transshipment nodes. The flow along an arc is again measured in thousands of tons. The new restrictions can be transformed into arc capacities using the same approach that was used in part (c). The objective is now to maximize the combined flow into the three Russian cities.

The linear programming model describing the maximum flow problem appears as follows.

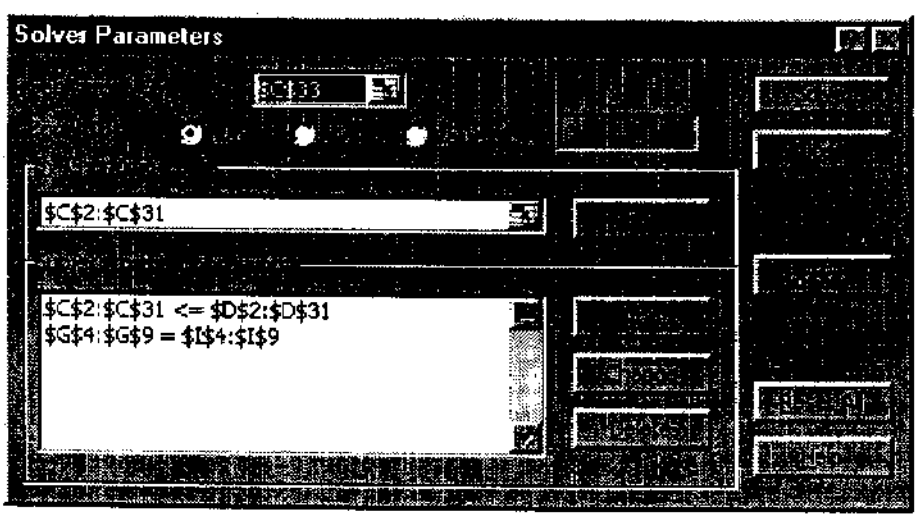
	A	B	C	D	E	F	G	H	I
1	From	To	Ship	Capacity (in 1000 tons)		Nodes	Net Flow		Supply/Demand
2	Boston	Hamburg	19.2	500		Boston	282.2		
3	Boston	Rotterdam	21.6	500		Jacksonville	240		
4	Boston	Napoli	46.4	500		Hamburg	0	=	0
5	Boston	London	75	75		Rotterdam	0	=	0
6	Boston	Berlin	45	45		Napoli	0	=	0
7	Boston	Istanbul	75	75		London	0	=	0
8	Jacksonville	Hamburg	0	500		Berlin	0	=	0
9	Jacksonville	Rotterdam	0	500		Istanbul	0	=	0
10	Jacksonville	Napoli	0	500		St. Petersburg	-225		
11	Jacksonville	London	90	90		Moscow	-104.8		
12	Jacksonville	Berlin	75	75		Rostov	-192.4		
13	Jacksonville	Istanbul	75	105					
14	Hamburg	St. Petersburg	0	0					
15	Rotterdam	St. Petersburg	0	0					
16	Napoli	St. Petersburg	0	0					
17	London	St. Petersburg	150	150					
18	Berlin	St. Petersburg	75	75					
19	Istanbul	St. Petersburg	0	0					
20	Hamburg	Moscow	11.2	11.2					
21	Rotterdam	Moscow	9.6	9.6					
22	Napoli	Moscow	24	24					
23	London	Moscow	0	30					
24	Berlin	Moscow	45	45					
25	Istanbul	Moscow	15	15					
26	Hamburg	Rostov	8	8					
27	Rotterdam	Rostov	12	12					
28	Napoli	Rostov	22.4	22.4					
29	London	Rostov	15	15					
30	Berlin	Rostov	0	0					
31	Istanbul	Rostov	135	135					
32									
33		Total Cost =	522.2						

The following formulas appear in the spreadsheet.

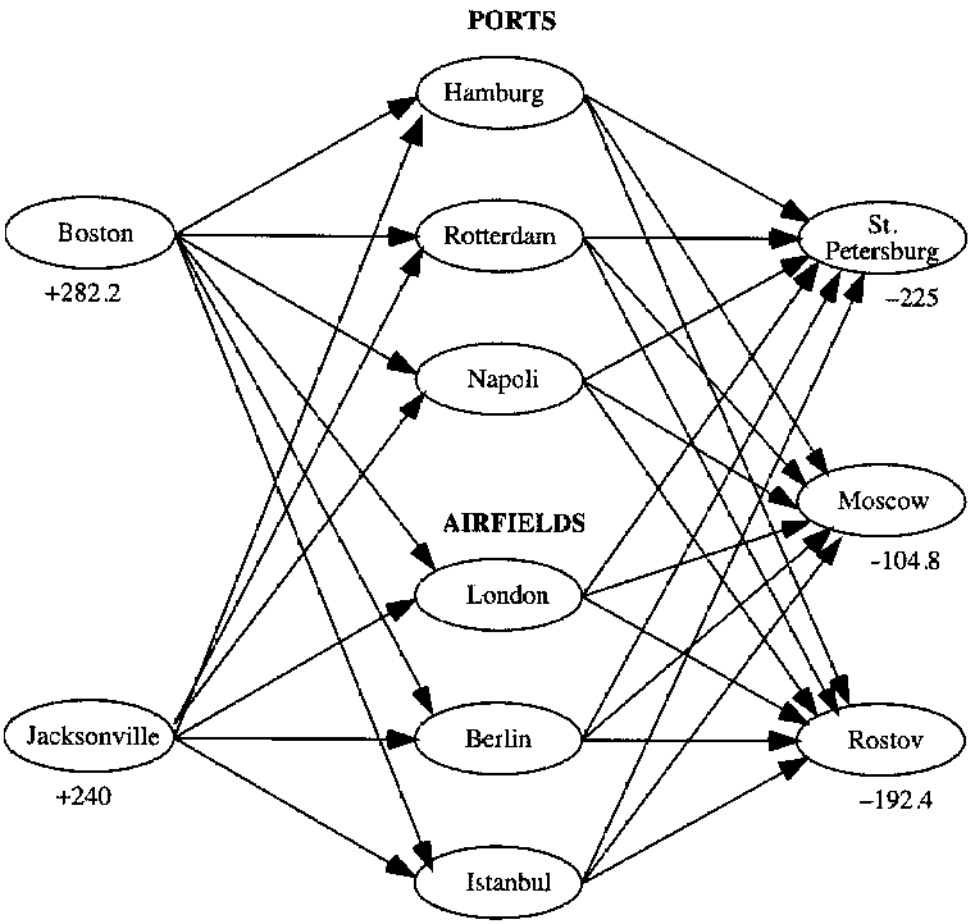
	G
1	Net Flow
2	=SUM(C2:C7)
3	=SUM(C8:C13)
4	=-C2-C8+C14+C20+C26
5	=-C3-C9+C15+C21+C27
6	=-C4-C10+C16+C22+C28
7	=-C5-C11+C17+C23+C29
8	=-C6-C12+C18+C24+C30
9	=-C7-C13+C19+C25+C31
10	=-SUM(C14:C19)
11	=-SUM(C20:C25)
12	=-SUM(C26:C31)
13	

	B	C
33	Total Cost =	SUM(G2:G3)

We use the following solver dialogue box.



The worksheet shows all the amounts that are shipped between the various cities. The total supply for Saint Petersburg, Moscow, and Rostov equals 225,000 tons, 104,800 tons, and 192,400 tons, respectively. The following network diagram highlights the paths used to ship supplies between the US and the Russian Federation.



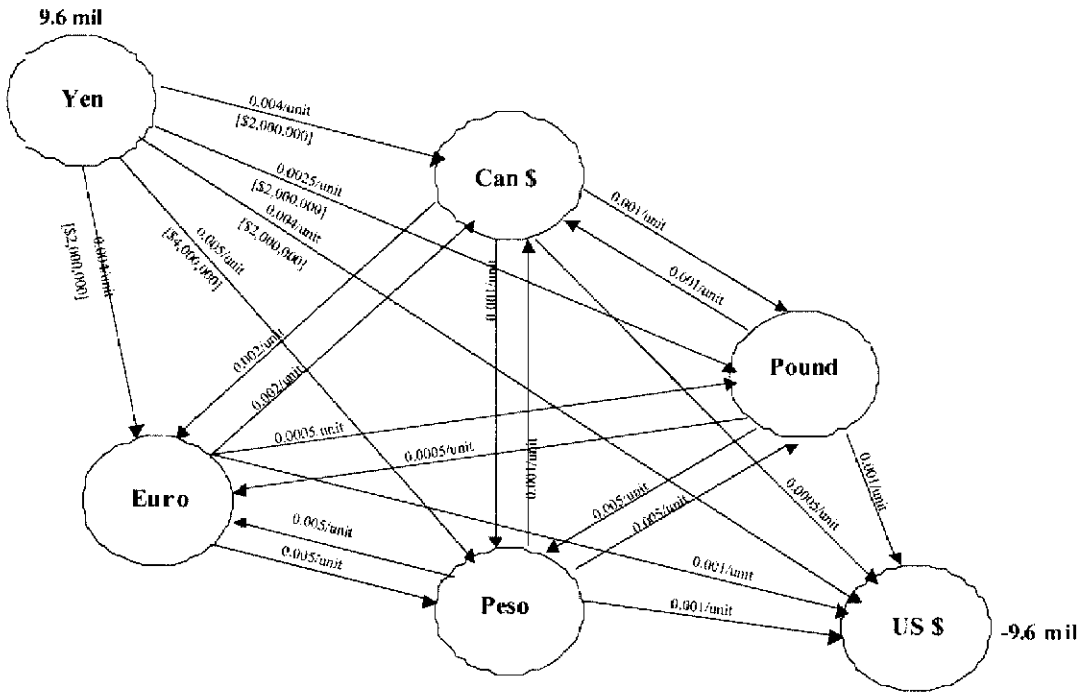
e) The creation of the new communications network is a minimum spanning tree problem. As usual, a greedy algorithm solves this type of problem.

Arcs are added to the network in the following order (one of several optimal solutions):

Rostov - Orenburg	120
Ufa - Orenburg	75
Saratov - Orenburg	95
Saratov - Samara	100
Samara - Kazan	95
Ufa – Yekaterinburg	125
Perm – Yekaterinburg	85

The minimum cost of reestablishing the communication lines is $695,000.

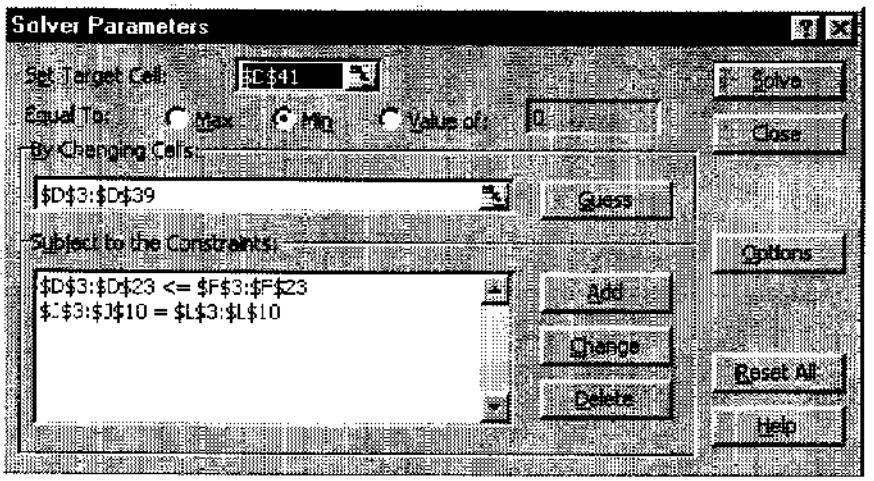
9-2 a) There are three supply nodes – the Yen node, the Rupiah node, and the Ringgit node. There is one demand node – the US$ node. Below, we draw the network originating from only the Yen supply node to illustrate the overall design of the network. In this network, we exclude both the Rupiah and Ringgit nodes for simplicity.



b) Since all transaction limits are given in the equivalent of 1000 dollars we define the flow variables as the amount in 1000's of dollars that Jake converts from one currency into another one. His total holdings in Yen, Rupiah, and Ringgit are equivalent to $9.6 million, $1.68 million, and $5.6 million, respectively. So, the supplies at the supply nodes Yen, Rupiah, and Ringgit are -$9.6 million, -$1.68 million, and -$5.6 million, respectively. The demand at the only demand node US$ equals $16.88 million. The transaction limits are capacity constraints for all arcs leaving from the nodes Yen, Rupiah, and Ringgit. The unit cost for every arc is given by the transaction cost for the currency conversion.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2		From	To	Ship		Capacity	Unit Cost		Nodes	Net Flow		Supply /Demand	
3		Yen	Rupiah	0	<=	5000	0.005		Yen	-9600	=	-9600	
4		Yen	Ringgit	0	<=	5000	0.005		Rupiah	-1680	=	-1680	
5		Yen	US$	2000	<=	2000	0.004		Ringgit	-5600	=	-5600	
6		Yen	Can$	2000	<=	2000	0.004		US$	16880	=	16880	
7		Yen	Euro	2000	<=	2000	0.004		Can$	0	=	0	
8		Yen	Pound	2000	<=	2000	0.0025		Euro	0	=	0	
9		Yen	Peso	1600	<=	4000	0.005		Pound	0	=	0	
10		Rupiah	Yen	0	<=	5000	0.005		Peso	0	=	0	
11		Rupiah	Ringgit	0	<=	2000	0.007						
12		Rupiah	US$	200	<=	200	0.005						
13		Rupiah	Can$	200	<=	200	0.003						
14		Rupiah	Euro	1000	<=	1000	0.003						
15		Rupiah	Pound	80	<=	500	0.0075						
16		Rupiah	Peso	200	<=	200	0.0075						
17		Ringgit	Yen	0	<=	3000	0.005						
18		Ringgit	Rupiah	0	<=	4500	0.007						
19		Ringgit	US$	1100	<=	1500	0.007						
20		Ringgit	Can$	0	<=	1500	0.007						
21		Ringgit	Euro	2500	<=	2500	0.004						
22		Ringgit	Pound	1000	<=	1000	0.0045						
23		Ringgit	Peso	1000	<=	1000	0.005						
24		Can$	US$	2200		-	0.0005						
25		Can$	Euro	0		-	0.002						
26		Can$	Pound	0		-	0.001						
27		Can$	Peso	0		-	0.001						
28		Euro	US$	5500		-	0.001						
29		Euro	Can$	0		-	0.002						
30		Euro	Pound	0		-	0.0005						
31		Euro	Peso	0		-	0.005						
32		Pound	US$	3080		-	0.001						
33		Pound	Can$	0		-	0.001						
34		Pound	Euro	0		-	0.0005						
35		Pound	Peso	0		-	0.005						
36		Peso	US$	2800		-	0.001						
37		Peso	Can$	0		-	0.001						
38		Peso	Euro	0		-	0.005						
39		Peso	Pound	0		-	0.005						
40													
41			Total Cost	$ 83,380.00									

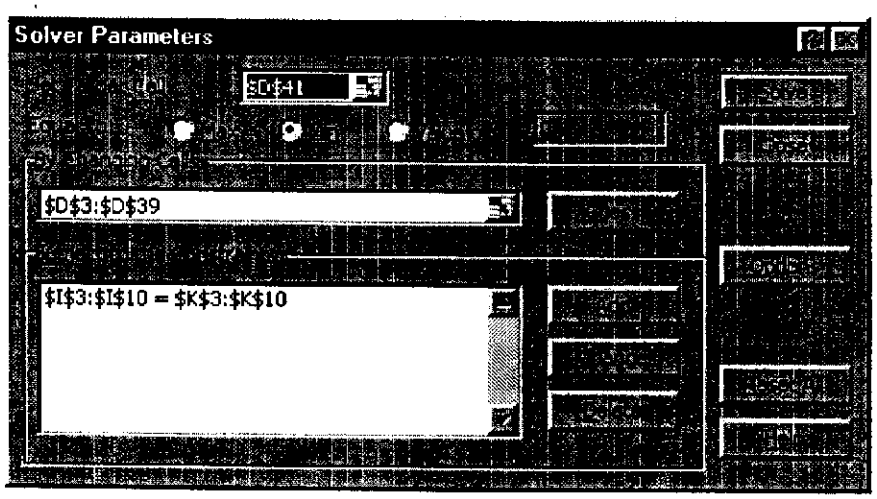
	J
1	
2	Net Flow
3	=-SUM(D3:D9)+D10+D17
4	=-SUM(D10:D16)+D3+D18
5	=-SUM(D17:D23)+D4+D11
6	=D5+D12+D19+D24+D28+D32+D36
7	=D6+D13+D20-SUM(D24:D27)+D29+D33+D37
8	=D7+D14+D21+D25-SUM(D28:D31)+D34+D38
9	=D8+D15+D22+D26+D30-SUM(D32:D35)+D39
10	=D9+D16+D23+D27+D31+D35-SUM(D36:D39)
11	



Jake should convert the equivalent of $2 million from Yen to each US$, Can$, Euro, and Pound. He should convert $1.6 million from Yen to Peso. Moreover, he should convert the equivalent of $200,000 from Rupiah to each US$, Can$, and Peso, $1 million from Rupiah to Euro, and $80,000 from Rupiah to Pound. Furthermore, Jake should convert the equivalent of $1.1 million from Ringgit to US$, $2.5 million from Ringgit to Euro, and $1 million from Ringgit to each Pound and Peso. Finally, he should convert all the money he converted into Can$, Euro, Pound, and Peso directly into US$. Specifically, he needs to convert into US$ the equivalent of $2.2 million, $5.5 million, $3.08 million, and $2.8 million Can$, Euro, Pound, and Peso, respectively. Assuming Jake pays for the total transaction costs of $83,380 directly from his American bank accounts he will have $16,880,000 dollars to invest in the US.

c) We eliminate all capacity restrictions on the arcs.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		From	To	Ship	Capacity	Unit Cost		Nodes	Net Flow		Supply /Demand	
3		Yen	Rupiah	0	-	0.005		Yen	-9600	=	-9600	
4		Yen	Ringgit	0	-	0.005		Rupiah	-1680	=	-1680	
5		Yen	US$	0	-	0.004		Ringgit	-5600	=	-5600	
6		Yen	Can$	0	-	0.004		US$	16880	=	16880	
7		Yen	Euro	0	-	0.004		Can$	0	=	0	
8		Yen	Pound	9600	-	0.0025		Euro	0	=	0	
9		Yen	Peso	0	-	0.005		Pound	0	=	0	
10		Rupiah	Yen	0	-	0.005		Peso	0	=	0	
11		Rupiah	Ringgit	0	-	0.007						
12		Rupiah	US$	0	-	0.005						
13		Rupiah	Can$	1680	-	0.003						
14		Rupiah	Euro	0	-	0.003						
15		Rupiah	Pound	0	-	0.0075						
16		Rupiah	Peso	0	-	0.0075						
17		Ringgit	Yen	0	-	0.005						
18		Ringgit	Rupiah	0	-	0.007						
19		Ringgit	US$	0	-	0.007						
20		Ringgit	Can$	0	-	0.007						
21		Ringgit	Euro	5600	-	0.004						
22		Ringgit	Pound	0	-	0.0045						
23		Ringgit	Peso	0	-	0.005						
24		Can$	US$	1680	-	0.0005						
25		Can$	Euro	0	-	0.002						
26		Can$	Pound	0	-	0.001						
27		Can$	Peso	5600	-	0.001						
28		Euro	US$	0	-	0.001						
29		Euro	Can$	0	-	0.002						
30		Euro	Pound	0	-	0.0005						
31		Euro	Peso	9600	-	0.005						
32		Pound	US$	0	-	0.001						
33		Pound	Can$	0	-	0.001						
34		Pound	Euro	0	-	0.0005						
35		Pound	Peso	0	-	0.005						
36		Peso	US$	0	-	0.001						
37		Peso	Can$	0	-	0.001						
38		Peso	Euro	0	-	0.005						
39		Peso	Pound	0	-	0.005						
40												
41			Total Cost	$ 67,480.00								
42												



Jake should convert the entire holdings in Japan from Yen into Pound and then into US$, the entire holdings in Indonesia from Rupiah into Can$ and then into US$, and the entire holdings in Malaysia from Ringgit into Euro and then into US$. Without the capacity limits the transaction costs are reduced to $67,480.00.

d) We multiply all unit cost for Rupiah by 6.

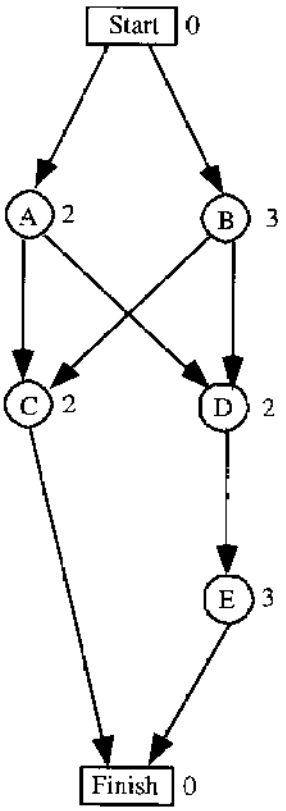
	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		From	To	Ship	Capacity	Unit Cost		Nodes	Net Flow		Supply/ Demand	
3		Yen	Rupiah	0	-	0.005		Yen	-9600	=	-9600	
4		Yen	Ringgit	0	-	0.005		Rupiah	-1680	=	-1680	
5		Yen	US$	0	-	0.004		Ringgit	-5600	=	-5600	
6		Yen	Can$	0	-	0.004		US$	16880	=	16880	
7		Yen	Euro	0	-	0.004		Can$	0	=	0	
8		Yen	Pound	9600	-	0.0025		Euro	0	=	0	
9		Yen	Peso	0	-	0.005		Pound	0	=	0	
10		Rupiah	Yen	0	-	0.03		Peso	0	=	0	
11		Rupiah	Ringgit	0	-	0.042						
12		Rupiah	US$	0	-	0.03						
13		Rupiah	Can$	1680	-	0.018						
14		Rupiah	Euro	0	-	0.018						
15		Rupiah	Pound	0	-	0.045						
16		Rupiah	Peso	0	-	0.045						
17		Ringgit	Yen	0	-	0.005						
18		Ringgit	Rupiah	0	-	0.007						
19		Ringgit	US$	0	-	0.007						
20		Ringgit	Can$	0	-	0.007						
21		Ringgit	Euro	5600	-	0.004						
22		Ringgit	Pound	0	-	0.0045						
23		Ringgit	Peso	0	-	0.005						
24		Can$	US$	1680	-	0.0005						
25		Can$	Euro	0	-	0.002						
26		Can$	Pound	0	-	0.001						
27		Can$	Peso	0	-	0.001						
28		Euro	US$	5600	-	0.001						
29		Euro	Can$	0	-	0.002						
30		Euro	Pound	0	-	0.0005						
31		Euro	Peso	0	-	0.005						
32		Pound	US$	9600	-	0.001						
33		Pound	Can$	0	-	0.001						
34		Pound	Euro	0	-	0.0005						
35		Pound	Peso	0	-	0.005						
36		Peso	US$	0	-	0.001						
37		Peso	Can$	0	-	0.001						
38		Peso	Euro	0	-	0.005						
39		Peso	Pound	0	-	0.005						
40												
41			Total Cost	$ 92,680.00								
42												

The optimal routing for the money doesn't change, but the total transaction costs are now increased to $92,680.

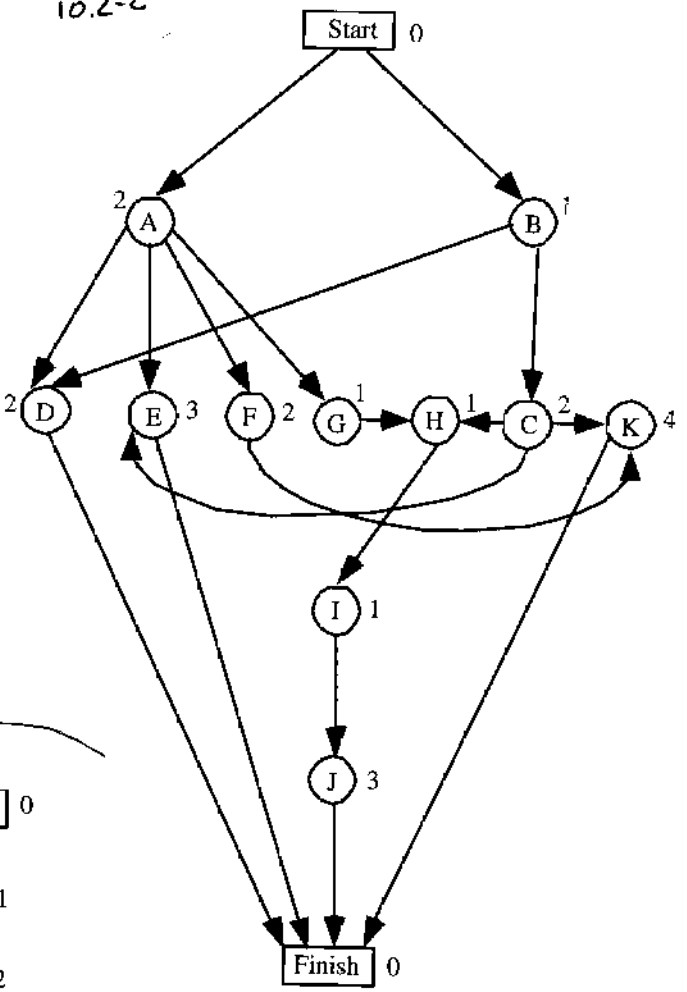
e) In the described crisis situation the currency exchange rates might change every minute. Jake should carefully check the exchange rates again when he performs the transactions.

The European economies might be more insulated from the Asian financial collapse than the US economy. To impress his boss Jake might want to explore other investment opportunities in safer European economies that provide higher rates of return than US bonds.

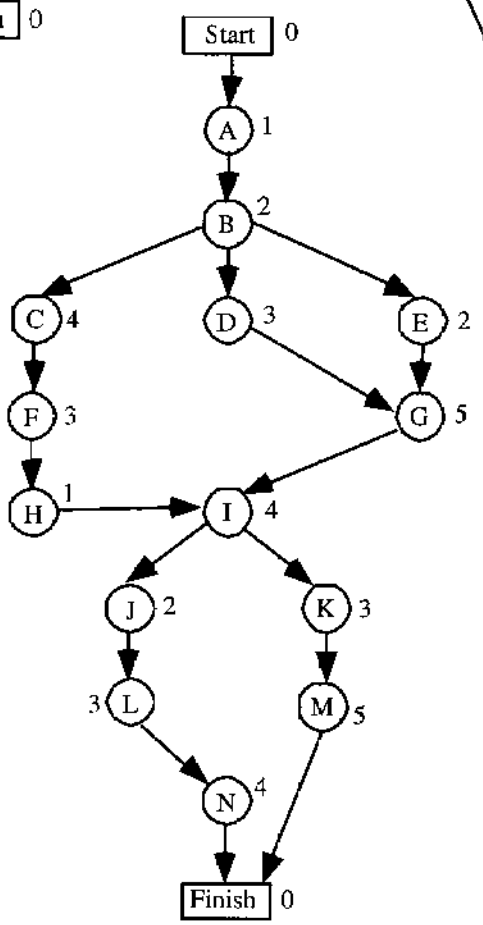
10.2-1



10.2-2

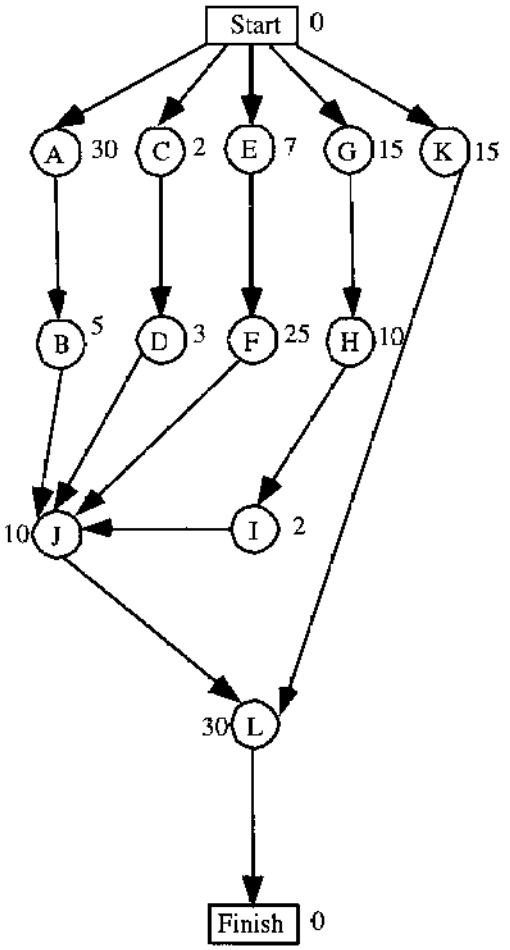


10.2-3



10.3-1

a)



b) Start → A → B → J → L → Finish — Length = 75 minutes
*critical path

Start → C → D → J → L → Finish — Length = 45 minutes

Start → E → F → J → L → Finish — Length = 72 minutes

Start → G → H → I → J → L → Finish — Length = 67 minutes

Start → K → L → Finish — Length = 45 minutes

c, d & e)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	30	0	30	0	Yes
B	30	35	30	35	0	Yes
C	0	2	30	32	30	No
D	2	5	32	35	30	No
E	0	7	3	10	3	No
F	7	32	10	35	3	No
G	0	15	8	23	8	No
H	15	25	23	33	8	No
I	25	27	33	35	8	No
J	35	45	35	45	0	Yes
K	0	15	30	45	30	No
L	45	75	45	75	0	Yes
Finish	75	75	75	75	0	Yes

Critical Path: Start → A → B → J → L → Finish

f) Dinner will be delayed 3 minutes because of the phone call. If the food processor is used then dinner will not be delayed because there was 3 minutes of slack and 5 minutes of cutting time saved and the call only used 6 minutes of the 8 total.

10.3-2 a) Start → A → C → Finish — Length = 4 weeks
Start → A → D → E → Finish — Length = 7 weeks
Start → B → C → Finish — Length = 5 weeks
Start → B → D → E → Finish — Length = 8 weeks
*critical path

b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	2	1	3	1	No
B	0	3	0	3	0	Yes
C	3	5	6	8	3	No
D	3	5	3	5	0	Yes
E	5	8	5	8	0	Yes
Finish	8	8	8	8	0	Yes

Critical Path: Start → B → D → E → Finish

c) No, this will not shorten the length of the project because the activity is not on the critical path.

10.3-3 a)

Start → A → D → Finish	Length = 4 weeks
Start → A → E → Finish	Length = 5 weeks
Start → A → F → K → Finish *critical path	Length = 8 weeks
Start → A → G → H → I →J → Finish *critical path	Length = 8 weeks
Start → B → D → Finish	Length = 3 weeks
Start → B → C → E → Finish	Length = 6 weeks
Start → B → C → H → I →J → Finish *critical path	Length = 8 weeks
Start → B → C → K → Finish	Length = 7 weeks

b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	2	0	2	0	Yes
B	0	1	0	1	0	Yes
C	1	3	1	3	0	Yes
D	2	4	6	8	4	No
E	3	6	5	8	2	No
F	2	4	2	4	0	Yes
G	2	3	2	3	0	Yes
H	3	4	3	4	0	Yes
I	4	5	4	5	0	Yes
J	5	8	5	8	0	Yes
K	4	8	4	8	0	Yes
Finish	8	8	8	8	0	Yes

Critical Paths: Start → A → F → K → Finish
Start → A → G → H → I →J → Finish
Start → B → C → H → I →J → Finish

c) No, this will not shorten the length of the project because A is not on all of the critical paths.

10.3-4 a)

Start → A → D → H → M → Finish	Length = 19 weeks
Start → B → E → J → M → Finish *critical path	Length = 20 weeks
Start → C → F → K → N → Finish	Length = 16 weeks
Start → A → I → M → Finish	Length = 17 weeks
Start → C → G → L →N → Finish *critical path	Length = 20 weeks

b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	6	1	7	1	No
B	0	3	0	3	0	Yes
C	0	4	0	4	0	Yes
D	6	10	7	11	1	No
E	3	10	3	10	0	Yes
F	4	8	8	12	4	No
G	4	10	4	10	0	Yes
H	10	13	11	14	1	No
I	6	11	9	14	3	No
J	10	14	10	14	0	Yes
K	8	11	12	15	4	No
L	10	15	10	15	0	Yes
M	14	20	14	20	0	Yes
N	15	20	15	20	0	Yes
Finish	20	20	20	20	0	Yes

Ken will be able to meet his deadline if no delays occur.

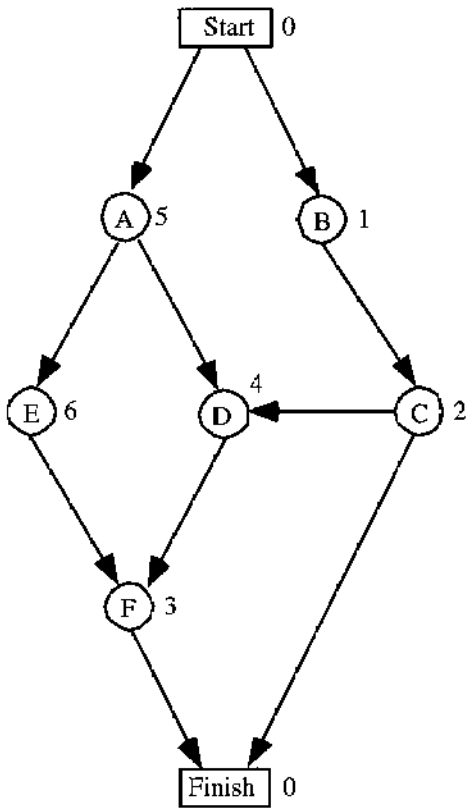
c) Critical Paths: Start → B → E → J → M → Finish
Start → C → G → L → N → Finish

Focus attention on activities with 0 slack.

d) If activity I takes 2 extra weeks there will be no delay because its slack is 3. If activity H takes 2 extra weeks then there will be a delay of 1 week because its slack is 1. If activity J takes 2 extra weeks there will be a delay of 2 weeks because it has no slack.

10.3-5

a)



b)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	5	0	5	0	Yes
B	0	1	11	12	11	No
C	1	3	12	14	11	No
D	5	9	7	11	2	No
E	5	11	5	11	0	Yes
F	11	14	11	14	0	Yes
Finish	14	14	14	14	0	Yes

Critical Path: Start → A → E → F → Finish Length = 14 months

c) 6 months

10.3-6

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	3	0	3	0	Yes
B	3	11	3	11	0	Yes
C	11	29	11	29	0	Yes
D	29	39	29	39	0	Yes
E	29	34	30	35	1	No
F	34	44	35	45	1	No
G	39	50	39	50	0	Yes
H	50	67	50	67	0	Yes
I	29	38	36	45	7	No
J	44	53	45	54	1	No
K	53	57	57	61	4	No
L	53	60	54	61	1	No
M	67	70	67	70	0	Yes
N	60	69	61	70	1	No
Finish	70	70	70	70	0	Yes

Critical Path: Start → A → B → C → D → G → H → M → Finish
Total duration = 70 weeks

10.3-7

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	1	0	1	0	Yes
B	1	3	1	3	0	Yes
C	3	9	3	9	0	Yes
D	9	13	11	15	2	No
E	9	10	9	10	0	Yes
F	10	14	10	14	0	Yes
G	13	18	15	20	2	No
H	18	23	20	25	2	No
I	9	12	11	14	2	No
J	14	17	14	17	0	Yes
K	17	21	17	21	0	Yes
L	17	18	20	21	3	No
M	23	24	25	26	2	No
N	21	26	21	26	0	Yes
Finish	26	26	26	26	0	Yes

Critical Path: Start → A → B → C → E → F → J → K → N→ Finish
Total duration = 26 weeks

10.3-8

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	1	0	1	0	Yes
B	1	3	1	3	0	Yes
C	3	10	3	10	0	Yes
D	10	14	13	17	3	No
E	10	13	10	13	0	Yes
F	13	16	13	16	0	Yes
G	14	18	17	21	3	No
H	18	24	21	27	3	No
I	10	15	11	16	1	No
J	16	22	16	22	0	Yes
K	22	25	22	25	0	Yes
L	22	25	22	25	0	Yes
M	24	25	27	28	3	No
N	25	28	25	28	0	Yes
Finish	28	28	28	28	0	Yes

Critical Paths: Start → A → B → C → E → F → J → K → N→ Finish
Start → A → B → C → E → F → J → L → N→ Finish
Total duration = 28 weeks

10.4-1

$$\mu = \frac{o+4m+p}{6} = \frac{30+(4)(36)+48}{6} = 37$$

$$\sigma^2 = \left(\frac{p-o}{6}\right)^2 = \left(\frac{48-30}{6}\right)^2 = 9$$

10.4-2 a) Start → A → E → I → Finish — Length = 17 months
Start → A → C → F → I → Finish — Length = 17 months
Start → B → D → G → J → Finish — Length = 17 months
Start → B → H → J → Finish — Length = 18 months
*critical path

b) $\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{22-18}{\sqrt{31}} = 0.718 \quad \Rightarrow \quad P(T \le 22) \approx 0.77$

c) Start → A → E → I → Finish

$$\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{22-17}{\sqrt{25}} = 1 \quad \Rightarrow \quad P(T \le 22) \approx 0.84$$

Start → A → C → F → I → Finish

$$\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{22-17}{\sqrt{27}} = 0.962 \quad \Rightarrow \quad P(T \le 22) \approx 0.84$$

Start → B → D → G → J → Finish

$$\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{22-17}{\sqrt{28}} = 0.945 \quad \Rightarrow \quad P(T \le 22) \approx 0.84$$

d) There is approximately a 77% chance that the drug will be ready in 22 weeks.

10.4-3 Start → B → H → J → Finish

Mean Critical Path

μ =	**18.4166667**
σ^2 =	**31.2013889**
P(TŠd)=	**0.73940284**
where d =	**22**

Start → A → E → I → Finish

Mean Critical Path

μ =	**17.0833333**
σ^2 =	**25.3402778**
P(TŠd)=	**0.83564332**
where d =	**22**

Start → A → C → F → I → Finish

Mean Critical Path

μ =	**17.5833333**
σ^2 =	**27.3680556**
P(TŠd)=	**0.80073605**
where d =	**22**

Start → B → D → G → J → Finish

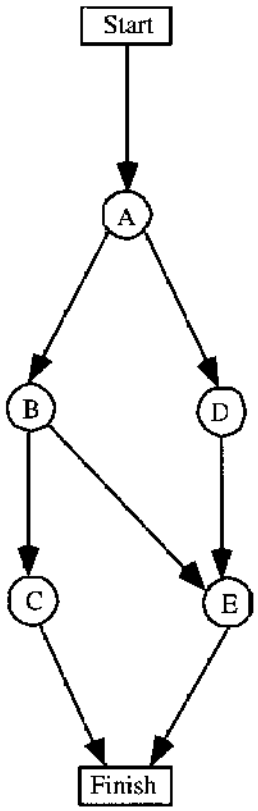
Mean Critical Path

μ =	**17.8333333**
σ^2 =	**28.0416667**
P(TŠd)=	**0.78431252**
where d =	**22**

There is approximately a 73% chance that the drug will be ready in 22 weeks.

10.4-4

a)



b)

Activity	μ	σ^2
A	4	0.111
B	2	0
C	4.83	0.25
D	3	0.444
E	3.17	0.25

c) Start → A → B → C → Finish Length = 10.83 weeks
*critical path
Start → A → B → E → Finish Length = 9.17 weeks
Start → A → D → E → Finish Length = 10.17 weeks

d) $\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{11-10.83}{\sqrt{.361}} = 0.028 \quad \Rightarrow \quad P(T \le 11) = 0.6$

e) Make the bid since there is approximately a 60% chance that the project will be completed in 11weeks or less.

10.4-5 a)

Activity	μ	σ^2
A	12	0
B	23	16
C	15	1
D	27	9
E	18	4
F	6	4

b) Start $\rightarrow$ A $\rightarrow$ C $\rightarrow$ E $\rightarrow$ F $\rightarrow$ Finish Length = 51 days
*critical path
Start $\rightarrow$ B $\rightarrow$ D $\rightarrow$ Finish Length =50 days

c) $\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{57-51}{\sqrt{9}} = 2 \quad \Rightarrow \quad P(T \le 57) = 0.9772$ (from the Normal table)

d) $\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{57-50}{\sqrt{25}} = 1.4 \quad \Rightarrow \quad P(T \le 57) = 0.9192$ (from the Normal table)

e) (0.9772)(0.9192)=0.8982 This answer tells us that the procedure used in part c) overestimates the probability of completing within 57 days.

10.4-6 a)

Activity	μ	σ^2
A	32	1.78
B	27.7	2.78
C	36	11.1
D	16	0.444
E	32	0
F	53.7	32.1
G	16.7	4
H	20.3	2.78
I	34	7.11
J	17.7	9

b) Start → A → C → J → Finish — Length = 85.7 weeks
Start → B → F → J → Finish — Length =99.1 weeks
*critical path
Start → B → E → H → Finish — Length = 80 weeks
Start → B → E → I → Finish — Length =93.7 weeks
Start → B → D → G → H → Finish — Length = 80.7 weeks
Start → B → D → G → I → Finish — Length =94.4 weeks

c) $\frac{d-\mu_p}{\sqrt{\sigma_p^2}} = \frac{100-99.1}{\sqrt{43.89}} = 0.136 \quad \Rightarrow \quad P(T \le 100) = 0.4443$ (from the Normal table)

d) Higher

10.5-1 a) True. The optimistic and pessimistic estimates are meant to lie at the extremes of what is possible. (Pg. 33)

b) False. The form of the probability distribution is a beta distribution. (Pg. 33)

c) False. The mean critical path will turn out to be the longest path through the project network. (Pg. 40)

10.5-1

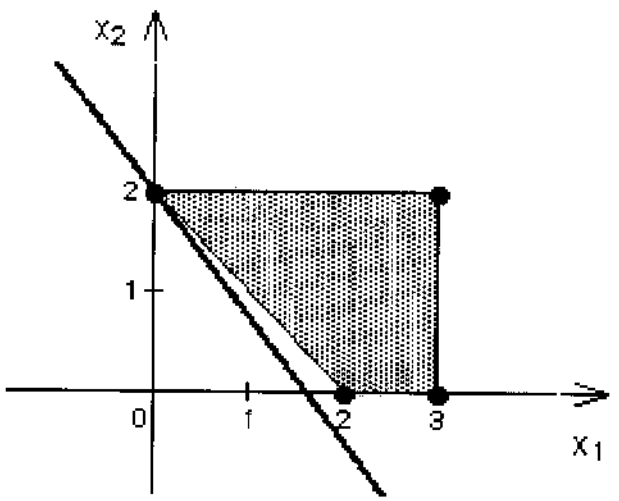
Activity to Crash	Crash Cost	Length of Path A – C	Length of Path B - D
		14	16
B	$5,000	14	15
B	$5,000	14	15
D	$6,000	14	14
C	$4,000	13	14
D	$6,000	13	13
C	$4,000	12	13
D	$6,000	12	12

10.5-2

a)

Let x_A = reduction in A due to crashing
Let x_C = reduction in C due to crashing

Minimize $C = 5,000x_A + 4,000x_C$,
subject to $x_A \le 3$
$x_C \le 2$
$x_A + x_C \ge 2$
and $x_A \ge 0,\ x_C \ge 0.$ (CONT'D)

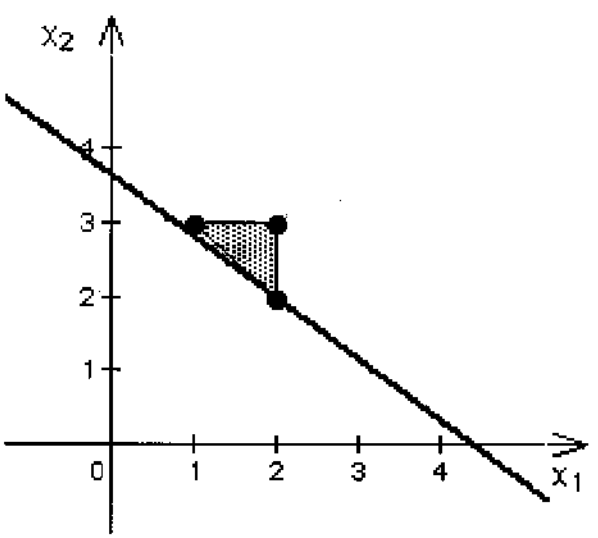


Optimal solution $(x_A, x_C) = (0, 2)$ and $C = 8{,}000$.

b)

Let x_B = reduction in B due to crashing
Let x_D = reduction in D due to crashing

Minimize $C = 5{,}000x_B + 6{,}000x_D$,
subject to $x_B \le 2$
$x_D \le 3$
$x_B + x_D \ge 4$
and $x_B \ge 0,\ x_D \ge 0.$



Optimal solution $(x_B, x_D) = (2, 2)$ and $C = 22{,}000$.

c)

Let x_A = reduction in A due to crashing
Let x_B = reduction in B due to crashing
Let x_C = reduction in C due to crashing
Let x_D = reduction in D due to crashing

Minimize $C = 5{,}000x_A + 5{,}000x_B + 4{,}000x_C + 6{,}000x_D$,

subject to
$$x_A \le 3$$
$$x_B \le 2$$
$$x_C \le 2$$
$$x_D \le 3$$
$$x_A + x_C \ge 2$$
$$x_B + x_D \ge 4$$

and $x_A \ge 0,\ x_B \ge 0,\ x_c \ge 0,\ x_D \ge 0.$

Optimal solution $(x_A, x_B, x_c x_D) = (0, 2, 2, 2)$ and $C = 30{,}000$.

d)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	**0**	**0**	8
B	9	7	$20000	$30000	2	$5000	**0**	**2**	7
C	6	4	$16000	$24000	2	$4000	**8**	**2**	12
D	7	4	$27000	$45000	3	$6000	**7**	**2**	12

Finish Time = 12
Total Cost = $ 118,000

e) Deadline of 11 months:

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	**0**	**0**	7
B	9	7	$20000	$30000	2	$5000	**0**	**2**	7
C	6	4	$16000	$24000	2	$4000	**7**	**2**	11
D	7	4	$27000	$45000	3	$6000	**7**	**3**	11

Finish Time = 11
Total Cost = $ 129,000

9)

Deadline of 13 months:

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	8	5	$25000	$40000	3	$5000	**0**	**0**	8
B	9	7	$20000	$30000	2	$5000	**0**	**2**	7
C	6	4	$16000	$24000	2	$4000	**8**	**1**	13
D	7	4	$27000	$45000	3	$6000	**7**	**1**	13

Finish Time = 13
Total Cost = $ 108,000

10.5-3

a)

Activity to Crash	Crash Cost	Length of Path B – D
		50
B	$10,000	49
B	$10,000	48
B	$10,000	47

b)

Activity to Crash	Crash Cost	Length of Path A – C – E – F
		51
C	$10,000	50
C	$10,000	49
C	$10,000	48
E	$15,000	47

c)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	12	9	$210000	$270000	3	$20000	**0**	**0**	12
B	23	18	$410000	$460000	5	$10000	**0**	**3**	20
C	16	12	$290000	$320000	3	$10000	**12**	**3**	24
D	27	21	$440000	$500000	6	$10000	**20**	**0**	47
E	18	14	$350000	$410000	4	$15000	**24**	**1**	41
F	6	4	$160000	$210000	2	$25000	**41**	**0**	47

Finish Time = 47
Total Cost = $ 193,500

10.5-4

a)

Activity	ES	EF	LS	LF	Slack	Critical Path
Start	0	0	0	0	0	Yes
A	0	3	0	3	0	Yes
B	3	7	4	8	1	No
C	3	8	3	8	0	Yes
D	7	10	9	12	2	No
E	8	12	8	12	0	Yes
Finish	12	12	12	12	0	Yes

Critical Path: Start → A → C → E → Finish
Total duration = 12 weeks

b)

Activity to Crash	Crash Cost	Length of Path A – B – D	A – B – E	A – C - E
		10	11	12
C	$1,333	10	11	11
E	$2,500	10	10	10
D & E	$4,000	9	9	9
B & C	$4,333	8	8	8

New Plan:

Activity	Duration	Cost
A	3 weeks	$54,000
B	3 weeks	$65,000
C	3 weeks	$68,666
D	2 weeks	$41,500
E	2 weeks	$80,000

$7834 is saved by this crashing schedule.

c)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**4**	**0**	8
C	5	2	$66000	$70000	3	$1333	**3**	**0**	8
D	3	1	$40000	$43000	2	$1500	**9**	**0**	12
E	4	2	$75000	$80000	2	$2500	**8**	**0**	12

Finish Time = 12
Total Cost = $ 297,000

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**3**	**0**	7
C	5	2	$66000	$70000	3	$1333	**3**	**1**	7
D	3	1	$40000	$43000	2	$1500	**8**	**0**	11
E	4	2	$75000	$80000	2	$2500	**7**	**0**	11

Finish Time = 11
Total Cost = $ 298,333

(CONT'D)

(CONT'D)

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**3**	**0**	7
C	5	2	$66000	$70000	3	$1333	**3**	**1**	7
D	3	1	$40000	$43000	2	$1500	**7**	**1.22E-15**	10
E	4	2	$75000	$80000	2	$2500	**7**	**1**	10

Finish Time = 10
Total Cost = $ 300,833

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**0**	3
B	4	3	$62000	$65000	1	$3000	**3**	**4.66E-12**	7
C	5	2	$66000	$70000	3	$1333	**3**	**1**	7
D	3	1	$40000	$43000	2	$1500	**7**	**1**	9
E	4	2	$75000	$80000	2	$2500	**7**	**2**	9

Finish Time = 9
Total Cost = $ 304,833

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**3.66E-11**	3
B	4	3	$62000	$65000	1	$3000	**3**	**1**	5
C	5	2	$66000	$70000	3	$1333	**3**	**2**	5
D	3	1	$40000	$43000	2	$1500	**6**	**1**	8
E	4	2	$75000	$80000	2	$2500	**6**	**2**	8

Finish Time = 8
Total Cost = $ 309,167

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	3	2	$54000	$60000	1	$6000	**0**	**1**	2
B	4	3	$62000	$65000	1	$3000	**2**	**1**	5
C	5	2	$66000	$70000	3	$1333	**2**	**2**	5
D	3	1	$40000	$43000	2	$1500	**5**	**1**	7
E	4	2	$75000	$80000	2	$2500	**5**	**2**	7

Finish Time = 7
Total Cost = $ 315,167

Crash to 8 weeks.

10.5-5

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	5	3	$20	$30	2	$5	**0**	**2**	3
B	3	2	$10	$20	1	$10	**0**	**1**	2
C	4	2	$16	$24	2	$4	**3**	**0**	7
D	6	3	$25	$43	3	$6	**3**	**0**	9
E	5	4	$22	$30	1	$8	**2**	**0**	7
F	7	4	$30	$48	3	$6	**2**	**0**	9
G	9	5	$25	$45	4	$5	**7**	**1**	15
H	8	6	$30	$44	2	$7	**9**	**2**	15

Finish Time = 15
Total Cost = $ 217

10.5-6

Activity	Time Normal	Time Crash	Cost Normal	Cost Crash	Maximum Time Reduction	Crash Cost per Week saved	Start Time	Time Reduction	Finish Time
A	32	28	$160	$180	4	$5	**8**	**0**	40
B	28	25	$125	$146	3	$7	**0**	**3**	25
C	36	31	$170	$210	5	$8	**40**	**0**	76
D	16	13	$60	$72	3	$4	**25**	**0**	41
E	32	27	$135	$160	5	$5	**25**	**0**	57
F	54	47	$215	$257	7	$6	**25**	**3**	76
G	17	15	$90	$96	2	$3	**41**	**0**	58
H	20	17	$120	$132	3	$4	**58**	**0**	78
I	34	30	$190	$226	4	$9	**58**	**0**	92
J	18	16	$80	$84	2	$2	**76**	**2**	92

Finish Time = 92
Total Cost = $ 1,388

10.6-1

a)

Activity	ES	EF
Start	0	0
A	0	3
B	3	6
C	3	6
D	6	8
E	6	8
Finish	8	8

Total duration = 8 weeks

b, c & d)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost per week of it's duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	3	$54,000	0	**$ 18,000**	**18000**	**18000**	**18000**	**0**	**0**	**0**	**0**	**0**
B	3	$65,000	3	**$ 21,667**	**0**	**0**	**0**	**21667**	**21667**	**21667**	**0**	**0**
C	3	$68,666	3	**$ 22,889**	**0**	**0**	**0**	**22889**	**22889**	**22889**	**0**	**0**
D	2	$41,500	6	**$ 20,750**	**0**	**0**	**0**	**0**	**0**	**0**	**20750**	**20750**
E	2	$80,000	6	**$ 40,000**	**0**	**0**	**0**	**0**	**0**	**0**	**40000**	**40000**
					0	**0**	**0**	**0**	**0**	**0**	**0**	**0**
				Weekly Project Cost	**1800**	**1800**	**1800**	**44555**	**44555**	**44555**	**60750**	**60750**
				Cumulative Project Cost	**1800**	**3600**	**1800**	**5400**	**98555**	**14311**	**187666**	**309166**

e)

Activity	Budgeted Cost	Percent Completed	Value Completed	Actual Cost To Date	Cost Overrun To Date
A	$54,000	100%	$54,000	$65,000	$11,000
B	$65,000	100%	$65,000	$55,000	-$10,000
C	$68,666	33%	$22,660	$44,000	$21,340
Total	$187,666		$141,660	$164,000	$22,340

Michael should concentrate his efforts on activity C since it is not yet completed.

10.6-2 a)

Activity	ES	EF	LS	LF	Slack
Start	0	0	0	0	0
A	0	6	0	6	0
B	0	2	4	6	4
C	6	10	9	13	3
D	6	11	6	11	0
E	10	17	13	20	3
F	11	20	11	20	0
Finish	20	20	20	20	0

The earliest finish time for the project is 20 weeks.

b)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost per week of it's duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$420,000	0	$ 70,000	7000	7000	7000	7000	7000	7000	0	0
B	2	$180,000	0	$ 90,000	9000	9000	0	0	0	0	0	0
C	4	$540,000	6	$ 135,000	0	0	0	0	0	0	135000	135000
D	5	$360,000	6	$ 72,000	0	0	0	0	0	0	72000	72000
E	7	$590,000	10	$ 84,286	0	0	0	0	0	0	0	0
F	9	$630,000	11	$ 70,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0
				Weekly Project Cost	160000	160000	70000	70000	70000	70000	207000	207000
				Cumulative Project Cost	160000	320000	390000	460000	530000	600000	807000	1014000

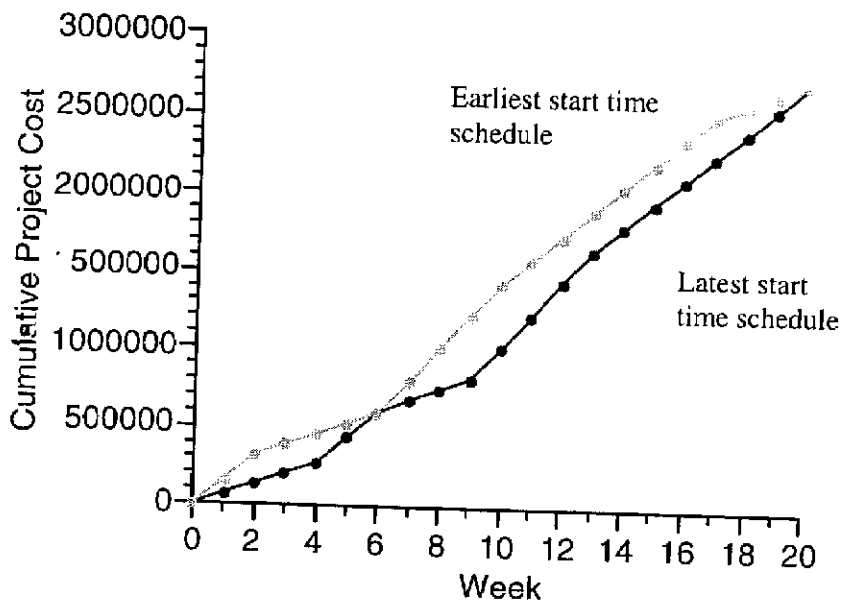
Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
135000	135000	0	0	0	0	0	0	0	0	0	0
72000	72000	72000	0	0	0	0	0	0	0	0	0
0	0	84286	84285.7	84285.7	84285.7	84285.7	84285.7	84285.7	0	0	0
0	0	0	7000	7000	7000	7000	7000	7000	7000	7000	7000
0	0	0	0	0	0	0	0	0	0	0	0
207000	207000	156285.7	154286	154286	154286	154286	154286	154286	70000	70000	70000
1221000	1428000	1584286	1738571	1892857	2047143	2201429	2355714	2510000	2580000	2650000	2720000

c)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost Per Week of Its Duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$420,000	0	$70,000	70000	70000	70000	70000	70000	70000	0	0
B	2	$180,000	4	$90,000	0	0	0	0	90000	90000	0	0
C	4	$540,000	9	$135,000	0	0	0	0	0	0	0	0
D	5	$360,000	6	$72,000	0	0	0	0	0	0	72000	72000
E	7	$590,000	13	$84,286	0	0	0	0	0	0	0	0
F	9	$630,000	11	$70,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0

Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	135000	135000	135000	135000	0	0	0	0	0	0	0
72000	72000	72000	0	0	0	0	0	0	0	0	0
0	0	0	0	0	84285.7	84285.7	84285.7	84285.7	84285.7	84285.7	84285.7
0	0	0	70000	70000	70000	70000	70000	70000	70000	70000	70000
0	0	0	0	0	0	0	0	0	0	0	0
72000	207000	207000	205000	205000	154286	154286	154286	154286	154286	154286	154286
816000	1023000	1230000	1435000	1640000	1794286	1948571	2102857	2257143	2411429	2565714	2720000

d)



e)

Activity	Budgeted Cost	Percent Completed	Value Completed	Actual Cost To Date	Cost Overrun To Date
A	$420,000	50%	$210,000	$200,000	-$10,000
B	$180,000	100%	$180,000	$200,000	$20,000
D	$360,000	50%	$180,000	$210,000	$30,000
Total	$960,000		$570,000	$610,000	$40,000

The project manager should focus attention on activity D since it is not yet finished and is running over budget.

10.6-3

a)

Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost Per Week of Its Duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$180,000	0	$30,000	30000	30000	30000	30000	30000	30000	0	0
B	3	$75,000	0	$25,000	25000	25000	25000	0	0	0	0	0
C	4	$120,000	0	$30,000	30000	30000	30000	30000	0	0	0	0
D	4	$140,000	6	$35,000	0	0	0	0	0	0	35000	35000
E	7	$175,000	3	$25,000	0	0	0	25000	25000	25000	25000	25000
F	4	$80,000	4	$20,000	0	0	0	0	20000	20000	20000	20000
G	6	$210,000	4	$35,000	0	0	0	0	35000	35000	35000	35000
H	3	$45,000	10	$15,000	0	0	0	0	0	0	0	0
I	5	$125,000	6	$25,000	0	0	0	0	0	0	25000	25000
J	4	$100,000	10	$25,000	0	0	0	0	0	0	0	0
K	3	$60,000	8	$20,000	0	0	0	0	0	0	0	0
L	5	$50,000	10	$10,000	0	0	0	0	0	0	0	0
M	6	$90,000	14	$15,000	0	0	0	0	0	0	0	0
N	5	$150,000	15	$30,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0
				Weekly Project Cost	85000	85000	85000	85000	110000	110000	140000	140000
				Cumulative Project Cost	85000	170000	255000	340000	450000	560000	700000	840000

Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
35000	35000	0	0	0	0	0	0	0	0	0	0
25000	25000	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
35000	35000	0	0	0	0	0	0	0	0	0	0
0	0	15000	15000	15000	0	0	0	0	0	0	0
25000	25000	25000	0	0	0	0	0	0	0	0	0
0	0	25000	25000	25000	25000	0	0	0	0	0	0
20000	20000	20000	0	0	0	0	0	0	0	0	0
0	0	10000	10000	10000	10000	10000	0	0	0	0	0
0	0	0	0	0	0	15000	15000	15000	15000	15000	15000
0	0	0	0	0	0	0	30000	30000	30000	30000	30000
0	0	0	0	0	0	0	0	0	0	0	0
140000	140000	95000	50000	50000	35000	25000	45000	45000	45000	45000	45000
980000	1120000	1215000	1265000	1315000	1350000	1375000	1420000	1465000	1510000	1555000	1600000

b)

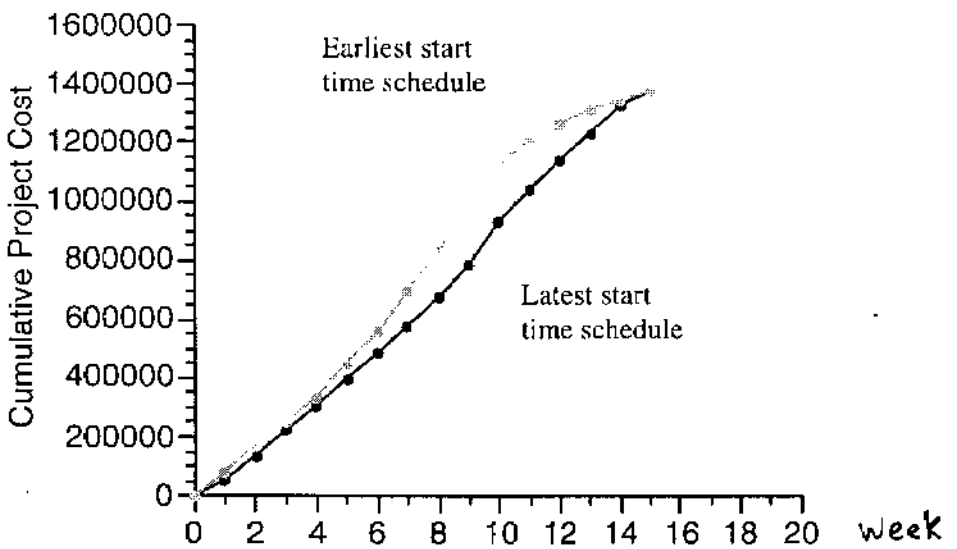
Activity	Estimated Duration (weeks)	Estimated Cost	Start Time	Cost Per Week of Its Duration	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7	Week 8
A	6	$180,000	1	$30,000	0	30000	30000	30000	30000	30000	30000	0
B	3	$75,000	0	$25,000	25000	25000	25000	0	0	0	0	0
C	4	$120,000	0	$30,000	30000	30000	30000	30000	0	0	0	0
D	4	$140,000	7	$35,000	0	0	0	0	0	0	0	35000
E	7	$175,000	3	$25,000	0	0	0	25000	25000	25000	25000	25000
F	4	$80,000	8	$20,000	0	0	0	0	0	0	0	0
G	6	$210,000	4	$35,000	0	0	0	0	35000	35000	35000	35000
H	3	$45,000	11	$15,000	0	0	0	0	0	0	0	0
I	5	$125,000	9	$25,000	0	0	0	0	0	0	0	0
J	4	$100,000	10	$25,000	0	0	0	0	0	0	0	0
K	3	$60,000	12	$20,000	0	0	0	0	0	0	0	0
L	5	$50,000	10	$10,000	0	0	0	0	0	0	0	0
M	6	$90,000	14	$15,000	0	0	0	0	0	0	0	0
N	5	$150,000	15	$30,000	0	0	0	0	0	0	0	0
					0	0	0	0	0	0	0	0
				Weekly Project Cost	55000	85000	85000	85000	90000	90000	90000	95000
				Cumulative Project Cost	55000	140000	225000	310000	400000	490000	580000	675000

(CONT'D)

10.6-3 (CONT'D)

Week 9	Week 10	Week 11	Week 12	Week 13	Week 14	Week 15	Week 16	Week 17	Week 18	Week 19	Week 20
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
35000	35000	35000	0	0	0	0	0	0	0	0	0
25000	25000	0	0	0	0	0	0	0	0	0	0
20000	20000	20000	20000	0	0	0	0	0	0	0	0
35000	35000	0	0	0	0	0	0	0	0	0	0
0	0	0	15000	15000	15000	0	0	0	0	0	0
0	25000	25000	25000	25000	25000	0	0	0	0	0	0
0	0	25000	25000	25000	25000	0	0	0	0	0	0
0	0	0	0	20000	20000	20000	0	0	0	0	0
0	0	10000	10000	10000	10000	10000	0	0	0	0	0
0	0	0	0	0	0	15000	15000	15000	15000	15000	15000
0	0	0	0	0	0	0	30000	30000	30000	30000	30000
0	0	0	0	0	0	0	0	0	0	0	0
115000	140000	115000	95000	95000	95000	45000	45000	45000	45000	45000	45000
790000	930000	1045000	1140000	1235000	1330000	1375000	1420000	1465000	1510000	1555000	1600000

c)



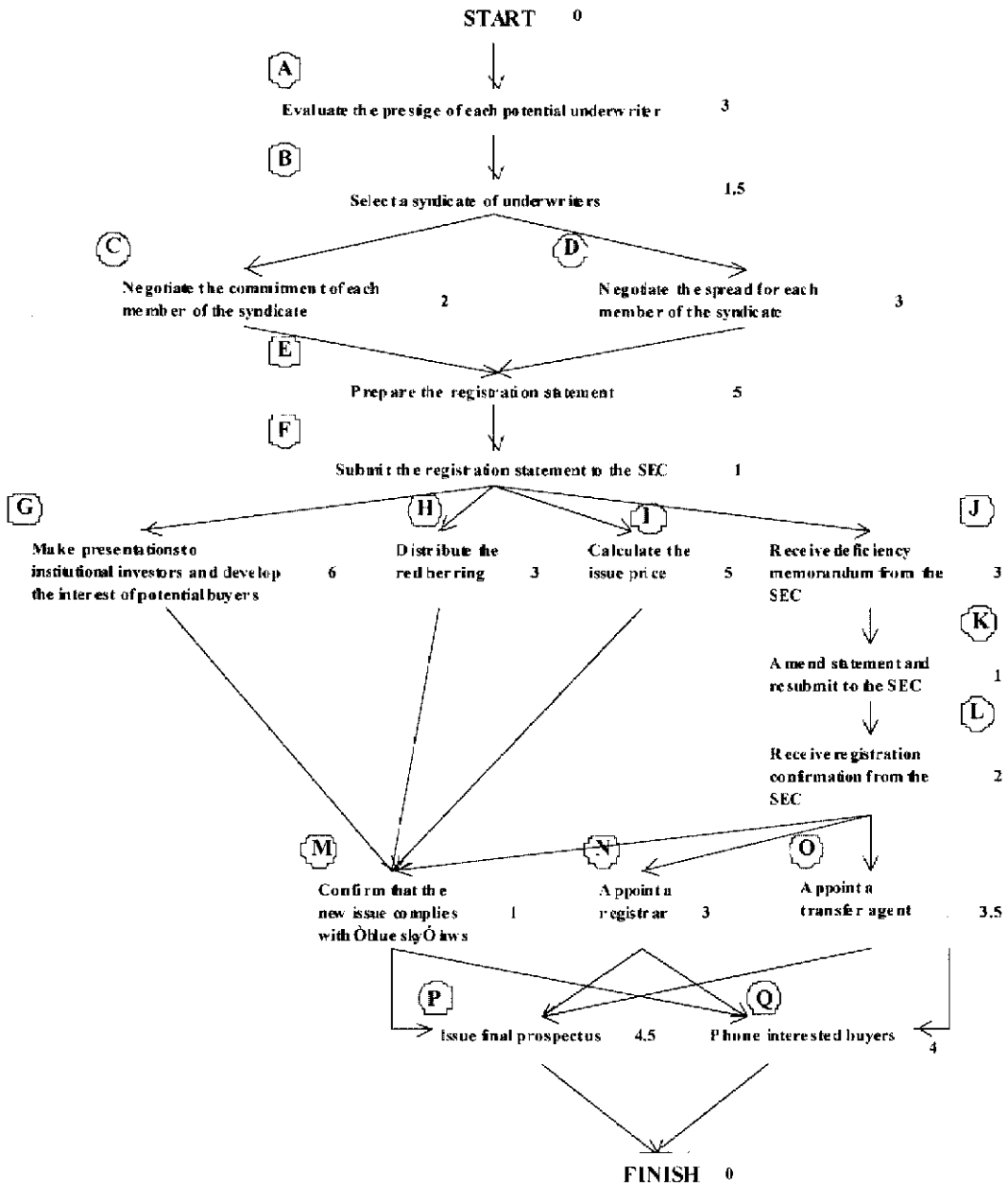
d)

Activity	Budgeted Cost	Percent Complete d	Value Completed	Actual Cost To Date	Cost Overrun To Date
A	$180,000	100%	$180,000	$190,000	$10,000
B	$75,000	100%	$75,000	$70,000	-$5,000
C	$120,000	100%	$120,000	$150,000	$30,000
D	$140,000	40%	$56,000	$70,000	$14,000
E	$175,000	50%	$87,500	$100,000	$12,500
F	$80,000	60%	$48,000	$45,000	-$3,000
G	$210,000	25%	$52,500	$50,000	-$2,500
I	$125,000	20%	$25,000	$35,000	$10,000
Total	$1,105,000		$644,000	$710,000	$66,000

The project manager should investigate activities D, E and I since they are not yet finished and running over budget.

Cases

10-1 a) A diagram of the project network appears below.



The estimated project duration equals the length of the longest path through the project network. To calculate the length of the longest path, we use the layout of the worksheet titled "Reliable" in the "Ch. 7 – Project Management" workbook. We need to modify the spreadsheet to reflect the precedents unique to this case. The formulas in the new spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Evaluate prestige	3	0	=E4+D4	=H4-D4	=G5	=H4-F4	=IF(I4=0,"Yes","No")
5		B	Select syndicate	1.5	=MAX(F4)	=E5+D5	=H5-D5	=MIN(G6,G7)	=H5-F5	=IF(I5=0,"Yes","No")
6		C	Negotiate commitment	2	=MAX(F5)	=E5+D6	=H6-D6	=G8	=H6-F6	=IF(I6=0,"Yes","No")
7		D	Negotiate spread	3	=MAX(F5)	=E7+D7	=H7-D7	=G8	=H7-F7	=IF(I7=0,"Yes","No")
8		E	Prepare registration	5	=MAX(F6,F7)	=E8+D8	=H8-D8	=G9	=H8-F8	=IF(I8=0,"Yes","No")
9		F	Submit registration	1	=MAX(F8)	=E9+D9	=H9-D9	=MIN(G10,G11,G12,G13)	=H9-F9	=IF(I9=0,"Yes","No")
10		G	Present	6	=MAX(F9)	=E10+D10	=H10-D10	=G16	=H10-F10	=IF(I10=0,"Yes","No")
11		H	Distribute red herring	3	=MAX(F9)	=E11+D11	=H11-D11	=G15	=H11-F11	=IF(I11=0,"Yes","No")
12		I	Calculate price	5	=MAX(F9)	=E12+D12	=H12-D12	=G15	=H12-F12	=IF(I12=0,"Yes","No")
13		J	Receive deficiency	3	=MAX(F9)	=E13+D13	=H13-D13	=G14	=H13-F13	=IF(I13=0,"Yes","No")
14		K	Amend statement	1	=MAX(F13)	=E14+D14	=H14-D14	=G15	=H14-F14	=IF(I14=0,"Yes","No")
15		L	Receive registration	2	=MAX(F14)	=E15+D15	=H15-D15	=MIN(G16,G17,G16)	=H15-F15	=IF(I15=0,"Yes","No")
16		M	Confirm blue sky	1	=MAX(F10,F11,F12,F15)	=E16+D16	=H16-D16	=MIN(G19,G20)	=H16-F16	=IF(I16=0,"Yes","No")
17		N	Appoint registrar	3	=MAX(F15)	=E17+D17	=H17-D17	=MIN(G19,G20)	=H17-F17	=IF(I17=0,"Yes","No")
18		O	Appoint transfer	3.5	=MAX(F15)	=E18+D18	=H18-D18	=MIN(G19,G20)	=H18-F18	=IF(I18=0,"Yes","No")
19		P	Issue prospectus	4.5	=MAX(F16,F17,F18)	=E19+D19	=H19-D19	=F22	=H19-F19	=IF(I19=0,"Yes","No")
20		Q	Phone buyers	4	=MAX(F16,F17,F18)	=E20+D20	=H20-D20	=F22	=H20-F20	=IF(I20=0,"Yes","No")
21										
22					Project Duration	=MAX(F19,F20)				

The values in the new spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Evaluate prestige	3	0	3	0	3	0	Yes
5		B	Select syndicate	1.5	3	4.5	3	4.5	0	Yes
6		C	Negotiate commitment	2	4.5	6.5	5.5	7.5	1	No
7		D	Negotiate spread	3	4.5	7.5	4.5	7.5	0	Yes
8		E	Prepare registration	5	7.5	12.5	7.5	12.5	0	Yes
9		F	Submit registration	1	12.5	13.5	12.5	13.5	0	Yes
10		G	Present	6	13.5	19.5	16	22	2.5	No
11		H	Distribute red herring	3	13.5	16.5	19	22	5.5	No
12		I	Calculate price	5	13.5	18.5	17	22	3.5	No
13		J	Receive deficiency	3	13.5	16.5	13.5	16.5	0	Yes
14		K	Amend statement	1	16.5	17.5	16.5	17.5	0	Yes
15		L	Receive registration	2	17.5	19.5	17.5	19.5	0	Yes
16		M	Confirm blue sky	1	19.5	20.5	22	23	2.5	No
17		N	Appoint registrar	3	19.5	22.5	20	23	0.5	No
18		O	Appoint transfer	3.5	19.5	23	19.5	23	0	Yes
19		P	Issue prospectus	4.5	23	27.5	23	27.5	0	Yes
20		Q	Phone buyers	4	23	27	23.5	27.5	0.5	No
21										
22						Project Duration = 27.5				

The initial public offering process is 27.5 weeks long. The critical path is:
START → A → B → D → E → F → J → K → L → O → P → FINISH

b) We explore each change independently.

i) Negotiating the commitment (step C) is performed parallel to negotiating the spread (step D). In part (a) above, negotiating the spread is on the critical path since it takes three days to complete while negotiating the commitment takes only two days to complete. We now increase the time to negotiate the commitment from two days to three days, and negotiating the commitment now takes as much time as negotiating the spread. Thus, there are now two critical paths through the network:

START → A → B → C → E → F → J → K → L → O → P → FINISH
START → A → B → D → E → F → J → K → L → O → P → FINISH

The project duration is still 27.5 weeks.

ii) In part (a) above, calculating the issue price is not on the critical path. Thus, decreasing the time to calculate the price does not change the solution found in part (a). The critical path remains the same, and the project duration is still 27.5 weeks.

iii) In part (a) above, the step to amend the statement and resubmit it to the SEC (step K) is on the critical path. Therefore, increasing the time for the step increases the project duration. The project duration increases to 29 weeks, and the critical path remains the same.

iv) In part (a) above, the step to confirm that the new issue complies with "blue sky" laws (step M) occurs in parallel to appointing a registrar (step N) and to appointing a transfer agent (step O). Step M is not on the critical path since it only takes one week while step O takes 3.5 weeks. When we increase the time to complete step M from one week to four weeks, we change the critical path since step M now takes longer than step O. We also change the project duration. The project duration is now 28 weeks. Two new critical paths appear:

START → A → B → D → E → F → G → M → P → FINISH
START → A → B → D → E → F → J → K → L → M → P → FINISH

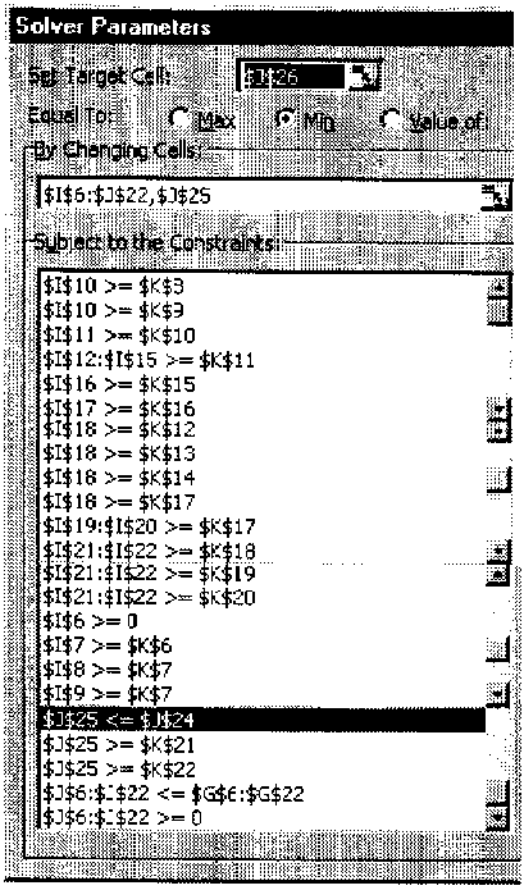
c) We formulate a linear programming problem to make the crashing decisions. We use a worksheet similar to the worksheet titled "Reliable's Time Cost" in the "Ch. 7 – Project Management" workbook. The formulas we use for the worksheet appear below.

	A	B	C	D	E	F	G	H	I	J	K
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	3	1.5	8000	14000	=C6-D6	=(F6-E6)/G6	0	1.5	=6+C6-J6
7		B	1.5	0.5	4500	8000	=C7-D7	=(F7-E7)/G7	1.5	1	=7+C7-J7
8		C	2	2	9000	0	=C8-D8	0	2	0	=8+C8-J8
9		D	3	3	12000	0	=C9-D9	0	2	0	=9+C9-J9
10		E	5	4	50000	95000	=C10-D10	=(F10-E10)/G10	5	0	=10+C10-J10
11		F	1	1	1000	0	=C11-D11	0	10	0	=11+C11-J11
12		G	6	4	25000	60000	=C12-D12	=(F12-E12)/G12	11	0	=12+C12-J12
13		H	3	2	15000	22000	=C13-D13	=(F13-E13)/G13	14	0	=13+C13-J13
14		I	5	3.5	12000	31000	=C14-D14	=(F14-E14)/G14	11	0	=14+C14-J14
15		J	3	3	0	0	=C15-D15	0	11	0	=15+C15-J15
16		K	1	0.5	8000	9000	=C16-D16	=(F16-E16)/G16	14	0.5	=16+C16-J16
17		L	2	2	0	0	=C17-D17	0	14.5	0	=17+C17-J17
18		M	1	0.5	5000	8300	=C18-D18	=(F18-E18)/G18	17	0	=18+C18-J18
19		N	3	1.5	12000	19000	=C19-D19	=(F19-E19)/G19	16.5	1.5	=19+C19-J19
20		O	3.5	1.5	13000	21000	=C20-D20	=(F20-E20)/G20	16.5	2	=20+C20-J20
21		P	4.5	2	40000	99000	=C21-D21	=(F21-E21)/G21	18	0.5	=21+C21-J21
22		Q	4	1.5	9000	20000	=C22-D22	=(F22-E22)/G22	18	0	=22+C22-J22
23											
24									Desired Finish	22	
25									Finish Time =	22	
26									Total Cost	=SUM(E6:E22)+SUMPRODUCT(H6:H22,J6:J22)	

The values used in the spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J	K
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	3	1.5	$8000	$14000	1.5	$4000	0.0	1.5	1.5
7		B	1.5	0.5	$4500	$8000	1	$3500	1.5	1	2
8		C	2	2	$9000	$0	0	$0	2.0	0	4
9		D	3	3	$12000	$0	0	$0	2.0	0	5
10		E	5	4	$50000	$95000	1	$45000	5.0	0	10
11		F	1	1	$1000	$0	0	$0	10.0	0	11
12		G	6	4	$25000	$60000	2	$17500	11.0	0	17
13		H	3	2	$15000	$22000	1	$7000	14.0	0	17
14		I	5	3.5	$12000	$31000	1.5	$12667	11.0	0	16
15		J	3	3	$0	$0	0	$0	11.0	0	14
16		K	1	0.5	$6000	$9000	0.5	$6000	14.0	0.5	14.5
17		L	2	2	$0	$0	0	$0	14.5	0	16.5
18		M	1	0.5	$5000	$8300	0.5	$6600	17.0	0	18
19		N	3	1.5	$12000	$19000	1.5	$4667	16.5	1.5	18
20		O	3.5	1.5	$13000	$21000	2	$4000	16.5	2.0	18
21		P	4.5	2	$40000	$99000	2.5	$23600	18.0	0.5	22
22		Q	4	1.5	$9000	$20000	2.5	$4400	18.0	0	22
23											
24									Desired Finish	22	
25									Finish Time =	22	
26									Total Cost =	$260800	

The Solver settings for the linear programming appear below.



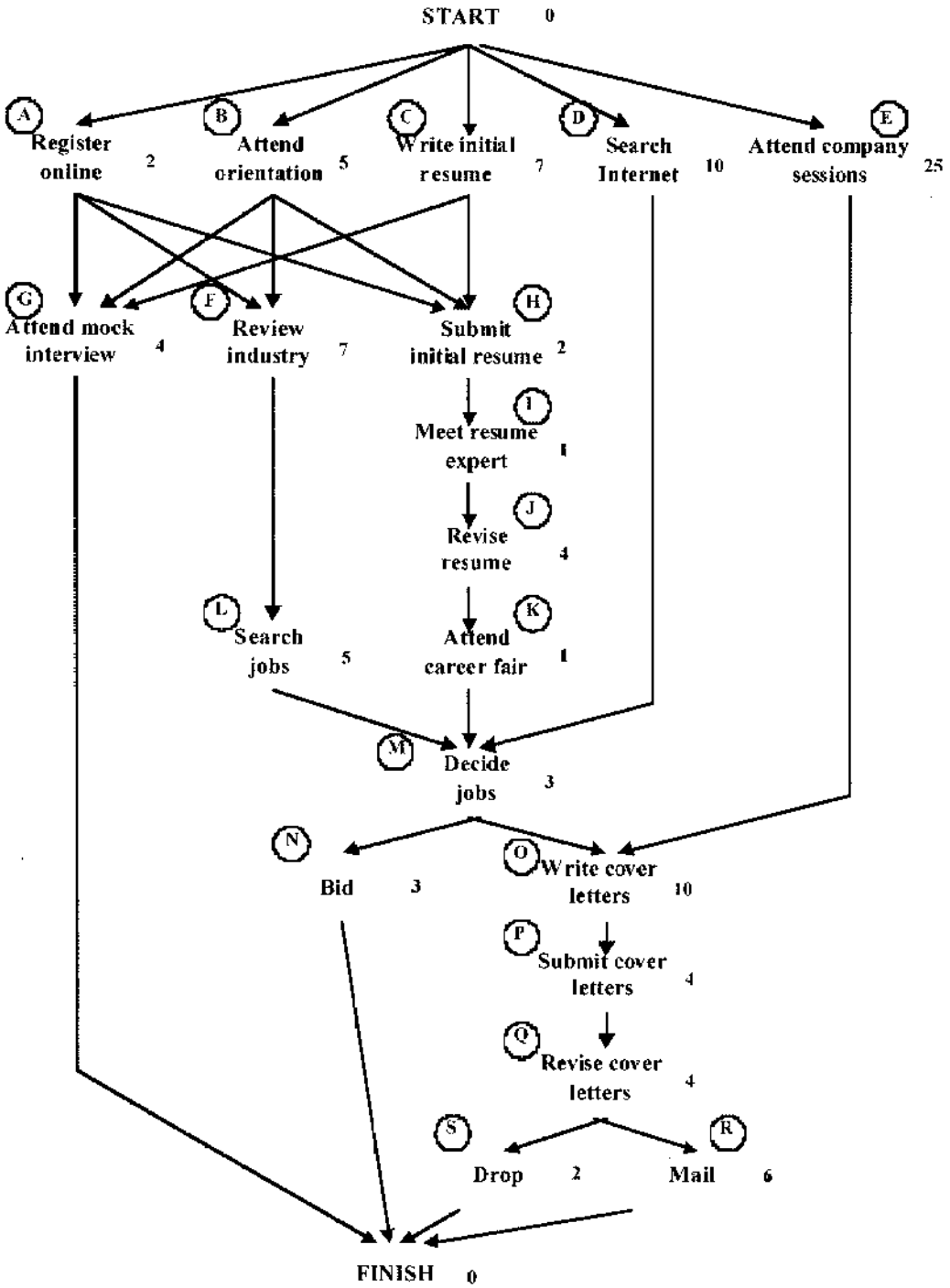
Janet and Gilbert should reduce the time for step A (evaluating the prestige of each potential underwriter) by 1.5 weeks, the time for step B (selecting a syndicate of underwriters) by one week, the time for step K (amending statement and resubmitting it to the SEC) by 0.5 weeks, the time for step N (appointing a registrar) by 1.5 weeks, the time for step O (appointing a transfer agent) by two weeks, and the time for step P (issuing final prospectus) by 0.5 weeks. Janet and Gilbert can now meet the new deadline of 22 weeks at a total cost of $260,800.

d) We use the same model formulation that was used in part (c). We change one constraint, however. The project duration now has to be greater than or equal to 24 weeks instead of 22 weeks. We obtain the following solution in Excel.

	A	B	C	D	E	F	G	H	I	J	K
3							Maximum	Crash Cost			
4			Time		Cost		Time	per Week	Start	Time	Finish
5		Activity	Normal	Crash	Normal	Crash	Reduction	saved	Time	Reduction	Time
6		A	3	1.5	$8000	$14000	1.5	$4000	0.0	1.5	1.5
7		B	1.5	0.5	$4500	$8000	1	$3500	1.5	1	2
8		C	2	2	$9000	$0	0	$0	2.0	0	4
9		D	3	3	$12000	$0	0	$0	2.0	0	5
10		E	5	4	$50000	$95000	1	$45000	5.0	0	10
11		F	1	1	$1000	$0	0	$0	10.0	0	11
12		G	6	4	$25000	$60000	2	$17500	12.5	0	18.5
13		H	3	2	$15000	$22000	1	$7000	15.5	0	18.5
14		I	5	3.5	$12000	$31000	1.5	$12667	11.0	0	16
15		J	3	3	$0	$0	0	$0	11.0	0	14
16		K	1	0.5	$6000	$9000	0.5	$6000	14.0	0.5	14.5
17		L	2	2	$0	$0	0	$0	14.5	0	16.5
18		M	1	0.5	$5000	$8300	0.5	$6600	18.5	0	19.5
19		N	3	1.5	$12000	$19000	1.5	$4667	16.5	0	19.5
20		O	3.5	1.5	$13000	$21000	2	$4000	16.5	0.5	19.5
21		P	4.5	2	$40000	$99000	2.5	$23600	19.5	0	24
22		Q	4	1.5	$9000	$20000	2.5	$4400	19.5	0	23.5
23											
24									Desired Finish	24	
25									Finish Time =	24	
26									Total Cost =	$236000	

Janet and Gilbert should reduce the time for step A (evaluating the prestige of each potential underwriter) by 1.5 weeks, the time for step B (selecting a syndicate of underwriters) by one week, the time for step K (amending statement and resubmitting it to the SEC) by 0.5 weeks, and the time for step O (appointing a transfer agent) by 0.5 weeks. Janet and Gilbert can now meet the new deadline of 24 weeks at a total cost of $236,000.

10-2 a) The project network appears below.



The estimated project duration equals the length of the longest path through the project network. To calculate the length of the longest path, we use the layout of the worksheet titled "Reliable" in the "Ch. 7 – Project Management" workbook. We need to modify the spreadsheet to reflect the network unique to this case. The formulas in the new spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2	0	=E4+D4	=H4-D4	=MIN(G9,G10,G11)	=H4-F4	=IF(I4=0,"Yes","No")
5		B	Attend orientation	5	0	=E5+D5	=H5-D5	=MIN(G9,G10,G11)	=H5-F5	=IF(I5=0,"Yes","No")
6		C	Write initial resume	7	0	=E6+D6	=H6-D6	=MIN(G11,G13)	=H6-F6	=IF(I6=0,"Yes","No")
7		D	Search Internet	10	0	=E7+D7	=H7-D7	=G16	=H7-F7	=IF(I7=0,"Yes","No")
8		E	Attend company sessions	25	0	=E8+D8	=H8-D8	=G18	=H8-F8	=IF(I8=0,"Yes","No")
9		F	Review industry, etc.	7	=MAX(F4,F5)	=E9+D9	=H9-D9	=G15	=H9-F9	=IF(I9=0,"Yes","No")
10		G	Attend mock interview	4	=MAX(F4,F5,F6)	=E10+D10	=H10-D10	=F25	=H10-F10	=IF(I10=0,"Yes","No")
11		H	Submit initial resume	2	=MAX(F4,F5,F6)	=E11+D11	=H11-D11	=G12	=H11-F11	=IF(I11=0,"Yes","No")
12		I	Meet resume expert	1	=F11	=E12+D12	=H12-D12	=G13	=H12-F12	=IF(I12=0,"Yes","No")
13		J	Revise resume	4	=F12	=E13+D13	=H13-D13	=G14	=H13-F13	=IF(I13=0,"Yes","No")
14		K	Attend career fair	1	=F13	=E14+D14	=H14-D14	=G16	=H14-F14	=IF(I14=0,"Yes","No")
15		L	Search jobs	5	=F9	=E15+D15	=H15-D15	=G16	=H15-F15	=IF(I15=0,"Yes","No")
16		M	Decide jobs	3	=MAX(F15,F14,F7)	=E16+D16	=H16-D16	=MIN(G17,G18)	=H16-F16	=IF(I16=0,"Yes","No")
17		N	Bid	3	=F16	=E17+D17	=H17-D17	=F25	=H17-F17	=IF(I17=0,"Yes","No")
18		O	Write cover letters	10	=MAX(F16,F8)	=E18+D18	=H18-D18	=G19	=H18-F18	=IF(I18=0,"Yes","No")
19		P	Submit cover letters	4	=F18	=E19+D19	=H19-D19	=G20	=H19-F19	=IF(I19=0,"Yes","No")
20		Q	Revise cover letters	4	=F19	=E20+D20	=H20-D20	=MIN(G21,G22)	=H20-F20	=IF(I20=0,"Yes","No")
21		R	Mail	6	=F20	=E21+D21	=H21-D21	=F25	=H21-F21	=IF(I21=0,"Yes","No")
22		S	Drop	2	=F20	=E22+D22	=H22-D22	=F25	=H22-F22	=IF(I22=0,"Yes","No")
23										
24										
25					Project Duration	=MAX(F22,F21,F17,F13)				

The values in the new spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2	0	2	8	10	8	No
5		B	Attend orientation	5	0	5	5	10	5	No
6		C	Write initial resume	7	0	7	7	14	7	No
7		D	Search Internet	10	0	10	12	22	12	No
8		E	Attend company sessions	25	0	25	0	25	0	Yes
9		F	Review industry, etc.	7	5	12	10	17	5	No
10		G	Attend mock interview	4	7	11	45	49	38	No
11		H	Submit initial resume	2	7	9	14	16	7	No
12		I	Meet resume expert	1	9	10	16	17	7	No
13		J	Revise resume	4	10	14	17	21	7	No
14		K	Attend career fair	1	14	15	21	22	7	No
15		L	Search jobs	5	12	17	17	22	5	No
16		M	Decide jobs	3	17	20	22	25	5	No
17		N	Bid	3	20	23	46	49	26	No
18		O	Write cover letters	10	25	35	25	35	0	Yes
19		P	Submit cover letters	4	35	39	35	39	0	Yes
20		Q	Revise cover letters	4	39	43	39	43	0	Yes
21		R	Mail	6	43	49	43	49	0	Yes
22		S	Drop	2	43	45	47	49	4	No
23										
24										
25					Project Duration	=49				

Brent can start the interviews in 49 days. The critical steps in the process are:

START → E → O → P → Q → R → FINISH

b) We substitute first the pessimistic and then the optimistic estimates for the time values used in the part (a) spreadsheet.

The spreadsheet estimating the longest path for the pessimistic values appears below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	4	0	4	6	10	6	No
5		B	Attend orientation	10	0	10	0	10	0	Yes
6		C	Write initial resume	14	0	14	4	18	4	No
7		D	Search Internet	12	0	12	20	32	20	No
8		E	Attend company sessions	30	0	30	6	36	6	No
9		F	Review industry, etc.	12	10	22	10	22	0	Yes
10		G	Attend mock interview	8	14	22	66	74	52	No
11		H	Submit initial resume	6	14	20	18	24	4	No
12		I	Meet resume expert	1	20	21	24	25	4	No
13		J	Revise resume	6	21	27	25	31	4	No
14		K	Attend career fair	1	27	28	31	32	4	No
15		L	Search jobs	10	22	32	22	32	0	Yes
16		M	Decide jobs	4	32	36	32	36	0	Yes
17		N	Bid	8	36	44	66	74	30	No
18		O	Write cover letters	12	36	48	36	48	0	Yes
19		P	Submit cover letters	7	48	55	48	55	0	Yes
20		Q	Revise cover letters	9	55	64	55	64	0	Yes
21		R	Mail	10	64	74	64	74	0	Yes
22		S	Drop	3	64	67	71	74	7	No
23										
24										
25					Project Duration =	74				

Under the worst-case scenario, Brent will require 74 days before he is ready to begin interviewing. The critical path changes to:

START → B → F → L → M → O → P → Q → R → FINISH

The spreadsheet estimating the longest path for the optimistic values appears below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	1	0	1	9	10	9	No
5		B	Attend orientation	3	0	3	7	10	7	No
6		C	Write initial resume	5	0	5	7	12	7	No
7		D	Search Internet	7	0	7	11	18	11	No
8		E	Attend company sessions	20	0	20	0	20	0	Yes
9		F	Review industry, etc.	5	3	8	10	15	7	No
10		G	Attend mock interview	3	5	8	29	32	24	No
11		H	Submit initial resume	1	5	6	12	13	7	No
12		I	Meet resume expert	1	6	7	13	14	7	No
13		J	Revise resume	3	7	10	14	17	7	No
14		K	Attend career fair	1	10	11	17	18	7	No
15		L	Search jobs	3	8	11	15	18	7	No
16		M	Decide jobs	2	11	13	18	20	7	No
17		N	Bid	2	13	15	30	32	17	No
18		O	Write cover letters	3	20	23	20	23	0	Yes
19		P	Submit cover letters	2	23	25	23	25	0	Yes
20		Q	Revise cover letters	3	25	28	25	28	0	Yes
21		R	Mail	4	28	32	28	32	0	Yes
22		S	Drop	1	28	29	31	32	3	No
23										
24										
25					Project Duration =	32				

Under the best-case scenario, Brent will require 32 days before he is ready to begin interviewing. The critical path remains the same as that in part (a).

c) The mean critical is the path through the project network that would be the critical path if the duration of each activity equals its mean. We therefore first need to determine the mean duration of each activity given the optimistic, most likely, and pessimistic length estimates. To calculate the mean duration of each activity, we use the PERT template in the "Ch. 7 – Project Management" workbook.

The formulas for the calculation of the mean duration of each activity appear below.

	A	B	C	D	E	F	G
3			Time Estimates				
4		Activity	o	m	p	μ	σ²
5		A	1	2	4	=IF(C5="","",(C5+4*D5+E5)/6)	=IF(C5="","",((E5-C5)/6)^2)
6		B	3	5	10	=IF(C6="","",(C6+4*D6+E6)/6)	=IF(C6="","",((E6-C6)/6)^2)
7		C	5	7	14	=IF(C7="","",(C7+4*D7+E7)/6)	=IF(C7="","",((E7-C7)/6)^2)
8		D	7	10	12	=IF(C8="","",(C8+4*D8+E8)/6)	=IF(C8="","",((E8-C8)/6)^2)
9		E	20	25	30	=IF(C9="","",(C9+4*D9+E9)/6)	=IF(C9="","",((E9-C9)/6)^2)
10		F	5	7	12	=IF(C10="","",(C10+4*D10+E10)/6)	=IF(C10="","",((E10-C10)/6)^2)
11		G	3	4	8	=IF(C11="","",(C11+4*D11+E11)/6)	=IF(C11="","",((E11-C11)/6)^2)
12		H	1	2	6	=IF(C12="","",(C12+4*D12+E12)/6)	=IF(C12="","",((E12-C12)/6)^2)
13		I	1	1	1	=IF(C13="","",(C13+4*D13+E13)/6)	=IF(C13="","",((E13-C13)/6)^2)
14		J	3	4	6	=IF(C14="","",(C14+4*D14+E14)/6)	=IF(C14="","",((E14-C14)/6)^2)
15		K	1	1	1	=IF(C15="","",(C15+4*D15+E15)/6)	=IF(C15="","",((E15-C15)/6)^2)
16		L	3	5	10	=IF(C16="","",(C16+4*D16+E16)/6)	=IF(C16="","",((E16-C16)/6)^2)
17		M	2	3	4	=IF(C17="","",(C17+4*D17+E17)/6)	=IF(C17="","",((E17-C17)/6)^2)
18		N	2	3	8	=IF(C18="","",(C18+4*D18+E18)/6)	=IF(C18="","",((E18-C18)/6)^2)
19		O	3	10	12	=IF(C19="","",(C19+4*D19+E19)/6)	=IF(C19="","",((E19-C19)/6)^2)
20		P	2	4	7	=IF(C20="","",(C20+4*D20+E20)/6)	=IF(C20="","",((E20-C20)/6)^2)
21		Q	3	4	9	=IF(C21="","",(C21+4*D21+E21)/6)	=IF(C21="","",((E21-C21)/6)^2)
22		R	4	6	10	=IF(C22="","",(C22+4*D22+E22)/6)	=IF(C22="","",((E22-C22)/6)^2)
23		S	1	2	3	=IF(C23="","",(C23+4*D23+E23)/6)	=IF(C23="","",((E23-C23)/6)^2)

The values for the mean duration of each activity appear below.

	A	B	C	D	E	F	G
3			Time Estimates				
4		Activity	o	m	p	μ	σ²
5		A	1	2	4	2.167	0.25
6		B	3	5	10	5.5	1.36
7		C	5	7	14	7.833	2.25
8		D	7	10	12	9.833	0.69
9		E	20	25	30	25	2.77
10		F	5	7	12	7.5	1.36
11		G	3	4	8	4.5	0.69
12		H	1	2	6	2.5	0.69
13		I	1	1	1	1	0
14		J	3	4	6	4.167	0.25
15		K	1	1	1	1	0
16		L	3	5	10	5.5	1.36
17		M	2	3	4	3	0.11
18		N	2	3	8	3.667	1
19		O	3	10	12	9.167	2.25
20		P	2	4	7	4.167	0.69
21		Q	3	4	9	4.667	1
22		R	4	6	10	6.333	1
23		S	1	2	3	2	0.11

We now substitute the mean duration for each activity for the time values used in the part (a) spreadsheet. We obtain the following values:

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2.17	0.00	2.17	6.83	9.00	6.833	No
5		B	Attend orientation	5.5	0.00	5.50	3.50	9.00	3.5	No
6		C	Write initial resume	7.83	0.00	7.83	5.50	13.33	5.5	No
7		D	Search Internet	9.83	0.00	9.83	12.17	22.00	12.17	No
8		E	Attend company sessions	25	0.00	25.00	0.00	25.00	0	Yes
9		F	Review industry, etc.	7.5	5.50	13.00	9.00	16.50	3.5	No
10		G	Attend mock interview	4.5	7.83	12.33	44.83	49.33	37	No
11		H	Submit initial resume	2.5	7.83	10.33	13.33	15.83	5.5	No
12		I	Meet resume expert	1	10.33	11.33	15.83	16.83	5.5	No
13		J	Revise resume	4.17	11.33	15.50	16.83	21.00	5.5	No
14		K	Attend career fair	1	15.50	16.50	21.00	22.00	5.5	No
15		L	Search jobs	5.5	13.00	18.50	16.50	22.00	3.5	No
16		M	Decide jobs	3	18.50	21.50	22.00	25.00	3.5	No
17		N	Bid	3.67	21.50	25.17	45.67	49.33	24.17	No
18		O	Write cover letters	9.17	25.00	34.17	25.00	34.17	0	Yes
19		P	Submit cover letters	4.17	34.17	38.33	34.17	38.33	0	Yes
20		Q	Revise cover letters	4.67	38.33	43.00	38.33	43.00	0	Yes
21		R	Mail	6.33	43.00	49.33	43.00	49.33	0	Yes
22		S	Drop	2	43.00	45.00	47.33	49.33	4.333	No
23										
24										
25					Project Duration =	49.33				

The mean critical path is the same as that in part (a).

To calculate the variance of the project duration, we again use the PERT template. The formulas for the mean and variance calculations are shown below.

	J	K
5		Mean Critical
6		Path
7	μ =	=SUMIF(H5:H23,"*",F5:F23)
8	σ^2 =	=SUMIF(H5:H23,"*",G5:G23).

The mean and variance of the mean critical path are:

	J	K
5	Mean Critical	
6	Path	
7	μ =	49.3333
8	σ^2 =	7.72222

d) We use the PERT template used in part (c). We substitute the value 60 for d in cell K12. The formulas to calculate the probability that Brent will be ready to interview within 60 days are shown below.

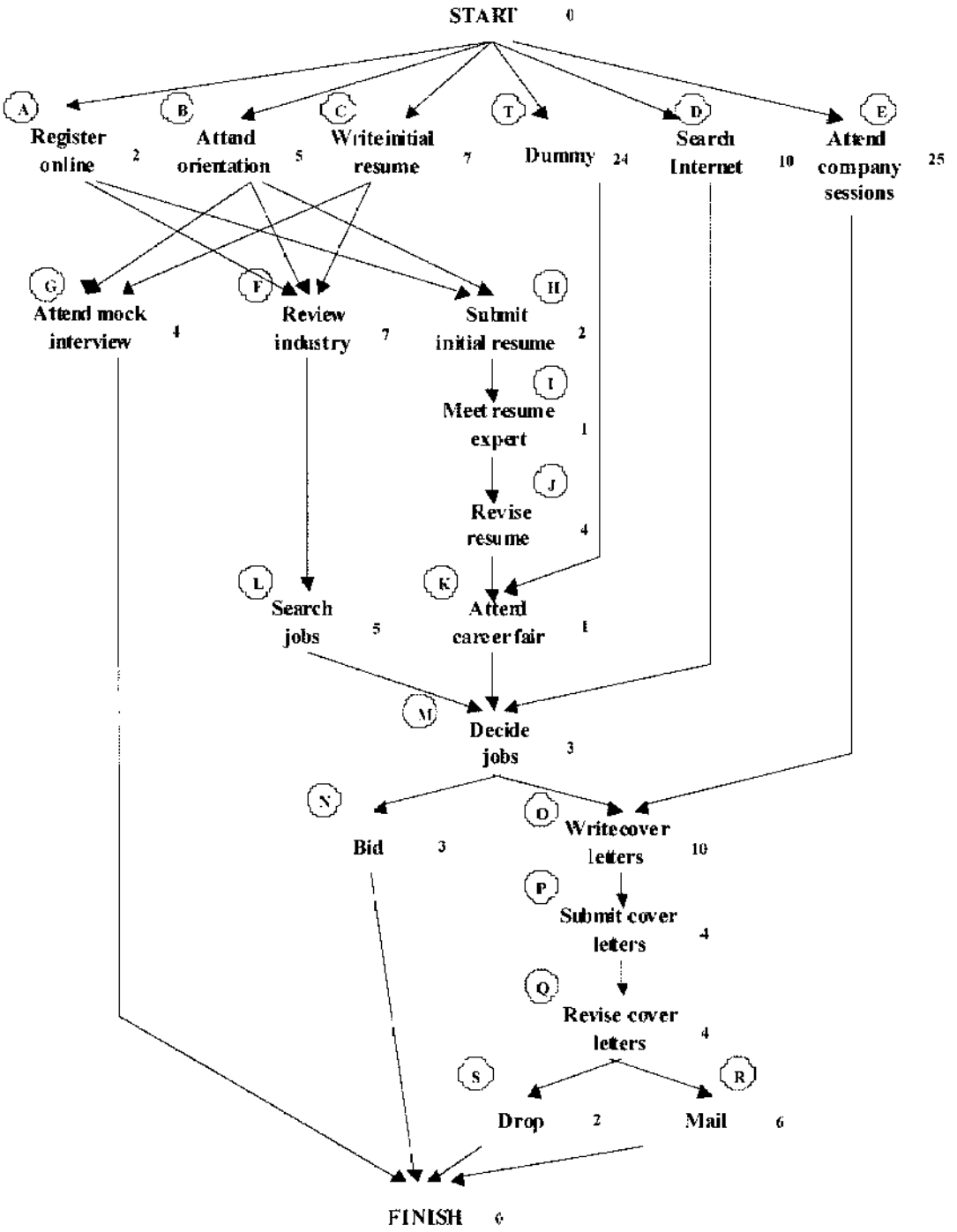
	J	K
5		Mean Critical
6		Path
7	μ =	=SUMIF(H5:H23,"*",F5:F23)
8	σ^2 =	=SUMIF(H5:H23,"*",G5:G23)
9		
10	P(T=d) =	=NORMDIST(K12,K7,SQRT(K8),1)
11	where	
12	d =	60

The values obtained from the above calculations are shown below.

	J	K
5	Mean Critical	
6	Path	
7	μ =	49.3333
8	σ^2 =	7.72222
9		
10	P(T=d) =	0.99994
11	where	
12	d =	60

Brent will be ready for his interviews within 60 days with 99.994 percent probability. Good for him!

e) Now the earliest start time for the career fair is day 24, and the career fair itself still lasts one day. To ensure that the earliest start time for the career fair is day 24, we add a dummy node T to the project network that has a duration of 24 days, directly follows the START node, and directly precedes the career fair node K. The updated project network appears below.



f) To obtain the mean critical path for the new network and the probability that Brent will complete his project within 60 days, we first use the PERT template to calculate the mean duration for each activity. We add the new node T to the list of activities, and the values of the mean duration for each activity appear below.

	A	B	C	D	E	F	G
3		Time Estimates					
4		Activity	o	m	p	μ	σ^2
5		A	1	2	4	2.17	0.25
6		B	3	5	10	5.50	1.36111111
7		C	5	7	14	7.83	2.25
8		D	7	10	12	9.83	0.69444444
9		E	20	25	30	25.00	2.77777777
10		T	24	24	24	24.00	0
11		F	5	7	12	7.50	1.36111111
12		G	3	4	8	4.50	0.69444444
13		H	1	2	6	2.50	0.69444444
14		I	1	1	1	1.00	0
15		J	3	4	6	4.17	0.25
16		K	1	1	1	1.00	0
17		L	3	5	10	5.50	1.36111111
18		M	2	3	4	3.00	0.11111111
19		N	2	3	8	3.67	1
20		O	3	10	12	9.17	2.25
21		P	2	4	7	4.17	0.69444444
22		Q	3	4	9	4.67	1
23		R	4	6	10	6.33	1
24		S	1	2	3	2.00	0.11111111

We next substitute these mean duration values for the time values used to calculate the critical path. We need to alter our spreadsheet used to calculate the critical path in part (a) to reflect the addition of the new node T. The formulas used in the new critical path spreadsheet appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2.166	0	=E4+D4	=H4-D4	=MIN(G10,G11,G12)	=H4-F4	=IF(I4=0,"Yes","No")
5		B	Attend orientation	5.5	0	=E5+D5	=H5-D5	=MIN(G10,G11,G12)	=H5-F5	=IF(I5=0,"Yes","No")
6		C	Write initial resume	7.833	0	=E6+D6	=H6-D6	=MIN(G12,G11)	=H6-F6	=IF(I6=0,"Yes","No")
7		D	Search Internet	9.833	0	=E7+D7	=H7-D7	=G17	=H7-F7	=IF(I7=0,"Yes","No")
8		E	Attend company sessions	25	0	=E8+D8	=H8-D8	=G19	=H8-F8	=IF(I8=0,"Yes","No")
9		T	Dummy for career fair	24	0	=E9+D9	=H9-D9	=G15	=H9-F9	=IF(I9=0,"Yes","No")
10		F	Review industry etc	7.5	=MAX(F4,F5)	=E10+D10	=H10-D10	=G16	=H10-F10	=IF(I10=0,"Yes","No")
11		G	Attend mock interview	4.5	=MAX(F4,F5,F6)	=E11+D11	=H11-D11	=F26	=H11-F11	=IF(I11=0,"Yes","No")
12		H	Submit initial resume	2.5	=MAX(F4,F5,F6)	=E12+D12	=H12-D12	=G13	=H12-F12	=IF(I12=0,"Yes","No")
13		I	Meet resume expert	1	=F12	=E13+D13	=H13-D13	=G14	=H13-F13	=IF(I13=0,"Yes","No")
14		J	Revise resume	4.166	=F13	=E14+D14	=H14-D14	=G15	=H14-F14	=IF(I14=0,"Yes","No")
15		K	Attend career fair	1	=MAX(F14,F9)	=E15+D15	=H15-D15	=G17	=H15-F15	=IF(I15=0,"Yes","No")
16		L	Search jobs	5.5	=F10	=E16+D16	=H16-D16	=G17	=H16-F16	=IF(I16=0,"Yes","No")
17		M	Decide jobs	0	=MAX(F16,F15,F7)	=E17+D17	=H17-D17	=MIN(G18,G19)	=H17-F17	=IF(I17=0,"Yes","No")
18		N	Bid	3.666	=F17	=E18+D18	=H18-D18	=F26	=H18-F18	=IF(I18=0,"Yes","No")
19		O	Write cover letters	9.166	=MAX(F17,F8)	=E19+D19	=H19-D19	=G20	=H19-F19	=IF(I19=0,"Yes","No")
20		P	Submit cover letters	4.166	=F19	=E20+D20	=H20-D20	=G21	=H20-F20	=IF(I20=0,"Yes","No")
21		Q	Revise cover letters	4.666	=F20	=E21+D21	=H21-D21	=MIN(G22,G23)	=H21-F21	=IF(I21=0,"Yes","No")
22		R	Mail	6.333	=F21	=E22+D22	=H22-D22	=F26	=H22-F22	=IF(I22=0,"Yes","No")
23		S	Drop	2	=F21	=E23+D23	=H23-D23	=F26	=H23-F23	=IF(I23=0,"Yes","No")
24										
25										
26					Project Duration	=MAX(F23,F22,F18,F11)				

The values used in the calculation of the new critical path appear below.

	A	B	C	D	E	F	G	H	I	J
3		Activity	Description	Time	ES	EF	LS	LF	Slack	Critical?
4		A	Register online	2.17	0.00	2.17	9.83	12.00	9.833	No
5		B	Attend orientation	5.50	0.00	5.50	6.50	12.00	6.5	No
6		C	Write initial resume	7.83	0.00	7.83	8.50	16.33	8.5	No
7		D	Search Internet	9.83	0.00	9.83	15.17	25.00	15.17	No
8		E	Attend company sessions	25.00	0.00	25.00	3.00	28.00	3	No
9		T	Dummy for career fair	24.00	0.00	24.00	0.00	24.00	0	Yes
10		F	Review industry, etc.	7.50	5.50	13.00	12.00	19.50	6.5	No
11		G	Attend mock interview	4.50	7.83	12.33	47.83	52.33	40	No
12		H	Submit initial resume	2.50	7.83	10.33	16.33	18.83	8.5	No
13		I	Meet resume expert	1.00	10.33	11.33	18.83	19.83	8.5	No
14		J	Revise resume	4.17	11.33	15.50	19.83	24.00	8.5	No
15		K	Attend career fair	1.00	24.00	25.00	24.00	25.00	0	Yes
16		L	Search jobs	5.50	13.00	18.50	19.50	25.00	6.5	No
17		M	Decide jobs	3.00	25.00	28.00	25.00	28.00	0	Yes
18		N	Bid	3.67	28.00	31.67	48.67	52.33	20.67	No
19		O	Write cover letters	9.17	28.00	37.17	28.00	37.17	0	Yes
20		P	Submit cover letters	4.17	37.17	41.33	37.17	41.33	0	Yes
21		Q	Revise cover letters	4.67	41.33	46.00	41.33	46.00	0	Yes
22		R	Mail	5.33	46.00	52.33	46.00	52.33	0	Yes
23		S	Drop	2.00	46.00	48.00	50.33	52.33	4.333	No
24										
25										
26					Project Duration =	52.33				

The mean project duration is now 52.33 days, and the new mean critical path is START $\rightarrow$ T $\rightarrow$ K $\rightarrow$ M $\rightarrow$ O $\rightarrow$ P $\rightarrow$ Q $\rightarrow$ R $\rightarrow$ FINISH. We specify this new critical path in the PERT spreadsheet to obtain the probability that Brent will complete his project within 60 days.

	J	K
5	Mean Critical	
6	Path	
7	μ =	52.3333
8	σ^2 =	5.05556
9		
10	P(T=d) =	0.9996
11	where	
12	d =	60

Brent will be ready for his interviews within 60 days with 99.967 percent probability. The probability that Brent will complete his project decreases slightly over the probability obtained in part (d). The probability decreases because this new network has a longer mean project duration than the network in part (d). The probability decreases only slightly, however, since the variance of the project duration for the new network is smaller than the variance in part (d).

11.4-2 cont.

$n=1$:

s_1 \ x_1	0	A	B	$f_1^*(s_1)$	x_1^*
5000	8400	9800	8150	9800	A

The optimal policy is to always invest in A with an expected fortune after three years of \$9800.

(b) Let x_n and s_n be as in part (a).
Let $f_n(s_n, x_n)$ be the maximum probability of having at least \$10,000 after 3 years given s_n and x_n.

$n=3$:

s_3 \ x_3	0	A	B	$f_3^*(s_3)$	x_3^*
$0 \le s_3 < 5000$	0	—	—	0	0
$5000 \le s_3 < 10000$	0	.7	.1	.7	A
$10000 \le s_3 < 15000$	1	.7	1	1	0 or B
$15000 \le s_3$	1	1	1	1	0, A or B

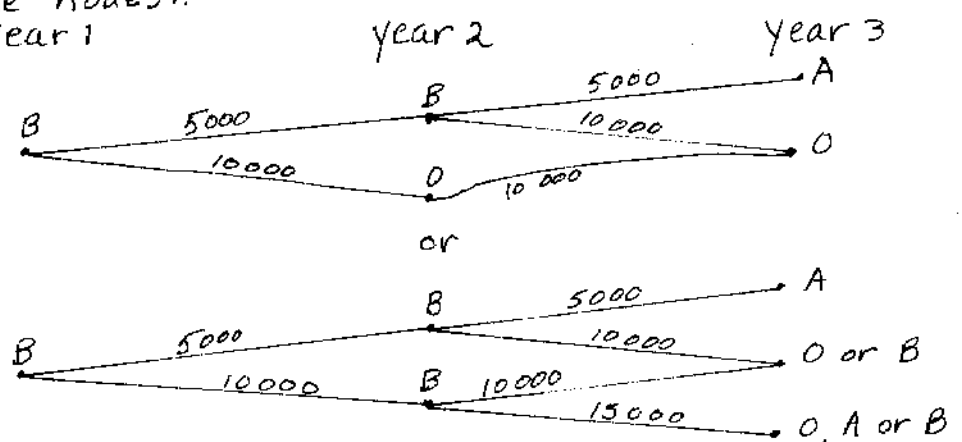
$n=2$:

s_2 \ x_2	0	A	B	$f_2^*(s_2)$	x_2^*
$0 \le s_2 < 5000$	0	—	—	0	0
$5000 \le s_2 < 10000$	.7	.7	.73	.73	B
$10000 \le s_2$	1	.73	1	1	0 or B

$n=1$:

s_1 \ x_1	0	A	B	$f_1^*(s_1)$	x_1^*
5000	.73	.7	.757	.757	B

Hence, the optimal policies are (using the numbers on the arcs to represent the return on the investment indicated at the nodes).



and the maximum probability of having at least \$10,000 at the end of three years is .757.

11.4-3 $f_n(1, x_n) = K + x_n + \left(\frac{1}{3}\right)^{x_n} - f_{n+1}^*(1) + \left[1 - \left(\frac{1}{3}\right)^{x_n}\right] f_{n+1}^*(0)$

$= K + x_n + \left(\frac{1}{3}\right)^{x_n} f_{n+1}^*(1)$ since $f_{n+1}^*(0) = 0$

where $f_3^*(1) = 16$, $f_3^*(0) = 0$ and $K = \begin{cases} 0 & \text{if } x_n = 0 \\ 3 & \text{if } x_n > 0 \end{cases}$

11.4-3 (CONT)

$n=2$:

s_2 \ x_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	1	2	3	4	0	0

$n=1$:

s_1 \ x_1	0	1	2	3	4	$f_1^*(s_1)$	x_1^*
1	6.77	6.26	5.75	6.25	7.08	5.75	2

The optimal policy is to produce two on the first run and if none are acceptable, produce two on the second run. The minimum expected cost = \$5.75.

11.4-4 $f_n^*(s_n) = \max_{0 \le x_n} \left\{ \frac{1}{3} f_{n+1}^*(s_n - x_n) + \frac{2}{3} f_{n+1}^*(s_n + x_n) \right\}$

where $f_6^*(s_6) = 0$ for $s_6 < 5$ and $f_6^*(s_6) = 1$ for $s_6 \ge 5$

$n=5$:

s_5	$f_5^*(s_5)$	x_5^*
0	0	0
1	0	0
2	0	0
3	2/3	$x_5^* \ge 2$
4	2/3	$x_5^* \ge 1$
$s_5 \ge 5$	1	$x_5^* \le s_5 - 5$

$n=4$:

s_4 \ x_4	0	1	2	3	4	$f_4^*(s_4)$	x_4^*
0	0	—	—	—	—	0	0
1	0	0	—	—	—	0	0
2	0	4/9	4/9	—	—	4/9	1,2
3	2/3	4/9	2/3	2/3	—	2/3	0,2,3
4	2/3	8/9	2/3	2/3	2/3	8/9	1
$s_4 \ge 5$	1	—	—	—	—	1	$\le s_4 - 5$

$n=3$:

s_3 \ x_3	0	1	2	3	4	$f_3^*(s_3)$	x_3^*
0	0	—	—	—	—	0	0
1	0	8/27	—	—	—	8/27	1
2	4/9	4/9	16/27	—	—	16/27	2
3	2/3	20/27	2/3	2/3	—	20/27	1
4	8/9	8/9	23/27	2/3	2/3	23/27	0,1
$s_3 \ge 5$	1	—	—	—	—	1	$\le s_3 - 5$

$n=2$:

s_2 \ x_2	0	1	2	3	4	$f_2^*(s_2)$	x_2^*
0	0	—	—	—	—	0	0
1	8/27	32/81	—	—	—	32/81	1
2	16/27	48/81	48/81	—	—	48/81	0,1,2
3	20/27	64/81	62/81	2/3	—	64/81	1
4	24/27	74/81	70/81	62/81	2/3	74/81	1
$s_2 \ge 5$	1	—	—	—	—	1	$\le s_2 - 5$

$n=1$:

s_1 \ x_1	0	1	2	$f_1^*(s_1)$	x_1^*
2	48/81	160/243	124/243	160/243	1

So Probability {winning bet using above policy} = $\frac{160}{243} = .658$

11.4-5

Let $x_n = a$ and $x_n = d$ denote the decision to advertise or discontinue the product, respectively, for quarter n ($n = 1, 2, 3$).

Let s_n denote the level of sales above ($s_n \ge 0$) or below ($s_n \le 0$) the breakeven point (in millions) for quarter $(n-1)$.

Let $f_n(s_n, x_n)$ denote the maximum expected discounted profit (in millions) from the beginning of period n onward given state s_n and decision x_n.

11.4-5 cont. Then $f_n(s_n, x_n) = -30 + 5\left[s_n + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} t\,dt\right] + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s+t)\,dt$

where a_n and b_n are given by

n	a_n	b_n
1	1	5
2	0	4
3	-1	3

Hence, for $1 \le n \le 3$, $f_n(s_n, a) = -30 + 5\left[s_n + \frac{a_n + b_n}{2}\right] + \frac{1}{b_n - a_n}\int_{a_n}^{b_n} f_{n+1}^*(s_n + t)\,dt$

$f_n(s_n, d) = -20$ and the process stops.

Then $f_n^*(s_n) = \max\{f_n(s_n, a), f_n(s_n, d)\}$ and $f_4^*(s_4) = \begin{cases} -20 & \text{if } 0 > s_4 \\ -40 s_4 & \text{if } s_4 \ge 0 \end{cases}$

n=3:

$f_3(s_3, d) = -20$

$f_3(s_3, a) = -30 + 5(s_3 + 1) + \frac{1}{4}\int_{-1}^{3} f_4^*(s_3 + t)\,dt$

For $-3 \le s_3 \le 1$, $f_3(s_3, a) = -30 + 5(s_3+1) + \frac{1}{4}\left[\int_{-1}^{-s_3} -20\,dt + \frac{1}{4}\int_{-s_3}^{3} 40(s_3 + t)\,dt\right]$

$= 5(s_3 + 4)^2 - 65$

For $1 \le s_3 \le 5$, $f_3(s_3, a) = -30 + 5(s_3+1) + \frac{1}{4}\int_{-1}^{3} 40(s_3 + t)\,dt = 15 + 45 s_3$.

Hence, for $-3 \le s_3 \le 1$, $f_3^*(s_3) = \max\{-20, 5(s_3+4)^2 - 65\}$

$= \begin{cases} -20 & \text{if } -3 \le s_3 \le -1,\ x_3^* = d \\ 5(s_3+4)^2 - 65 & \text{if } -1 \le s_3 \le 1,\ x_3^* = a \end{cases}$

summarizing:

s_3	$f_3^*(s_3)$	x_3^*
$-3 \le s_3 \le -1$	-20	d
$-1 < s_3 \le 1$	$5(s_3+4)^2 - 65$	a
$1 < s_3 \le 5$	$15 + 45 s_3$	a

n=2:

for $-3 \le s_2 \le -1$,

$f_2(s_2, a) = -30 + 5(s_2 + 2) + \frac{1}{4}\left[\int_{0}^{-s_2 - 1} -20\,dt + \int_{-s_2 - 1}^{1 - s_2} (5(s_2 + t + 4)^2 - 65)\,dt + \int_{1 - s_2}^{4} (45(s_2 + t) + 15)\,dt\right]$

$= \frac{5}{4}\left(\frac{9}{2} s_2^2 + 47 s_2 + \frac{427}{6}\right)$

Since $f_2(-3, a) < f_2(s_2, d) = -20 < f_2(-1, a)$, we must find where $f_2(s_2, a) = f_2(s_2, d)$; that is, $\frac{5}{4}\left(\frac{9}{2} s_2^2 + 47 s_2 + \frac{427}{6}\right) = -20$.

$s_2^* = \frac{-47 + 8\sqrt{10}}{9} \doteq -2.411$

11.4-5 (cont)

For $-1 \le s_2 \le 1$, $f_2(s_2, a) = -30 + 5(s_2+2) + \frac{1}{4}\left[\int_0^{1-s_2} (5(s_2+t+4)^2 - 65)\,dt + \int_{1-s_2}^{4} (45(s_2+t)+15)\,dt\right]$

$= \frac{5}{4}\left(-(s_2+4)^3/3 + 9(s_2+4)^2/2 + 20s_2 + 103/6\right)$

Since $f_2(-1, a) = 35\ 5/6$, and $f_2(s_2, a)$ is an increasing function of s_2 over $-1 \le s_2 \le 1$, $x_2^* = a$ is the optimal decision in this interval.

s_2	$f_2^*(s_2)$	x_2^*
$-3 \le s_2 \le s_2^*$	-20	d
$s_2^* < s_2 \le -1$	$5/4\left(\frac{9}{2}s_2^2 + 47s_2 + \frac{427}{6}\right)$	a
$-1 < s_2 \le 1$	$5/4\left[-(s_2+4)^3/3 + \frac{9}{2}(s_2+4)^2 + 20s_2 + \frac{103}{6}\right]$	a

n=1:

$f_1(-20, d) = -20$

$f_1(-20, a) = -30 + 5(-4+3) + \frac{1}{4}\int_1^5 f_2^*(-4+t)\,dt$

$= -35 + \frac{1}{4}\left[\int_1^{s_2^*+4} -20\,dt + \frac{5}{4}\int_{s_2^*+4}^{3}\left(\frac{9}{2}(-4+t)^2 + 47(-4+t) + \frac{427}{6}\right)dt\right]$

$+ \frac{1}{4}\left[\frac{5}{4}\int_3^5 \left(\frac{t^3}{3} + \frac{9t^2}{2} + 20(-4+t) + \frac{103}{6}\right)dt\right] = 4.77$

Summarizing:

s_1	$f_1^*(s_1)$	x_1^*
-4	4.77	a

So the optimal policy is:

First quarter - advertise

Second quarter - if $s_2 \le s_2^*$, discontinue. If $s_2 > s_2^*$, advertise

Third quarter - if $s_3 \le -1$, discontinue. If $s > -1$, advertise.

where $s_2^* \doteq -2.411$.

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CHAPTER 12

12.1-1 a)

Let F_{LA} = 1 if build a factory in LA; 0 otherwise

F_{SF} = 1 if build a factory in SF; 0 otherwise

F_{SD} = 1 if build a factory in SD; 0 otherwise

W_{LA} = 1 if build a warehouse in LA; 0 otherwise

W_{SF} = 1 if build a warehouse in SF; 0 otherwise

W_{SD} = 1 if build a warehouse in SD; 0 otherwise

Maximize $NPV = 9F_{LA} + 5F_{SF} + 7F_{SD} + 6W_{LA} + 4W_{SF} + 5W_{SD}$

subject to $6F_{LA} + 3F_{SF} + 4F_{SD} + 5W_{LA} + 2W_{SF} + 3W_{SD} \le 10$

$W_{LA} + W_{SF} + W_{SD} \le 1$

$-F_{LA} + W_{LA} \le 0$

$-F_{SF} + W_{SF} \le 0$

$-F_{SD} + W_{SD} \le 0$

and $F_{LA}, F_{SF}, F_{SD}, W_{LA}, W_{SF}, W_{SD}$ are binary variables.

b & c)

	Yes-or-No Question								
Constraint	Warehouse in LA?	Factory in LA?	Warehouse in SD?	Factory in SD?	Warehouse in SF?	Factory in SF?	Totals		Right-Hand Side
Capital ($millions)	5	6	3	4	2	3	10	≤	10
≤ 1 Warehouse	1	0	1	0	1	0	1	≤	1
NPV ($millions)	6	9	5	7	4	5	17		
Solution	0	0	1	1	0	1			

12.1-2 a)

Let M_E = 1 if Eve does marketing; 0 if not

M_S = 1 if Steven does marketing; 0 if not

C_E = 1 if Eve does cooking; 0 if not

C_S = 1 if Steven does cooking; 0 if not

D_E = 1 if Eve does dishwashing; 0 if not

D_S = 1 if Steven does dishwashing; 0 if not

L_E = 1 if Eve does laundry; 0 if not

L_S = 1 if Steven does laundry; 0 if not.

Minimize $T = 4.5M_E + 7.8C_E + 3.6D_E + 2.9L_E$
$+4.9M_S + 7.2C_S + 4.3D_S + 3.1L_S,$

subject to $M_E + C_E + D_E + L_E = 2$

$M_S + C_S + D_S + L_S = 2$

$M_E + M_S = 1$

$C_E + C_S = 1$

$D_E + D_S = 1$

$L_E + L_S = 1,$

and $M_E, M_S, C_E, C_S, D_E, D_S, L_E, L_S$ are binary variables.

b & c

	Yes-or-No Question										
Constraint	Marketing by Eve	Marketing by Steven	Cooking by Eve	Cooking by Steven	Dishes by Eve	Dishes by Steven	Laundry by Eve	Laundry by Steven	Total		Right-Hand Side
Eve's Chores	1	0	1	0	1	0	1	0	2	=	2
Steven's Chores	0	1	0	1	0	1	0	1	2	=	2
Marketing	1	1	0	0	0	0	0	0	1	=	1
Cooking	0	0	1	1	0	0	0	0	1	=	1
Dishwashing	0	0	0	0	1	1	0	0	1	=	1
Laundry	0	0	0	0	0	0	1	1	1	=	1
Time Needed	4.5	4.9	7.8	7.2	3.6	4.3	2.9	3.1	18.4	hours	
Solution	1	0	0	1	1	0	0	1			

12.1-3 a)

Let x_1 = 1 if invest in project 1; 0 if not

x_2 = 1 if invest in project 2; 0 if not

x_3 = 1 if invest in project 3; 0 if not

x_4 = 1 if invest in project 5; 0 if not

x_5 = 1 if invest in project 5; 0 if not

Maximize $NPV = x_1 + 1.8x_2 + 1.6x_3 + 0.8x_4 + 1.4x_5$

subject to $6x_1 + 12x_2 + 10x_3 + 4x_4 + 8x_5 \le 20$

and x_1, x_2, x_3, x_4, x_5, are binary variables.

b & c)

Constraint	Yes-or-No Question: Project 1	Project 2	Project 3	Project 4	Project 5	Total		Right-Hand Side
Capital	6	12	10	4	8	20	≤	20
Net Present Value	1	1.8	1.6	0.8	1.4	**$ 3.4 MILLION**		
Solution	1	0	1	1	0			

12.1-4

a. Let $x_i = \begin{cases} 1 & \text{if invest in opportunity } i \\ 0 & \text{otherwise} \end{cases}$ for $i = 1, \ldots, 7$.

p_i: estimated profit of opportunity i

c_i: capital required by opportunity i

The objective is to maximize $\sum_{i=1}^{7} x_i p_i$

subject to:

$\sum_{i=1}^{7} x_i c_i \le 100M$

$x_1 + x_2 \le 1$

$x_3 + x_4 \le 1$

$x_3 \le x_1 + x_2$

$x_4 \le x_1 + x_2$

$x_i = 0$ or 1; $i = 1, \ldots, 7$

b)

	Investment opportunity 1	2	3	4	5	6	7
Estimated profit	17	10	15	19	7	13	9
Capital required	43	28	34	48	17	32	23
Invest or not	**1**	**0**	**1**	**0**	**0**	**0**	**1**
Total profit	41						

Total capital req.	100	100
x1+x2<=1	1	1
x3+x4<=1	1	1
x3<=x1+x2	1	1
x4<=x1+x2	0	1

Sol^n: do projects 1, 3 and 7.

12.1-5

Unit Cost (Seconds)

		Task (Swimmer)					
		Carl	Chris	David	Tony	Ken	Supply
Assignee (Stroke)	Back	37.7	32.9	33.8	37	35.4	1
	Breast	43.4	33.1	42.2	34.7	41.8	1
	Fly	33.3	28.5	38.9	30.4	33.6	1
	Free	29.2	26.4	29.6	28.5	31.1	1
Demand		1	1	1	1	1	

Assignment

		Task (Swimmer)							
		Carl	Chris	David	Tony	Ken	Totals		Supply
Assignee (Stroke)	Back	0	0	1	0	0	1	=	1
	Breast	0	0	0	1	0	1	=	1
	Fly	0	1	0	0	0	1	=	1
	Free	1	0	0	0	0	1	=	1
Totals		1	1	1	1	0			
		≤	≤	≤	≤	≤	Total Cost	=	126.20
Demand		1	1	1	1	1			

Each swimmer can only swim one stroke. Each stroke can only be swum by one swimmer.

12.1-6 a)

Let T = the number of tow bars produced

Let S = the number of stabilizer bars produced

Maximize $P = 130T + 150S$,

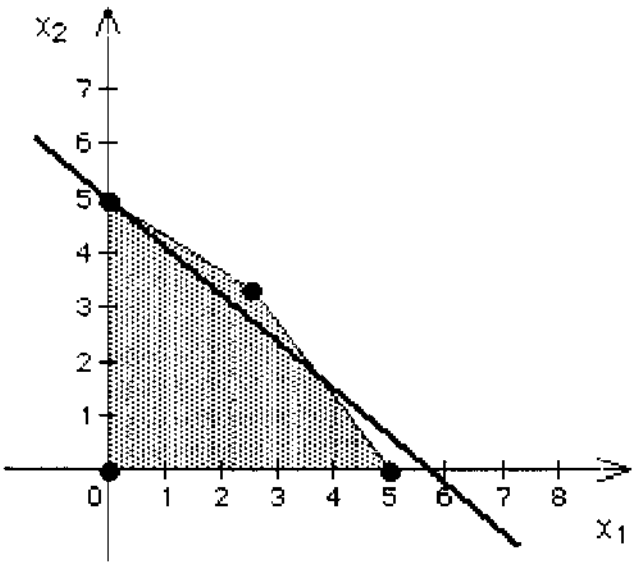
subject to $3.2T + 2.4S \le 16$

$2T + 3S \le 15$

and $T \ge 0,\ S \ge 0$

$T,\ S$ are integers.

12.1-6 b) Optimal solution: $(T,S) = (0,5)$, $P = \$750$.



c)

	Resource Usage Per Unit of Each Activity				Resource
Resource	Tow Bars	Stabilizer Bars	Totals		Available
Machine 1	3.2	2.4	12	≤	16
Machine 2	2	3	15	≤	15
Unit Profit	$ 130.00	$ 150.00	$750		
Solution	0	5			

12.1-7

a) Let $A =$ the number of Model A copiers to buy

Let $B =$ the number of Model B copiers to buy

Minimize $C = 6000A + 4000B$,

subject to $A + B \geq 6$

$A \geq 1$

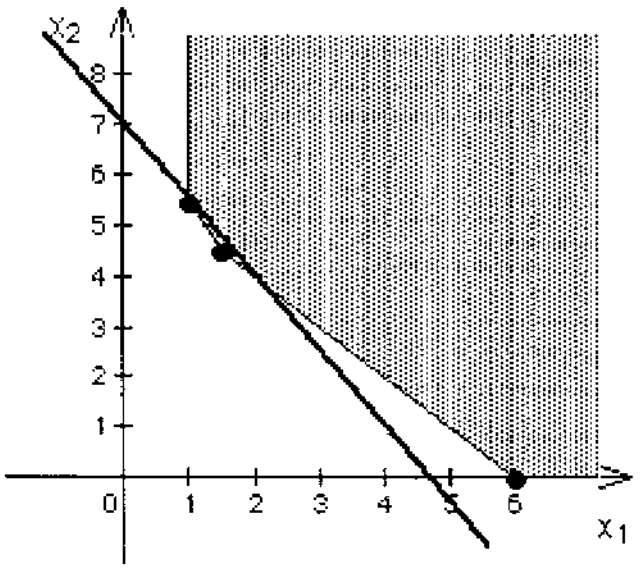
$20{,}000A + 10{,}000B \geq 75{,}000$

and $A \geq 0, \ B \geq 0$

$A, \ B$ are integers.

12.1-7 Optimal solution: $(A,B) = (2,4)$, $C = \$28{,}000$.

b)



c)

Requirements	Requirement Per Unit of Each Activity: Model A	Model B	Totals		Required Amount
Total Need	1	1	6	≥	6
High-Sped Need	1	0	2	≥	1
Capacity	20000	10000	80000	≥	75000
Cost	$ 6,000.00	$ 4,000.00	$28000		
Solution	2	4			

12.1-8 a)

Let x_{ij} = the number of trucks hauling from pit i to site j (for i = N,S; j = 1,2,3)

y_{ij} = tons of gravel hauled from pit i to site j (for i = N,S; j = 1,2,3).

$$\text{Minimize } C = 130y_{N1} + 160y_{N2} + 150y_{N3} + 180y_{S1} + 150y_{S2} + 160y_{S3} + 50x_{N1} + 50x_{N2} + 50x_{N3} + 50x_{S1} + 50x_{S2} + 50x_{S3}$$

subject to

$$y_{N1} + y_{N2} + y_{N3} \le 18$$

$$y_{S1} + y_{S2} + y_{S3} \le 14$$

$$y_{ij} \le 5x_{ij} \qquad \text{(for i = N,S; j = 1,2,3)}$$

$$y_{N1} + y_{S1} \ge 10$$

$$y_{N2} + y_{S2} \ge 5$$

$$y_{N3} + y_{S3} \ge 10$$

and

$$y_{ij} \ge 0, \; x_{ij} \ge 0,$$

x_{ij} are integers (for i = N,S; j = 1,2,3).

12.1-8

b)

Resource Usage Per Unit of Each Activity

	Truck						Gravel								
Resource	N1	N2	N3	S1	S2	S3	N1	N2	N3	S1	S2	S3	Totals		Resource Available
No. supply	0	0	0	0	0	0	1	1	1	0	0	0	15	≤	18
So. supply	0	0	0	0	0	0	0	0	0	1	1	1	10	≤	14
cap. N1	5	0	0	0	0	0	-1	0	0	0	0	0	0	≥	0
cap. N2	0	5	0	0	0	0	0	-1	0	0	0	0	0	≥	0
cap. N3	0	0	5	0	0	0	0	0	-1	0	0	0	0	≥	0
cap. S1	0	0	0	5	0	0	0	0	0	-1	0	0	0	≥	0
cap. S2	0	0	0	0	5	0	0	0	0	0	-1	0	0	≥	0
cap. S3	0	0	0	0	0	5	0	0	0	0	0	-1	0	≥	0
Site 1 need	0	0	0	0	0	0	1	0	0	1	0	0	10	≥	10
Site 2 need	0	0	0	0	0	0	0	1	0	0	1	0	5	≥	5
Site 3 need	0	0	0	0	0	0	0	0	1	0	0	1	10	≥	10
Cost	$ 50	$ 50	$ 50	$ 50	$ 50	$ 50	$ 130	$ 160	$ 150	$ 180	$ 150	$ 160	$3850		
Solution	2	0	1	0	1	1	10	0	5	0	5	5			

12.2-1 See the articles in Interfaces.

12.3-1 a) maximize $70x_1 - 50000y_1 + 60x_2 - 40000y_2 + 90x_3 - 70000y_3 + 80x_4 - 60000y_4$

subject to:

$y_1 + y_2 + y_3 + y_4 \le 2$

$y_3 \le y_1 + y_2 \qquad y_4 \le y_1 + y_2$

$5x_1 + 3x_2 + 6x_3 + 4x_4 \le 6000 + My_5$

$4x_1 + 6x_2 + 3x_3 + 5x_4 \le 6000 + M(1 - y_5)$

$x_i \le My_i$; $i = 1, \ldots, 4$

$x_i \ge 0$ $i = 1, \ldots, 4$

y_i binary $i = 1, \ldots, 5$

$M = 100$ Million.

b)

Constraint	Product 1	Product 2	Product 3	Product 4	Totals		Modified Right-Hand Side	Original Right-Hand Side
First	5	3	6	4	6000	≤	6000	6000
Second	4	6	3	5	12000	≤	105999	6000
Marginal revenue	$70	$60	$90	$80	$80000			
Solution	0	2000	0	0				
	≤	≤	≤	≤				
	0	9999	0	0				
Set Up?	0	1	0	0	1	≤	2	
Start-up Cost	$50,000	$40,000	$70,000	$60,000				

Contingency Constraints:

Product 3:	0	≤	1	:Product 1 or 2
Product 4:	0	≤	1	:Product 1 or 2

Which Constraint (0 = First, 1 = Second): 0

12.3-2 $x_1 - x_2 = 0y_1 + 3y_2 - 3y_3 + 6y_4 - 6y_5$

y_i binary ; $i = 1, \ldots, 5$.

12.3-3

1. $x_1+x_2+x_3+x_4 \le 4+My_1$

$3x_1-x_2-x_3+x_4 \le 3+M(1-y_1)$

y_1 binary

2. $5x_1+3x_2+3x_3-x_4 \le 10+My_2$

$2x_1+5x_2-x_3+3x_4 \le 10+My_3$

$-x_1+3x_2+5x_3+3x_4 \le 10+My_4$

$3x_1-x_1+3x_3+5x_4 \le 10+My_5$

$y_2+y_3+y_4+y_5 \le 2$

y_i binary $\quad i=2,\dots,5$

12.3-4

a) Let y_1, y_2 be binary variables indicating whether or not toys 1 and 2 are produced. Let x_1, x_2 be the number of toys 1 and 2 produced. Let $z = \begin{cases} 0 & \text{if factory 1 is used} \\ 1 & \text{if factory 2 is used} \end{cases}$.

IP: max. $10x_1+15x_2-50000y_1-80000y_2$

s.t. $x_1 \le My_1$

$x_2 \le My_2$

$x_1/50 + x_2/40 \le 500+Mz$

$x_1/40 + x_2/25 \le 700+M(1-z)$

$x_1, x_2 \ge 0$, integer $\quad y_1, y_2, z$ binary

12.3-4

b)

Constraint	Toy 1	Toy 2	Totals		Modified Right-Hand Side	Original Right-Hand Side
Factory 1	0.02	0.025	560	≤	10499	500
Factory 2	0.025	0.04	700	≤	700	700
Unit Profit	$10	$15	$230000			
Solution	28000	0				
	≤	≤				
	99999	0				
Set Up?	1	0				
Setup Cost	$50,000	$80,000				

Which factory? (0=Factory 1, 1=Factory 2)	1

12.3-5

a)

Let L = the number of long - range jets to buy

Let M = the number of medium - range jets to buy

Let S = the number of short - range jets to buy.

Maximize $P = 4.2L + 3M + 2.3S$,

subject to $67L + 50M + 35S \le 1500$

$L + M + S \le 30$

$\frac{5}{3}L + \frac{4}{3}M + S \le 40$

and $L \ge 0,\ M \ge 0,\ S \ge 0$

$L,\ M,\ S$ are integers.

b)

	Resource Usage Per Unit of Each Activity					Resource
Resource	Long-range	Medium-range	Short-range	Totals		Available
Money	67	50	35	1498	≤	1500
Pilots	1	1	1	30	≤	30
Maintenance	1.667	1.333	1	39.338	≤	40
Profit	$ 4.20	$ 3.00	$ 2.30	$95.6		
Solution	14	0	16			

12.3-5

c) $L \le \min\left\{\left[\frac{1500}{67}\right], \frac{30}{1}, \left[\frac{40}{5/3}\right]\right\} = 20$

$M \le \min\left\{\frac{1500}{50}, \frac{30}{1}, \left[\frac{40}{1/3}\right]\right\} = 30$

$S \le \min\left\{\left[\frac{1500}{35}\right], \frac{30}{1}, \frac{40}{1}\right\} = 30$

where $[x] \triangleq$ integer part of x

$L = 2^0 l_0 + 2^1 l_1 + 2^2 l_2 + 2^3 l_3 + 2^4 l_4$

$M = 2^0 m_0 + 2^1 m_1 + 2^2 m_2 + 2^3 m_3 + 2^4 m_4$

$S = 2^0 s_0 + 2^1 s_1 + 2^2 s_2 + 2^3 s_3 + 2^4 s_4$

and the problem may be rewritten as follows:

maximize $P = 4.2 \sum_{i=0}^{4} 2^i l_i + 3 \sum_{i=0}^{4} 2^i m_i + 2.3 \sum_{i=0}^{4} 2^i s_i$

subject to:

$$67 \sum_{i=0}^{4} 2^i l_i + 50 \sum_{i=0}^{4} 2^i m_i + 35 \sum_{i=0}^{4} 2^i s_i \le 1500$$

$$\frac{5}{3} \sum_{i=0}^{4} 2^i l_i + \frac{4}{3} \sum_{i=0}^{4} 2^i m_i + 1 \cdot \sum_{i=0}^{4} 2^i s_i \le 40$$

$$\sum_{i=0}^{4} 2^i l_i + 1 \cdot \sum_{i=0}^{4} 2^i m_i + 1 \cdot \sum_{i=0}^{4} 2^i s_i \le 30$$

l_i, m_i, s_i binary For $i = 0, \ldots, 4$

d) Solution: $l_0=0$ $l_1=1$ $l_2=1$ $l_3=1$ $l_4=0$ $\sum_{i=0}^{4} 2^i l_i = 14$

$m_0=0$ $m_1=0$ $m_2=0$ $m_3=0$ $m_4=0$ $\sum_{i=0}^{4} 2^i m_i = 0$

$s_0=0$ $s_1=0$ $s_2=0$ $s_3=0$ $s_4=1$ $\sum_{i=0}^{4} 2^i s_i = 16$

$P = \$95.6$ same as before

12.3-6 (a) Let $x_1 = y_{11} + 2y_{12}$; $x_2 = y_{21} + 2y_{22}$

The BIP formulation is:

$$\text{Maximize } z = y_{11} + 2y_{12} + 5y_{21} + 10y_{22}$$

$$\text{subject to: } y_{11} + 2y_{12} + 10y_{21} + 20y_{22} \leq 20$$

$$y_{11} + 2y_{12} \leq 2$$

y_{ij} binary for $i = 1,2$ $j = 1,2$.

b) Solution:

x_1	= 0	$0 = x_1$
x_2	= 0	
x_3	= 0	$0 + 2 = 2 = x_2$
x_4	= 1	
z	= 10	

12.3-7

a- Let x_i = # units to produce of product i, $i = 1,2,3$.

$$y_i = \begin{cases} 1 & \text{if product } i \text{ is produced} \\ 0 & \text{otherwise} \end{cases}$$

$$\max \quad 2x_1 + 3x_2 + 0.8x_3 - 3y_1 - 2y_2$$

s.t.

$$0.2x_1 + 0.4x_2 + 0.2x_3 \leq 1$$

$$x_1 \leq My_1$$

$$x_2 \leq My_2$$

$$x_1 \leq 3 \ , \ x_2 \leq 2 \ , \ x_3 \leq 5$$

$$x_1, x_2, x_3 \geq 0 \text{ integer} \ ; \ y_1, y_2 \text{ binary}$$

12.3-7

b)

Constraint	Amt. For Customer 1	Amt. For Customer 2	Amt. For Customer 3	Totals		Right-Hand Side
Capacity	0.2	0.4	0.2	1	≤	1
Max. Sales - Customer 1	1	0	0	0	≤	3
Max. Sales - Customer 2	0	1	0	2	≤	2
Max. Sales - Customer 3	0	0	1	1	≤	5
Marginal Net Revenue	$2	$3	$1	$4.8	million	
Solution	0	2	1			
	≤	≤	≤			
	0	99	99			
Set Up?	0	1	1			
Setup Cost(millions)	$3	$2	$0			

12.4-1

a) let $y_{ij} = \begin{cases} 1 & \text{if } x_i = j \text{ (i.e. produce } j \text{ units of } i) \\ 0 & \text{if not} \end{cases}$

$i = 1,2,3$, $j = 1,2,3,4,5$

max. $-1y_{11} + 2y_{12} + 4y_{13} + 1y_{21} + 5y_{22} + 1y_{31} + 3y_{32} + 5y_{33} + 6y_{34} + 7y_{35}$

s.t. $y_{11} + y_{12} + y_{13} \le 1$

$y_{21} + y_{22} \le 1$

$y_{31} + y_{32} + y_{33} + y_{34} + y_{35} \le 1$

$1y_{11} + 2y_{12} + 3y_{13} + 2y_{21} + 4y_{22} + 1y_{31} + 2y_{32} + 3y_{33} + 4y_{34} + 5y_{35} \le 5$

y_{ij} binary

b)

Solution:

$X_1 = 0$
$X_2 = 0$
$X_3 = 0$
$X_4 = 0$
$X_5 = 0$
$X_6 = 0$
$X_7 = 0$
$X_8 = 0$
$X_9 = 0$
y_{35} $X_{10} = 1$ $\Rightarrow X_3 = 5$

$Z = 7$

d)

Solution:

$X_1 = 0$
$X_2 = 0$
$X_3 = 0$
$X_4 = 0$
$X_5 = 0$
y_{31} $X_6 = 1$
y_{32} $X_7 = 1$ $\Rightarrow X_3 = 5$
y_{33} $X_8 = 1$
y_{34} $X_9 = 1$
y_{35} $X_{10} = 1$

$Z = 7$

12.4-1

c) let $y_{ij} = \begin{cases} 1, & x_i \geq j \\ 0, & \text{if not} \end{cases}$

$$\max. \; -1y_{11} + 3y_{12} + 2y_{13} + 1y_{21} + 4y_{22} + 1y_{31} + 2y_{32} + 2y_{33} + 1y_{34} + 1y_{35}$$

s.t. $y_{12} \leq y_{11}$, $y_{13} \leq y_{12}$, $y_{22} \leq y_{21}$

$y_{32} \leq y_{31}$, $y_{33} \leq y_{32}$, $y_{34} \leq y_{33}$, $y_{35} \leq y_{34}$

$y_{11} + y_{12} + y_{13} + 2y_{21} + 2y_{22} + y_{31} + y_{32} + y_{33} + y_{34} + y_{35} \leq 5$

y_{ij} binary

12.4-2 Wyndor Glass Co.

introduce y_1, y_2 binary variables

add constraints: $x_1 \leq My_1$

$x_2 \leq My_2$

(all the old constraints remain) $y_1 + y_2 = 1$

y_1, y_2 binary

12.4-3 a) introduce binary variables y_1, y_2, y_3 to represent positive (nonzero) production levels

New MIP:

$$\max. \; z = 50x_1 + 20x_2 + 25x_3$$

s.t.

$$9x_1 + 3x_2 + 5x_3 \leq 500$$
$$5x_1 + 4x_2 \leq 350$$
$$3x_1 + 2x_3 \leq 150$$
$$x_3 \leq 20$$
$$x_1 \leq My_1, \; x_2 \leq My_2, \; x_3 \leq My_3$$
$$y_1 + y_2 + y_3 \leq 2$$

$x_1, x_2, x_3 \geq 0$ $\quad y_1, y_2, y_3$ binary

12.4-3
b)

Constraint	Product 1	Product 2	Product 3	Total		Right-Hand Sice
Milling	9	3	5	498	≤	500
Lathe	5	4	0	349	≤	350
Grinder	3	0	2	135	≤	150
Sales Potential	0	0	1	0	≤	20
Unit Profit	50	20	25	$2870		
Solution	45	31	0			
	≤	≤	≤			
	999	999	0			
Produce?	1	1	0	2	≤	2

12.4-4

a) let $y_{ij} = \begin{cases} 1, & x_i = j \\ 0, & \text{if not} \end{cases}$ $i = 1, 2$ $j = 1, 2, 3$

It's simple to work out by hand the objective function contribution for $x_1, x_2 = 0, 1, 2, 3$. This gives:

max. $3y_{11} + 8y_{12} + 9y_{13} + 9y_{21} + 24y_{22} + 9y_{23}$

s.t. $y_{11} + y_{12} + y_{13} \le 1$, $y_{21} + y_{22} + y_{23} \le 1$

$y_{11} + y_{23} \le 1$ $\quad y_{13} + y_{23} \le 1$

$y_{12} + y_{23} \le 1$ $\quad y_{13} + y_{22} \le 1$

$y_{12} + y_{22} \le 1$ $\quad y_{13} + y_{21} \le 1$

y_{ij} binary

b)
Solution:
$x_1 = 1$ y_{11}
$x_2 = 0$
$x_3 = 0$
$x_4 = 0$
$x_5 = 1$ y_{22}
$x_6 = 0$
$\Rightarrow x_1 = 1, x_2 = 2$
$Z = 27$

c) let $y_{ij} = \begin{cases} 1, & x_i \ge j \\ 0, & \text{if not} \end{cases}$ $i = 1, 2$ $j = 1, 2, 3$

max. $3y_{11} + 5y_{12} + 1y_{13} + 9y_{21} + 15y_{22} - 15y_{23}$

s.t. $y_{12} \le y_{11}$, $y_{13} \le y_{12}$

$y_{22} \le y_{21}$, $y_{23} \le y_{22}$

$y_{11} + y_{23} \le 1$
$y_{12} + y_{22} \le 1$
$y_{13} + y_{21} \le 1$
← this is sufficient due to the first four constraints

y_{ij} binary

d)
Solution:
$x_1 = 1$ y_{11}
$x_2 = 0$
$x_3 = 0$
$x_4 = 1$ y_{21}
$x_5 = 1$ y_{22}
$x_6 = 0$
$\Rightarrow x_1 = 1, x_2 = 2$
$Z = 27$

12.4-5

a) let $y_{ij} = \begin{cases} 1, & X_i = \frac{1}{j+1} \\ 0, & \text{if not} \end{cases}$ $i = 1, 2$ $j = 1, 2, 3, 4$

max. $3/4\, y_{11} + 5/9\, y_{12} + 7/16\, y_{13} + 9/25\, y_{14}$
$+ 3/4\, y_{21} + 2/3\, y_{22} + 9/16\, y_{23} + 12/25\, y_{24}$

s.t. $\sum_{j=1}^{4} y_{ij} \le 1$, $i = 1, 2$

$y_{11} + y_{21} \le 1$
$y_{11} + y_{22} \le 1$
$y_{12} + y_{21} \le 1$

y_{ij} binary

i)
```
Solution:
X1 = 1 y11
X2 = 0
X3 = 0
X4 = 0
X5 = 0
X6 = 0
X7 = 1 y13
X8 = 0
```
$\Rightarrow X_1 = \frac{1}{2}$
$X_2 = \frac{1}{4}$

12.4-6 a) Let $x_{ij} = \begin{cases} 1 & \text{if arc from node } i \text{ to node } j \text{ is used in shortest route} \\ 0 & \text{otherwise} \end{cases}$

Then the problem is:

Minimize $3x_{12} + 6x_{13} + 6x_{24} + 5x_{25} + 4x_{34} + 3x_{35} + 3x_{46} + 2x_{56}$

$x_{12} + x_{13} = 1$
$x_{24} + x_{25} + x_{34} + x_{35} = 1$ } at each stage (between col.) exactly one arc is used
$x_{46} + x_{56} = 1$

$x_{24} + x_{25} \le x_{12}$
$x_{34} + x_{35} \le x_{13}$
$x_{46} \le x_{24} + x_{34}$
$x_{56} \le x_{25} + x_{35}$ } route leaves node i only if route enters node i

x_{ij} is binary

b)
```
Solution:
X1 = 1 x12
X2 = 0
X3 = 0
X4 = 1 x25
X5 = 0
X6 = 0
X7 = 0
X8 = 1 x56
Z = 10
```
path: $1 \to 2 \to 5 \to 6$

The first three equations represent mutually exclusive alternatives and the last four equations represent contingent decisions.

12.4-7

Let $x_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is in the critical path} \\ 0 & \text{otherwise} \end{cases}$

$$\text{Maximize } z = 5x_{12} + 3x_{13} + 4x_{24} + 2x_{25} + 3x_{35} + 1x_{46} + 3x_{47} + 6x_{57} + 2x_{58} + 5x_{69} + 4x_{79} + 7x_{89}$$

$$\text{subject to: } \sum_i x_{ij} - \sum_k x_{jk} = \begin{cases} -1 & \text{for } j=1 \\ 1 & \text{for } j=9 \\ 0 & \text{otherwise} \end{cases}$$

x_{ij} binary.

12.4-8 a)

Let $y_j = \begin{cases} 1 \text{ if route } j \text{ is chosen} \\ 0 \text{ otherwise} \end{cases}$; $j = 1, \ldots, 10$

$x_{ij} = i^{th}, j^{th}$ element of the location/route matrix; $i = A, \ldots, I$; $j = 1, \ldots, 10$

c_j = cost of route j ; $j = 1, \ldots, 10$

$$\text{minimize } \sum_{j=1}^{10} c_j y_j$$

$$\text{s.t. } \sum_{j=1}^{10} x_{ij} y_j \geq 1 \quad ; \quad i = A, \ldots, I$$

$$\sum_{j=1}^{10} y_j = 3$$

y_j binary ; $j = 1, \ldots, 10$

12.4-8
b)

Delivery Location	Yes-or-No? Possible Routes 1	2	3	4	5	6	7	8	9	10	Total		Right-Hand Side
A	1	0	0	0	1	0	0	0	1	0	1	≥	1
B	0	0	0	1	0	1	0	0	1	1	1	≥	1
C	0	0	1	1	0	0	1	0	1	0	1	≥	1
D	1	0	0	0	0	1	0	1	0	0	1	≥	1
E	0	0	1	1	0	1	0	0	0	0	1	≥	1
F	0	0	0	0	1	0	0	0	0	0	1	≥	1
G	1	0	0	0	0	0	1	1	0	1	1	≥	1
H	0	0	1	0	1	0	0	0	0	1	1	≥	1
I	0	0	0	1	0	0	1	0	0	0	1	≥	1
3 routes	1	0	1	1	1	1	1	1	1	1	3	=	3
Cost	6	4	7	5	4	5	5	3	7	6	12		
Solution	0	0	0	1	1	0	0	1	0	0			

12.4-9

Let $x_{ij} = \begin{cases} 1 & \text{if tract } j \text{ is assigned to station located in tract } i \\ 0 & \text{otherwise} \end{cases}$

a_{ij} = response time to a fire in tract j, if that tract is served by a station located in tract i (entries in the table are the a_{ij}'s)

Then the problem is:

$$\text{Minimize } 2\sum_{i=1}^{5} a_{i1}x_{i1} + \sum_{i=1}^{5} a_{i2}x_{i2} + 3\sum_{i=1}^{5} a_{i3}x_{i3} + \sum_{i=1}^{5} a_{i4}x_{i4} + 3\sum_{i=1}^{5} a_{i5}x_{i5}$$

subject to

$\sum_{i=1}^{5} x_{ii} = 2$ (have to locate 2 stations)

mutually exclusive alternatives → $\sum_{i=1}^{5} x_{ij} = 1$ for $j = 1, \ldots, 5$ (each tract is assigned to a station)

contingent decisions → $x_{ij} \le x_{ii}$ for $i = 1, \ldots, 5$; $j = 1, \ldots, 5$ (assign tract j to station in tract i only if there is a station located in tract i)

x_{ij} is binary

12.4-10 a) let $x_i = \begin{cases} 1, & \text{if a station is located in tract } i \\ 0, & \text{if not} \end{cases}$ $i = 1, 2, 3, 4, 5$

min. $200x_1 + 250x_2 + 400x_3 + 300x_4 + 500x_5$

s.t. $x_1 + x_3 + x_5 \geq 1$

$x_1 + x_2 + x_4 \geq 1$

$x_2 + x_3 + x_5 \geq 1$

$x_2 + x_3 + x_4 + x_5 \geq 1$

$x_1 + x_3 + x_4 + x_5 \geq 1$

x_i binary, $i = 1, 2, 3, 4, 5$

c)
```
Solution:
  X1  = 1
  X2  = 1
  X3  = 0
  X4  = 0
  X5  = 0

  Z   = 450
```

b) Yes, this is a set covering problem. The activities are locating stations, and the characteristics are the fires. S_i is the set of all locations that could cover a fire in tract i, so $S_1 = \{1, 3, 5\}$ for example. We have $\sum_{j \in S_i} x_j \geq 1$ $\forall i$.

12.4-11 Let $x_j = \begin{cases} 1 & \text{if district } j \text{ is chosen} \\ 0 & \text{if not} \end{cases}$

y_j = auxiliary variable, will be zero for every j, except for the index of the district with largest c_j that is chosen, will be one for this index.

The problem is:

Minimize $\sum_{j=1}^{N} c_j y_j$

subject to
$$y_j \le x_j \quad j = 1, \ldots, N$$
$$\sum_{j=1}^{N} y_j = 1$$
$$\sum_{j=1}^{N} c_j y_j \ge c_i x_i \quad i = 1, \ldots, N$$
$$\sum_{j=1}^{N} x_j = R$$
$$\sum_{j=1}^{N} a_{ij} x_j = 1 \quad i = 1, \ldots, D$$
x_j and y_j are binary

This is a set partitioning problem with additional constraints.

12.4-12

Let $s_i = \begin{cases} 1 & \text{if skirt } i \text{ is taken} \\ 0 & \text{otherwise} \end{cases}$ $i = 1, 2, 3$

$p_i = \begin{cases} 1 & \text{if slacks } i \text{ is taken} \\ 0 & \text{otherwise} \end{cases}$ $i = 1, 2, 3$

$t_i = \begin{cases} 1 & \text{if top } i \text{ is taken} \\ 0 & \text{otherwise} \end{cases}$ $i = 1, 2, 3, 4$

$t_5 = 1$ to indicate the use of the Icelandic sweater "top #5"

$d_i = \begin{cases} 1 & \text{if dress } i \text{ is taken} \\ 0 & \text{otherwise} \end{cases}$ $i = 1, 2, 3$

$x_{ij} = \begin{cases} 1 & \text{if both skirt } i \text{ and top } j \text{ are taken} \\ 0 & \text{otherwise} \end{cases}$

$y_{ij} = \begin{cases} 1 & \text{if both slacks } i \text{ and top } j \text{ are taken} \\ 0 & \text{otherwise} \end{cases}$

The BIP model is:

$$\text{Maximize } z = x_{11} + x_{12} + x_{15} + x_{21} + x_{24} + x_{32} + x_{33} + x_{34} + x_{35}$$
$$+ y_{11} + y_{13} + y_{21} + y_{22} + y_{24} + y_{25} + y_{33} + y_{34} + y_{35}$$
$$+ d_1 + d_2 + d_3$$

(cont.)

12.4-12 (CONT.)

Subject to: $600 s_1 + 450 s_2 + 700 s_3 + 600 p_1 + 550 p_2 + 500 p_3 + 350 t_1 + 300 t_2 + 300 t_3 + 450 t_4 + 600 d_1 + 700 d_2 + 800 d_3 \le 4000$

$5000 s_1 + 3500 s_2 + 3000 s_3 + 3500 p_1 + 6000 p_2 + 4000 p_3 + 4000 t_1 + 3500 t_2 + 3000 t_3 + 5000 t_4 + 6000 d_1 + 5000 d_2 + 4000 d_3 \le 32000$

$x_{ij} \le \frac{1}{2}(s_i + t_j)$

$y_{ij} \le \frac{1}{2}(p_i + t_j)$

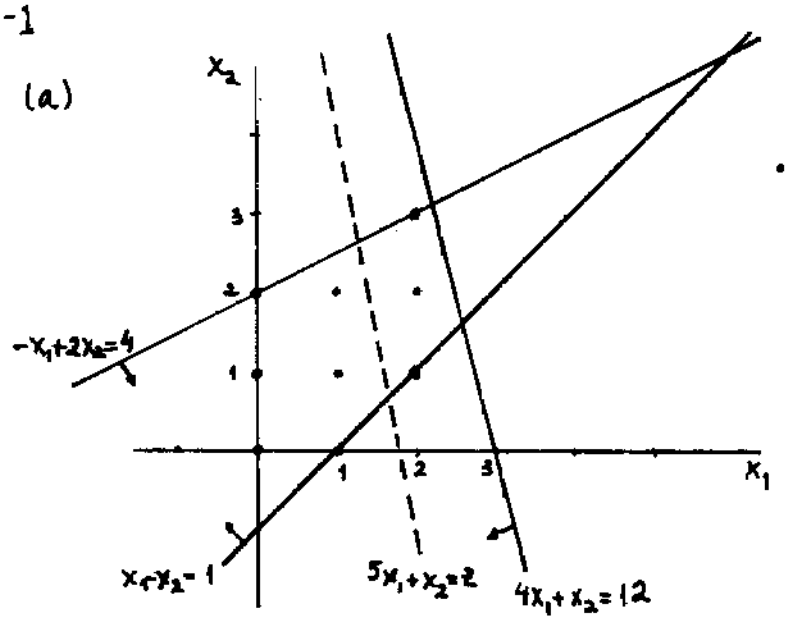
s_i, p_i, d_i binary for $i = 1, 2, 3$

t_i binary for $i = 1, 2, 3, 4$

x_{ij}, y_{ij} binary for $i = 1, 2, 3$ $j = 1, 2, 3, 4, 5$

12.5-1

(a)



• – feasible solutions

Optimal solution: $(x_1, x_2) = (2, 3)$

$z = 5x_1 + x_2 = 13$

(b) Solving LP-relaxation graphically we obtain the optimal solution $(x_1, x_2) = (\frac{13}{5}, \frac{8}{5}) = (2.6, 1.6)$

$z = 13 + \frac{8}{5} = 14.6$

(cont.)

12.5-1 (CONT.) Rounding the optimal solution to the LP-relaxation:

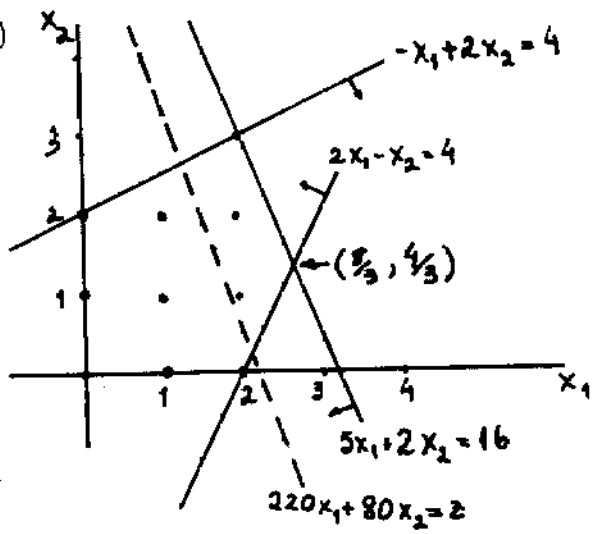
$(\bar{x}_1, \bar{x}_2) = (3, 2)$ is not feasible $4 \cdot 3 + 2 \nleq 12$

Rounded Solutions	Constraints violated	Z
(3,2)	3rd	-
(3,1)	2nd and 3rd	-
(2,2)	none	12
(2,1)	none	11

So, none of the feasible rounded solutions is optimal for the IP problem.

12.5-2

(a)



Optimal solution: $(x_1, x_2) = (2, 3)$

$z = 220x_1 + 80x_2 = 680$

(b) Optimal solution to LP-relaxation:

$(x_1, x_2) = (8/3, 4/3)$

$z = 2080/3 = 693\frac{1}{3}$

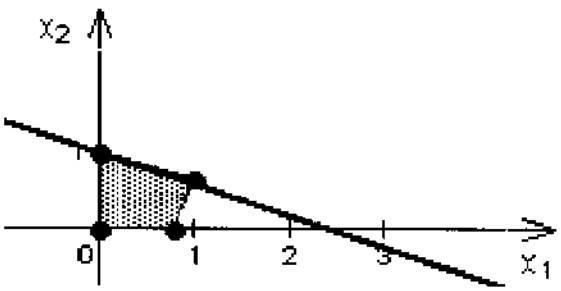
Nearest integer point is (3,1), which is not feasible

Rounded Solution	Constraints violated	z
(3,2)	2nd	-
(3,1)	2nd and 3rd	-
(2,2)	none	600
(2,1)	none	520

12.5-3 a)

Solution	Feasible?	$P = 2x_1 + 5x_2$	Optimal?
(0,0)	Yes	0	
(1,0)	No		
(0,1)	Yes	5	***
(1,1)	No		

b) Optimal solution: $(x_1, x_2) = (0,1)$.



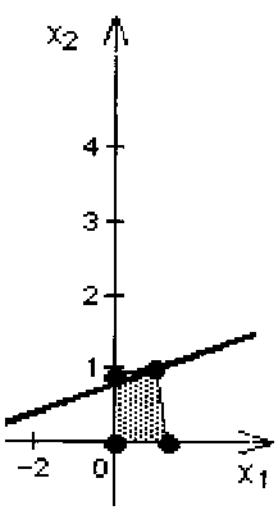
c) Solving the LP relaxation graphically, the optimal solution is $(x_1, x_2) = (1, 0.667)$. This rounds to (1,1) which is not a feasible solution.

d) Rounding down results in a solution of (1,0) which is also not a feasible solution.

12.5-4 a)

Solution	Feasible?	$P = -5x_1 + 25x_2$	Optimal?
(0,0)	Yes	0	
(1,0)	Yes	-5	
(0,1)	No		
(1,1)	Yes	20	***

b) Optimal solution: $(x_1, x_2) = (1,1)$.



c) Solving the LP relaxation graphically, the optimal solution is $(x_1, x_2) = (0, 0.9)$. This rounds to (0,1) which is not a feasible solution.

d) Rounding down results in a solution of (0,0) which is a feasible solution but yields Z=0 which in not an optimal solution.

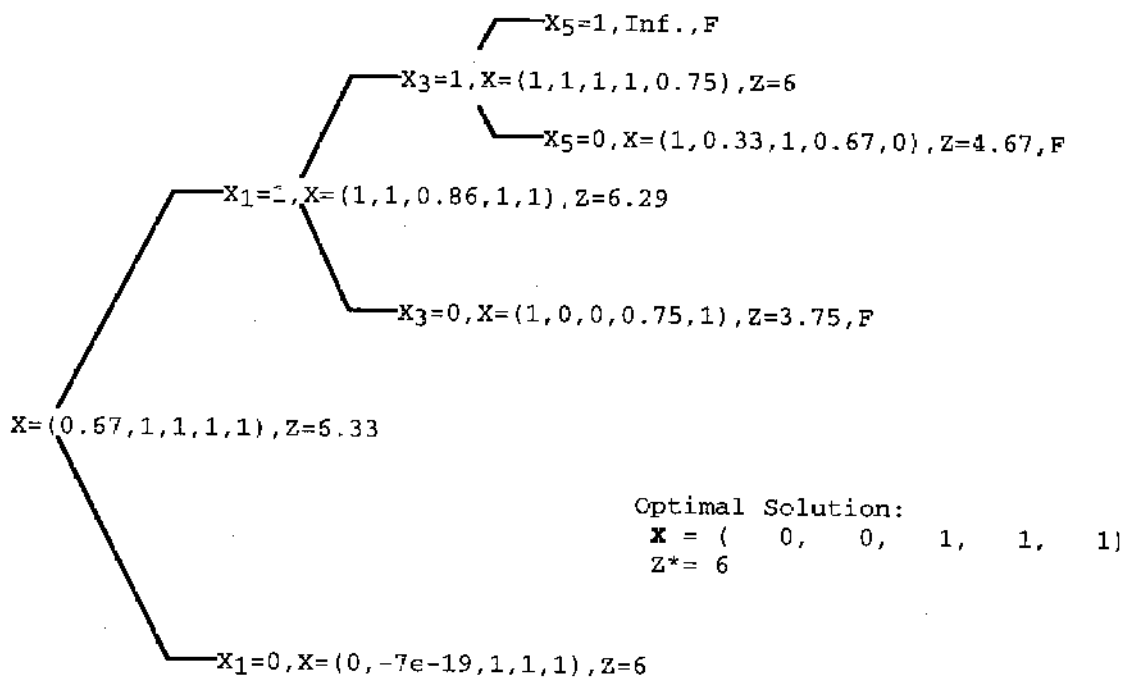
12.5-5

a) TRUE. Section 12.5, 3rd paragraph, last line

b) TRUE. Section 12.5, last paragraph before rounding examples

c) False. The result will not necessarily be Feasible. See problems 12.6-1 and 12.5-2 For examples of this. Also see the first "pitfall" mentioned in section 12.5, rounding.

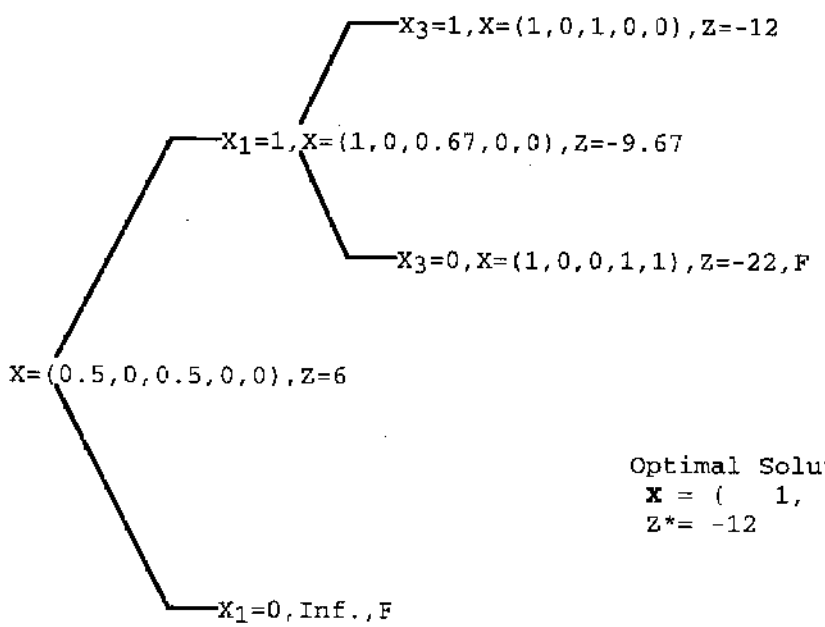
12.6-1 Solution Tree:



Optimal Solution:
X = (0, 0, 1, 1, 1)
Z*= 6

12.6-2

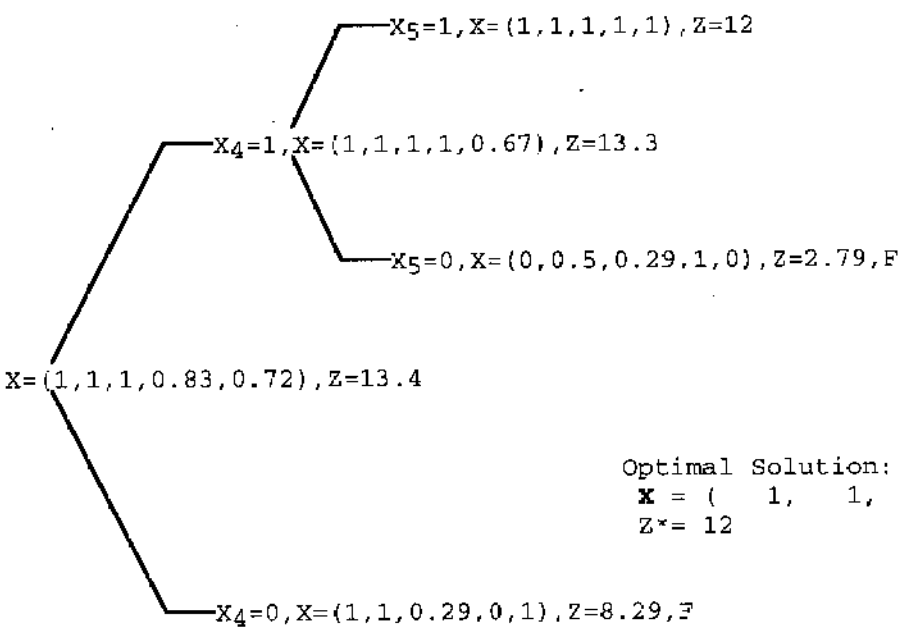
Solution Tree:



Optimal Solution:
X = (1, 0, 1, 0, 0)
Z*= -12

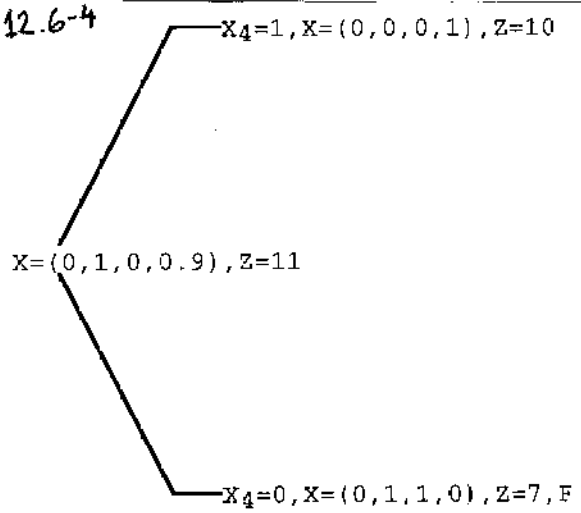
12.6-3

Solution Tree:



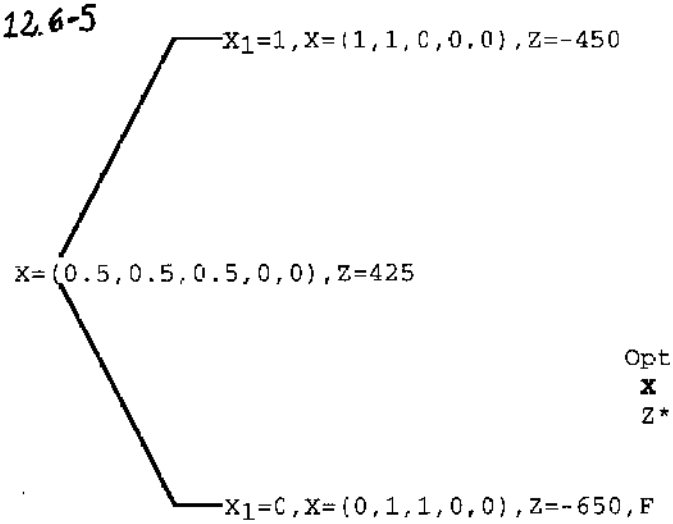
Optimal Solution:
X = (1, 1, 1, 1, 1)
Z*= 12

12.6-4



Optimal Solution:
X = (0, 0, 0, 1)
Z*= 10

12.6-5



Optimal Solution:
X = (1, 1, 0, 0, 0)
Z*= -450

12.6-6

a) False. It's the other way around. That's why it's called a "relaxation," because it relaxes (expands) the feasible region.

b) True. If you then restrict the feasible region to integers you've only eliminated points that weren't optimal anyway. You can't do better.

c) False! See section 12.5 and its problems for examples.

505

12.6-7

(a) Initialization Step: Set $Z^*=+\infty$. Apply the bounding and fathoming steps and the optimality test as dexcribed below on the whole problem. If the whole problem is not fathomed, it becomes the initial subproblem for the first iteration below.

Iteration:

1. Branching: Choose the most recently created unfathomed subproblem (breaking ties by selecting the one with the smallest bound). Among the assignees not yet assigned for the current subproblem, choose the first one in the natural ordering to be the branching variable. Subproblems will correspond to each of the possible remaining assignments for the branching assignee.
   Form a subproblem for each remaining assignment by deleting the constraint that each of the unassigned assignees must perform exactly one assignment.

2. Bounding: For each new subproblem, obtain its bound by choosing the cheapest assignee for each remaining assignment and totaling the costs.

3. Fathoming: For each new subproblem, apply the two fathoming test below:
   Test 1. Its bound>= Z^*
   Test 2. The optimal solution for its relaxation is a feasible assignment (if this solution is better than the incumbent, it becomes the new incumbent and Test 1 is reapplied to all unfathomed subproblems with the new smaller Z^*).

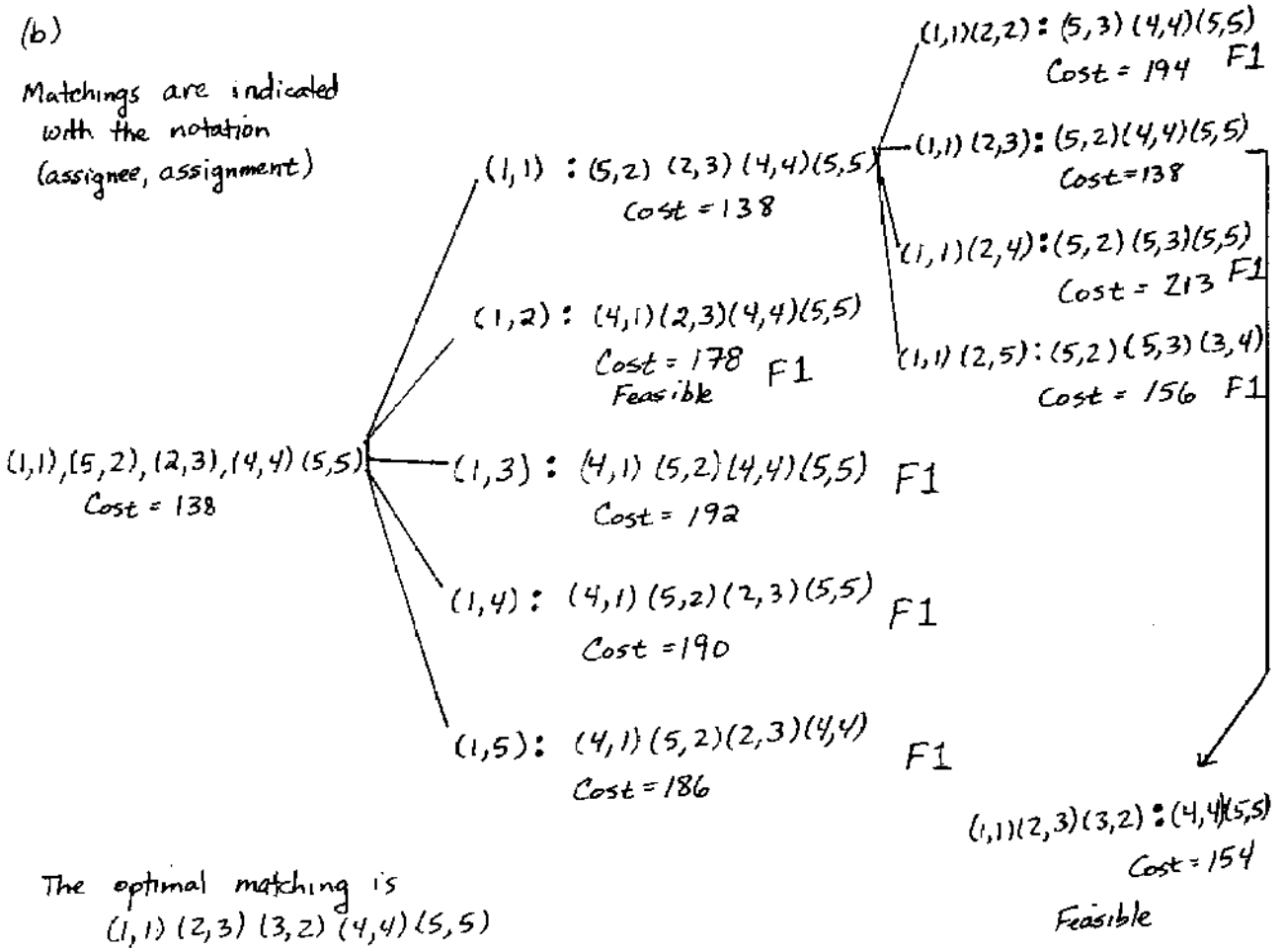
Optimality Test: Identical to that in the text.

(cont.)

12.6-7 -

(b)

Matchings are indicated with the notation (assignee, assignment)



The optimal matching is
(1,1) (2,3) (3,2) (4,4) (5,5)
with Cost = 154.

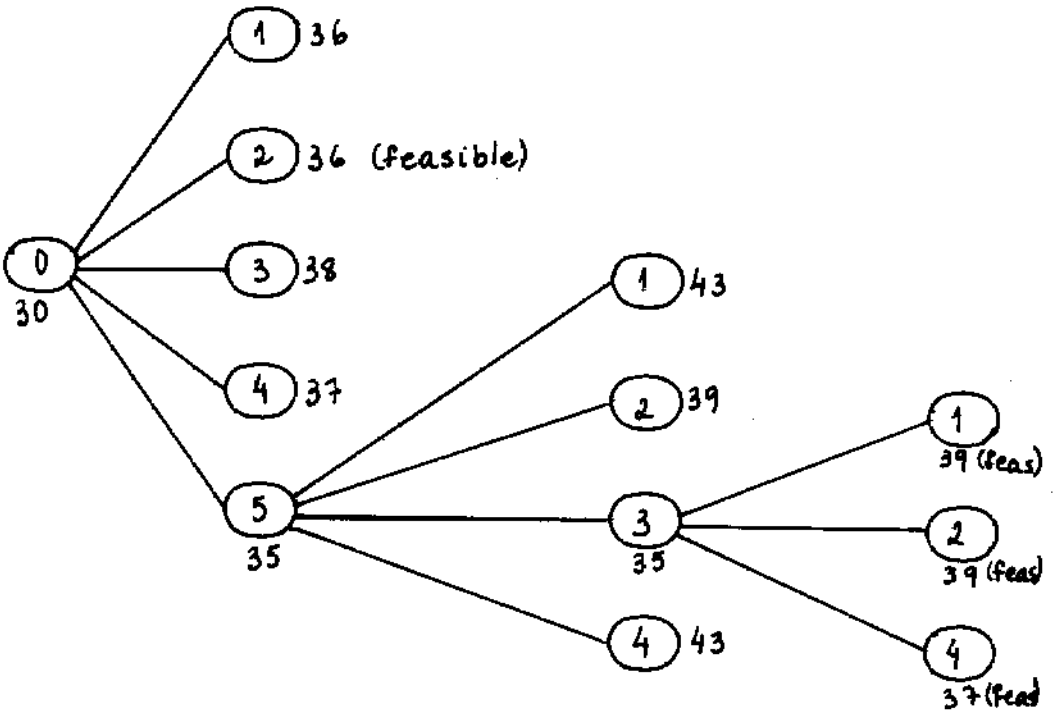
12.6-8

(a) Branch step : Best bound rule

Bound step : Given a partial sequencing $J_1, \ldots, J_k$ of the first K jobs, a lower bound on the time for the set-up of the remaining (5-K) jobs is found by adding the minimum elements of the columns corresponding to the remaining jobs, excluding those elements in rows "None", $J_1, J_2, \ldots, J_{k-1}$.

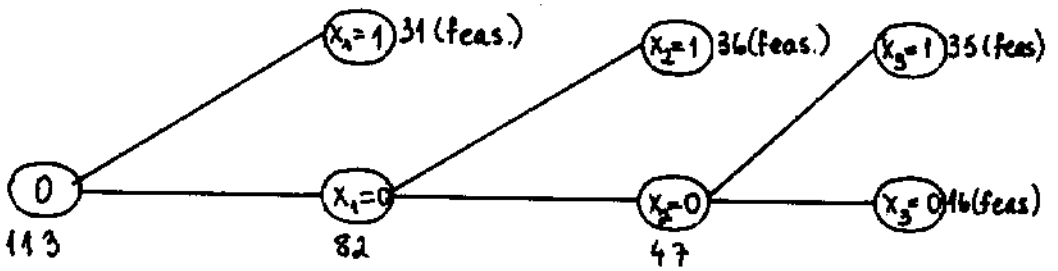
Fathoming step: As specified in the summary of the Branch-and-Bound technique given in Section 13.4. (CONT.)

12.6-8 (b)



Thus the optimal sequence is 2-1-4-5-3 requiring a total set-up time of 36.

12.6-9



So $(x_1, x_2, x_3, x_4) = (0, 1, 1, 0)$ is optimal, with $z^* = 36$.

12.6-10 a) x is feasible for an MIP. Since the Lagrangian relaxation has no constraints, other than nonnegativity or integer constraints, x is trivially feasible for this problem too.

b) x^* is optimal for MIP $\Rightarrow$ x^* feasible, i.e. $Ax^* \le b$ or $Ax^* - b \le 0$. Since $\lambda \ge 0$, $cx^* - \lambda(Ax^* - b) \ge cx^* = z$.

From (a), x^* is feasible for the Lagrangian relaxation, so its objective value is a lower bound on z_R^* [we are maximizing].

So we have $z \le cx^* - \lambda(Ax^* - b) \le z_R^*$

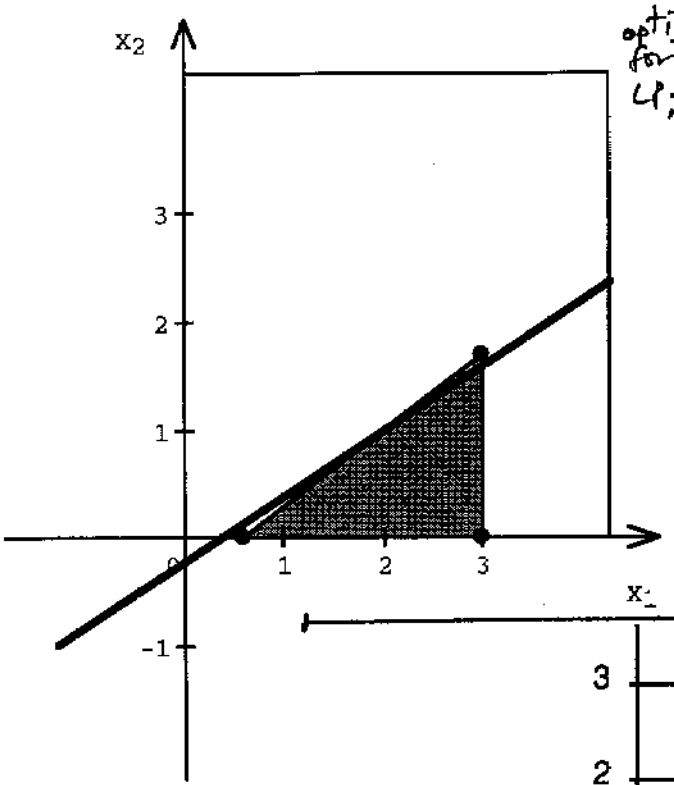
$$z \le z_R^*$$

12.7-1
Solve Interactively by the Graphical Method:

a)

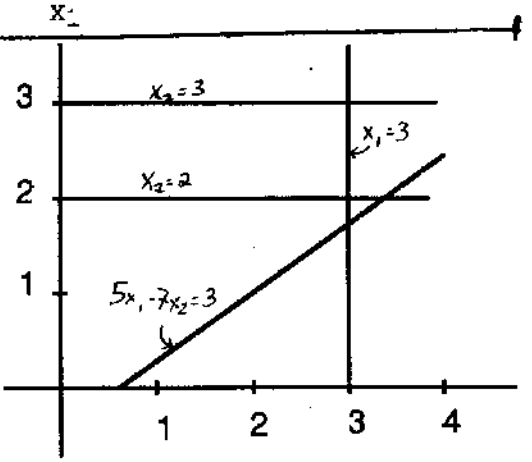
Corner Points	Z
(3, 1.7143)	-0.429
(0.6, 0)	-1.8
(3, 0)	-9

optimal for LP relaxation



Optimal value of Z: -1
Optimal integer solution: (2,1)

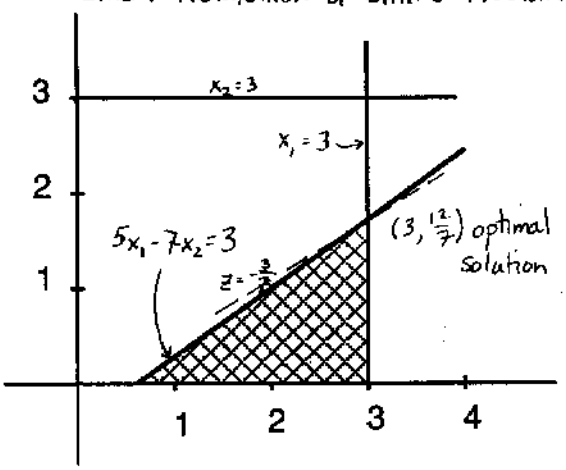
$x_2 \geq 2$ Branch



Infeasible Subproblem
Branch is Fathomed.

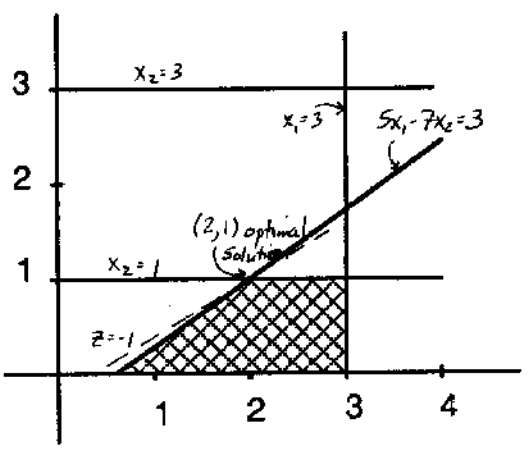
b)

Linear Relaxation of Entire Problem



optimal solution $(x_1,x_2)=(3,\frac{12}{7})$
$z=-\frac{3}{7}$

$x_2 \leq 1$ Branch



optimal solution $(x_1,x_2)=(2,1)$
$z=-1$

(Cont.)

12.7-1 c) Let $x_1 = y_{11} + 2y_{12}$

$x_2 = y_{21} + 2y_{22}$

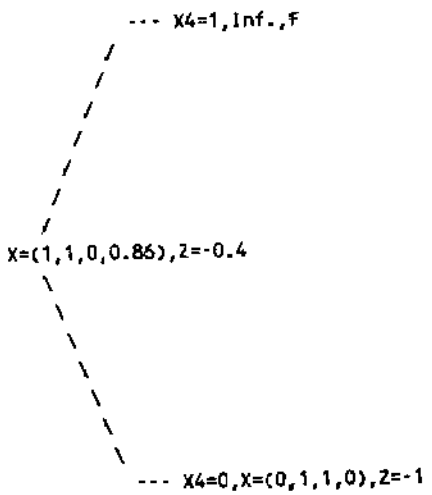
The BIP formulation is:

Maximize $z = -3y_{11} - 6y_{12} + 5y_{21} + 10y_{22}$

subject to: $5y_{11} + 10y_{12} - 7y_{21} - 14y_{22} \geq 3$

$y_{11}, y_{12}, y_{21}, y_{22}$ binary.

d)



The optimal solution is:

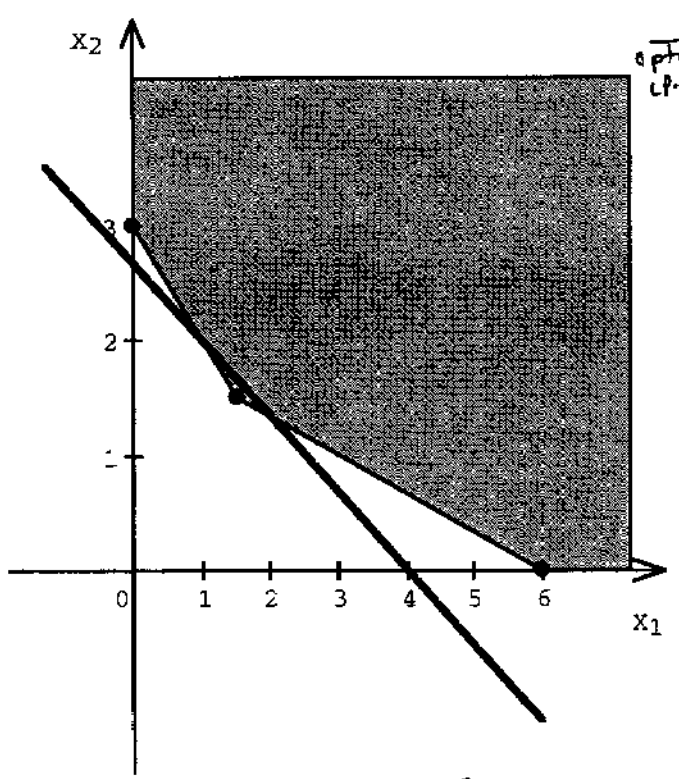
$(y_{11}, y_{12}, y_{21}, y_{22}) = (0, 1, 1, 0)$

$Z = -1$

so $x_1 = 2$, $x_2 = 1$ as in part (a).

12.7-2 Solve Interactively by the Graphical Method:

(a)



Corner Points			Z
(	1.5,	1.5)	7.5
(	0,	3)	9
(	6,	0)	12

optimal LP-relax.

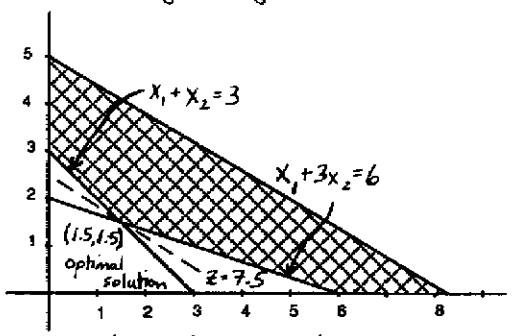
Optimal value of Z: 8

Optimal integer solution: (1,7)

(cont.)

12.7-2 b)

Linear Programming Relaxation of Entire Problem

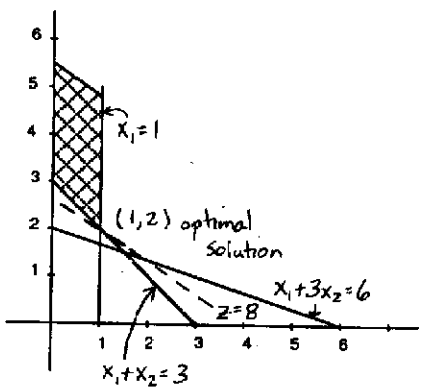


optimal solution $(X_1, X_2) = (1.5, 1.5)$

$Z = 7.5$

$X_1 \leq 1$ Branch

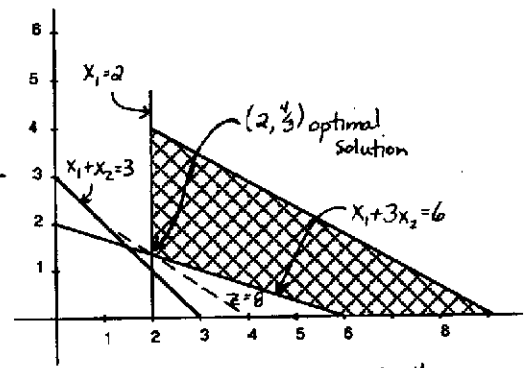
$X_1 \geq 2$ Branch



optimal solution $(X_1, X_2) = (1, 2)$

$Z = 8$

Feasible for original problem

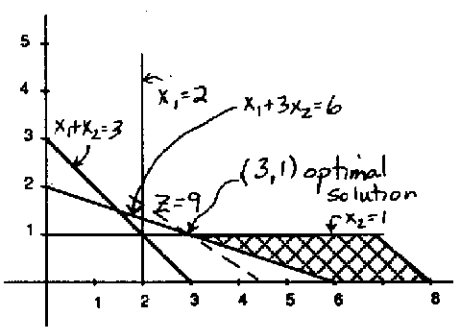


optimal solution $(X_1, X_2) = (2, \frac{4}{3})$

$Z = 8$

$X_2 \leq 1$ Branch

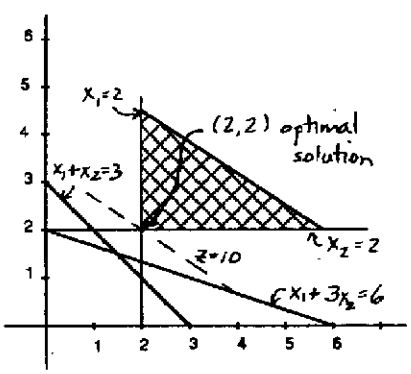
$X_2 \geq 2$ Branch



optimal solution $(X_1, X_2) = (3, 1)$

$Z = 9$

F1



optimal solution $(X_1, X_2) = (2, 2)$

$Z = 10$

F1

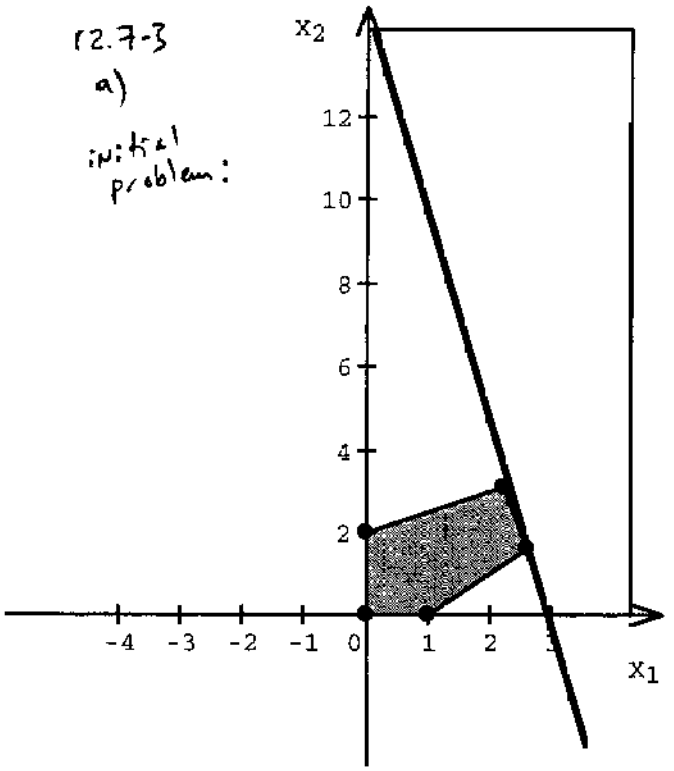
The optimal solution to the original problem is:

$(X_1, X_2) = (1, 2)$ with $Z = 8$

12.7-3

a)

initial problem:



Corner Points	Z
(2.2222, 3.1111)	14.222
(0, 2)	2
(2.6, 1.6)	14.6*
(1, 0)	5
(0, 0)	0

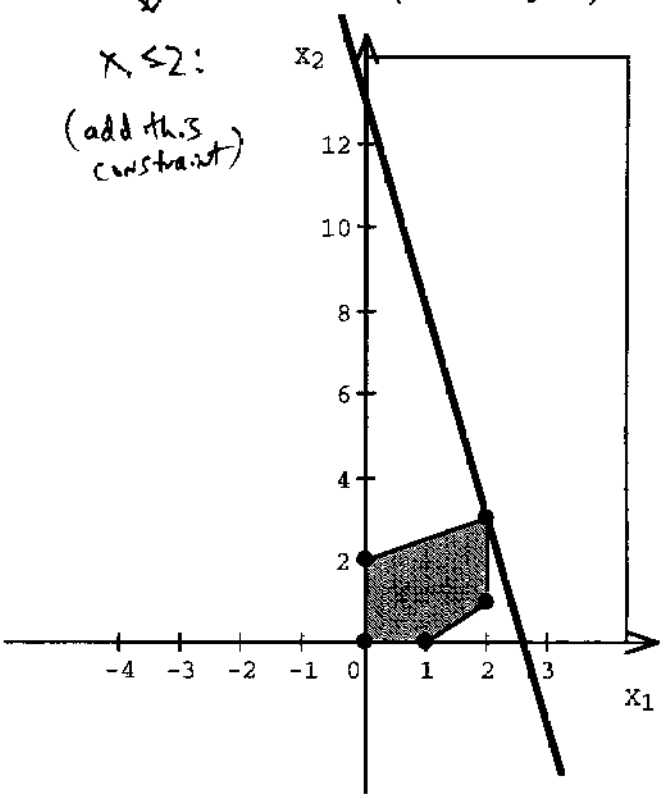
Optimal value of Z: 14.6

Optimal solution: (2.6, 1.6)

Branch: $x_1 \geq 3$ infeasible (empty feasible region)

$x_1 \leq 2$:

(add this constraint)



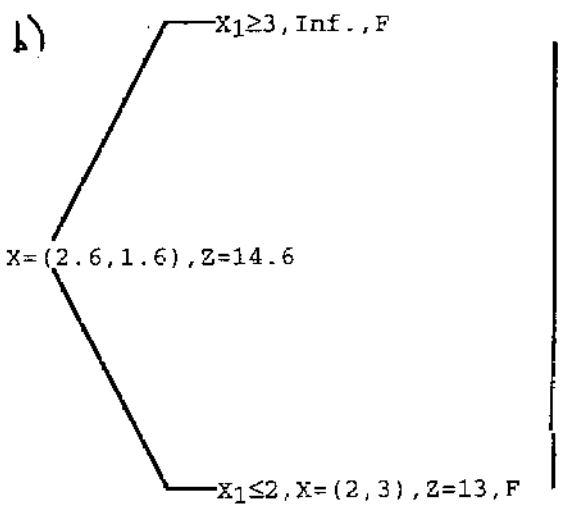
Corner Points	Z
(2, 3)	13*
(0, 2)	2
(2, 1)	11
(1, 0)	5
(0, 0)	0

Optimal value of Z: 13

Optimal solution: (2, 3)

This is integer, so we're done.

b)



Optimal Solution:
X = (2, 3)
Z* = 13

c) Solution:

$X_1 = 2$

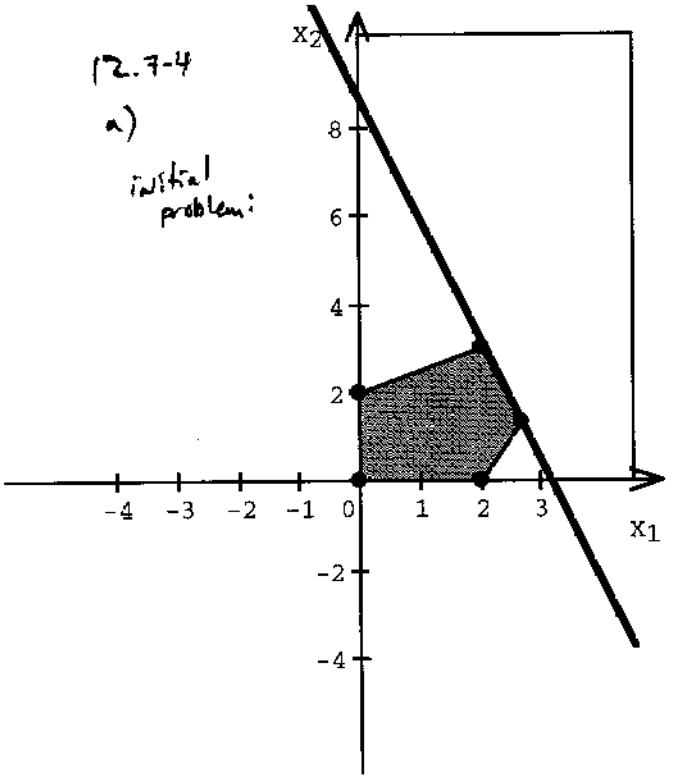
$X_2 = 3$

$Z = 13$

12.7-4

a)

initial problem:



Corner Points	Z
(2.6667, 1.3333)	693.33*
(2, 3)	680
(2, 0)	440
(0, 2)	160
(0, 0)	0

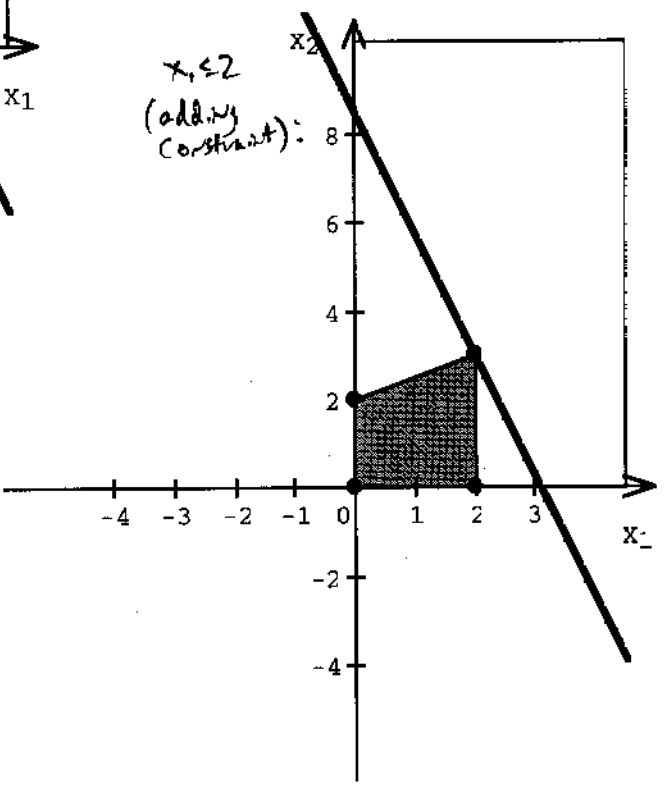
Optimal value of Z: 693.33

Optimal solution: (2.6667,1.3333)

Branch: $X_1 \geq 3$ infeasible

$\downarrow$

$X_1 \leq 2$ (adding constraint):



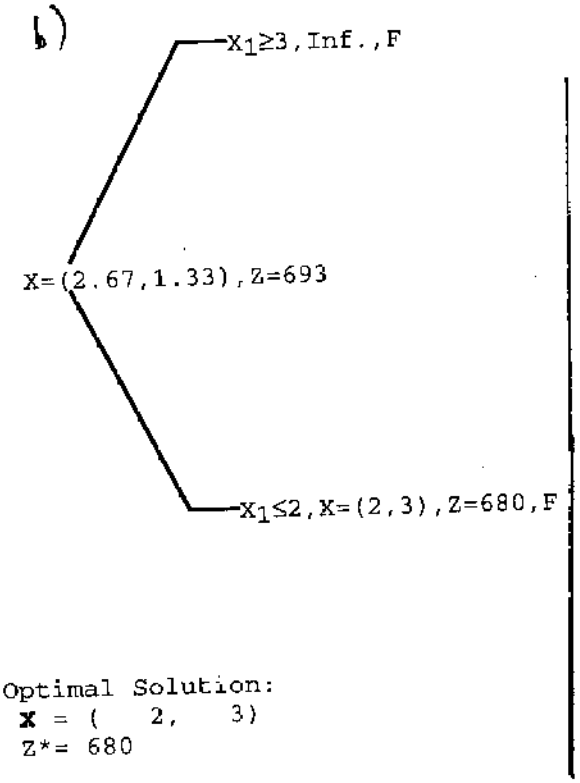
Corner Points	Z
(2, 3)	680*
(2, 0)	440
(0, 2)	160
(0, 0)	0

Optimal value of Z: 680

Optimal solutions: (2, 3)
(2, 3)
(2, 3)

integer → finished

b)



Optimal Solution:
x = (2, 3)
Z*= 680

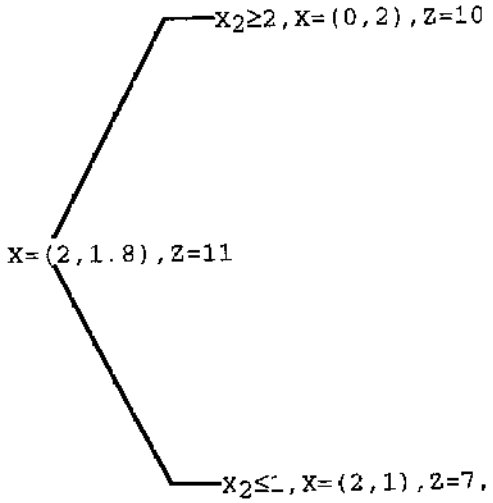
c) Solution:

$X_1 = 2$

$X_2 = 3$

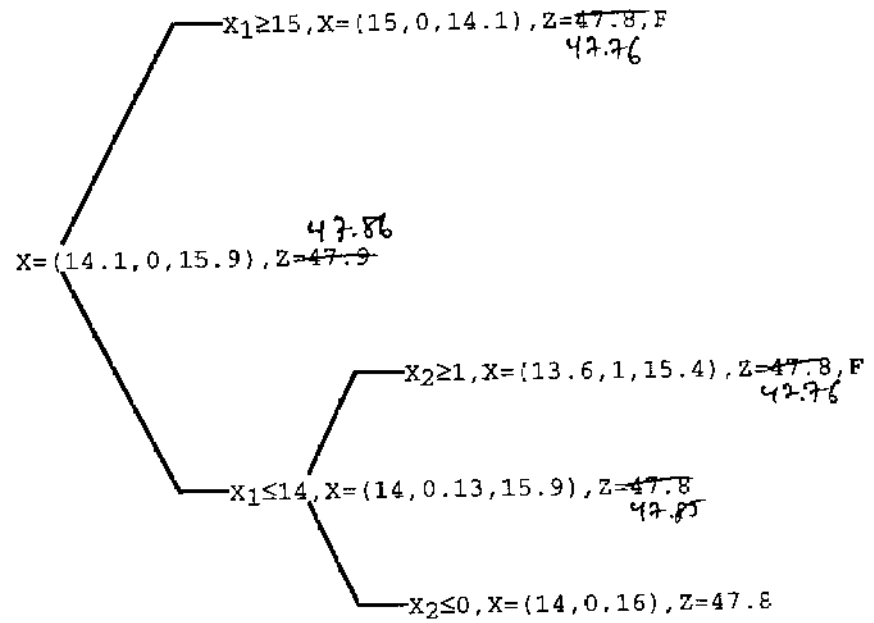
$Z = 680$

12.7-5



Optimal Solution:
X = (0, 2)
Z*= 10

12.7-6



Optimal Solution:
X = (14, 0, 16)
Z*= 47.8

12.7-7 a) let x_i = # of 1/4 units of product i to produce, i=1,2

max. $4x_1 + 2.5x_2$

s.t. $3/4 x_1 + 1/2 x_2 \leq 8$

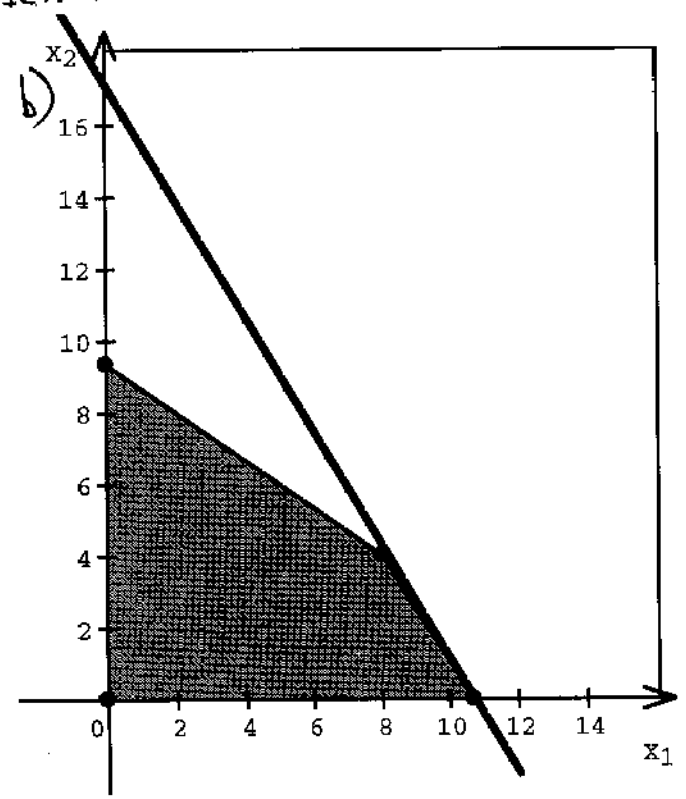
$1/2 x_1 + 3/4 x_2 \leq 7$

$x_1, x_2 \geq 0$, integer

(cont.)

12.7-7

b)



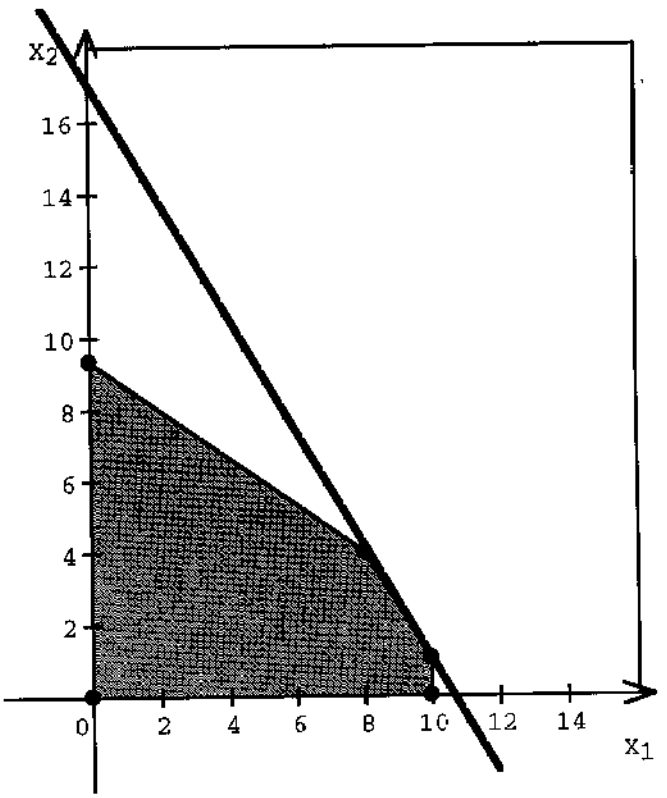
Corner Points	Z
(8, 4)	42
(10.667, 0)	42.667*
(0,9.3333)	23.333
(0, 0)	0

Optimal value of Z: 42.667

Optimal solution: (10.667, 0)

c) $X_1 \geq 11 \rightarrow$ infeasible

$X_1 \leq 10$ ↓



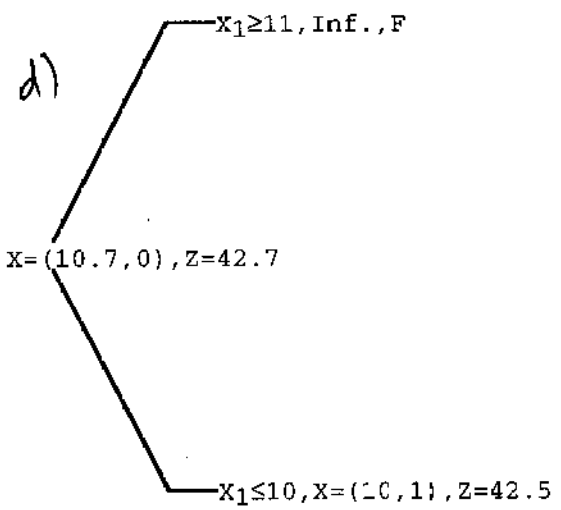
Corner Points	Z
(8, 4)	42
(10, 1)	42.5*
(0,9.3333)	23.333
(10, 0)	40
(0, 0)	0

Optimal value of Z: 42.5

Optimal solution: (10, 1)

integer ⇒ optimal for IP

d)



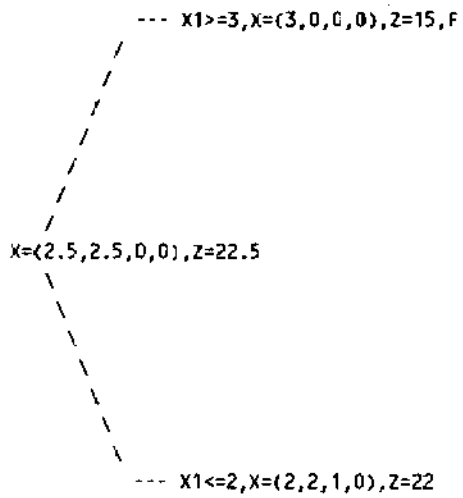
Optimal Solution:
X = (10, 1)
Z*= 42.5

e) Solution:

X_1 = 10
X_2 = 1

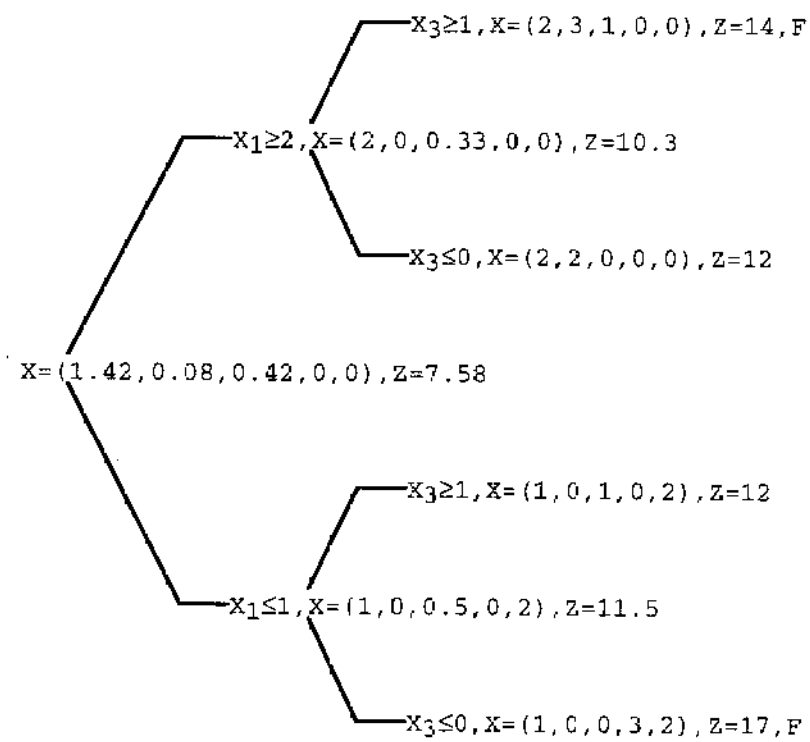
Z = 42.5

12.7-8



The optimal solution is
$(x_1, x_2, x_3, x_4) = (2, 2, 1, 0)$
with $z = 22$.

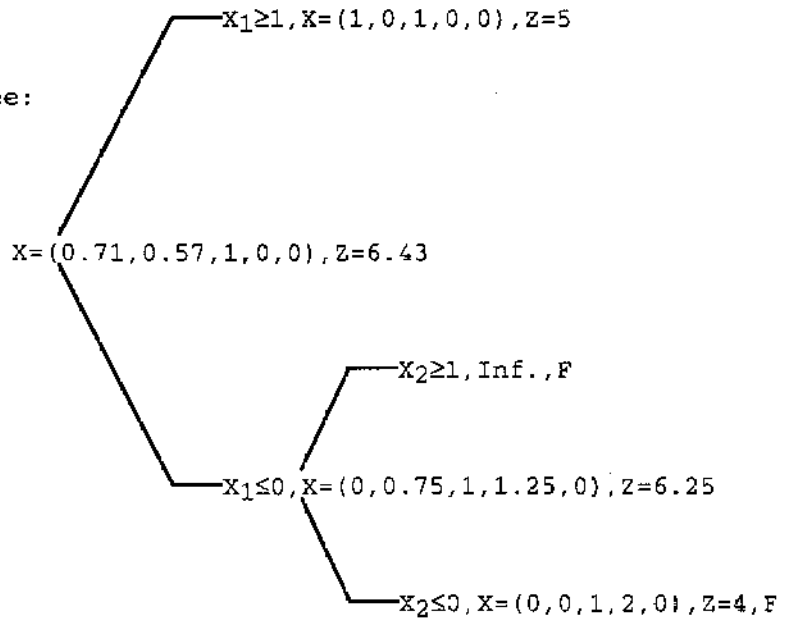
Solution Tree:



Optimal Solution:
x = (1, 0, 1, 0, 2) , (2,2,0,0,0)
Z*= 12

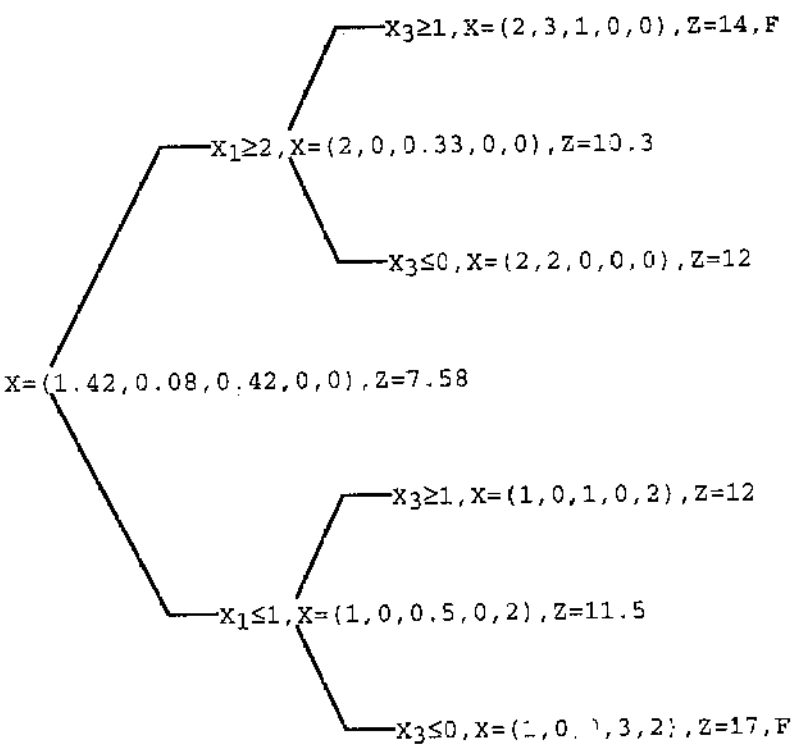
12.7-9.

Solution Tree:



Optimal Solution:
X = (1, 0, 1, 0, 0)
Z*= 5

12.7-10 Solution Tree:



Optimal Solution:
X = (1, 0, 1, 0, 2)
Z*= 12 or x=(2,2,0,0,0)

12.7-11

a) (1) Use the quadratic programming (QP) relaxation:

$$\max. \; Z = 2x_1 - x_1^2 + 3x_2 - 3x_2^2$$

s.t. $x_1 + x_2 \le 3/4$ | same

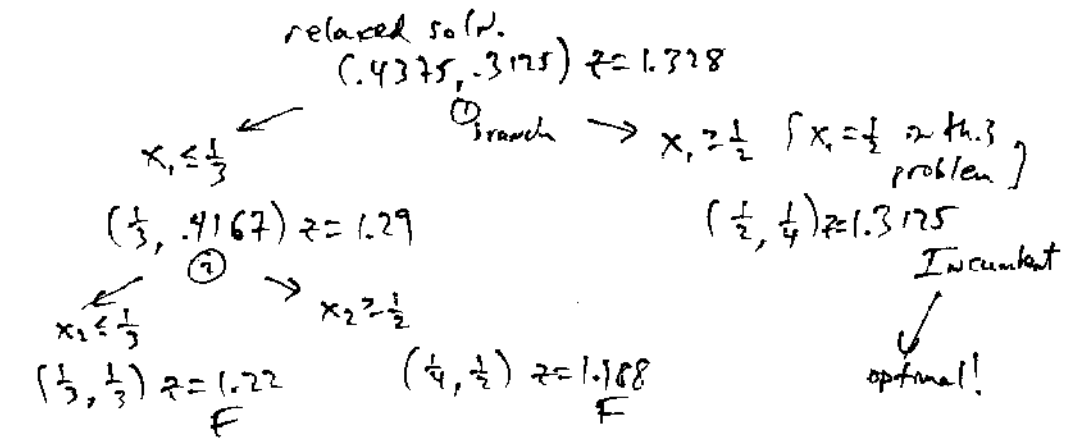
$\frac{1}{5} \le x_1 \le \frac{1}{2}$, $\frac{1}{5} \le x_2 \le \frac{1}{2}$ | relaxed

(2) The fathoming tests are the same as those stated in the chapter. A subproblem is fathomed if it is infeasible, has value $\le Z^*$ for the current incumbent, or has a qualifying solution, where to qualify here we need $x_1, x_2 \in \{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$.

(3) To branch from a value for x_j^* where $(m+1)^{-1} < x_j^* < m^{-1}$ for some $m = 2, 3, 4$, we use the two alternative constraints

$$x_j \le (m+1)^{-1} \quad \underline{\text{or}} \quad x_j \ge m^{-1}$$

b) The branch & bound tree will look like this:



So $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{4}$, $Z = 1.3125$ is optimal, agreeing with 12.3-5.

12.8-1

a) $x_1 = 0$, $x_3 = 0$

b) $x_1 = 0$

c) $x_1 = 1$, $x_3 = 1$

12.8-2

a) $x_1 = 0$

b) $x_1 = 1$, $x_2 = 0$

c) $x_1 = 0$, $x_2 = 1$

12.8-3 From eqn. 1, $x_3 = 0$; this makes eqn. 1 redundant.
From eqn. 3, $x_5 = 0$, $x_6 = 1$; now eqn. 3 is redundant
Since $x_6 = 1$, from eqn. 2 $x_2 = x_4 = 0$, and this is redundant.
Finally eqn. 4 reduces to $x_1 = 0$, which leaves all equations redundant. We have $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, $x_6 = 1$, x_7 ?

12.8-4

a) redundant, because even if all variables are 1, $2+1+2 \leq 5$

b) not redundant, as $(1,0,1)$ violates this ($8 \not\leq 5$)

c) not redundant, as $(0,0,0)$ (for example) violates

d) redundant, because $(0,1,1) \rightarrow 0-1-2-2-4$ still; this is the worst case, because we let variables with positive coefficients $= 0$ and variables with negative coefficients $= 1$ to try to violate the ≥ -4 condition, and we can't do it.

12.8-5

$$4x_1 - 3x_2 + x_3 + 2x_4 \leq 5$$

$b = 5$ $\quad S = 7$ and $S < b + a_1$

so $\bar{a}_1 = S - b = 2$

$\bar{b} = S - a_1 = 3$

so we have:

$$2x_1 - 3x_2 + x_3 + 2x_4 \leq 3 \qquad (\Delta)$$

$b = 3$ $\quad S = 5$ and $S < b + |a_2|$

so $a_2 = b - S = -2$

thus in (Δ):

$$2x_1 - 2x_2 + x_3 + 2x_4 \leq 3$$

now $S = 5$ $\quad b = 3$

and $S \geq b + |a_j|$ for $j = 1, \ldots, 4$.

12.8-6

$3x_1 - 2x_2 + x_3 \le 3$

$S = 4 < 3 + |a_1| = 6$

$\bar{a}_1 = S - b = 1 \qquad \bar{b} = S - a_1 = 1$

$\rightarrow x_1 - 2x_2 + x_3 \le 1$

$S = 2 < 1 + |a_2|$

$a_2 := b - S = -1$

$\rightarrow$ $\boxed{x_1 - x_2 + x_3 \le 1}$, done.

Check for yourself that the same binary (x_1, x_2, x_3) vectors satisfy both the original and the tightened constraint.

12.8-7

$x_1 - x_2 + 3x_3 + 4x_4 \ge 1$

$-x_1 + x_2 - 3x_3 - 4x_4 \le -1$

$S = 1 < -1 + |a_3| = 2$

$\rightarrow -x_1 + x_2 - 2x_3 - 4x_4 \le -1$

$S = 1 < -1 + |a_4| = 3$

$\rightarrow$ $\boxed{-x_1 + x_2 - 2x_3 - 2x_4 \le -1}$, done.

(these steps could have been done at the same time, since they involve $a_j < 0$)

12.8-8

a) $x_1 + 3x_2 - 4x_3 \leq 7$

$S = \{4 < 2 + |a_2|\}$

$\rightarrow x_1 + 2x_2 - 4x_3 \leq 1$

$S = \{3 < 1 + |a_3|\}$

$\rightarrow \boxed{x_1 + 2x_2 - 2x_3 \leq 1}$

b) $3x_1 - x_2 + 4x_3 \geq 1$

$-3x_1 + x_2 - 4x_3 \leq -1$

$S = \{1 < -1 + |a_1|, -1 + |a_3|\}$

$\rightarrow \boxed{-2x_1 + x_2 - 2x_3 \leq -1}$

12.8-9 $2x_1 + 3x_2 \leq 4$

minimum cover is $\{x_1, x_2\}$

$\longrightarrow \boxed{x_1 + x_2 \leq 1}$ ✓same as with tightening

12.8-10 $x_1 + 3x_2 + 2x_3 + 4x_4 \leq 5$

minimum covers: $\{x_2, x_4\} \rightarrow x_2 + x_4 \leq 1$

$\{x_3, x_4\} \rightarrow x_3 + x_4 \leq 1$

$\{x_1, x_2, x_3\} \rightarrow x_1 + x_2 + x_3 \leq 2$

12.8-11

$3x_1 + 4x_2 + 2x_3 + 5x_4 \leq 7$

minimum covers: $\{x_1, x_4\} \rightarrow x_1 + x_4 \leq 1$

$\{x_2, x_4\} \rightarrow x_2 + x_4 \leq 1$

$\{x_1, x_2, x_3\} \rightarrow x_1 + x_2 + x_3 \leq 2$

12.8-12 $3x_1 + 5x_2 + 4x_3 + 8x_4 \leq 10$

cutting planes:

$x_1 + x_4 \leq 1$

$x_2 + x_4 \leq 1$

$x_3 + x_4 \leq 1$

$x_1 + x_2 + x_3 \leq 2$

12.8-13 $5x_1 + 3x_2 + 7x_3 + 4x_4 + 6x_5 \le 9$

cutting planes:

$x_1 + x_3 \le 1$

$x_1 + x_5 \le 1$

$x_2 + x_3 \le 1$

$x_3 + x_4 \le 1$

$x_3 + x_5 \le 1$

$x_4 + x_5 \le 1$

$x_1 + x_2 + x_4 \le 2$

12.8-14

1) $3x_2 + x_4 + x_5 \ge 3 \Rightarrow x_2 = 1$, leaving a redundant constraint

2) $x_1 + x_2 \le 1$, with $x_2 = 1$ we get $x_1 = 0$, redundant now

3) $x_2 + x_4 - x_5 - x_6 \le -1$, $x_2 = 1 \Rightarrow x_4 = 0$, $x_5 = x_6 = 1$, eliminate

4) $x_2 + 2x_6 + 3x_7 + x_8 + 2x_9 \ge 4$

$x_2 = x_6 = 1 \rightarrow 3x_7 + x_8 + 2x_9 \ge 1$

tighten: $\rightarrow x_7 + x_8 + x_9 \ge 1$

5) $-x_3 + 2x_5 + x_6 + 2x_7 - 2x_8 + x_9 \le 5$

$x_5 = x_6 = 1 \rightarrow -x_3 + 2x_7 - 2x_8 + x_9 \le 2$

tighten: $\rightarrow -x_3 + x_7 - x_8 + x_9 \le 1$

Now we have

$$\max\ x_3 + 2x_7 + x_8 + 3x_9$$

$$\text{s.t.}\quad x_2 + x_7 + x_9 \ge 1$$

$$-x_3 + x_7 - x_8 + x_9 \le 1$$

Since we're maximizing & all variables have positive coefficients in objective, try $(1,1,1,1) \rightarrow 3 \ge 1$ —

$0 \le 1$ — feasible, so optimal!

$x_2 = x_5 = x_6 = x_3 = x_7 = x_8 = x_9 = 1 \Rightarrow Z = 15$

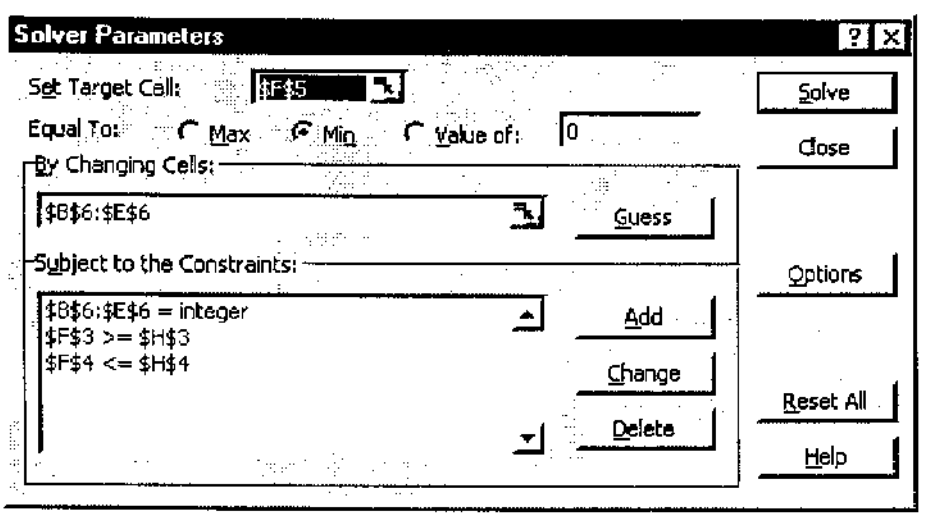
Cases

12-1 a) With this approach, we need to formulate an integer program for each month and optimize each month individually.

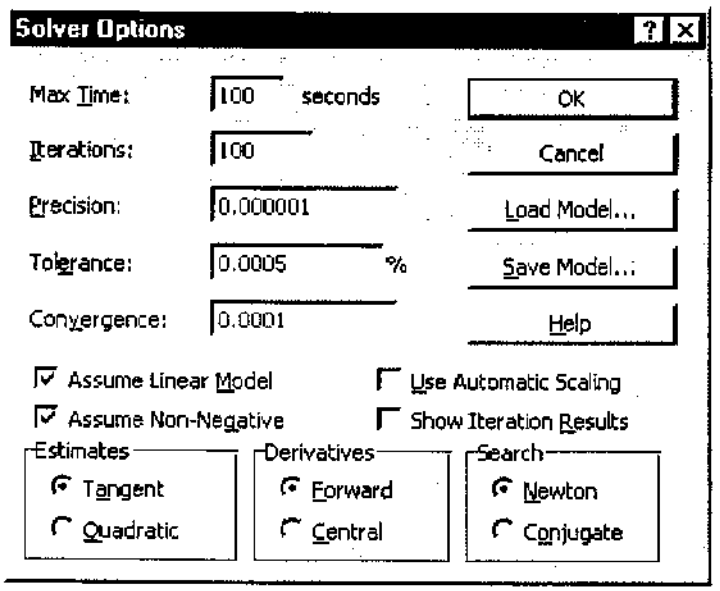
In the first month, Emily does not buy any servers since none of the departments implement the intranet in the first month. In the second month she must buy computers to ensure that the Sales Department can start the intranet. Emily can formulate her decision problem as an integer problem. Her objective is to minimize the purchase cost. She has to satisfy to constraints. She cannot spend more than $9500 (she still has her entire budget for the first two months since she didn't buy any computers in the first month) and the computer(s) must support at least 60 employees. She solves her integer programming problem using the Excel solver.

	A	B	C	D	E	F	G	H
1								
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		
3	Support	30	80	200	2000	80	>=	60
4	Capital m2	$2500	$5000	$9000	$18750	$5000	<=	$9500
5	Capital	$2500	$5000	$9000	$18750	$5000		
6	Solution	0	1	0	0			
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				
10		Formula in cell F5:		"=SUMPRODUCT(B5:E5,B6:E6)"				

The solver dialogue box looks as follows:



Emily uses the following options throughout her entire analysis of this case:

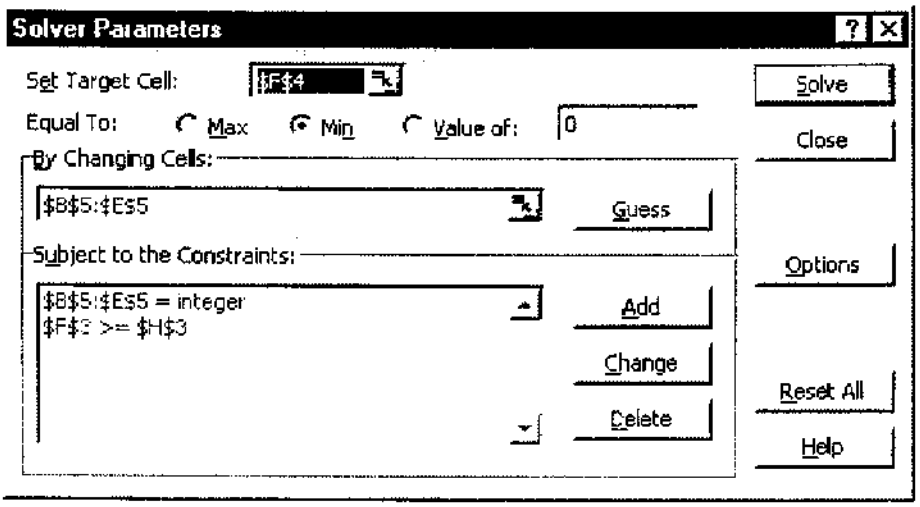


Notice that the price of the SGI server is 10 percent less than the actual price and the price of the Sun server is 25 percent less than the actual price because these two manufacturers offer discounts in the second month. Emily decides to buy one enhanced PC in the second month.

Note, that there is a second optimal solution to this integer programming problem. For the same amount of money Emily could buy two standard PC's that would also support 60 employees. However, since Emily knows that she needs to support more employees in the near future, she decides to buy the enhanced PC since it supports more users.

For the third month Emily needs to support 260 users. Since she has already computing power to support 80 users, she now needs to figure out how to support additional 180 users at minimum cost. She can disregard the constraint that the Manufacturing Department needs one of the three larger servers, since she already bought such a server in the previous month. Her task leads her to the following integer programming problem:

	A	B	C	D	E	F	G	H
1								Support
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		Needed
3	Support	30	80	200	2000	200	>=	180
4	Capital	$2,500	$5,000	$10,000	$25,000	$10,000		
5	Solution	0	0	1	0			
6								
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				



Emily decides to buy one SGI Workstation in month 3. The network is now able to support 280 users.

In the fourth month Emily needs to support a total of 290 users. Since she has already computing power to support 280 users, she now needs to figure out how to support additional 10 users at minimum cost. This task leads her to the following integer programming problem:

	A	B	C	D	E	F	G	H
1								Support
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		Needed
3	Support	30	80	200	2000	30	>=	10
4	Capital	$2,500	$5,000	$10,000	$25,000	$2,500		
5	Solution	1	0	0	0			
6								
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				

The solver dialogue box appears just as in her previous problem. Emily decides to buy a standard PC in the fourth month. The network is now able to support 310 users.

Finally, in the fifth and last month Emily needs to support the entire company with a total of 365 users. Since she has already computing power to support 310 users, she now needs to figure out how to support additional 55 users at minimum cost. This task leads her to the following integer programming problem:

	A	B	C	D	E	F	G	H
1								Support
2	Server	Std. PC	Enh. PC	SGI	Sun	Totals		Needed
3	Support	30	80	200	2000	80	>=	55
4	Capital	$2,500	$5,000	$10,000	$25,000	$5,000		
5	Solution	0	1	0	0			
6								
7								
8		Formula in cell F3:		"=SUMPRODUCT(B3:E3,B6:E6)"				
9		Formula in cell F4:		"=SUMPRODUCT(B4:E4,B6:E6)"				

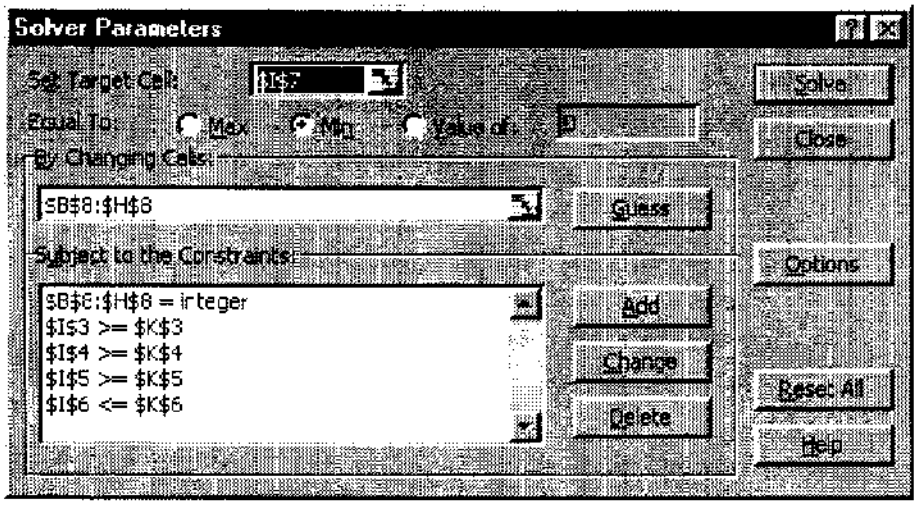
Again, the solver dialogue box has not changed. Emily decides to buy another enhanced PC in the fifth month. (Note that again she could have also bought two standard PC's, but clearly the enhanced PC provides more room for the workload of the system to grow.) The entire network of CommuniCorp consists now of 1 standard PC, 2 enhanced PC's and 1 SGI workstation and it is able to support 390 users. The total purchase cost for this network is $22,500.

b) Emily realizes that she will not be able to buy a Sun workstation during the first and second month, since the cost of such a server exceeds her budget even after the 25% discount. However, she could buy any one of the other three servers during the first two months. Due to the budget restriction she faces in the first two months she needs to distinguish between the computers she buys in those early months and in the later months. Therefore, Emily introduces two variables for each of the first three servers but only one for the Sun workstation:

Std. PC m2 = number of standard PCs bought during the first two months
Std. PC = number of standard PCs bought during the later three months
Enh. PC m2 = number of enhanced PCs bought during the first two months
Enh. PC = number of enhanced PCs bought during the later three months
SGI m2 = number of SGI workstations bought during the first two months
SGI = number of SGI workstations bought during the later three months
Sun = number of Sun workstations bought during the later three months

Emily essentially faces four constraints. First, she must support the 60 users in the sales department in the second month. She realizes that, since she no longer buys the computers sequentially after the second month, that it suffices to include only the constraint on the network to support the all users in the entire company. This second constraint requires her to support a total of 365 users. The third constraint requires her to buy at least one of the three large servers. Finally, Emily has to make sure that she stays within her budget during the second month.

	A	B	C	D	E	F	G	H	I	J	K
1											
2	Server	Stdd. PC m2	Stdd. PC	Enh. PC m2	Enh. PC	SGI m2	SGI	Sun	Totals		
3	Support m2	30	0	80	0	200	0	0	200	>=	60
4	Support	30	30	80	80	200	200	2000	400	>=	365
5	Large Comp.	0	0	1	1	1	1	1	2	>=	1
6	Capital m2	$2500	$0	$5,000	$0	$9,000	$0	$0	$9000	<=	$9500
7	Capital	$2,500	$2,500	$5,000	$5,000	$9,000	$10,000	$25,000	$19,000		
8	Solution	0	0	0	0	1	1	0			
9											
10											
11		Formula for cell I3:		"=SUMPRODUCT(B3:H3,B8:H8)"							
12		Formula for cell I4:		"=SUMPRODUCT(B4:H4,B8:H8)"							
13		Formula for cell I5:		"=SUMPRODUCT(B5:H5,B8:H8)"							
14		Formula for cell I6:		"=SUMPRODUCT(B6:H6,B8:H8)"							
15		Formula for cell I7:		"=SUMPRODUCT(B7:H7,B8:H8)"							



Emily should purchase a discounted SGI workstation in the second month, and another regular priced one in the third month. The total purchase cost is $19,000.

c) Emily's second method in part (b) finds the cost for the best overall purchase policy. The method in part (a) only finds the best purchase policy for the given month, ignoring the fact that the decision in a particular month has an impact on later decisions. The method in (a) is very short-sighted and thus yields a worse result that the method in part (b).

d) Installing the intranet will incur a number of other costs. These costs include:

Training cost,
Labor cost for network installation,
Additional hardware cost for cabling, network interface cards, necessary hubs, etc.,
Salary and benefits for a network administrator and web master,
Cost for establishing or outsourcing help desk support.

e) The intranet and the local area network are complete departures from the way business has been done in the past. The departments may therefore be concerned that the new technology will eliminate jobs. For example, in the past the manufacturing department has produced a greater number of pagers than customers have ordered. Fewer employees may be needed when the manufacturing department begins producing only enough pagers to meet orders. The departments may also become territorial about data and procedures, fearing that another department will encroach on their business. Finally, the departments may be concerned about the security of their data when sending it over the network.

12-2 a) We want to maximize the number of pieces displayed in the exhibit. For each piece, we therefore need to decide whether or not we should display the piece. Each piece becomes a binary decision variable. The decision variable is assigned 1 if we want to display the piece and assigned 0 if we do not want to display the piece.

We group our constraints into four categories – the artistic constraints imposed by Ash, the personal constraints imposed by Ash, the constraints imposed by Celeste, and the cost constraint. We now step through each of these constraint categories.

Artistic Constraints Imposed by Ash

Ash imposes the following constraints that depend upon the type of art that is displayed. The constraints are as follows:

1. Ash wants to include only one collage. We have four collages available: "Wasted Resources" by Norm Marson, "Consumerism" by Angie Oldman, "My Namesake" by Ziggy Lite, and "Narcissism" by Ziggy Lite. This constraint forces us to include only one of these four pieces.

2. Ash wants at least one wire-mesh sculpture displayed if a computer-generated drawing is displayed. We have three wire-mesh sculptures available and two computer-generated drawings available. Thus, if we include either one or two computer-generated drawings, we have to include at least one wire-mesh sculpture.

3. Ash wants at least one computer-generated drawing displayed if a wire-mesh sculpture is displayed. We have two computer-generated drawings available and three wire-mesh sculptures available. Thus, if we include one, two, or three wire-mesh sculptures, we have to include either one or two computer-generated drawings.

4. Ash wants at least one photo-realistic painting displayed. We have three photo-realistic paintings available: "Storefront Window" by David Lyman, "Harley" by David Lyman, and "Rick" by Rick Rawls. At least one of these three paintings has to be displayed.

5. Ash wants at least one cubist painting displayed. We have three cubist paintings available: "Rick II" by Rick Rawls, "Study of a Violin" by Helen Row, and "Study of a Fruit Bowl" by Helen Row. At least one of these three paintings has to be displayed.

6. Ash wants at least one expressionist painting displayed. We have only one expressionist painting available: "Rick III" by Rick Rawls. This painting has to be displayed.

7. Ash wants at least one watercolor painting displayed. We have six watercolor paintings available: "Serenity" by Candy Tate, "Calm Before the Storm" by Candy Tate, "All That Glitters" by Ash Briggs, "The Rock" by Ash Briggs, "Winding Road" by Ash Briggs, and "Dreams Come True" by Ash Briggs. At least one of these six paintings has to be displayed.

8. Ash wants at least one oil painting displayed. We have five oil paintings available: "Void" by Robert Bayer, "Sun" by Robert Bayer, "Beyond" by Bill Reynolds, "Pioneers" by Bill Reynolds, and "Living Land" by Bear Canton. At least one of these five paintings has to be displayed.

9. Finally, Ash wants the number of paintings to be no greater than twice the number of other art forms. We have 18 paintings available and 16 other art forms available. We classify the following pieces as paintings: "Serenity," "Calm Before the Storm," "Void," "Sun," "Storefront Window," "Harley," "Rick," "Rick II," "Rick III," "Beyond," "Pioneers," "Living Land," "Study of a Violin," "Study of a Fruit Bowl," "All That Glitters," "The Rock," "Winding Road," and "Dreams Come True." The maximum number of these paintings that we display has to be less than or equal to twice the number of other art forms we display.

Personal Constraints Imposed by Ash

1. Ash wants all of his own paintings included in the exhibit, so we must include "All That Glitters," "The Rock," "Winding Road," and "Dreams Come True."

2. Ash wants all of Candy Tate's work included in the exhibit, so we must include "Serenity" and "Calm Before the Storm."

3. Ash wants to include at least one piece from David Lyman, so we have to include one or more of the following pieces: "Storefront Window" and "Harley."

4. Ash wants to include at least one piece from Rick Rawls, so we have to include one or more of the following pieces: "Rick," "Rick II," and "Rick III."

5. Ash wants to display as many pieces from David Lyman as from Rick Rawls. Because the number of displayed pieces from David Lyman has to equal the number of displayed pieces from Rick Rawls and because David Lyman only has two pieces available, we can only display a maximum of two pieces from each of these artists.

6. Finally, Ash wants at most one piece from Ziggy Lite displayed. We can therefore include either none or one of the following pieces: "My Namesake" and "Narcissism."

Constraints Imposed by Celeste

1. Celeste wants to include at least one piece from a female artist for every two pieces included from a male artist. We have 11 pieces by female artists available: "Chaos Reigns" by Rita Losky, "Who Has Control?" by Rita Losky, "Domestication" by Rita Losky, "Innocence" by Rita Losky, "Serenity" by Candy Tate, "Calm Before the Storm" by Candy Tate, "Consumerism" by Angie Oldman, "Reflection" by Angie Oldman, "Trojan Victory" by Angie Oldman, "Study of a Violin" by Helen Row, and "Study of a Fruit Bowl" by Helen Row. One or more of these pieces has to be displayed for every two pieces by male artists displayed.

2. Celeste wants either one or both of the pieces "Aging Earth" and "Wasted Resources" displayed.

12-50

5 20

3. Celeste wants to include at least one piece by Bear Canton, so we must include one or more of the following pieces: "Wisdom," "Superior Powers," and "Living Land."

4. Celeste wants to include one or more of the following pieces: "Chaos Reigns," "Who Has Control," "Beyond," and "Pioneers."

5. Celeste knows that the museum only has enough floor space for four sculptures. We have six sculptures available: "Perfection" by Colin Zweibell, "Burden" by Colin Zweibell, "The Great Equalizer" by Colin Zweibell, "Aging Earth" by Norm Marson, "Reflection" by Angie Oldman, and "Trojan Victory" by Angie Oldman. We can only include a maximum of four of these six sculptures.

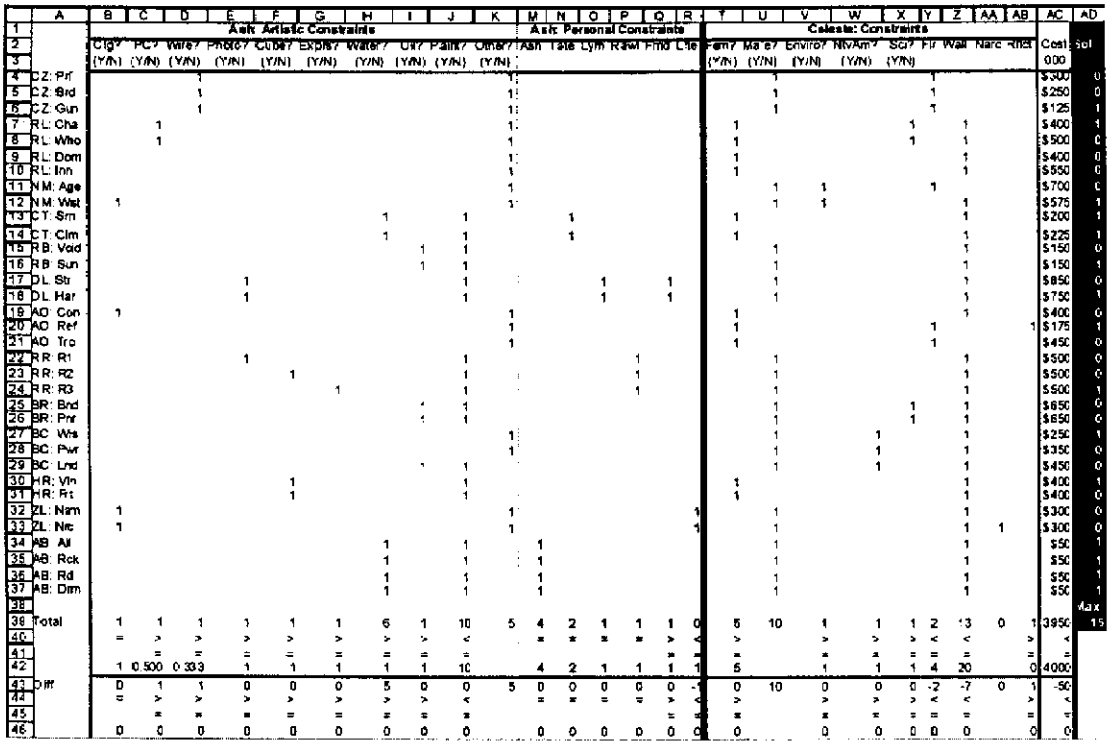
6. Celeste also knows that the museum only has enough wall space for 20 paintings, collages, and drawings. We have 28 paintings, collages, and drawings available: "Chaos Reigns," "Who Has Control," "Domestication," "Innocence," "Wasted Resources," "Serenity," "Calm Before the Storm," "Void," "Sun," "Storefront Window," "Harley," "Consumerism," "Rick," "Rick II," "Rick III," "Beyond," "Pioneers," "Wisdom," "Superior Powers," "Living Land," "Study of a Violin," "Study of a Fruit Bowl," "My Namesake," "Narcissism," "All That Glitters," "The Rock," "Winding Road," and "Dreams Come True." We can only include a maximum of 20 of these 28 wall pieces.

7. Finally, Celeste wants "Narcissism" displayed if "Reflection" is displayed. So if the decision variable for "Reflection" is 1, the decision variable for "Narcissism" must also be 1. However, the decision variable for "Narcissism" can still be 1 even if the decision variable for "Reflection" is 0.

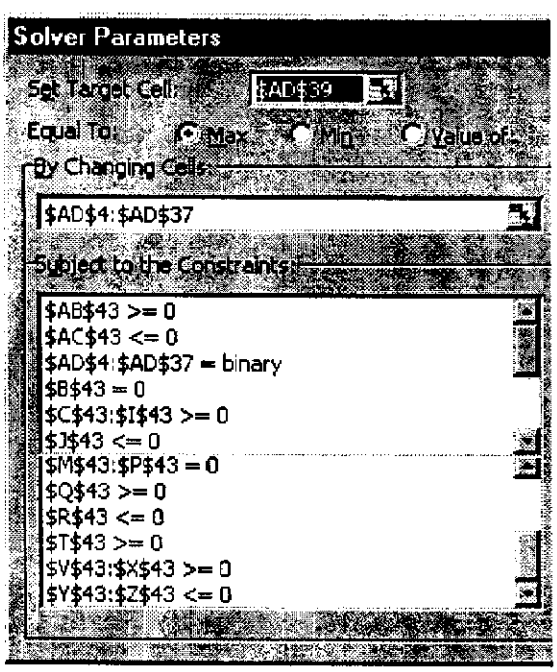
Cost Constraint

The cost of all of the pieces displayed has to be less than or equal to $4 million.

The problem formulation in an Excel spreadsheet follows.



The Solver settings used in the problem are shown below.

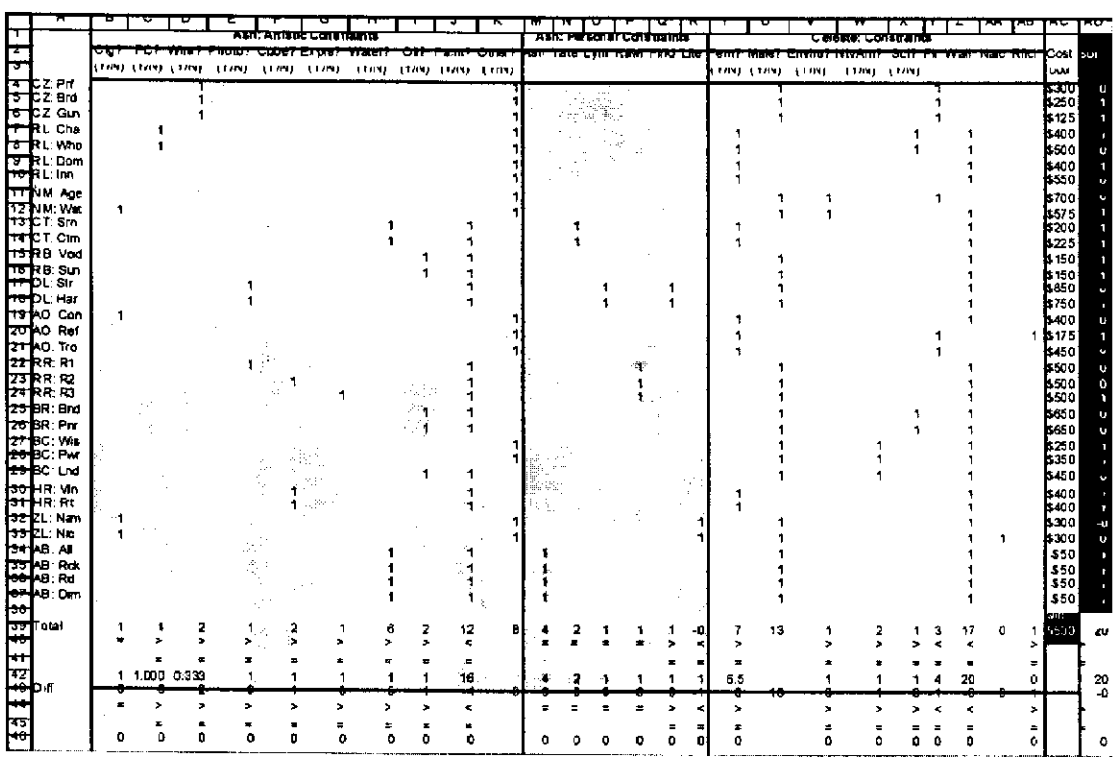


In the optimal solution, 15 pieces are displayed at a cost of $3.95 million. The following pieces are displayed:

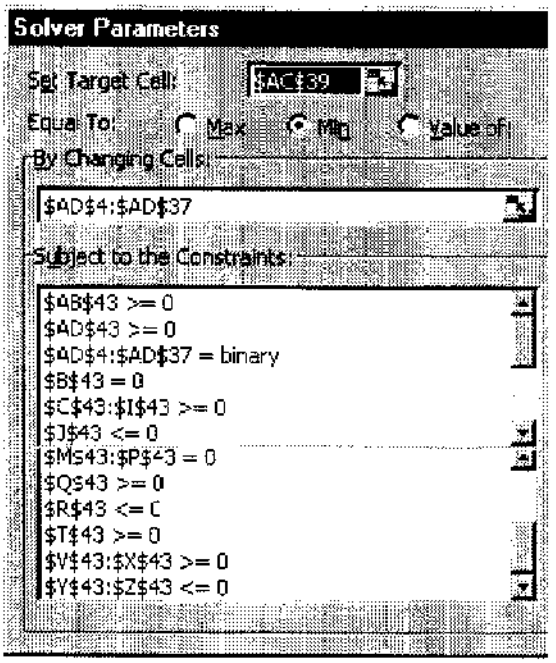
1. "The Great Equalizer" by Colin Zweibell
2. "Chaos Reigns" by Rita Losky
3. "Wasted Resources" by Norm Marson
4. "Serenity" by Candy Tate
5. "Calm Before the Storm" by Candy Tate
6. "Sun" by Robert Bayer
7. "Harley" by David Lyman
8. "Reflection" by Angie Oldman
9. "Rick III" by Rick Rawls
10. "Wisdom" by Bear Canton
11. "Study of a Violin" by Helen Row
12. "All That Glitters" by Ash Briggs
13. "The Rock" by Ash Briggs
14. "Winding Road" by Ash Briggs
15. "Dreams Come True" by Ash Briggs

b) The formulation of this problem is the same as the formulation in part (a) except that the objective function from part (a) now becomes a constraint and the cost constraint from part (a) now becomes the objective function. Thus, we have the new constraint that we need to select 20 or more pieces to display in the exhibit. We also have the new objective to minimize the cost of the exhibit.

The new formulation of the problem in an Excel spreadsheet appears below.



The Solver settings used in the problem appear below.

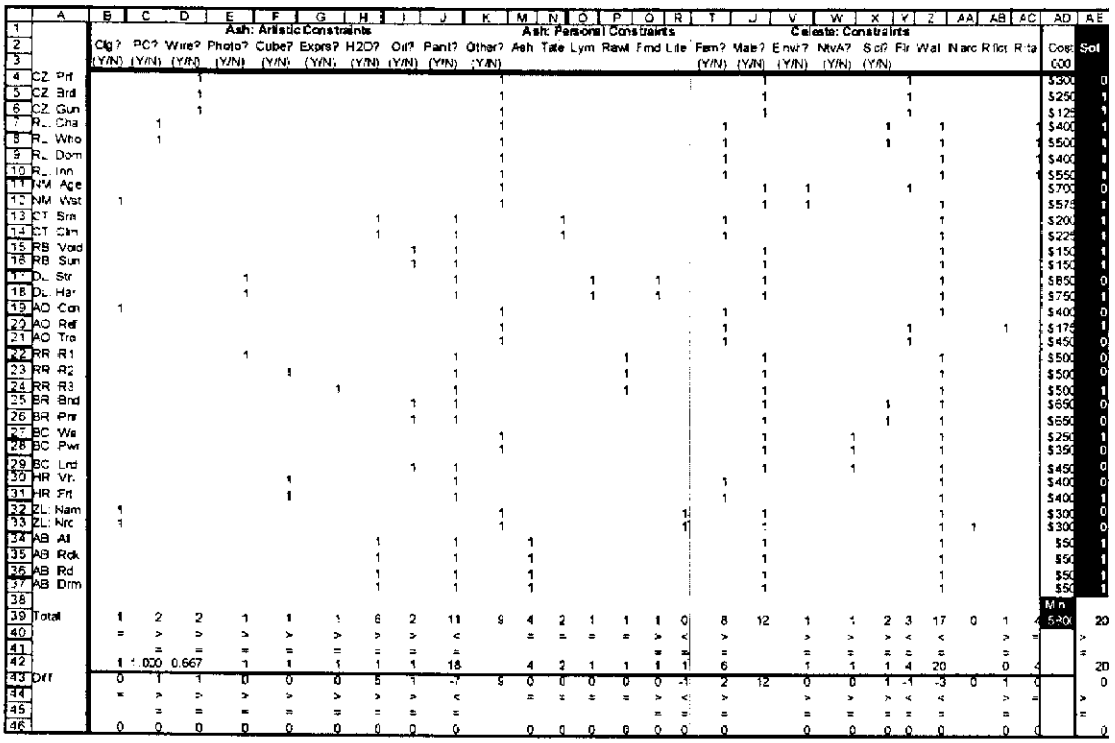


In the optimal solution, exactly 20 pieces are displayed at a cost of $5.5 million – $1.5 million more than Ash decided to allocate in part (a). All pieces from part (a) are displayed in addition to the following five new pieces:

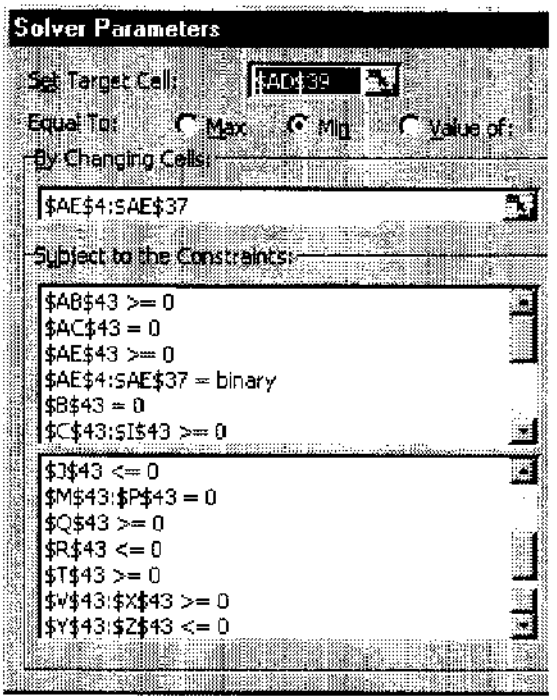
1. "Burden" by Colin Zweibell
2. "Domestication" by Rita Losky
3. "Void" by Robert Bayer
4. "Superior Powers" by Bear Canton
5. "Study of a Fruit Bowl" by Helen Row

c) This problem is also a cost minimization problem. The problem formulation is the same as that used in part (b). A new constraint is added, however. The patron wants all of Rita's pieces displayed. Rita has four pieces: "Chaos Reigns," "Who Has Control?," "Domestication," and "Innocence." All of these four pieces must be displayed.

The problem formulation in Excel appears below.



The Solver settings used in this problem appear below.



In the optimal solution, exactly 20 pieces are displayed at a total cost of $5.8 million. The patron has to pay $1.8 million. The following pieces are displayed:

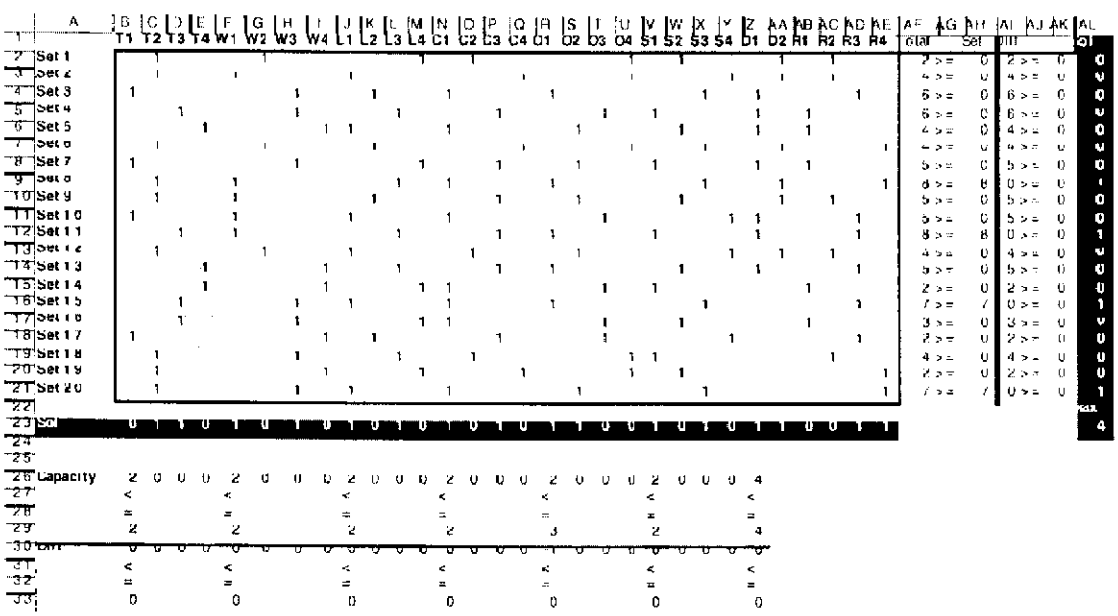
1. "Burden" by Colin Zweibell
2. "The Great Equalizer" by Colin Zweibell
3. "Chaos Reigns" by Rita Losky
4. "Who Has Control?" by Rita Losky
5. "Domestication" by Rita Losky
6. "Innocence" by Rita Losky
7. "Wasted Resources" by Norm Marson
8. "Serenity" by Candy Tate
9. "Calm Before the Storm" by Candy Tate
10. "Void" by Robert Bayer
11. "Sun" by Robert Bayer
12. "Harley" by David Lyman
13. "Reflection" by Angie Oldman
14. "Rick III" by Rick Rawls
15. "Wisdom" by Bear Canton
16. "Study of a Fruit Bowl" by Helen Row
17. "All That Glitters" by Ash Briggs
18. "The Rock" by Ash Briggs
19. "Winding Road" by Ash Briggs
20. "Dreams Come True" by Ash Briggs

12-3

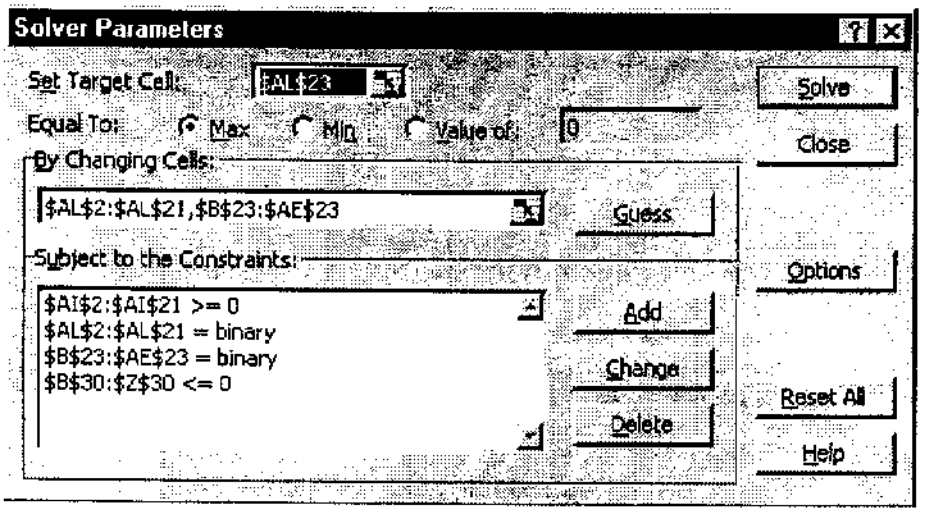
a) We want to maximize the total number of kitchen sets, so each of the 20 kitchen sets becomes a decision variable. But the kitchen sets are not our only decision variables. Because we assume that any particular item composing a kitchen set is replenished immediately, we only need to stock one of each item. A particular item may compose multiple kitchen sets. For example, tile T1 is part of kitchen sets 3, 7, 10, and 17. So a kitchen set exists when all of the items composing that kitchen set are in stock. Therefore, each of 30 items also becomes a decision variable. These decision variables are binary decision variables. If a kitchen set or item is in stock, the decision variable is 1. If a kitchen set or item is not in stock, the decision variable is 0.

A handful of constraints exist in this problem.
1. We cannot indicate that a kitchen set is in stock unless all the items composing that kitchen set are also in stock. Thus, a kitchen set decision variable is 1 only if all the decision variables for the items composing that kitchen set are also 1. For example, for set 1 this constraint equals 8*(Set 1) <= T2+W2+L4+C2+O4+S2+D2+R2
2. Each kitchen set requires 20 square feet of tile. Thus, if a particular tile is in stock, 20 square feet of that tile are in stock. The warehouse can only hold 50 square feet of tile, so only a maximum of two different styles of tile can be in stock.
3. Each kitchen set requires five rolls of wallpaper. Thus, if a particular style of wallpaper is in stock, five rolls of that wallpaper are in stock. The warehouse can only hold 12 rolls of wallpaper, so only a maximum of two different styles of wallpaper can be in stock.
4. A maximum of two different styles of light fixtures can be in stock.
5. A maximum of two different styles of cabinets can be in stock.
6. A maximum of three different styles of countertops can be in stock.
7. A maximum of two different sinks can be in stock.
8. A combination of four different styles of dishwashers and ranges can be held in stock.

The problem formulated in an Excel spreadsheet follows.



The solver settings are the following:



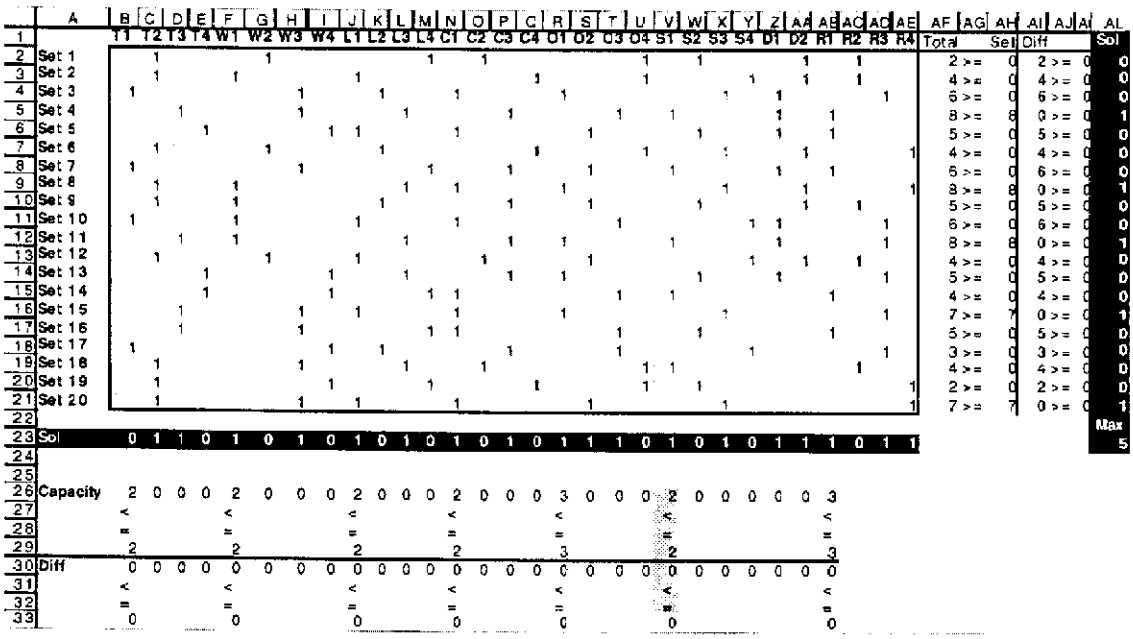
b) Four different kitchen sets are in stock. We should stock the following number of items:

Item	Quantity
T1	0
T2	1
T3	1
T4	0
W1	1
W2	0
W3	1
W4	0
L1	1
L2	0
L3	1
L4	0
C1	1
C2	0
C3	1
C4	0
O1	1
O2	1
O3	0
O4	0
S1	1
S2	0
S3	1
S4	0
D1	1
D2	1
R1	0
R2	0
R3	1
R4	1

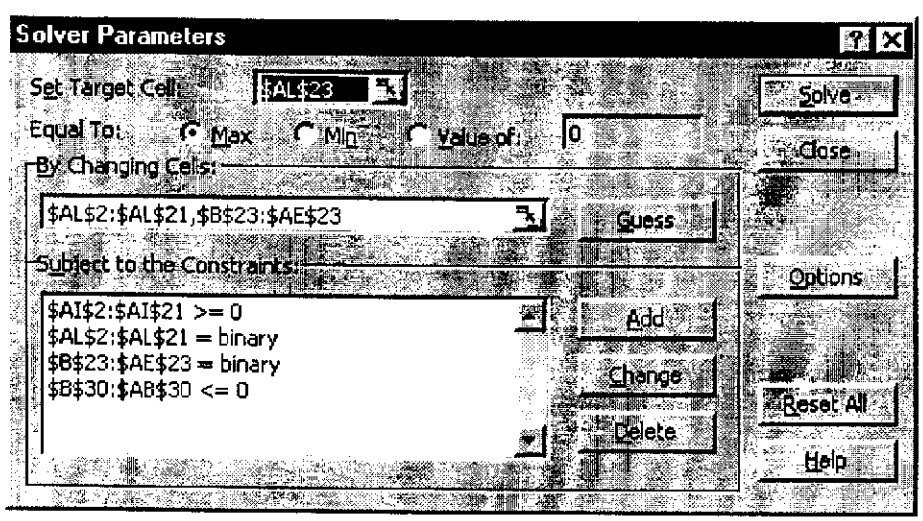
Note that the above optimal solution is not a unique solution. The value of the objective function is always four complete kitchen sets, but the specific items and kitchen sets stocked may be different. Throughout this solution, we will refer to the optimal solution shown above, but because other optimal solutions exist, student answers may differ from the solution somewhat.

c) We model this new problem by changing the capacity constraint for the dishwashers and ranges. Now, instead of being able to stock a combination of only four different styles of dishwashers and ranges, we can stock a maximum of two different styles of dishwashers and a maximum of three different styles of ranges. Because we only have two different styles of dishwashers available, we now effectively do not have a constraint on the number of dishwashers we can carry.

The formulation of the problem in Excel follows:



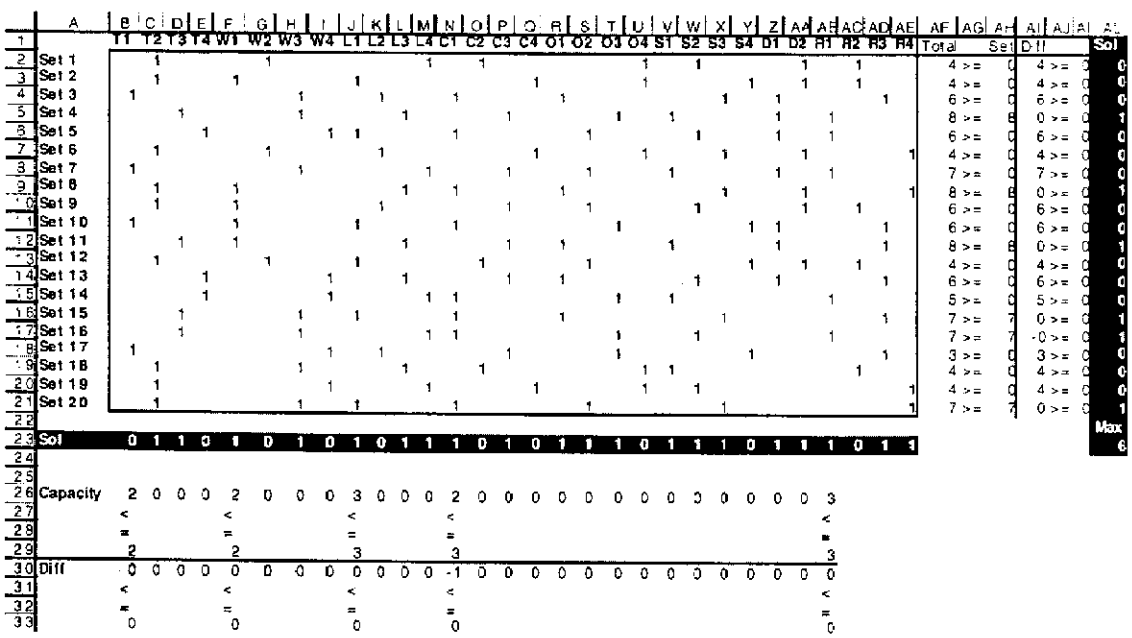
The Solver settings for this problem follow.



With the extra space, the number of kitchen sets we can stock increases from four to five. In part (a), the constraint on the number of different styles of dishwashers and ranges was binding, and we could only stock four different kitchen sets – sets 8, 11, 15, and 20. With the extra space, we can add set 4 to our stock. Set 4 requires two items that are not required by sets 8, 11, 15, and 20 – a different countertop O3 and a different range R1. In part (a), the constraint limiting the maximum number of different styles of countertops was non-binding, so we can add a new countertop style to our stock. The new space vacated by the nursery department provides us with the space to stock the new range.

d) With the additional space, our constraints change. We eliminate the constraints limiting the maximum number of different styles of sinks and countertops we can stock. Instead of stocking two of the four styles of light fixtures, we can now stock three of the four styles of light fixtures. Finally, instead of stocking only two of the four cabinet styles, we can now stock three of the four cabinet styles.

The problem formulated in Excel follows.



The Solver settings are the same settings used in part (c).

With the extra space, we are now able to stock six complete kitchen kits. In part (c), we stocked five sets – sets 4, 8, 11, 15, and 20. Now we add set 16 to the inventory. All of the extra space does not increase the number of complete sets we can stock significantly because the capacity constraints on the countertops, and cabinets were not binding in part (c). Only the capacity constraints on the sinks and light fixtures were binding. Set 16 requires both a sink and a light fixture that sets 4, 8, 11, 15, and 20 do not require. We therefore add an additional style of sink and light fixture to our stock. We still have space for one additional sink, so the constraint is not binding in this problem.

e) If the items composing a kitchen set could not be replenished immediately, we could not formulate this problem as a binary integer program. We would have to formulate the problem as an integer program since we may have to store more than one kitchen component or kitchen set to ensure that we meet demand.

The assumption of immediate replenishment is justified if the average time to replenish the component is less than the average time between demands for that component.

12-4 a)

Let $x_{ij} = 1$ if students from area i are assigned to school j; 0 if not

C_{ij} = bussing cost

S_i = student population of area i

K_j = capacity of school j

P_{ik} = % of students in area i in grade k

(for $i = 1,2,3,4,5,6$ $j = 1,2,3$ and $k = 6,7,8$).

$$\text{Minimize Cost} = \sum_{i=1}^{6}\sum_{j=1}^{3}(C_{ij})(S_i)(x_{ij})$$

subject to $$\sum_i S_i x_{ij} \le K_j$$

$$\sum_j x_{ij} = 1$$

$$0.30\sum_i S_i x_{ij} \le \sum_i P_{ik} S_i x_{ij} \le 0.36\sum_i S_i x_{ij}, \qquad (k = 6,7,8)$$

and x_{ij} are binary variables (for $i = 1,2,3,4,5,6$ and $j = 1,2,3$).

Note $x_{21} = x_{43} = x_{52} = 0$ due to infeasibility.

b) The models really aren't too different. x_{ij} are binary here, which amounts to forcing their value in the LP of Case 4-3 to be either 0 or S_i. We can leave out the three variables known to be 0, and also 9 redundant constraints. The LP-relaxation of this model, with $0 \le x_{ij} \le 1$ would allow us to interpret x_{ij} as the fraction of students from area i to be assigned to school j. This obviously would be a more general model, equivalent to that in Case 4-3.

c)

Area	Number of Students	Percentage in 6th grade	Percentage in 7th grade	Percentage in 8th grade	Busing Cost ($/student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	9999	400	500
3	550	0.3	0.32	0.38	600	300	200
4	350	0.28	0.4	0.32	200	500	9999
5	500	0.39	0.34	0.27	0	9999	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	950	1150	1050

Solution:

Area	Area Assignments School 1	School 2	School 3	Total			
1	0	1	0	1 =	1	**Total Bussing Cost =**	845000
2	0	1	0	1 =	1		
3	0	0	1	1 =	1		
4	1	0	0	1 =	1		
5	0	0	1	1 =	1		
6	1	0	0	1 =	1		

Area	Number of Students School 1	School 2	School 3
1	0	450	0
2	0	600	0
3	0	0	550
4	350	0	0
5	0	0	500
6	450	0	0
Total	800	1050	1050
	<=	<=	<=
Capacity	950	1150	1050

Grade Constraints:

	School 1	School 2	School 3
6th graders	251	366	360
7th graders	266	339	346
8th graders	283	345	344
30% of total	240	315	315
36% of total	288	378	378

d)

Without prohibiting the splitting of residential areas, the total cost was $555,555. Thus, adding this restriction increases the cost by $845000 - $555,555 = $289,445

e)

Area	Number of Students	Percentage in 6th grade	Percentage in 7th grade	Percentage in 8th grade	Bussing Cost ($/student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	300	0	700
2	600	0.37	0.28	0.35	9999	400	500
3	550	0.3	0.32	0.38	600	300	0
4	350	0.28	0.4	0.32	0	500	9999
5	500	0.39	0.34	0.27	0	9999	400
6	450	0.34	0.28	0.38	500	300	0
				Capacity:	950	1150	1050

Solution:

Area	Area Assignments School 1	School 2	Schoo 3	Total			
1	0	1	0	1 =	1	**Total Bussing Cost =**	665000
2	0	1	0	1 =	1		
3	0	0	1	1 =	1		
4	1	0	0	1 =	1		
5	0	0	1	1 =	1		
6	1	0	0	1 =	1		

Area	Number of Students School 1	School 2	School 3
1	0	450	0
2	0	600	0
3	0	0	550
4	350	0	0
5	0	0	500
6	450	0	0
Total	800	1050	1050
	<=	<=	<=
Capacity	950	1150	1050

Grade Constraints:

	School 1	School 2	School 3
6th graders	251	366	360
7th graders	266	339	346
8th graders	283	345	344
30% of total	240	315	315
36% of total	288	378	378

As shown in the spreadsheet, the solution remains the same, but the busing costs are reduced to $665000.

F)

Area	Number of Students	Percentage in 5th grade	Percentage in 7th grade	Percentage in 8th grade	Bussing Cost ($/student) School 1	School 2	School 3
1	450	0.32	0.38	0.3	0	0	700
2	600	0.37	0.28	0.35	9999	400	500
3	550	0.3	0.32	0.38	600	0	0
4	350	0.28	0.4	0.32	0	500	9999
5	500	0.39	0.34	0.27	0	9999	400
6	450	0.34	0.28	0.38	500	0	0
				Capacity:	950	1150	1050

Solution:

Area	Area Assignments School 1	School 2	School 3	Total		
1	0	1	0	1 =	1	
2	0	1	0	1 =	1	
3	0	0	1	1 =	1	
4	1	0	0	1 =	1	
5	0	0	1	1 =	1	
6	1	0	0	1 =	1	

Total Bussing Cost = 665000

Area	Number of Students School 1	School 2	School 3
1	0	450	0
2	0	600	0
3	0	0	550
4	350	0	0
5	0	0	500
6	450	0	0
Total	800	1050	1050
	<=	<=	<=
Capacity	950	1150	1050

Grade Constraints:

	School 1	School 2	School 3
6th graders	251	366	360
7th graders	266	339	345
8th graders	283	345	344
30% of total	240	315	315
36% of total	288	378	378

As shown in the spreadsheet, the solution and the bussing cost remains the same as for Option 1.

g)

For all three options, the assignments are identical. For the current alternative, the bussing costs are $845000. For option 1, the bussing costs are $665000 (a reduction of $180000). This savings results from the fact that students from area 3 would no longer be bussed to school 3 and the same happens for area 4 and school 1.
For option 2, the bussing costs are the same as those for option 1, the reason being that under the optimal assignment policy there are no students bused 1.5 to 2 miles.

h) Arguments can be made for the current situation and option 1. Option 2 can be discarded since it provides NO improvement over option 1.

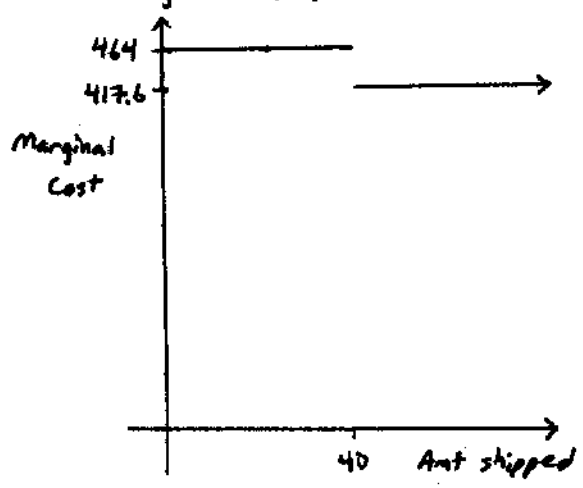
Chapter 13: Nonlinear Programming

13.1-1. Maximize $f(x) = 100x_1^{2/3} + 10x_1 + 40x_2^{3/4} + 5x_2 + 50x_3^{1/2} + 5x_3$

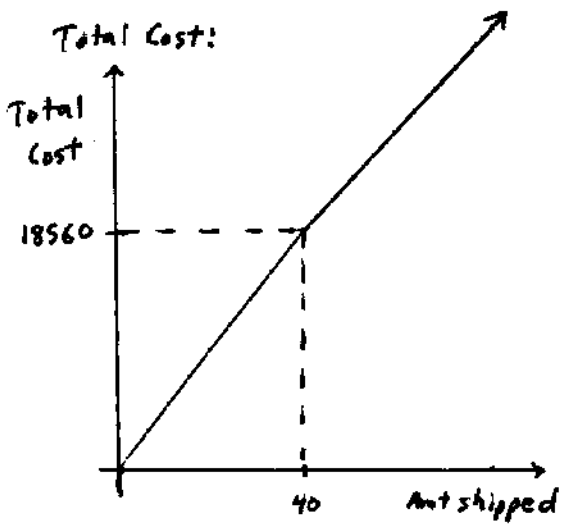
subject to:

$$9x_1 + 3x_2 + 5x_3 \le 500$$
$$5x_1 + 4x_2 \le 350$$
$$3x_1 + 2x_3 \le 150$$
$$x_3 \le 20$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

13.1-2. Marginal Cost:



Total Cost:



Each term in the objective function changes (as above) from $a_{ij}x_{ij}$ to

$$a_{ij}x_{ij} - 0.1\,a_{ij}(x_{ij}-40)\,S(x_{ij}-40)$$

where a_{ij} is the shipping cost from cannery i to warehouse j

and $S(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$

otherwise, the formulation is the same.

13.1-3

Let S_1 = number of blocks of stock 1 to purchase

S_2 = number of blocks of stock 2 to purchase.

Minimize Risk $= 4S_1^2 + 100S_2^2 + 5S_1S_2$,

subject to $20S_1 + 30S_2 \le 50$

$5S_1 + 10S_2 \ge$ minimum acceptable expected return

and $S_1 \ge 0,\ S_2 \ge 0.$

13.2-1

$f(x) = f_1(x_1) + f_2(x_2) + f_3(x_3)$

with $f_1(x_1) = 100x_1^{2/3} + 10x_1$

$f_2(x_2) = 40x_2^{3/4} + 5x_2$

$f_3(x_3) = 50x_3^{1/2} + 5x_3$

$\frac{d^2f_1}{dx_1^2} = -\frac{200}{9}x_1^{-4/3} \le 0$ for $x_1 \ge 0$

$\frac{d^2f_2}{dx_2^2} = -\frac{120}{16}x_2^{-5/4} \le 0$ for $x_2 \ge 0$

$\frac{d^2f_3}{dx_3^2} = -\frac{50}{4}x_3^{-3/2} \le 0$ for $x_3 \ge 0$

f_1, f_2 and f_3 are concave over the non-negative orthant so f must be concave in the same region. The constraints are linear and, therefore, convex so the problem is a convex programming problem.

13.2-2

$$\left.\begin{array}{l} \frac{\partial^2 f}{\partial s_1^2} = 8 \geq 0 \\ \frac{\partial^2 f}{\partial s_2^2} = 200 \geq 0 \\ \frac{\partial^2 f}{\partial s_1 \partial s_2} = 5 \end{array}\right\} \frac{\partial^2 f}{\partial s_1^2}\frac{\partial^2 f}{\partial s_2^2} - \left[\frac{\partial^2 f}{\partial s_1 s_2}\right]^2 = 1595$$

$\Rightarrow f$ is convex everywhere

13.2-3. Objective function: $z = 3x_1 + 5x_2$

$\Rightarrow x_2 = -3/5\, x_1 + 1/5\, z$

$\Rightarrow$ slope is $-3/5$

Constraint boundary: $9x_1^2 + 5x_2^2 = 216$

$\Rightarrow x_2 = \sqrt{1/5(216 - 9x_1^2)}$

$\Rightarrow \frac{\partial x}{\partial x_1} = -\frac{1}{5}\frac{9x_1}{\sqrt{1/5(216-9x_1^2)}}$

$= -3/5$ for $x_1 = 2$

So the objective function is tangent to this constraint at $(x_1, x_2) = (2, 6)$

13.2-4. Constraint boundary: $3x_1 + 2x_2 = 18$

$\Rightarrow g(x_1) = x_2 = -3/2\, x_1 + 9$

$\Rightarrow \frac{dg(x_1)}{dx_1} = -3/2$

Objective function at $(8/3, 5)$:

$(9x_1^2 - 126x_1 + 857) - 182x_2 + 13x^2 = 0$

$\Rightarrow f(x_1) = x_2 = \frac{182 - 2\sqrt{-2860 + 1638x_1 - 117x_1^2}}{26}$

$\Rightarrow \frac{df(x_1)}{dx_1} = -\frac{1}{26}\frac{1638 - 234x_1}{\sqrt{-2860 + 1638x_1 - 117x_1^2}}$

$\Rightarrow \frac{df(8/3)}{dx_1} = -3/2$

and $f(8/3) = 5$, $g(8/3) = 5$

so the objective function is tangent to this constraint at $(x_1, x_2) = (8/3, 5)$

13.2-5. (a) $\frac{df}{dx} = 48 - 120x + 3x^2 = 0$

$\Rightarrow x^* = \frac{120 \pm \sqrt{120^2 - 4 \cdot 3 \cdot 48}}{6}$

$= .4041$ or 39.596

$\frac{d^2 f(.4041)}{dx^2} = -117.6 \Rightarrow f(.4041) = 2.475$ is a local maximum

$\frac{d^2 f(39.596)}{dx^2} = 117.6 \Rightarrow f(39.596) = .0253$ is a local minimum

13.2-5. (b) for $x > 39.596$ $\frac{df}{dx} > 0$ and $\frac{d^2f}{dx^2} = 6x - 120 > 0$
$\Rightarrow f$ is unbounded above.
for $x < .4041$ $\frac{df}{dx} < 0$ and $\frac{d^2f}{dx^2} < 0$
$\Rightarrow f$ is unbounded below.

13.2-6. (a) $\frac{d^2f}{dx^2} = -2 \quad \forall x \Rightarrow f$ is concave

(b) $\frac{d^2f}{dx^2} = 12x^2 + 12 > 0 \quad \forall x \Rightarrow f$ is convex

(c) $\frac{d^2f}{dx^2} = 12x - 6 \begin{cases} > 0 & \text{for } x > \frac{1}{2} \\ < 0 & \text{for } x < \frac{1}{2} \end{cases} \Rightarrow f$ is neither convex nor concave

(d) $\frac{d^2f}{dx^2} = 12x^2 + 2 > 0 \quad \forall x \Rightarrow f$ is convex

(e) $\frac{d^2f}{dx^2} = 6x + 12x^2 \begin{cases} > 0 & \text{for } x < -\frac{1}{2} \text{ or } x > 0 \\ < 0 & \text{for } -\frac{1}{2} < x < 0 \end{cases}$
$\Rightarrow f$ is neither convex nor concave.

13.2-7. (a) $\frac{\partial^2 f}{\partial x_1^2} = -2 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = -2 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 f}{\partial x_1^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 4 - 1^2 = 3 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is concave

(b) $\frac{\partial^2 f}{\partial x_1^2} = 4 > 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = 2 > 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 8 - (2)^2 = 4 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is convex

(c) $\frac{\partial^2 f}{\partial x_1^2} = 2 > 0 \quad \forall (x_1, x_2) \quad \frac{\partial^2 f}{\partial x_2^2} = 4 > 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = -1 < 0$

$\Rightarrow f$ is neither convex nor concave

(d) $\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$

$\Rightarrow \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 0$

$\Rightarrow f$ is both convex and concave

13.2-7. (e) $\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2} = 0 \quad \forall (x_1, x_2)$

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = -1 < 0$$

$\Rightarrow f$ is neither convex nor concave

13.2-8. Let $f = f_1 + f_2 + f_{34} + f_{56} + f_{67}$

where $f_1 = 5x_1$

$f_2 = 2x_2^2$

$f_{34} = x_3^2 - 3x_3x_4 + 4x_4^2$

$f_{56} = x_5^2 + 3x_5x_6 + 3x_6^2$

$f_{67} = 3x_6^2 + 3x_6x_7 + x_7^2$

$\frac{d^2 f_1}{dx_1^2} = 0 \quad \forall x_1 \Rightarrow f_1$ is convex (and concave).

$\frac{d^2 f_2}{dx_2^2} = 4 \quad \forall x_2 \Rightarrow f_2$ is convex

$\frac{\partial^2 f_{34}}{\partial x_3^2} = 2 > 0 \quad \forall (x_3, x_4) \quad \frac{\partial^2 f_{34}}{\partial x_4^2} = 8 > 0 \quad \forall (x_3, x_4)$

$$\frac{\partial^2 f_{34}}{\partial x_3^2}\frac{\partial^2 f_{34}}{\partial x_4^2} - \left[\frac{\partial^2 f_{34}}{\partial x_3 \partial x_4}\right]^2 = 16 - 3^2 = 7 > 0 \quad \forall (x_3, x_4)$$

$\Rightarrow f_{34}$ is convex

$\frac{\partial^2 f_{56}}{\partial x_5^2} = 2 > 0 \quad \forall (x_5, x_6) \quad \frac{\partial^2 f_{56}}{\partial x_6^2} = 6 > 0 \quad \forall (x_5, x_6)$

$$\frac{\partial^2 f_{56}}{\partial x_5^2}\frac{\partial^2 f_{56}}{\partial x_6^2} - \left[\frac{\partial^2 f_{56}}{\partial x_5 \partial x_6}\right]^2 = 12 - 3^2 = 3 > 0 \quad \forall (x_5, x_6)$$

$\Rightarrow f_{56}$ is convex

$f_{67}(x_6, x_7) = f_{56}(x_7, x_6) \Rightarrow f_{67}$ is convex.

$\Rightarrow f$ is convex

13.2-9. (a) maximize $f(\underline{x}) = x_1 + x_2$

subject to $g(\underline{x}) = x_1^2 + x_2^2 \le 1$

$\underline{x} \ge 0$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial x_2^2} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 0$$

$\Rightarrow f$ is concave (and convex)

$\frac{\partial^2 g}{\partial x_1^2} = 2 > 0 \quad \forall (x_1, x_2) \quad \frac{\partial^2 g}{\partial x_2^2} = 2 > 0 \quad \forall (x_1, x_2)$

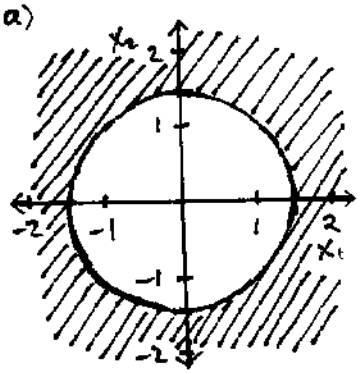
$$\frac{\partial^2 g}{\partial x_1^2}\frac{\partial^2 g}{\partial x_2^2} - \left[\frac{\partial^2 g}{\partial x_1 \partial x_2}\right]^2 = 4 - 0^2 = 4 > 0 \quad \forall (x_1, x_2)$$

$\Rightarrow g_1$ is convex

$\Rightarrow$ the problem is a convex programming problem.

13.2-9. (b)

13.2-10. a)



$x_1^2 + x_2^2 \geq 2$

Clearly, this is <u>not</u> a convex feasible region (Take $(0, \sqrt{2})$ and $(0, -\sqrt{2})$, $(0,0) = \frac{1}{2}(0,\sqrt{2}) + \frac{1}{2}(0,-\sqrt{2})$ is not feasible)

b) Feasible region: $-x_1^2 - x_2^2 \leq -2$

$g_1(x_1) = -x_1^2$ and $g_2(x_2) = -x_2^2$ are both <u>concave</u> functions, not convex, so the feasible region does not need to be convex.

$\left(\frac{\partial^2}{\partial x_1^2} g_1(x_1) = -1 < 0\right)$

To <u>prove</u> that the feasible region is not convex, we need to show there are y and z both feasible, but $\alpha y + (1-\alpha) z$ not feasible $(0<\alpha<1)$, as shown in (a).

13.3-1. Since we are <u>minimizing</u> a <u>concave</u> function (see 13.1-2.) this is a nonconvex programming problem.

13.3-2. $\frac{df}{dx} = -6 + 6x - 6x^2 = 0$

$\Rightarrow x = \frac{-6 \pm \sqrt{36 - 4 \cdot 36}}{12}$ has no real solution

$\frac{d^2 f}{dx^2} = 6 - 12x \begin{cases} > 0, & x < \frac{1}{2} \\ < 0, & x > \frac{1}{2} \end{cases}$

So the slope of f increases from -6 at $x = 0$ to $-3/2$ at $x = \frac{1}{2}$ and decreases for all x thereafter.

Thus, $x^* = 0$ is optimal.

13.3-3. a) Linearly constrained convex program:

$\frac{\partial^2 f}{\partial x_1^2} = -12x_1^2 - 4 < 0 \quad \forall (x_1, x_2)$ $\frac{\partial^2 f}{\partial x_2^2} = -8 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right] = 96x_1^2 + 32 - (-2)^2 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is concave

g_1 and g_2 are linear and, hence, convex.

Geometric Program:

$f(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}} + c_3 x_1^{a_{31}} x_2^{a_{32}} + c_4 x_1^{a_{41}} x_2^{a_{42}}$

with $c_1 = 1 \quad a_{11} = 4 \quad a_{12} = 0$
$c_2 = 2 \quad a_{21} = 2 \quad a_{22} = 0$
$c_3 = 2 \quad a_{31} = 1 \quad a_{32} = 1$
$c_4 = 4 \quad a_{41} = 0 \quad a_{42} = 2$

$g_1(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$

with $c_1 = -2 \quad a_{11} = 1 \quad a_{12} = 0$
$c_2 = -1 \quad a_{21} = 0 \quad a_{22} = 1$

$g_2(\underline{x}) = c_1 x_1^{a_{11}} x_2^{a_{12}} + c_2 x_1^{a_{21}} x_2^{a_{22}}$

with $c_1 = -1 \quad a_{11} = 1 \quad a_{12} = 0$
$c_2 = -2 \quad a_{21} = 0 \quad a_{22} = 1$

Fractional Program:

$f' = f_1/f_2$ where $f_1 = f$ & $f_2 = 1$

b) Let $y_1 = x_1 - 1$ and $y_2 = x_2 - 1$

$\Rightarrow$ Minimize $y_1^4 + 4y_1^3 + 8y_1^2 + 10y_1 + 2y_1y_2 + 4y_2^2 + 10y_2$
subject to $2y_1 + y_2 \geq 7$
$y_1 + 2y_2 \geq 7$
$y_1 \geq 0, \; y_2 \geq 0$

13.3-4. a) Let $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$

$\Rightarrow$ Minimize $2e^{2y_1 - y_2} + e^{-y_1 - 2y_2}$
subject to $4e^{y_1 + y_2} + e^{2y_1 + 2y_2} \leq 12$
$e^{y_1} > 0, \; e^{y_2} > 0$

b) $\frac{\partial^2 f}{\partial y_1^2} = -8e^{2y_1 - y_2} - e^{-y_1 - 2y_2} < 0 \quad \forall (y_1, y_2)$

$\frac{\partial^2 f}{\partial y_2^2} = -2e^{2y_1 - y_2} - 4e^{-y_1 - 2y_2} < 0 \quad \forall (y_1, y_2)$

$\frac{\partial^2 f}{\partial y_1^2}\frac{\partial^2 f}{\partial y_2^2} - \left[\frac{\partial^2 f}{\partial y_1 \partial y_2}\right]^2 = 18e^{2y_1 - 3y_2} > 0 \quad \forall (y_1, y_2)$

so f is concave

13.3-4. b) (cont') for $g(y_1, y_2) = 4e^{y_1+y_2} + e^{2y_1+2y_2} - 12$

$$\frac{\partial^2 g}{\partial y_1^2} = 4e^{y_1+y_2} + 4e^{2y_1+2y_2} \geq 0 \quad \forall (y_1, y_2)$$

$$\frac{\partial^2 g}{\partial y_2^2} = 4e^{y_1+y_2} + 4e^{2y_1+2y_2} \geq 0 \quad \forall (y_1, y_2)$$

$$\frac{\partial^2 g}{\partial y_1^2}\frac{\partial^2 g}{\partial y_2^2} - \left[\frac{\partial^2 g}{\partial y_1 \partial y_2}\right]^2 = 0 \quad \forall (y_1, y_2)$$

so g is convex

$e^{y_1} \geq 0$, $e^{y_2} \geq 0$ $\forall (y_1, y_2)$

Now there is, in fact, no non-negativity constraints on the y's, and so, strictly speaking, according to the definition in the book this isn't quite a convex program. However, in general, the entire theory of convex programming applies as long as the feasible region is convex which is the case here.

13.3-5. (a) Maximize $10y_1 + 20y_2 + 10t$

subject to
$$y_1 + 3y_2 - 50t \leq 0$$
$$3y_1 + 2y_2 - 80t \leq 0$$
$$3y_1 + 4y_2 + 20t = 1$$
$$y_1 \geq 0, \; y_2 \geq 0, \; t \geq 0$$

(b)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	Right side
								1M	
Z	0	1	3.269	0	0	1.385	0	3.962	3.962
X2	1	0	0.654	1	0	0.077	0	0.192	0.192
X5	2	0	3.231	0	0	-1.38	1	0.538	0.538
X3	3	0	0.019	0	1	-0.02	0	0.012	0.012

Variables (x_1, x_2, x_3) from the courseware solution correspond to (y_1, y_2, t) for part (a) variables, respectively. Therefore, the optimal solution has $(y_1, y_2, t) = (0, 0.192, 0.012)$ with $z = 3.962$. The optimal solution to the original problem is $(x_1, x_2) = (0, 16.67)$ with $f(x) = 3.962$.

13.3-6. The KKT conditions may be rewritten:

$$Qx + A^T u - c = y$$
$$-Ax + b = v$$
$$x \geq 0, \; u \geq 0, \; y \geq 0, \; v \geq 0$$
$$x^T(Qx + A^T u - c) + u^T(-Ax + b) = 0$$

and this is the Linear Complementarity Problem with $z = \begin{pmatrix} x \\ u \end{pmatrix}$ $M = \begin{bmatrix} Q & A^T \\ -A & 0 \end{bmatrix}$ $q = \begin{pmatrix} -c \\ b \end{pmatrix}$

and $w = \begin{bmatrix} Qx + A^T u - c \\ -Ax + b \end{bmatrix}$

13.4-1.

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2.4	1.2	0.7296
1	-0.208	0	1.2	0.6	0.6636
2	+0.464	0.6	1.2	0.9	0.745
3	+0.101	0.9	1.2	1.05	0.7487
4	- 0.05	0.9	1.05	0.975	0.7497
5	+0.025	0.975	1.05	1.0125	0.7499
Stop					

13.4-2. a)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	4.8	2.4	8.64
1	+ 1.2	2.4	4.8	3.6	8.64
2	- 1.2	2.4	3.6	3	9
3	+ 0	3	3.6	3.3	8.91
4	- 0.6	3	3.3	3.15	8.9775
5	- 0.3	3	3.15	3.075	8.9944
6	- 0.15	3	3.075	3.0375	8.9986
Stop					

b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		-4	1	-1.5	-1.688
1	- 1.5	-1.5	1	-0.25	-1.121
2	+3.188	-1.5	-0.25	-0.875	-1.984
3	+0.258	-1.5	-0.875	-1.188	-1.964
4	-0.401	-1.188	-0.875	-1.031	-1.999
5	-0.063	-1.031	-0.875	-0.953	-1.998
6	+0.094	-1.031	-0.953	-0.992	-2
Stop					

13.4-3.

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		-1	4	1.5	-16.69
1	- 100	-1	1.5	0.25	0.3047
2	+0.156	0.25	1.5	0.875	0.2482
3	-0.923	0.25	0.875	0.5625	0.3125
4	-0.001	0.25	0.5625	0.4063	0.3124
5	+0.004	0.4063	0.5625	0.4844	0.3125
Stop					

13.4-4.

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2	1	25
1	+ 13	1	2	1.5	20.109
2	-44.81	1	1.5	1.25	26.068
3	-6.748	1	1.25	1.125	26.146
4	+4.844	1.125	1.25	1.1875	26.288
Stop					

13.4-5

(a) $f'(x) = 4x^3 + 2x - 4$

$f'(0) = -4$

$f'(1) = 2$

$f'(2) = 32$

since $f'(x)$ is continuous there must be a point in $0 \le x \le 1$ such that $f'(x) = 0$, and so since f is a concave function (this is a convex program) the optimal solution must be in the interval $0 \le x \le 1$.

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	2	1	-2
1	+ 2	0	1	0.5	-1.688
2	- 2.5	0.5	1	0.75	-2.121
3	-0.813	0.75	1	0.875	-2.148
4	+ 0.43	0.75	0.875	0.8125	-2.154
5	-0.229	0.8125	0.875	0.8438	-2.156
6	+ 0.09	0.8125	0.8438	0.8281	-2.156
Stop					

13.4-6

a) Claim: $(\bar{x}_{n+1} - \underline{x}_{n+1}) = \frac{1}{2}(\bar{x}_n - \underline{x}_n)$. To see this consider the two cases:

Case 1: $\bar{x}_{n+1} = \bar{x}_n$, $\underline{x}_{n+1} = x_n'$, then $\bar{x}_{n+1} - \underline{x}_{n+1} = \bar{x}_n - x_n' = \bar{x}_n - \frac{1}{2}(\bar{x}_n + \underline{x}_n)$
$= \frac{1}{2}(\bar{x}_n - \underline{x}_n)$

Case 2: $\bar{x}_{n+1} = x_n'$, $\underline{x}_{n+1} = \underline{x}_n$, then $\bar{x}_{n+1} - \underline{x}_{n+1} = x_n' - \underline{x}_n = \frac{1}{2}(\bar{x}_n + \underline{x}_n) - \underline{x}_n$
$= \frac{1}{2}(\bar{x}_n - \underline{x}_n)$

Therefore, we have $(\bar{x}_{n+1} - \underline{x}_{n+1}) = \frac{1}{2}(\bar{x}_n - \underline{x}_n) = \dots = \frac{1}{2^{n+1}}(\bar{x}_0 - \underline{x}_0)$

so $\lim_{n\to\infty} (\bar{x}_{n+1} - \underline{x}_{n+1}) = \lim_{n\to\infty} 2^{-n}(\bar{x}_0 - \underline{x}_0) = 0$

Suppose the points generated by this procedure do not converge, then we know there is $\varepsilon > 0$ such that, no matter what N we choose, there will be $n \ge N$ and $m \ge N$ so that $|x_n' - x_m'| > \varepsilon$. But if we choose N so that $2^{-N}(\bar{x}_0 - \underline{x}_0) < \varepsilon$ then for all $n \ge N$ $x_n' \in [\underline{x}_N, \bar{x}_N]$ and so for $n \ge N$ and $m \ge N$ and $|x_n' - x_m'| > \varepsilon$ we have $\varepsilon < |x_n' - x_m'| \le |\bar{x}_N - \underline{x}_N| = 2^{-N}(\bar{x}_0 - \underline{x}_0) < \varepsilon$, a contradiction. Thus, the sequence of trial solutions must converge.

(b) Let $\bar{x}$ be the limiting solution. Then we know $f'(x) \ge 0$ for $x < \bar{x}$ and $f'(x) \le 0$ for $x > \bar{x}$. Suppose $\bar{x}$ is not the global maximum, but that $\hat{x}$ is. Then $f(\hat{x}) > f(\bar{x})$.

Case 1: $\hat{x} > \bar{x}$. By the Mean Value Theorem there is a z with $\hat{x} > z > x$ such that $f(\hat{x}) - f(x) = (\hat{x} - x) f'(z)$. The right hand side is non-positive, since $z > \bar{x}$, $f'(z) \le 0$ and $\hat{x} > x \Rightarrow (\hat{x} - x) > 0$, so $f(\hat{x}) - f(x) \le 0$, a contradiction.

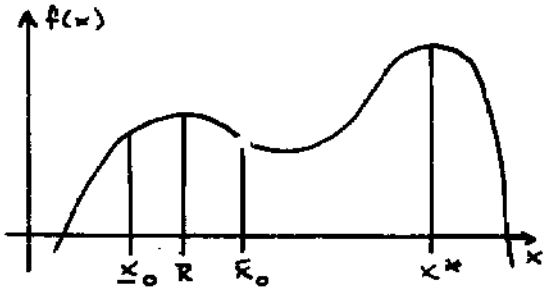
Case 2: $\hat{x} < \bar{x}$. Using the Mean Value Theorem, we have z with $\hat{x} < z < \bar{x}$, $f(\bar{x}) - f(\hat{x}) = (\bar{x} - \hat{x}) f'(z) \ge 0$ which implies $f(\bar{x}) \ge f(\hat{x})$, a contradiction.

So $\bar{x}$ must be the global maximum.

13.4-6

(c) The argument follows as in part (b) by observing that the z of the Mean Value Theorem (which is between $\bar{x}$ and $\hat{x}$) is in the part of the domain where f is concave.

(d)



The procedure would converge to $\bar{x}$, not to x^* which is the global maximum.

(e) Suppose $f'(x) < 0$, but that a global maximum occurs at $\hat{x}$. Let $x < \hat{x}$. By the Mean Value Theorem there exists a z with $\hat{x} > z > x$ and $f(\hat{x}) - f(x) = (\hat{x} - x) f'(z)$ or $f(x) = f(\hat{x}) + (x - \hat{x}) f'(z) > f(x')$, a contradiction. So a global maximumum cannot occur.

Suppose $f'(x) > 0$, but that a global maximum occurs at $\hat{x}$. Let $x > \hat{x}$. By the Mean Value Theorem, there exists a z with $x > z > \hat{x}$ and $f(\hat{x}) - f(x) = (\hat{x} - x) f'(z)$ or $f(x) = f(\hat{x}) + (x - \hat{x}) f'(z) > f(x')$, a contradiction. So a global maximum cannot occur.

(f) Suppose $f(x)$ is concave and $\lim_{x \to -\infty} f'(x) < 0$, but that an $\underline{x}_0$ exists. So $f'(\underline{x}_0) \geq 0$. But $f'(x)$ is monotone decreasing, so for $x < \underline{x}_0$, $f'(x) \geq 0$ and $\lim_{x \to -\infty} f'(x) \geq 0$, a contradiction. So no $\underline{x}_0$ exists.

Suppose $f(x)$ is concave and $\lim_{x \to \infty} f'(x) > 0$, but that an $\bar{x}_0$ exists. So $f'(\bar{x}_0) \leq 0$. But $f'(x)$ is monotone decreasing, so for $x > \bar{x}_0$, $f'(x) \leq 0$, and $\lim_{x \to \infty} f'(x) \leq 0$, a contradiction. So no $\bar{x}_0$ exists.

In either case, we know there is no global maximum from part (a).

13.4-7. $f(x) = f_1(x_1) + f_2(x_2)$

$f_1(x_1) = 32x_1 - x_1^4$

$f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$

$\frac{df_1}{dx_1} = 32 - 4x_1^3 \equiv 0 \Rightarrow x_1 = 2$

Using the automatic one-dimensional search procedure ($\varepsilon = 0.001$) with initial bounds 0 and 4, and

$f(x) = 48 + 50x_2 - 10x_2^2 + x_2^3 - x_2^4$

we get: $x_2 = 1.8076$, $f(x) = 100.936$

$3x_1 + x_2 = 7.8076 < 11$

$2x_1 + 5x_2 = 13.038 < 16$

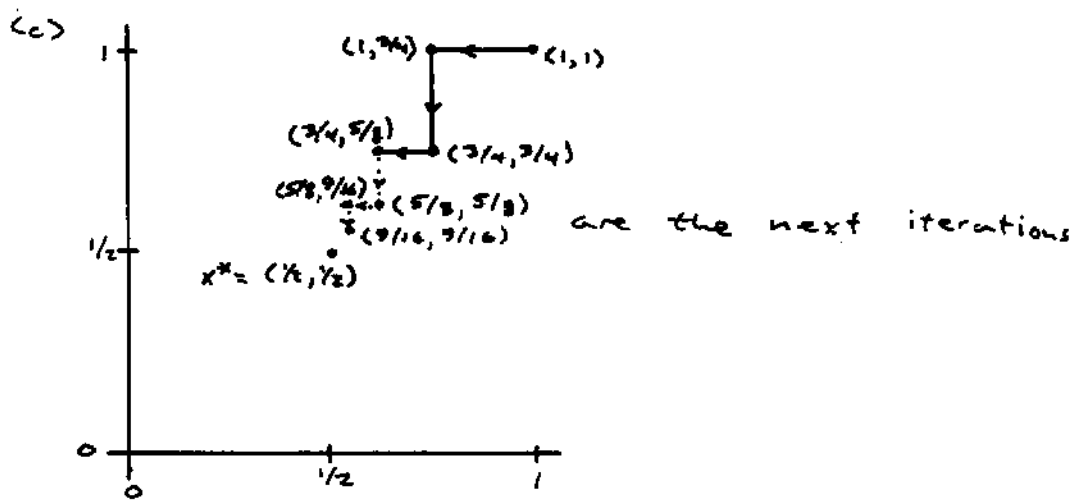
Since the optimum for the unconstrained problem is in the interior of the feasible region for the constrained problem, it is also optimal for the original constrained problem.

13.5-1. (a)

It.	X'	grad f(X')	x' + t[grad f(X')]	t*	X'+t[grad f]
1	(1, 1)	(0, -1)	(1+ 0t, 1- 1t)	0.25	(1, 0.75)
2	(1, 0.75)	(-0.5, 0)	(1- 0.5t, 0.75+ 0t)	0.5	(0.75, 0.75)
3	(0.75, 0.75)	(0, -0.5)	(0.75+ 0t, 0.75- 0.5t)	0.25	(0.75,0.625)
4	(0.75,0.625)	(-0.25, 0)			

(b) $-2x_1 + 2x_2 = 0$
$-2x_1 + 4x_2 = 1$
$\Rightarrow$ $x_2 = 1/2$, $x_1 = 1/2$ so $(1/2, 1/2)$ is optimal.

(c)



are the next iterations

(d) The software gives: Solution:

$(X_1, X_2) = (0.508, 0.504)$

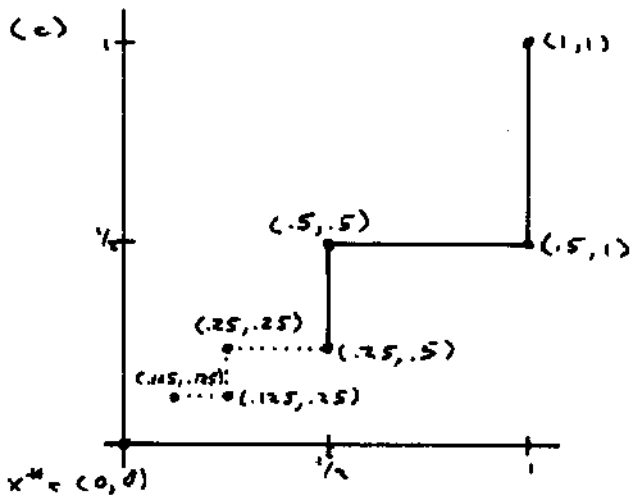
grad $f(X_1, X_2) = (-8e\text{-}3,\ 6e\text{-}8)$

13.5-2. (a)

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(1, 1)	(-2, 0)	(1- 2t, 1+ 0t)	0.25	(0.5, 1)
2	(0.5, 1)	(0, -1)	(0.5+ 0t, 1- 1t)	0.5	(0.5, 0.5)
3	(0.5, 0.5)	(-1, 0)	(0.5- 1t, 0.5+ 0t)	0.25	(0.25, 0.5)
4	(0.25, 0.5)	(0, -0.5)			

(b) $-4x_1 - 2x_2 = 0$
$2x_1 - 2x_2 = 0$ $\Rightarrow$ (0, 0) is optimal.

(c)



(d) with the automatic routine:

Solution:

(X_1, X_2) = (0.004, 0.008)

grad $f(X_1, X_2)$ = (1e-8, -8e-3)

13.5-3.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(1, 1)	(0, -2)	(1+ 0t, 1- 2t)	0.167	(1,0.667)
2	(1,0.667)	(-1.33, 0)	(1-1.33t,0.667+ 0t)	0.25	(0.667,0.667)
3	(0.667,0.667)	(0,-1.33)			

The automatic routine ($\varepsilon = 0.01$) gives:

Solution:

(X_1, X_2) = (0.005, 0.003)

grad $f(X_1, X_2)$ = (-7e-3, 3e-8)

$\nabla f = (4x_2 - 4x_1, 4x_1 - 6x_2)$

$\nabla f = 0 \Rightarrow (x_1, x_2) = (0,0)$ is the optimal solution.

13.5-4.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(8, -12)	(0+ 8t, 0- 12t)	0.191	(1.529,-2.29)
2	(1.529,-2.29)	(0.361,0.219)	(1.529+0.36t,-2.29+0.22t)	1.31	(2.002, -2)
3	(2.002, -2)	(-0,0.003)			

Automatic routine ($\varepsilon = 0.01$):

Solution:

$(X_1, X_2) = (1.997, -2)$

grad $f(X_1, X_2) = (0.002, 0.001)$

$\nabla f = (-2x_1 + 2x_2 + 8 \;,\; 2x_1 - 4x_2 - 12)$

$\nabla f = 0 \Rightarrow (x_1, x_2) = (2, -2)$ is the optimal solution

13.5-5.

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(6, -2)	(0+ 6t, 0- 2t)	0.2	(1.2, -0.4)
2	(1.2, -0.4)	(0.4, 1.2)	(1.2+ 0.4t, -0.4+ 1.2t)	1	(1.6, 0.8)
3	(1.6, 0.8)	(1.2, -0.4)			

Automatic routine ($\varepsilon = 0.01$)

Solution:

$(X_1, X_2) = (1.994, 0.989)$

grad $f(X_1, X_2) = (0.003, 0.01)$

$\nabla f = (-4x_1 + 2x_2 + 6 \;,\; 2x_1 - 2x_2 - 2)$

$\nabla f = 0 \Rightarrow (x_1, x_2) = (2, 1)$

13.5-6. $\nabla f(x_1, x_2) = (4 - 2x_1 - 4x_1^3 - 2x_2 \;,\; 2 - 2x_1 - 2x_2)$

Iter.	x_n	$\nabla f(x_n)$	$f(x_n + t\nabla f(x_n))$	Iter.	t'	$f(x_n)$
1	(0,0)	(4,2)	$20t - 36t^2 - 256t^4$	1	.5	-144
				2	.25	-14
					$t^* = .125$	
(2)	(1/2, 1/4)					

$\Rightarrow x + t^* \nabla f(x) = (0.500, 0.250)$ is our estimate.

13.5-7

a) We can rewrite f as $f = f_1(x_1, x_2) + f_2(x_2, x_3)$

with $f_1 = 3x_1x_2 - x_1^2 - 3x_2^2$

$f_2 = 3x_2x_3 - x_3^2 - 3x_2^2$

For any given x_2 (including the optimal one) we realize that by symmetry $x_1 = x_3$ at the maximizing point for f with x_2 at the given value. Therefore, to maximize f we can maximize f_1 (or f_2) and obtain (x_1, x_2). From the solution to maximize f_1 we can set $x_3 = x_1$ and $f = 2f_1$ to find the solution to our original problem. We can solve maximize f_1 with the courseware.

b) Using $f(x) = 3x_1x_2 - x_1^2 - 3x_2^2$:

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t*[grad f]
1	(1, -1)	(1, -3)	(1+ 1t, -1- 3t)	0.135	(1.135,0.595)
2	(1.135,0.595)	(-0.49,-0.16)	(1.14-0.49t,0.59-0.16t)	1.616	(0.343,0.336)
3	(0.343,0.336)	(0.323,-0.99)	(0.34+0.32t,0.34-0.99t)	0.135	(0.387,0.202)
4	(0.387,0.202)	(-0.17,-0.05)	(0.39-0.17t, 0.2-0.05t)	1.427	(0.144,0.131)
5	(0.144,0.131)	(0.103,-0.35)	(0.14+ 0.1t,0.13-0.35t)	0.139	(0.158,0.083)
6	(0.158,0.083)	(-0.07,-0.02)	(0.16-0.07t,0.08-0.02t)	1.361	(0.063,0.056)
7	(0.063,0.056)	(0.042,-0.15)	(0.06+0.04t,0.06-0.15t)	0.135	(0.069,0.036)

Final solution: $(X_1, X_2) = (0.069, 0.036)$

The estimated solution to the original problem will be

$(x_1, x_2, x_3) = (0.069, 0.036, 0.069)$

c) Automatic routine ($\varepsilon = 0.005$) gives:

Solution:

$(X_1, X_2) = (0.004, 0.002)$

grad $f(X_1, X_2) = (-2e-3, -6e-4)$

13.5-8

(a)

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(0, 3)	(0+ 0t, 0+ 3t)	0.5	(0, 1.5)
2	(0, 1.5)	(1.5, 0)	(0+ 1.5t, 1.5+ 0t)	0.5	(0.75, 1.5)
3	(0.75, 1.5)	(0, 0.75)			

Automatic routine ($\varepsilon = 0.01$) :

Solution:

$(X_1, X_2) = (0.996, 1.998)$

grad $f(X_1, X_2) = (0.005, -2e-8)$

(b)

It.	X'	grad f(X')	X' + t[grad f(X')]	t*	X'+t[grad f]
1	(0, 0)	(4, -4)	(0- 4t, 0- 4t)	0.193	(0.771,-0.77)
2	(0.771,-0.77)	(0.002, -0)			

Automatic routine ($\varepsilon = 0.01$):

(Note: min $f(x) \Leftrightarrow$ max $-f(x)$)

Solution:

$(X_1, X_2) = (0.771, -0.77)$

grad $f(X_1, X_2) = (8e-6, -8e-6)$

13.6-1. Converting min to max, the KKT conditions are:

1) $-4x^3-2x+4-u \le 0$
2) $x(-4x^3-2x+4-u)=0$
3) $x-2 \le 0$
4) $u(x-2)=0$
5) $x \ge 0$
6) $u \ge 0$ → (Also can get this from (1) and (6))

From 13.4-5. (a), $0 \le x \le 1$, so $x \neq 2$ and (4) $\Rightarrow u=0$.

From (1), we see that $x \neq 0$ ($4 \neq 0$), so from (2), we have $-4x^3-2x+4=0$ (or $2x^3+x-2=0$)

Solving for the root (real) of this cubic, we get

$$x = \sqrt[3]{\frac{1}{2}+\sqrt{\frac{53}{216}}} + \sqrt[3]{\frac{1}{2}-\sqrt{\frac{53}{216}}} = .83512$$

13.6-2. The KKT conditions are.

1 (j=1). $1-2ux_1 \le 0$
2 (j=1). $x_1(1-2ux_1)=0$
1 (j=2). $1-2ux_2 \le 0$
2 (j=2). $x_2(1-2ux_2)=0$
3. $x_1^2+x_2^2-1 \le 0$
4. $u(x_1^2+x_2^2-1)=0$
5. $x_1 \ge 0, x_2 \ge 0$
6. $u \ge 0$

For $x=\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, 2(j=1) $\Rightarrow 1-2u\frac{1}{\sqrt{2}}=0 \Rightarrow u=\frac{1}{\sqrt{2}}$

This satisfies all the KKT conditions, so $x=\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $u=\frac{1}{\sqrt{2}}$ is optimal.

13.6-3. KKT conditions:

1a) $-4x_1^3-4x_1-2x_2+2v_1+v_2 \le 0$
2a) $x_1(-4x_1^3-4x_1-2x_2+2v_1+v_2)=0$
1b) $-2x_1-8x_2+v_1+2v_2 \le 0$
2b) $x_2(-2x_1-8x_2+v_1+2v_2)=0$
3a) $2x_1+x_2 \ge 10$
3b) $x_1+2x_2 \ge 10$
4a) $v_1(-2x_1-x_2+10)=0$
4b) $v_2(-x_1-2x_2+10)=0$
5) $x_1 \ge 0, x_2 \ge 0$
6) $v_1 \ge 0, v_2 \ge 0$

for $(x_1, x_2)=(0,10)$ 2b) $\Rightarrow v_1+2v_2=80$
and 4b) $\Rightarrow v_2=0$ so $v_1=80$
but then 1a) does not hold
so $(x_1, x_2)=(0,10)$ is not optimal

13.6-4. (a) The KKT conditions:

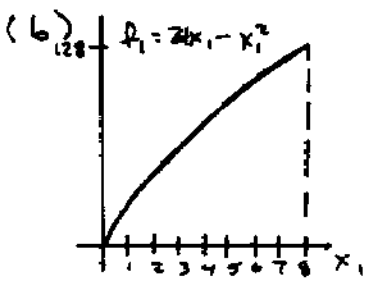
1a) $24 - 2x_1 - u_1 \le 0$
2a) $x_1(24 - 2x_1 - u_1) = 0$
1b) $10 - 2x_2 - u_2 \le 0$
2b) $x_2(10 - 2x_2 - u_2) = 0$
3) $x_1 \le 8, \; x_2 \le 7$
4a) $u_1(x_1 - 8) = 0$
4b) $u_2(x_2 - 7) = 0$
5) $x_1 \ge 0, \; x_2 \ge 0$
6) $u_1 \ge 0, \; u_2 \ge 0$

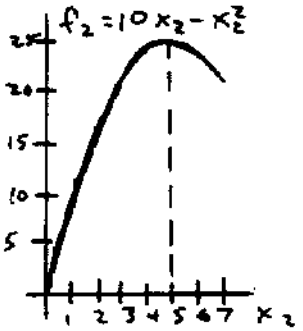
Take $x_1 = 8$ 2a) $\Rightarrow u_1 = 8$
then 1a), 3), 4a), 5) and 6) are satisfied

Take $u_2 = 0$ 2b) $\Rightarrow x_2 = 5$
then 1b), 3), 4b), 5) and 6) are satisfied

So $(x_1, x_2) = (8, 5)$ is optimal since this is a convex program.

(b)





$\frac{\partial f}{\partial x_1} = 24 - 2x_1 > 0 \quad \forall \, 0 \le x_1 \le 8 \Rightarrow x_1 = 8$ is the maximum over the feasible region.

$\frac{\partial f}{\partial x_2} = 10 - 2x_2 = 0$ at $x_2 = 5$ and $\frac{\partial^2 f}{\partial x_2} = -2 \le 0$

so $x_2 = 5$ is a global maximum.

13.6-5. a) $\frac{\partial^2 f}{\partial x_1^2} = -\frac{1}{(1 + x_1 + x_2)^2} \le 0 \quad \forall \, x_1, x_2 \ge 0$

$\frac{\partial^2 f}{\partial x_2^2} = -\frac{1}{(1 + x_1 + x_2)^2} \le 0 \quad \forall \, x_1, x_2 \ge 0$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial f}{\partial x_1 \partial x_2}\right]^2 = 0 \quad \forall \, x_1, x_2 \ge 0$

So f is concave for $x_1, x_2 \ge 0$

$g = x_1 + 2x_1 - 5$ is linear and, hence, convex

Thus, this is a convex program.

13.6-5, b) The KKT conditions are:

1a) $\frac{1}{1+X_1+X_2} - u \le 0$

2a) $X_1\left(\frac{1}{1+X_1+X_2} - u\right) = 0$

1b) $\frac{1}{1+X_1+X_2} - 2u \le 0$

2b) $X_2\left(\frac{1}{1+X_1+X_2} - 2u\right) = 0$

3) $X_1 + 2X_2 - 5 \le 0$

4) $u(X_1 + 2X_2 - 5) = 0$

5) $X_1 \ge 0, X_2 \ge 0$

6) $u \ge 0$

Trying $u \neq 0$, (4) $\Rightarrow X_1 + 2X_2 = 5$

Trying $X_2 = 0 \Rightarrow X_1 = 5$, then 2a) $\Rightarrow u = \frac{1}{1+X_1+X_2} = \frac{1}{5}$

Since $(X_1, X_2) = (5, 0)$, $u = \frac{1}{5}$ satisfies all KKT conditions, $(X_1, X_2) = (5, 0)$ is optimal.

c) Since $\ln(1 + X_1 + X_2)$ is a monotone strictly increasing function of $(1 + X_1 + X_2)$, the problem is equivalent to the linear program:

$$\begin{aligned} \text{maximize } & 1 + X_1 + X_2 \\ \text{subject to } & X_1 + 2X_2 \le 5 \\ & X_1 \ge 0, \ X_2 \ge 0 \end{aligned}$$

and so we expect an extreme point solution. The extreme points of the feasible region are $(5,0)$, $(0, \frac{5}{2})$ and $(0,0)$, and it is clear that, of the three, $(5,0)$ is optimal.

13.6-6. (a) $\frac{\partial^2 f}{\partial x_1^2} = -\frac{1}{(x_1+1)^2} \le 0 \quad \forall (x_1, x_2) : x_1 \neq -1$

$\frac{\partial^2 f}{\partial x_2^2} = -2 \le 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial f}{\partial x_1 \partial x_2}\right]^2 = \frac{2}{(x_1+1)^2} \ge 0 \quad \forall (x_1, x_2) : x_1 \neq -1$

$\Rightarrow$ f is concave

$g = x_1 + 2x_2 - 3$ is linear, and, hence, convex.

so this is a convex program.

SCI

13.6-6. (b) The KKT conditions:

1a) $\frac{1}{x_1+1} - v \le 0$

2a) $x_1\left[\frac{1}{x_1+1} - v\right] = 0$

1b) $-2x_2 - 2v \le 0$

2b) $x_2[-2x_2 - 2v] \le 0$

3) $x_1 + 2x_2 \le 3$

4) $v(x_1 + 2x_2 - 3) = 0$

5) $x_1 \ge 0;\ x_2 \ge 0$

6) $v \ge 0$

Trying $v \neq 0$ 4) $\Rightarrow$ $x_1 + 2x_2 = 3$

Trying $x_2 = 0 \Rightarrow x_1 = 3$ 2a) $\Rightarrow v = \frac{1}{4}$

and $(3, 0, \frac{1}{4})$ satisfies (a), 1b), 2b), 3), 5) & 6)

so $(x_1, x_2) = (3, 0)$ is optimal.

(c) Since $-x_2^2$ is monotonically strictly decreasing for $x_2 \ge 0$ and $\ln(x_1+1)$ is monotonically strictly increasing, it is intuitively clear that one would like to increase x_1 as much as possible and decrease x_2 toward 0 as much as possible in an optimal solution. The feasible region is "nice" in this respect since if we take F to be the feasible region:

$$\max_{x_1}[\min_{x_2} F] = \min_{x_2}[\max_{x_1} F] = \{(3,0)\}.$$

13.6-7. (a) The KKT conditions:

1a) $10 - 4x_1 - 3x_1^2 - v \le 0$

2a) $x_1(10 - 4x_1 - 3x_1^2 - v) = 0$

1b) $8 - 2x_2 - v \le 0$

2b) $x_2(8 - 2x_2 - v) = 0$

3) $x_1 + x_2 \le 2$

4) $v(x_1 + x_2 - 2) = 0$

5) $x_1 \ge 0,\ x_2 \ge 0$

6) $v \ge 0$

so if $(x_1, x_2) = (1, 1)$

2a) $\Rightarrow v = 3$

2b) $\Rightarrow v = 6$

thus, the KKT conditions cannot be met at $(x_1, x_2) = (1, 1)$ so $(1, 1)$ is not optimal.

(b) $(x_1, x_2) = (\sqrt{3} - 1, 3 - \sqrt{3})$ and $v = 2(1 + \sqrt{3})$ satisfies all the KKT conditions, and so since we have a convex program $(\sqrt{3} - 1, 3 - \sqrt{3})$ is optimal

13.6-8. The KKT conditions:

1a) $36 + 18x_1 - 18x_1^2 - u \le 0$

2a) $x_1(36 + 18x_1 - 18x_1^2 - u) = 0$

1b) $36 - 9x_2^2 - u \le 0$

2b) $x_2(36 - 9x_2^2 - u) = 0$

3) $x_1 + x_2 \le 3$

4) $u(x_1 + x_2 - 3) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

So for $(x_1, x_2) = (1, 2)$

2a) $\Rightarrow \quad u = 36$

2b) $\Rightarrow \quad u = 0$

Thus, the KKT conditions cannot be met at $(x_1, x_2) = (1, 2)$, and so $(1, 2)$ is not optimal.

13.6-9. (a) The KKT conditions:

1a) $\frac{1}{x_2 + 1} - u \le 0$

2a) $x_1\left[\frac{1}{x_2+1} - u\right] = 0$

1b) $-\frac{x_1}{(x_2+1)^2} + u \le 0$

2b) $x_2\left[-\frac{x_1}{(x_2+1)^2} + u\right] = 0$

3) $x_1 - x_2 \le 2$

4) $u[x_1 - x_2 - 2] = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

for $(x_1, x_2) = (4, 2)$

2a) $\Rightarrow u = 1/3$

2b) $\Rightarrow u = 4/9$

So $(x_1, x_2) = (4, 2)$ does not satisfy the KKT condition, and, hence, is not optimal.

(b) Try $x_2 = 0, u \ne 0$ 2a) $\Rightarrow u = 1$

4) $\Rightarrow x_1 = 2$

and $(2, 0, 1)$ satisfies 1a), 1b), 2a), 3), 5) & 6)

So $(x_1, x_2) = (2, 0)$ satisfies the KKT conditions

(a) $\frac{\partial^2 f}{\partial x_1^2} = 0 \qquad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = \frac{2x_1}{(x_2+1)^3} \ge 0 \qquad \forall x_1 \ge 0, x_2 \ge 0$

Thus, f cannot be concave, and so this is not a convex program.

13.6-9. (d) $f(x)$ is a monotone strictly increasing function of x_2 and is monotone strictly decreasing in x_2 for $x_2 > -1$, and, thus, any optimal solution in a bounded feasible region with $x_2 > -1$ will have x_1 increased as high as possible and x_2 decreased toward -1 as much as possible. The feasible region here allows x_1 to be increased without bound. However, then x_2 can only be decreased to the line $x_1 - x_2 = 2$ or $x_1 = 2 + x_2$:

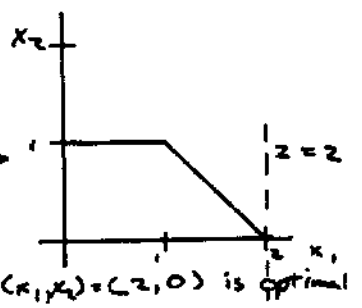
$$f(x_2+2, x_2) = \frac{x_2+3}{x_2+1} \longrightarrow 1 \quad \text{as} \quad x_2 \to \infty$$

$$= 2 \quad \text{at} \quad x_2 = 0$$

Conversely, if we then decrease x_2 to 0 we can increase x_1 to $x_1 = 2$. Thus, the feasible region is "nice" and the optimal solution is at $(x_1, x_2) = (2, 0)$.

(e) Maximize x_1
subject to $x_1 - x_2 - 2\epsilon \le 0$
$x_2 + \epsilon = 1$
$x_1 \ge 0, x_2 \ge 0, \epsilon \ge 0$

$\Leftrightarrow$ Maximize x_1
subject to $x_1 + x_2 \le 2$
$x_2 \le 1$
$x_1 \ge 0, x_2 \ge 0$



13.6-10. (a) The KKT conditions:

1a) $1 - u \le 0$
2a) $x_1(1 - u) = 0$
1b) $2 - 3x_2^2 - u \le 0$
2b) $x_2(2 - 3x_2^2 - u) = 0$
3) $x_1 + x_2 \le 1$
4) $u(x_1 + x_2 - 1) = 0$
5) $x_1 \ge 0, x_2 \ge 0$
6) $u \ge 0$

It is easy to check that $(x_1, x_2, u) = \left(1 - \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, 1\right)$ satisfies the KKT conditions. Thus, $(x_1, x_2) = \left(1 - \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ is optimal since this is a convex program.

13.6-10. (a) The KKT conditions:

1a) $20 - v_1 2x_1 - v_2 \le 0$
2a) $x_1(20 - v_1 2x_1 - v_2) = 0$
1b) $10 - v_1 2x_2 - 2v_2 \le 0$
2b) $x_2(10 - v_1 2x_2 - 2v_2) = 0$
3a) $x_1^2 + x_2^2 \le 1$
3b) $x_1 + 2x_2 \le 2$
4a) $v_1(x_1^2 + x_2^2 - 1) = 0$
4b) $v_2(x_1 + 2x_2 - 2) = 0$
5) $x_1 \ge 0, x_2 \ge 0$
6) $v_1 \ge 0, v_2 \ge 0$

It is easy to check that $(x_1, x_2, v_1, v_2) = \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 5\sqrt{5}, 0\right)$ satisfies the KKT conditions, and so since we have a convex program again $(x_1, x_2) = \left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ is optimal.

13.6-11. a) The KKT conditions:

1a) $4x_1 - 2u \le 0$
2a) $x_1(4x_1 - 2u) = 0$
1b) $2 - u \le 0$
2b) $x_2(2 - u) = 0$
1c) $4 - 2x_3 - u \le 0$
2c) $x_3(4 - 2x_3 - u) = 0$
3) $2x_1 + x_2 + x_3 - 4 \le 0$
4) $u(2x_1 + x_2 + x_3 - 4) = 0$
5) $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$
6) $u \ge 0$

$x_1 = 1$ so (2a) $\Rightarrow u = 2$
Since $(x_1, x_2, x_3) = (1, 1, 1)$ and $u = 2$ satisfies all the KKT conditions, this can be optimal.

b) No, the objective function is not convex.

13.6-12.

Minimize $f(\underline{x})$ subject to $g_i(\underline{x}) \ge b_i$, $\underline{x} \ge 0$ $\Longleftrightarrow$ Maximize $-f(\underline{x})$ subject to $-g_i(\underline{x}) \le -b_i$, $\underline{x} \ge 0$

has KKT conditions:

$$1) \sum_{i=1}^{m} v_i \frac{\partial g_i}{\partial x_j} - \frac{\partial f}{\partial x_j} \le 0$$
$$2) x_j^* \left(\sum_{i=1}^{m} v_i \frac{\partial g_i}{\partial x_j} - \frac{\partial f}{\partial x_j} \right) = 0$$

at $x_j = x_j^*$ for $j = 1, \ldots, n$

3) $g_i(x^*) \ge b_i$
4) $v_i(b_i - g_i(x^*)) = 0$

for $i = 1, \ldots, m$

5) $x_j^* \ge 0 \quad j = 1, \ldots, n$
6) $v_i \ge 0 \quad i = 1, \ldots, m$

13.6-13

(a) An equivalent nonlinear programming problem is:

$$\text{MAXIMIZE} \quad Z = -2x_1^2 - x_2^2$$

$$\text{subject to:} \quad x_1 + x_2 \le 10$$

$$-x_1 - x_2 \le -10 \qquad x_1 \ge 0 \quad x_2 \ge 0.$$

This nonlinear program fits the following types:

- Linearly Constrained Optimization Problem, because all constraints are linear.
- Quadratic Programming Problem, because all constraints are linear and the objective only involves the squares of variables.
- Convex Programming Problem, because the objective is concave as shown below and the constraints are linear and, therefore, convex

  Using the test in the appendix $\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = (-4)(-2) - 0$

  $= 8 \ge 0$

  so f is concave
- Geometric Programming Problem, because the first constraint can be written as $g_1(x_1, x_2) = c_1 P_1(x_1, x_2) + c_2 P_2(x_1, x_2)$ with $c_1 = 1 = c_2$ $P_1 = x_1$ and $P_2 = x_2$. Similarly for the second constraint and the objective function.
- Fractional Programming Problem, because $f(x_1, x_2) = -2x_1^2 - x_2^2 = \frac{f_1(x)}{f_2(x)}$ with $f_1(x) = -2x_1^2 - x_2^2$ and $f_2(x) = 1$.

b)

The KKT conditions are:

1(a) $-4x_1 - u_1 + u_2 \le 0$

(b) $-2x_2 - u_1 + u_2 \le 0$

2(a) $x_1(-4x_1 - u_1 + u_2) = 0$

(b) $x_2(-2x_2 - u_1 + u_2) = 0$

3(a) $x_1 + x_2 - 10 \le 0$

(b) $-x_1 - x_2 + 10 \le 0$

4(a) $u_1(x_1 + x_2 - 10) = 0$

(b) $u_2(-x_1 - x_2 + 10) = 0$

5 $x_1 \ge 0$, $x_2 \ge 0$

6 $u_1 \ge 0$, $u_2 \ge 0$

c) From 3(a)+(b), $x_1 + x_2 = 10$, so 4(a)+(b) are automatically satisfied. Let us try $x_1, x_2 \ne 0$. Then 2(a)+(b) give

$$-4x_1 - u_1 + u_2 = 0 = -2x_2 - u_1 + u_2$$

$$\Rightarrow x_2 = 2x_1$$

$$\Rightarrow x_1 + 2x_1 = 10 \Rightarrow x_1 = \frac{10}{3},\ x_2 = \frac{20}{3}$$

Now 2(a) gives $-u_1 + u_2 = \frac{40}{3}$

$u_1 = 0$, $u_2 = \frac{40}{3}$ satisfies this

(actually $u_1 = c \ge 0$, $u_2 = \frac{40}{3} + c$ works)

$(x_1, x_2) = \left(\frac{10}{3}, \frac{20}{3}\right)$, $(u_1, u_2) = \left(0, \frac{40}{3}\right)$

satisfies all KKT conditions, so

$(x_1, x_2) = \left(\frac{10}{3}, \frac{20}{3}\right)$ is optimal.

13.6-14. (a) An equivalent nonlinear programming problem is:

Maximize $f(y) = -(y_1+1)^3 - 4(y_2+1)^2 - 16(y_3+1)$

subject to: $y_1 + y_2 + y_3 \le 2$

$-y_1 - y_2 - y_3 \le -2$

$y_1 \ge 0 \quad y_2 \ge 0 \quad y_3 \ge 0$

(b) The KKT Conditions are:

1 (a) $-3(y_1+1)^2 - u_1 + u_2 \le 0$

(b) $-8(y_2+1) - u_1 + u_2 \le 0$

(c) $-16 - u_1 + u_2 \le 0$

2 (a) $y_1(-3(y_1+1)^2 - u_1 + u_2) = 0$

(b) $y_2(-8(y_2+1)^2 - u_1 + u_2) = 0$

(c) $y_3(-16 - u_1 + u_2) = 0$

3 (a) $y_1 + y_2 + y_3 - 2 \le 0$

(b) $-y_1 - y_2 - y_3 + 2 \le 0$

4 (a) $u_1(y_1 + y_2 + y_3 - 2) = 0$

(b) $u_2(-y_1 - y_2 - y_3 + 2) = 0$

5 $y_1 \ge 0 \quad y_2 \ge 0 \quad y_3 \ge 0$

6 $u_1 \ge 0 \quad u_2 \ge 0.$

(c) For $x = (2,1,2)$ we have $y = (1,0,1)$

2 (a) implies $-3(2^2) - u_1 + u_2 = 0$ or $-u_1 + u_2 = 12$

2 (c) implies $-16 - u_1 + u_2 = 0$ or $-u_1 + u_2 = 16$

The KKT Conditions are not satisfied by $x = (2,1,2)$

13.6-15. (a) The KKT conditions:

1a) $6 - 2x_1 - v \le 0$

2a) $x_1(6 - 2x_1 - v) = 0$

1b) $3 - 3x_2^2 - v \le 0$

2b) $x_2(3 - 3x_2^2 - v) = 0$

3) $x_1 + x_2 \le 1$

4) $v(x_1 + x_2 - 1) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $v \ge 0.$

(b) for $(x_1, x_2) = (1/2, 1/2)$

2a) $\Rightarrow v = 5$

2b) $\Rightarrow v = 9/4$

so $(x_1, x_2) = (1/2, 1/2)$ is not optimal.

(c) It is easy to check that $(x_1, x_2, v) = (1, 0, 4)$ satisfies the KKT condition, and once again we have a convex program so $(x_1, x_2) = (1, 0)$ is optimal.

13.6-16. (a) The KKT conditions:

1a) $8 - 2x_1 - u \le 0$
2a) $x_1(8 - 2x_1 - u) = 0$
1b) $2 - 3u \le 0$
2b) $x_2(2 - 3u) = 0$
1c) $1 - 2u \le 0$
2c) $x_3(1 - 2u) = 0$
3) $x_1 + 3x_2 + 2x_3 \le 12$
4) $u(x_1 + 3x_2 + 2x_3 - 12) = 0$
5) $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$
6) $u \ge 0$

for $(x_1, x_2, x_3) = (2, 2, 2)$ 2a) $\Rightarrow u = 4$
2b) $\Rightarrow u = 2/3$
2c) $\Rightarrow u = 1/2$

so (2,2,2) cannot be an optimal solution.

(b) It is easy to check that $(x_1, x_2, x_3, u) = (11/3, 25/9, 0, 2/3)$ satisfies the KKT conditions, and so $(x_1, x_2, x_3) = (11/3, 25/9, 0)$ is optimal since the program is convex.

13.6-17. The KKT conditions:

1a) $-2 - u(-2x_1) \le 0$
2a) $x_1(-2 + u2x_1) = 0$
1b) $-3x_2^2 - u(-4x_2) \le 0$
2b) $x_2(-3x_2^2 + u4x_2 = 0$
1c) $-2x_3 - u(-2x_3) \le 0$
2c) $x_3(-2x_3 + u2x_3) = 0$
3) $x_1^2 + 2x_2^2 + x_3^2 \ge 4$
4) $u(4 - x_1^2 - 2x_2^2 - x_3^2) = 0$
5) $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$
6) $u \ge 0$

For $(x_1, x_2, x_3) = (1, 1, 1)$ 2a) $\Rightarrow u = 1$
2b) $\Rightarrow u = 3/4$
2c) $\Rightarrow u = 1$

so $(x_1, x_2, x_3) = (1, 1, 1)$ cannot be optimal

13.6-18. KKT conditions:

1a) $-4x_1^3 + 2x_1 u \le 0$
2a) $x_1(-4x_1^3 + 2x_1 u) = 0$
1b) $-4x_2 + 2x_2 u \le 0$
2b) $x_2(-4x_2 + 2x_2 u) = 0$
3) $-x_1^2 - x_2^2 + 2 \le 0$
4) $u(-x_1^2 - x_2^2 + 2) = 0$
5) $x_1 \ge 0, x_2 \ge 0$
6) $u \ge 0$

If $x_1 = x_2 = 1$, 2a) $\Rightarrow -4 + 2u = 0$
or $u = 2$
$(x_1, x_2) = (1, 1), u = 2$ satisfies all KKT conditions, so (1,1) is optimal.

13.6-19. The KKT conditions:

1a) $32 - 4x_1^3 - 3u_1 - 2u_2 \le 0$

2a) $x_1(32 - 4x_1^3 - 3u_1 - 2u_2) = 0$

1b) $50 - 20x_2 + 3x_2^2 - 4x_2^3 - u_1 - 5u_2 \le 0$

2b) $x_2(50 - 20x_2 + 3x_2^2 - 4x_2^3 - u_1 - 5u_2) = 0$

3a) $3x_1 + x_2 \le 11$

4a) $u_1(3x_1 + x_2 - 11) = 0$

3b) $2x_1 + 5x_2 \le 16$

4b) $u_2(2x_1 + 5x_2 - 16) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u_1 \ge 0, u_2 \ge 0$

for $(x_1, x_2) = (2,2)$ 2a) & 6) $\Rightarrow u_1 = u_2 = 0$

2b) $\Rightarrow u_1 + 5u_2 = -10$

$\Rightarrow (x_1, x_2) = (2,2)$ is not optimal.

13.7-1. (a) $\frac{\partial^2 f}{\partial x_1^2} = -4 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_2^2} = -8 < 0 \quad \forall (x_1, x_2)$

$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 16 > 0 \quad \forall (x_1, x_2)$

$\Rightarrow f$ is strictly concave.

(b) $x^T Q x = 4x_1^2 - 8x_1x_2 + 8x_2^2 = 4(x_1 - x_2)^2 + 4x_2^2 > 0$

$\forall (x_1, x_2) \ne (0,0)$

$\Rightarrow Q$ is positive definite.

(c) The KKT conditions:

1a) $15 + 4x_2 - 4x_1 - v \le 0$

2a) $x_1(15 + 4x_2 - 4x_1 - v) = 0$

1b) $30 + 4x_1 - 8x_2 - 2v \le 0$

2b) $x_2(30 + 4x_1 - 8x_2 - 2v) = 0$

3) $x_1 + 2x_2 \le 30$

4) $v(x_1 + 2x_2 - 30) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $v \ge 0$

It is easy to verify that $(x_1, x_2, v) = (12, 9, 3)$ satisfies these conditions.

13.7-2. (a) The KKT conditions:

1 a) $8 - 2x_1 - v \le 0$

2 a) $x_1(8 - 2x_1 - v) = 0$

1 b) $4 - 2x_2 - v \le 0$

2 b) $x_2(4 - 2x_2 - v) = 0$

3) $x_1 + x_2 \le 2$

4) $v(x_1 + x_2 - 2) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $v \ge 0$

It is easy to verify that $(x_1, x_2, v) = (2, 0, 4)$ satisfies the above conditions; hence, since this is a convex program $(x_1, x_2) = (2, 0)$ is optimal.

(b) The objective function in vector notation is:

Maximize $(8, 4)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2}(x_1, x_2)\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Hence, the equivalent problem is:

Minimize $z_1 + z_2$

subject to

$$\begin{aligned} 2x_1 - y_1 + y_3 + z_1 &= 8 \\ 2x_2 - y_2 + y_3 + z_2 &= 4 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

$y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$

$z_1 \ge 0, z_2 \ge 0$

The complementarity constraint is: $x_1y_1 + x_2y_2 + x_3y_3 = 0$

(c)

(0)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	-2	-2	-2	1	1	0	0	0	-12
X6	1	0	2	0	1	-1	0	1	0	0	8
X7	2	0	0	2	1	0	-1	0	1	0	4
X8	3	0	1*	1	0	0	0	0	0	1	2

(1)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	-2	1	1	0	0	2	-8
X6	1	0	0	-2	1*	-1	0	1	0	-2	4
X7	2	0	0	2	1	0	-1	0	1	0	4
X1	3	0	1	1	0	0	0	0	0	1	2

13.7-2. (a) (cont.)

(2)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	-4	0	-1	1	2	0	-2	0
X3	1	0	0	-2	1	-1	0	1	0	-2	4
X7	2	0	0	4*	0	1	-1	-1	1	2	0
X1	3	0	1	1	0	0	0	0	0	1	2

(3)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	0	0	1	1	0	0
X3	1	0	0	0	1	-0.5	-0.5	0.5	0.5	-1	4
X2	2	0	0	1	0	0.25	-0.25	-0.25	0.25	0.5	0
X1	3	0	1	0	0	-0.25	0.25	0.25	-0.25	0.5	2

This provides the optimal solution $x_1 = 2, x_2 = 0, u = 4$.

d) Using the Excel Solver we get $X_1 = 2, X_2 = 0$

13.7-3. (a) The objective function in vector notation is:

Maximize $(20, 50)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{1}{2}(x_1, x_2)\begin{pmatrix} 40 & -20 \\ -20 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Hence, the equivalent problem is:

Minimize $z_1 + z_2$

subject to

$$\begin{aligned} 40x_1 - 20x_2 - y_1 + y_3 + y_4 + z_1 &= 20 \\ -20x_1 + 10x_2 - y_2 + y_3 + 4y_4 + z_2 &= 50 \\ x_1 + x_2 + x_3 &= 6 \\ x_1 + 4x_2 + x_4 &= 18 \end{aligned}$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, z_1 \geq 0, z_2 \geq 0$

The enforced complementarity constraint is:

$x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 = 0$

13.7-3 (b)

(0)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	-20	10	-2	-5	1	1	0	0	0	0	-70
X7	1	0	40*	-20	1	1	-1	0	1	0	0	0	20
X8	2	0	-20	10	1	4	0	-1	0	1	0	0	50
X9	3	0	1	1	0	0	0	0	0	0	1	0	6
Xe	4	0	1	4	0	0	0	0	0	0	0	1	18

(1)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	-1.5	-4.5	0.5	1	0.5	0	0	0	-60
X1	1	0	1	-0.5	0.025	0.025	-0.03	0	0.025	0	0	0	0.5
X8	2	0	0	0	1.5	4.5	-0.5	-1	0.5	1	0	0	60
X9	3	0	0	1.5*	-0.03	-0.03	0.025	0	-0.03	0	1	0	5.5
Xe	4	0	0	4.5	-0.03	-0.03	0.025	0	-0.03	0	0	1	17.5

13.7-3. (b) (cont.)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	-1.5	-4.5	0.5	1	0.5	0	0	0	-60
X1	1	0	1	0	0.017	0.017	-0.02	0	0.017	0	0.333	0	2.333
X8	2	0	0	0	1.5	4.5	-0.5	-1	0.5	1	0	0	60
X2	3	0	0	1	-0.02	-0.02	0.017	0	-0.02	0	0.667	0	3.667
Xe	4	0	0	0	0.05*	0.05	-0.05	0	0.05	0	-3	1	1

(2)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	0	-3	-1	1	2	0	-90	30	-30
X1	1	0	1	0	0	0	0	0	0	0	1.353	-0.34	1.993
X8	2	0	0	0	0	3*	1	-1	-1	1	90	-30	30
X2	3	0	0	1	0.003	0.003	-0	0	0.003	0	-0.53	0.4	4.067
X3	4	0	0	0	1	1	-1	0	1	0	-60	20	20

(3)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	0	0	0	0	1	1	0	0	0
X1	1	0	1	0	0	0	0	0	0	0	1.353	-0.34	1.993
X4	2	0	0	0	0	1	0.333	-0.33	-0.33	0.333	30	-10	10
X2	3	0	0	1	0.003	0	-0	0.001	0.004	-0	-0.63	0.433	4.033
X3	4	0	0	0	1	0	-1.33	0.333	1.333	-0.33	-90	30	10

(4)

This provides the optimal solution $(x_1, x_2) = (1.993, 4.033)$

$(u_1, u_2) = (10, 10)$

13.7-4. a) The KKT conditions:

1a) $2 - 2x_1 - u \le 0$

2a) $x_1(2 - 2x_1 - u) = 0$

1b) $4 - 3x_2 - u \le 0$

2b) $x_2(4 - 3x_2 - u) = 0$

3) $x_1 + x_2 \le 2$

4) $u(x_1 + x_2 - 2) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

If you plot the points obtained in (a) it is pretty clear that you are converging to a boundary point solution. So $x_1, x_2, u \neq 0$

$\Rightarrow (x_1, x_2, u) = (.8, 1.2, .4)$ satisfies the above conditions. So $(x_1, x_2) = (.8, 1.2)$ is optimal.

13.7-4. b) Minimize $z_1 + z_2$

subject to
$$\begin{aligned} 2x_1 - y_1 + y_3 + z_1 &= 2 \\ 3x_2 - y_2 + y_3 + z_2 &= 4 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$
$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0,\ z_1 \ge 0,\ z_2 \ge 0$

The enforced complementarity constraint is:
$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$

c) for $x_1 = .8$, $x_2 = 1.2$, $U = y_3 \le .4$
the first constraint becomes $-y_1 + z_1 = 0$
the second constraint becomes $-y_2 + z_2 = 0$
the third constraint becomes $x_3 = 0$
and, hence, the complementarity constraint becomes $.8y_1 + 1.2y_2 = 0$, but $y_1 \ge 0$ & $y_2 \ge 0$
$\Rightarrow y_1 = 0,\ y_2 = 0$

So $z_1 = z_2 = 0 \Rightarrow Z = 0$
Thus, $(.8, 1.2) = (x_1, x_2)$ is optimal

d)

(0)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	-2	-3	-2	1	1	0	0	0	-6
X6	1	0	2	0	1	-1	0	1	0	0	2
X7	2	0	0	3*	1	0	-1	0	1	0	4
X8	3	0	1	1	0	0	0	0	0	1	2

(1)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	-2	0	-1	1	0	0	1	0	-2
X6	1	0	2	0	1	-1	0	1	0	0	2
X2	2	0	0	1	0.333	0	-0.33	0	0.333	0	1.333
X8	3	0	1*	0	-0.33	0	0.333	0	-0.33	1	0.667

(2)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	-1.67	1	0.667	0	0.333	2	-0.67
X6	1	0	0	0	1.667*	-1	-0.67	1	0.667	-2	0.667
X2	2	0	0	1	0.333	0	-0.33	0	0.333	0	1.333
X1	3	0	1	0	-0.33	0	0.333	0	-0.33	1	0.667

13.7-4. d) (cont')

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	0	0	1	1	0	0
(3) X3	1	0	0	0	1	-0.6	-0.4	0.6	0.4	-1.2	0.4
X2	2	0	0	1	0	0.2	-0.2	-0.2	0.2	0.4	1.2
X1	3	0	1	0	0	-0.2	0.2	0.2	-0.2	0.6	0.8

This provides the optimal solution $(x_1, x_2) = (0.8, 1.2)$

$u = 0.4$

e) USING THE Excel Solver we get $x_1 = 0.8$, $x_2 = 1.2$

13.7-5. a) The KKT conditions:

1a) $126 - 18x_1 - v_1 - 3v_3 \le 0$

2a) $x_1(126 - 18x_1 - v_1 - 3v_3) = 0$

1b) $182 - 26x_2 - 2v_2 - 2v_3 \le 0$

2b) $x_2(182 - 26x_2 - 2v_2 - 2v_3) = 0$

3a) $x_1 \le 4$

4a) $v_1(x_1 - 4) = 0$

3b) $2x_2 \le 12$

4b) $v_2(2x_2 - 12) = 0$

3c) $3x_1 + 2x_2 \le 18$

4c) $v_3(3x_1 + 2x_2 - 18) = 0$

It is easy to verify that $(x_1, x_2, v_1, v_2, v_3) = (8/5, 5, 0, 0, 26)$ satisfies the above conditions, and, hence, $(x_1, x_2) = (8/3, 5)$ is optimal.

b) Minimize $z_1 + z_2$

subject to

$$18x_1 - y_1 + y_3 + 3y_5 + z_1 = 126$$
$$26x_2 - y_2 + 2y_4 + 2y_5 + z_2 = 182$$
$$x_1 + x_3 = 4$$
$$2x_2 + x_4 = 12$$
$$3x_1 + 2x_2 + x_5 = 18$$

$x_i \ge 0$ $i = 1, \ldots, 5$ $y_i \ge 0$ $i = 1, \ldots, 5$ $z_1 \ge 0, z_2 \ge 0$

13.7-5 c)

for $(8/3, 5, 26) = (x_1, x_2, y_3) = (x_1, x_2, y_5)$
the third constraint $\Rightarrow x_3 = 1/3$
the fourth constraint $\Rightarrow x_4 = 2$
the fifth constraint $\Rightarrow x_5 = 0$
the first constraint $\Rightarrow y_3 - y_1 + 2z_1 = 0$
the second constraint $\Rightarrow 2y_4 - y_2 + 2z_2 = 0$
the complementary constraint $\Rightarrow 8/3\, y_1 + 5y_2 + 1/3\, y_3 + 2y_4 = 0$
so non-negativity $\Rightarrow y_1 = y_2 = y_3 = y_4 = 0$
$\Rightarrow z_1 = z_2 = 0$
$\Rightarrow (8/3, 5) = (x_1, x_2)$ is optimal

13.7-6 a&b)

Minimum acceptable expected return = 13

Factor	Amount Per Block: Stock 1	Stock 2	Totals	Right-Hand Side
Budget	20	30	50	50
Expected Return	5	10	13.00	13
Risk	4	100	25.56	
Solution	2.2	0.2		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

Minimum acceptable expected return = 14

Factor	Amount Per Block: Stock 1	Stock 2	Totals	Right-Hand Side
Budget	20	30	50	50
Expected Return	5	10	14.00	14
Risk	4	100	51.04	
Solution	1.6	0.6		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

Minimum acceptable expected return = 15

Factor	Amount Per Block: Stock 1	Stock 2	Totals	Right-Hand Side
Budget	20	30	50	50
Expected Return	5	10	15.00	15
Risk	4	100	109.00	
Solution	1.0	1.0		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

Minimum acceptable expected return = 16

Factor	Amount Per Block: Stock 1	Stock 2	Totals	Right-Hand Side
Budget	20	30	50	50
Expected Return	5	10	16.00	16
Risk	4	100	199.44	
Solution	0.4	1.4		

Joint Risk	Stock 1	Stock 2
Stock 1		5
Stock 2		

c)

μ	σ	$\mu-\sigma$	$\mu-3\sigma$
13	5.06	7.94	-2.18
14	7.14	6.86	-7.42
15	10.44	4.56	-16.32
16	14.12	1.88	-26.36

13.7-7 (a) The KKT Conditions are:

1(a) $-2(x_1-1)+3-4u_1-u_2 \le 0$

(b) $-2(x_2-2)+3-u_1-4u_2 \le 0$

2(a) $x_1(-2(x_1-1)+3-4u_1-u_2)=0$

(b) $x_2(-2(x_2-2)+3-u_1-4u_2)=0$

3(a) $4x_1+x_2-20 \le 0$

(b) $x_1+4x_2-20 \le 0$

4(a) $u_1(4x_1+x_2-20)=0$

(b) $u_2(x_1+4x_2-20)=0$

5 $x_1 \ge 0 \quad x_2 \ge 0$

6 $u_1 \ge 0 \quad u_2 \ge 0$

(b) Since the optimal solution is not on a boundary from 2(a), 2(b), 4(a) and 4(b), respectively, we have:

$$-2(x_1-1)+3-4u_1-u_2=0$$
$$-2(x_2-2)+3-u_1-4u_2=0$$
$$u_1=0$$
$$u_2=0$$

Substituting the last 2 into the first 2 we get

$$-2x_1+5=0$$
$$-2x_2+7=0$$

So, the optimal solution is $x_1=5/2 \quad x_2=7/2$.

(c) Minimize z_1+z_2

subject to:

$$\begin{aligned} 2x_1 + 4u_1 + u_2 - y_1 + z_1 &= 5 \\ 2x_2 + u_1 + 4u_2 - y_2 + z_2 &= 7 \\ 4x_1 + x_2 + v_1 &= 20 \\ x_1 + 4x_2 + v_2 &= 20 \end{aligned}$$

$x_i \ge 0, u_i \ge 0, y_i \ge 0, v_i \ge 0, z_i \ge 0$ for $i=1,2$.

Complementarity Constraint is: $x_1y_1 + x_2y_2 + u_1v_1 + u_2v_2 = 0$

(d)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	-2	-2	-5	-5	1	1	0	0	0	0	-12
X7	1	0	2*	0	4	1	-1	0	1	0	0	0	5
(0) X8	2	0	0	2	1	4	0	-1	0	1	0	0	7
X9	3	0	4	1	0	0	0	0	0	0	1	0	20
X10	4	0	1	4	0	0	0	0	0	0	0	1	20

13.7-7

d) (cont')

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	-2	-1	-4	0	1	1	0	0	0	-7
X1	1	0	1	0	2	0.5	-0.5	0	0.5	0	0	0	2.5
(1) X8	2	0	0	2*	1	4	0	-1	0	1	0	0	7
X9	3	0	0	1	-8	-2	2	0	-2	0	1	0	10
Xe	4	0	0	4	-2	-0.5	0.5	0	-0.5	0	0	1	17.5

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	1	0	0	0	0	0	0	1	1	0	0	0
X1	1	0	1	0	2	0.5	-0.5	0	0.5	0	0	0	2.5
(2) X2	2	0	0	1	0.5	2	0	-0.5	0	0.5	0	0	3.5
X9	3	0	0	0	-8.5	-4	2	0.5	-2	-0.5	1	0	6.5
Xe	4	0	0	0	-4	-8.5	0.5	2	-0.5	-2	0	1	3.5

This shows $x_1 = 5/2$ $x_2 = 3/2$ is optimal with $u_1 = u_2 = 0$

e) Using the excel solver we get $x_1 = 2.5$ and $x_2 = 3.5$

13.8-1 a) Let $x_1 = x_{11} + x_{12}$ and $x_2 = x_{21} + x_{22}$, $f_1(x_1) = (x_1 - 1)^2 - 3x_1$ and $f_2(x_2) = (x_2 - 2)^2 - 3x_2$.

$f_1(0) = 1$, $f_1(2.5) = -5.25$, $f_1(5) = 1$, $f_2(0) = 4$, $f_2(2.5) = -7.25$ and $f_2(5) = -6$

so $s_{11} = \frac{1 + 5.25}{0 - 2.5} = -2.5$ $\quad s_{12} = \frac{-5.25 - 1}{2.5 - 5} = 2.5$

$s_{21} = \frac{4 + 7.25}{0 - 2.5} = -4.5$ $\quad s_{22} = \frac{-7.25 + 6}{2.5 - 5} = \frac{1}{2}$

The approximate linear programming model is:

Minimize $\quad -2.5x_{11} + 2.5x_{12} - 4.5x_{21} + \frac{1}{2}x_{22}$

subject to: $\quad 4x_{11} + 4x_{12} + x_{21} + x_{22} \le 20$

$x_{11} + x_{12} + 4x_{21} + 4x_{22} \le 20$

$0 \le x_{ij} \le 2.5$ for $i = 1, 2$ and $j = 1, 2$.

b)

Bas Var	Eq No	Z	X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
Z	0	-1	0	2.5	0	0.5	0	0	2.5	0	4.5	0	17.5
X5	1	0	0	4	0	1	1	0	-4	0	-1	0	7.5
X6	2	0	0	1	0	4	0	1	-1	0	-4	0	7.5
X1	3	0	1	0	0	0	0	0	1	0	0	0	2.5
X8	4	0	0	1	0	0	0	0	0	1	0	0	2.5
X3	5	0	0	0	1	0	0	0	0	0	1	0	2.5
Xe	6	0	0	0	0	1	0	0	0	0	0	1	2.5

The optimal solution is $x_{11} = 2.5$, $x_{12} = 0$, $x_{21} = 2.5$, $x_{22} = 0$

In terms of the original variables we have $x_1 = x_{11} + x_{12} = 2.5$, $x_2 = x_{21} + x_{22} = 2.5$

13.8-1. c) i) Let $X_1 = X_{11} + X_{12} + X_{13} + X_{14} + X_{15}$, $X_2 = X_{21} + X_{22} + X_{23} + X_{24} + X_{25}$

$f_1(X_1) = (X_1 - 1)^2 - 3X_1$, $f_2(X_2) = (X_2 - 2)^2 - 3X_2$

$f_1(0) = 1$, $f_1(1) = -3$, $f_1(2) = -5$, $f_1(3) = -5$, $f_1(4) = -3$, $f_1(5) = 1$

$f_2(0) = 4$, $f_2(1) = -2$, $f_2(2) = -6$, $f_2(3) = -8$, $f_2(4) = -8$, $f_2(5) = -6$

So, $S_{11} = \frac{1-(-3)}{0-1} = -4$, $S_{12} = \frac{-3-(-5)}{1-2} = -2$, $S_{13} = 0$, $S_{14} = 2$, $S_{15} = 4$

$S_{21} = -6$, $S_{22} = -4$, $S_{23} = -2$, $S_{24} = 0$, $S_{25} = 2$

The approximate linear programming model is:

Minimize $-4X_{11} - 2X_{12} + 2X_{14} + 4X_{15} - 6X_{21} - 4X_{22} - 2X_{23} + 2X_{25}$

subject to $4X_{11} + 4X_{12} + 4X_{13} + 4X_{14} + 4X_{15} + X_{21} + X_{22} + X_{23} + X_{24} + X_{25} \leq 20$

$X_{11} + X_{12} + X_{13} + X_{14} + X_{15} + 4X_{21} + 4X_{22} + 4X_{23} + 4X_{24} + 4X_{25} \leq 20$

$0 \leq X_{ij} \leq 1$ for $i = 1, 2$ and $j = 1, 2, 3, 4, 5$.

ii) Solving this using the Simplex Method, we get:

Optimal Solution

Value of the
Objective Function: Z = -18

Variable		Value
X_1	(X_{11})	1
X_2	(X_{12})	1
X_3	(X_{13})	0
X_4	(X_{14})	0
X_5	(X_{15})	0
X_6	(X_{21})	1
X_7	(X_{22})	1
X_8	(X_{23})	1
X_9	(X_{24})	0
X_{10}	(X_{25})	0

In terms of the original variable, we have

$X_1 = X_{11} + X_{12} + X_{13} + X_{14} + X_{15} = 2$

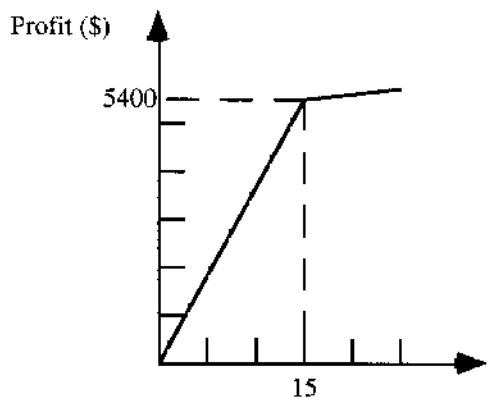
$X_2 = X_{21} + X_{22} + X_{23} + X_{24} + X_{25} = 3$

(Note that we improved by $\frac{1}{2}$; Z went from -17.5 to -18)

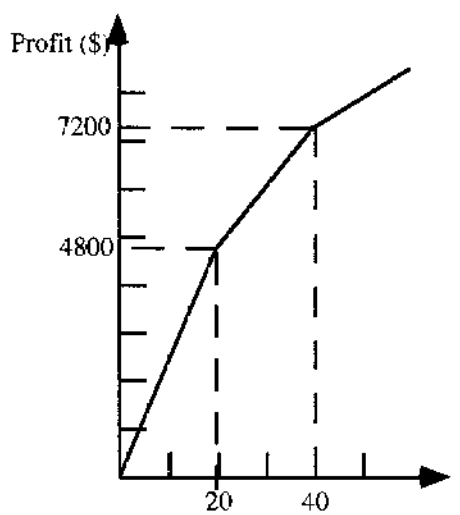
5'11

13.8-2

a) The profit graph for product 1 is shown below:



The profit graph for product 2 is shown below:

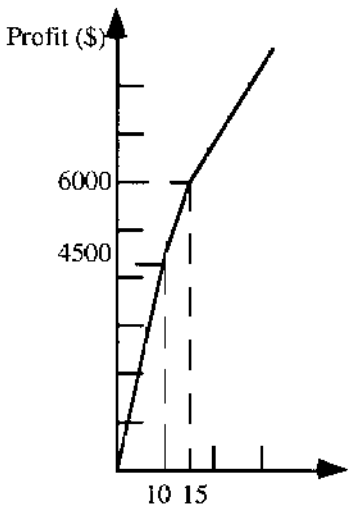


(CONT'D)

13B.2 (CONT'D)

a)

The profit graph for product 3 is shown below:



b) Maximize $360x_{11} + 30x_{12} + 240x_{21} + 120x_{22} + 90x_{23} + 450x_{31} + 300x_{32} + 180x_{33}$

subject to

$$x_{11} + x_{12} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 60$$
$$3x_{11} + 3x_{12} + 2x_{21} + 2x_{22} + 2x_{23} \leq 200$$
$$x_{11} + x_{12} + x_{31} + x_{32} + x_{33} \leq 70$$
$$0 \leq x_{11} \leq 15,\ 0 \leq x_{12}$$
$$0 \leq x_{21} \leq 20,\ 0 \leq x_{22} \leq 20,\ 0 \leq x_{23}$$
$$0 \leq x_{31} \leq 10,\ 0 \leq x_{32} \leq 5,\ 0 \leq x_{33}$$

where $x_1 = x_{11} + x_{12}$, $x_2 = x_{21} + x_{22} + x_{23}$ & $x_3 = x_{31} + x_{32} + x_{33}$

c) Solving automatically by the Simplex Method:

Optimal Solution

Value of the
Objective Function: Z = 18000

Variable		Value
X_1	(x_{11})	15
X_2	(x_{12})	0
X_3	(x_{21})	20
X_4	(x_{22})	0
X_5	(x_{23})	0
X_6	(x_{31})	10
X_7	(x_{32})	5
X_8	(x_{33})	10

(CONT'D)

13.8-2 c) ~~CONT'D~~)

In terms of the original variables, we have

$X_1 = X_{11} + X_{12} = 15$, $X_2 = X_{21} + X_{22} + X_{23} = 20$, $X_3 = X_{31} + X_{32} + X_{33} = 25$.

d) The restriction on profit from products 1 and 2 can be modeled by adding the following constraint to the linear program in part (b):

$$360X_{11} + 30X_{12} + 240X_{21} + 120X_{22} + 90X_{23} \geq 9000$$

e) The Simplex Method gives the optimal solution:

Value of the
Objective Function: Z = 18000

Variable	Value
X_1	15
X_2	0
X_3	20
X_4	0
X_5	0
X_6	10
X_7	5
X_8	10

Note that the optimal hasn't changed. This is because $\underline{X}$ still is feasible with the new constraint.
(i.e. $360x_{11} + 30x_{12} + 240x_{21} + 120x_{22} + 90x_{23} = 10200 > 9000$)

In terms of the original variables, we (still) have

$X_1 = 15$, $X_2 = 20$, $X_3 = 25$.

13.8-3. a) The KKT conditions:

1a) $4 - 3x_1^2 - u_1 - 5u_2 \leq 0$
2a) $x_1(4 - 3x_1^2 - u_1 - 5u_2) = 0$
1b) $6 - 4x_2 - 3u_1 - 2u_2 \leq 0$
2b) $x_2(6 - 4x_2 - 3u_1 - 2u_2) = 0$
3a) $x_1 + 3x_2 \leq 8$
4a) $u_1(x_1 + 3x_2 - 8) = 0$
3b) $5x_1 + 2x_2 \leq 14$
4b) $u_2(5x_1 + 2x_2 - 14) = 0$
5) $x_1 \geq 0$, $x_2 \geq 0$
6) $u_1 \geq 0$, $u_2 \geq 0$

It is easy to verify that $(x_1, x_2, u_1, u_2) = (\frac{2}{\sqrt{3}}, \frac{3}{2}, 0, 0)$ satisfies the above conditions, and so $(x_1, x_2) = (\frac{2}{\sqrt{3}}, \frac{3}{2})$ is optimal with $z = 7.58$

13.8-3

b) 3,000 power saws and 7,000 power drills should be produced in November.

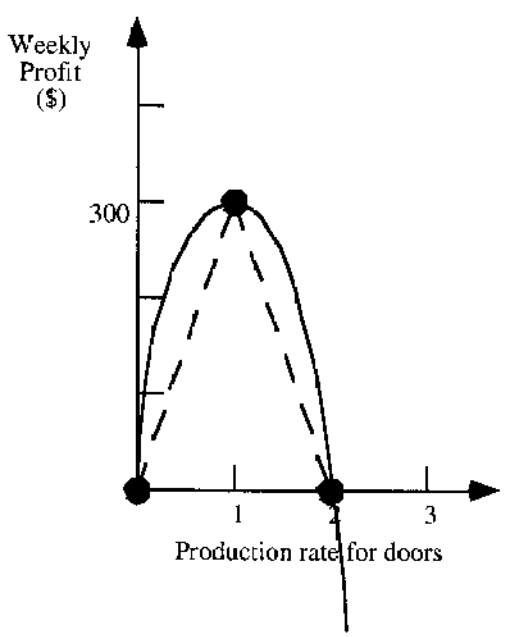
Profit data for doors when marketing costs are considered:

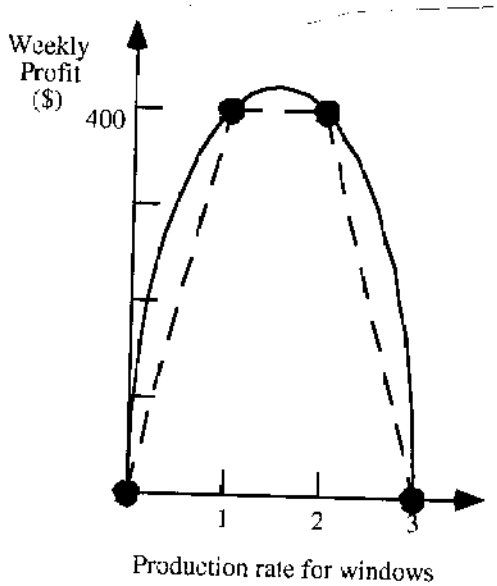
Production Rate	Gross Profit	Marketing Cost	Net Profit	Incremental Net Profit
0	0	0	0	—
1	$400	$100	S300	$300
2	$800	$800	$0	-$300
3	$1200	$2700	-$1900	-$1900
D	$4D$	D^3	$4D - D^3$	

Profit data for windows when marketing costs are considered:

Production Rate	Gross Profit	Marketing Cost	Net Profit	Incremental Net Profit
0	0	0	0	—
1	$600	S200	$400	$400
2	$1200	S800	$400	$0
3	$1800	$1800	0	-$400
W	$6W$	$2W^2$	$6W - 2W^2$	

c) The profit graphs for doors and windows are shown below:





13.8-3

d) Let $X_1 = X_{11} + X_{12} + X_{13}$, $X_2 = X_{21} + X_{22} + X_{23}$, $f_1(X_1) = 4X_1 - X_1^3$, $f_2(X_2) = 6X_2 - 2X_2^2$

$f_1(0) = 0$, $f_1(1) = 3$, $f_1(2) = 0$, $f_1(3) = -15$

$f_2(0) = 0$, $f_2(1) = 4$, $f_2(2) = 4$, $f_2(3) = 0$

$s_{11} = \frac{3-0}{1-0} = 3$, $s_{12} = \frac{0-3}{2-1} = -3$, $s_{13} = -15$

$s_{21} = 4$, $s_{22} = 0$, $s_{23} = -4$

The approximate linear programming model is:

Maximize $3X_{11} - 3X_{12} - 15X_{13} + 4X_{21} - 4X_{23}$

subject to: $X_{11} + X_{12} + X_{13} + 3X_{21} + 3X_{22} + 3X_{23} \le 8$

$5X_{11} + 5X_{12} + 5X_{13} + 2X_{21} + 2X_{22} + 2X_{23} \le 14$

$0 \le X_{ij} \le 1$ for $i = 1, 2$ and $j = 1, 2, 3$.

e) The Simplex method gives the solution:

Value of the
Objective Function: Z = 7

Variable		Value
X_1	(X_{11})	1
X_2	(X_{12})	0
X_3	(X_{13})	0
X_4	(X_{21})	1
X_5	(X_{22})	0
X_6	(X_{23})	0

} or $X_1 = X_{11} + X_{12} + X_{13} = 1$

$X_2 = X_{21} + X_{22} + X_{23} = 1$

(also $X_2 = 2$ gives optimal objective value $Z = 7$)

$X_{12} = 0 \Rightarrow X_{13} = 0$ (also $X_{11} = 0 \Rightarrow X_{12} = 0$ since $X_{11} \ne 0$)

$X_{22} = 0 \Rightarrow X_{23} = 0$ $X_{21} = 0 \Rightarrow X_{22} = 0$ since $X_{21} \ne 0$)

So special restriction for the model is satisfied.

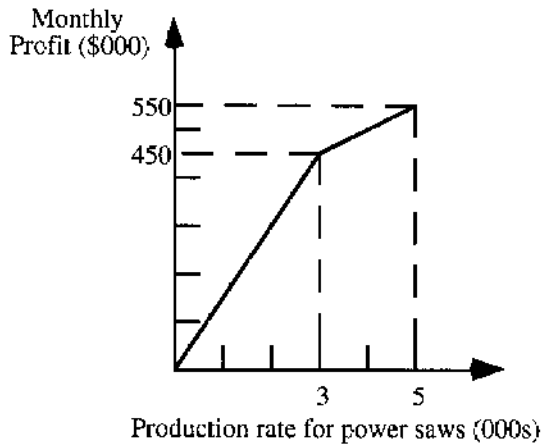
The approximate solution is fairly close: (1,1) or (1,2) $\approx$ (1.155, 1.5) ← optimal
* Z is close.

13.8-4

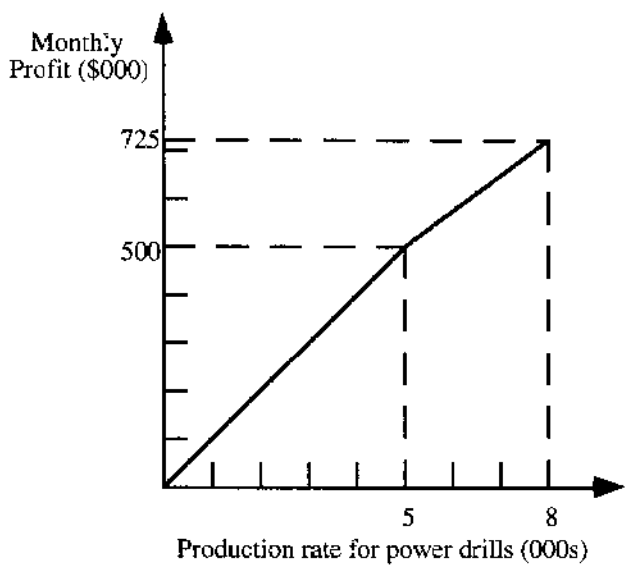
Since regular time production is cheaper than overtime, the objective of minimizing cost will force regular time to be used first. (If an increasing regular time by a small amount and decreasing overtime by the same amount will yield a feasible solution with a better objective value.)

13.8-5

a) The profit graph for power saws is shown below:



The profit graph for power drills is shown below:



b) Maximize $150x_{11} + 50x_{12} + 100x_{21} + 75x_{22}$

subject to:

$0 \le x_{11} \le 3000$; $0 \le x_{12} \le 2000$

$0 \le x_{21} \le 5000$; $0 \le x_{22} \le 3000$

$x_{11} + x_{12} + x_{21} + x_{22} \le 10{,}000$

$2x_{11} + 2x_{12} + x_{21} + x_{22} \le 15{,}000$

13.8-5

c)

Resource Usage Per Unit of Each Activity

Resource	Power Saws Regular	Power Saws Overtime	Power Drills Regular	Power Drills Overtime	Totals		Resource Available
power supplies	1	1	1	1	10000	≤	10000
gear assemblies	2	2	1	1	13000	≤	15000
Unit Profit	150	50	100	75	$ 1,100,000		
Solution	3000	0	5000	2000			
Maximum	3000	2000	5000	3000			

13.8-6

(a) Let $x_1 = x_{11} + x_{12} + x_{13}$, $x_2 = x_{21} + x_{22} + x_{23}$, $f_1(x_1) = 32x_1 - x_1^4$ and $f_2(x_2) = 50x_2 - 10x_2^2 + x_2^3 - x_2^4$.

We will have $s_{11} = 31$, $s_{12} = 17$, $s_{13} = -33$, $s_{21} = 40$, $s_{22} = 12$ and $s_{23} = -46$.

The approximate linear programming model is:

Maximize $31x_{11} + 17x_{12} - 33x_{13} + 40x_{21} + 12x_{22} - 46x_{23}$

subject to: $3x_{11} + 3x_{12} + 3x_{13} + x_{21} + x_{22} + x_{23} \le 11$

$2x_{11} + 2x_{12} + 2x_{13} + 5x_{21} + 5x_{22} + 5x_{23} \le 16$

$0 \le x_{ij} \le 1$ for $i = 1, 2$ and $j = 1, 2, 3$.

b) The automatic routine (Simplex method) gives the solution:

Value of the
Objective Function: Z = 100

Variable		Value
X_1	(X_{11})	1
X_2	(X_{12})	1
X_3	(X_{13})	0
X_4	(X_{21})	1
X_5	(X_{22})	1
X_6	(X_{23})	0

In terms of the original variables, we have:

$X_1 = X_{11} + X_{12} + X_{13} = 2$, $X_2 = X_{21} + X_{22} + X_{23} = 2$.

13.8-7

Let $f_1(x_1) = \begin{cases} 5x_1 & 0 \le x_1 \le 2 \\ 2+4x_1 & 2 \le x_1 \le 5 \\ 12+2x_1 & 5 \le x_1 \end{cases}$

$f_2(x_2) = \begin{cases} 4x_2 & 0 \le x_2 \le 3 \\ 9+x_2 & 3 \le x_2 \le 4 \end{cases}$

So Maximize $f_1(x_1) + f_2(x_2)$
subject to $3x_1 + 2x_2 \le 25$
$2x_1 - x_2 \le 10$
$x_2 \le 4$
$x_1 \ge 0, x_2 \ge 0$

Possibly, the $f_i(x_i)$ are piecewise-linear approximations of the true objective function.

13.8-8

a) Assume that in the optimal solution of the linear program there exists an x_{ij} such that $x_{ij} < v_{ij}$ and $x_{i(j+1)} > 0$. Create a new solution with $x'_{ij} = \min\{v_{ij}, x_{ij} + x_{i(j+1)}\}$, $x'_{i(j+1)} = \max\{0, x_{ij} + x_{i(j+1)} - v_{ij}\}$. this solution is feasible since the g_i's are all linear and $x_{ij} + x_{i(j+1)} = x'_{ij} + x'_{i(j+1)}$.

But $s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} = \begin{cases} s_{ij}(x_{ij} + x_{i(j+1)}) & x_{ij} + x_{i(j+1)} \le v_{ij} \\ s_{ij}v_{ij} + s_{i(j+1)}(x_{ij} + x_{i(j+1)} - v_{ij}) & \text{otherwise.} \end{cases}$

Clearly, $s_{ij}(x_{ij} + x_{i(j+1)}) > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$
since $s_{ij} \ge s_{i(j+1)}$

Furthermore, $(s_{ij} - s_{i(j+1)})v_{ij} > (s_{ij} - s_{i(j+1)})x_{ij}$ from $x_{ij} < v_{ij}$

$\Rightarrow s_{ij}v_{ij} + s_{i(j+1)}(x_{ij} - v_{ij}) > s_{ij}x_{ij}$

$\Rightarrow s_{ij}v_{ij} + s_{i(j+1)}(x_{ij} + x_{i(j+1)} - v_{ij}) > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$

So $s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$
Thus, the original solution was not optimal.

(b) Make the same assumptions as above and construct x' from x as above. For the linear approximate of g_i we will have... $a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)} \le b_i$ with $a_{ij} \le a_{i(j+1)}$ since g_i was convex. Thus, it can be shown by the analysis above if we reverse the inequalities in the appropriate places:

$a_{ij}x'_{ij} + a_{i(j+1)}x'_{i(j+1)} < a_{ij}x_{ij} + a_{i(j+1)}x_{i(j+1)}$

So x' is feasible. Further, by above
$s_{ij}x'_{ij} + s_{i(j+1)}x'_{i(j+1)} > s_{ij}x_{ij} + s_{i(j+1)}x_{i(j+1)}$
So x was not optimal.

13.8-9

Let $f_1(x_1) = \begin{cases} 15x_1 & 0 \le x_1 \le 2000 \\ 25x_1 - 20{,}000 & 2000 \le x_1 \end{cases}$

$f_2(x) = \begin{cases} 16x_2 & 0 \le x_2 \le 1000 \\ 24x_2 - 8000 & 1000 \le x_1 \end{cases}$

The (non-linear) programming problem is:

Maximize $z = x_1 + x_2$

subject to $f_1(x_1) + f_2(x_2) \le 60{,}000$

$x_1 \le 3000$

$x_2 \le 1500$

$x_1 \ge 0 \quad x_2 \ge 0$

(a) Let x_i^R, x_i^O denote regular and overtime production at plant i. The LP is:

Maximize $z = x_1^R + x_1^O + x_2^R + x_2^O$

subject to $15x_1^R + 25x_1^O + 16x_2^R + 24x_2^O \le 60{,}000$

$x_1^R \le 2000$

$x_1^O \le 1000$

$x_2^R \le 1000$

$x_2^O \le 500$

$x_1^R \ge 0, x_1^O \ge 0, x_2^R \ge 0, x_2^O \ge 0$

(b) Since overtime production is more costly than regular time, the objective of maximizing total production time will force the regular time to be used first.

13.8-10

(a) The objective function is linear and, therefore, concave.

$\frac{\partial^2 g_1}{\partial x_1^2} \cdot \frac{\partial^2 g_1}{\partial x_2^2} - \left(\frac{\partial^2 g_1}{\partial x_1 \partial x_2}\right)^2 = 4 \cdot 0 - 0^2 = 0 \ge 0$ so from the test in the appendix $g_1(x_1, x_2)$ is convex. Similarly

$\frac{\partial^2 g_2}{\partial x_1^2} \cdot \frac{\partial^2 g_2}{\partial x_2^2} - \left(\frac{\partial^2 g_2}{\partial x_1 \partial x_2}\right)^2 = 2 \cdot 0 - 0^2 = 0 \ge 0$ so $g_2(x_1, x_2)$ is also convex.

(b) Let $x_1 = x_{11} + x_{12} + x_{13}$. From constraint 1 we know $2x_1^2 \le 13$ or $x_1 \le \sqrt{\frac{13}{2}} \approx 2.55$, therefore, using integer breakpoints will require 3 linear pieces. Let $g_{11}(x_1) = 2x_1^2$, $g_{12}(x_2) = x_2$, $g_{21}(x_1) = x_1^2$ and $g_{22}(x_2) = x_2$

Constraint 1 = $g_{11}(x_1) + g_{12}(x_2)$ and Constraint 2 = $g_{21}(x_1) + g_{22}(x_2)$

$g_{11}(0) = 0$, $g_{11}(1) = 2$, $g_{11}(2) = 8$, $g_{11}(3) = 18$, $g_{21}(0) = 0$, $g_{21}(1) = 1$, $g_{21}(2) = 4$ and $g_{21}(3) = 9$.

Let $s_{11} = \frac{2-0}{1-0} = 2$ $s_{12} = \frac{8-2}{2-1} = 6$ $s_{13} = \frac{18-8}{3-2} = 10$

$s_{21} = \frac{1-0}{1-0} = 1$ $s_{22} = \frac{4-1}{2-1} = 3$ $s_{23} = \frac{9-4}{3-2} = 5$

The approximate linear program is:

Maximize $5x_{11} + 5x_{12} + 5x_{13} + x_2$

subject to: $2x_{11} + 6x_{12} + 10x_{13} + x_2 \le 13$

$x_{11} + 3x_{12} + 5x_{13} + x_2 \le 9$

$0 \le x_{11} \le 1$ $0 \le x_{12} \le 1$ $x_{13} \ge 0$ $x_2 \ge 0$.

We could have $0 \le x_{13} \le 1$ but the constraints will enforce the upper bound.

(c)

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	Right side
Z	0	1	0	0	0	0	0	1	4	2	15
X3	1	0	0	0	1	0	0.2	-0.2	-0.2	-0.6	0
X4	2	0	0	0	0	1	-1	2	0	0	5
X1	3	0	1	0	0	0	0	0	1	0	1
X2	4	0	0	1	0	0	0	0	0	1	1

In terms of the original variables, the optimal solution to the approximate problem is: $X_1 = X_{11} + X_{12} + X_{13} = 1 + 1 + 0 = 2$

$X_2 = 5$

13.8-11 a) Let $X_1 = X_{11} + X_{12} + X_{13}$, $X_2 = X_{21} + X_{22} + X_{23}$

Obj: $f_1(X_1) = 32X_1 - X_1^4$ $\left(\frac{\partial^2 f_1}{\partial X_1^2} = -12X_1^2 \le 0\right)$
$f_2(X_2) = 4X_2 - X_2^2$ $\left(\frac{\partial^2 f_2}{\partial X_2^2} = -2 < 0\right)$ } concave

$f_1(0) = 0$, $f_1(1) = 31$, $f_1(2) = 48$, $f_1(3) = 15$

$f_2(0) = 0$, $f_2(1) = 3$, $f_2(2) = 4$, $f_2(3) = 3$

So, $s_{11} = \frac{31-0}{1-0} = 31$, $s_{12} = 17$, $s_{13} = -33$

$s_{21} = 3$, $s_{22} = 1$, $s_{23} = -1$

Constraints: $g_{11}(X_1) = X_1^2$, $g_{12}(X_2) = X_2^2$ $\left(\frac{\partial^2 g_{11}}{\partial X_1^2} = \frac{\partial^2 g_{12}}{\partial X_2^2} = 2 > 0, \text{convex}\right)$

$g_{11}(0) = 0$, $g_{11}(1) = 1$, $g_{11}(2) = 4$, $g_{11}(3) = 9$

$g_{12}(0) = 0$, $g_{12}(1) = 1$, $g_{12}(2) = 4$, $g_{12}(3) = 9$

$t_{11} = \frac{1-0}{1-0} = 1$, $t_{12} = \frac{4-1}{2-1} = 3$, $t_{13} = 5$

$t_{21} = 1$, $t_{22} = 3$, $t_{23} = 5$

The approximate linear program is:

Maximize $31X_{11} + 17X_{12} - 33X_{13} + 3X_{21} + X_{22} - X_{23}$

subject to $X_{11} + 3X_{12} + 5X_{13} + X_{21} + 3X_{22} + 5X_{23} \le 9$

$X_{11} \ge 0$, $X_{12} \ge 0$, $X_{13} \ge 0$, $X_{21} \ge 0$, $X_{22} \ge 0$, $X_{23} \ge 0$

$X_{11} \le 1$, $X_{12} \le 1$, $(X_{13} \le 1)$, $X_{21} \le 1$, $X_{22} \le 1$, $(X_{23} \le 1)$

b) The Simplex Method gives the solution:

Value of the
Objective Function: Z = 52

Variable		Value
X_1	(X_{11})	1
X_2	(X_{12})	1
X_3	(X_{13})	0
X_4	(X_{21})	1
X_5	(X_{22})	1
X_6	(X_{23})	0

In terms of the original variables, we have

$X_1 = X_{11} + X_{12} + X_{13} = 2$, $X_2 = X_{21} + X_{22} + X_{23} = 2$.

c) The KKT conditions:

1 a) $32 - 4X_1^3 - 2X_1 u \le 0$

2 a) $X_1(32 - 4X_1^3 - 2X_1 u) = 0$

1 b) $4 - 2X_2 - 2X_2 u \le 0$

2 b) $X_2(4 - 2X_2 - 2X_2 u) = 0$

3) $X_1^2 + X_2^2 - 9 \le 0$

4) $u(X_1^2 + X_2^2 - 9) = 0$

5) $X_1 \ge 0$, $X_2 \ge 0$

6) $u \ge 0$

From (4), $X_1 = 2 = X_2 \Rightarrow u = 0$

These values satisfy all of the KKT conditions. So the solution to the linear approximation is, in fact, the optimal to the original problem.

13.8-12 a) $f(x) = f_1(x_1) + f_2(x_2)$

$f_1(x_1) = 3x_1^2 - x_1^3$

$f_2(x_2) = 5x_2^2 - x_2^3$

$\frac{\partial^2 f_1}{\partial x_1^2} = 6 - 6x_1 \quad \leftarrow \quad > 0 \text{ if } 0 \le x_1 < 1$

$\frac{\partial^2 f_2}{\partial x_2^2} = 10 - 6x_2 \quad \leftarrow \quad > 0 \text{ if } 0 \le x_2 < \frac{5}{3}$

Neither f_1 nor f_2 are concave (we only need to show one of these are not concave), so $f(x)$ is not concave.

b) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}$

$x_2 = x_{21} + x_{22}$

$f_1(0) = 0,\ f_1(1) = 2,\ f_1(2) = 4,\ f_1(3) = 0,\ f_1(4) = -16$

$f_2(0) = 0,\ f_2(1) = 4,\ f_2(2) = 12$

So, $s_{11} = \frac{2-0}{1-0} = 2,\ s_{12} = \frac{4-2}{2-1} = 2,\ s_{13} = -4,\ s_{14} = -16$

$s_{21} = 4,\ s_{22} = 8$

We need special restrictions

i) $x_{12} = 0$ if $x_{11} < 1$ iv) $x_{22} = 0$ if $x_{21} < 1$

ii) $x_{13} = 0$ if $x_{12} < 1$

iii) $x_{14} = 0$ if $x_{13} < 1$

Since $s_{12} > s_{13} > s_{14}$, (ii) and (iii) are automatically satisfied upon optimizing.

The approximate BIP formulation is:

Maximize $2x_{11} + 2x_{12} - 4x_{13} - 16x_{14} + 4x_{21} + 8x_{22}$

subject to: $x_{11} + x_{12} + x_{13} + x_{14} + 2x_{21} + 2x_{22} \le 4$

$-x_{11} + x_{12} \le 0$

$-x_{21} + x_{22} \le 0$

x_{ij} binary $(i,j) = (1,1), \ldots, (2,2)$

c) The solution (by BIP automatic routine) is:

$x_{11} = 0, x_{12} = 0, x_{13} = 0, x_{14} = 0, x_{21} = 1, x_{22} = 1$ and $Z = 12$

In terms of the original variables, we have

$x_1 = x_{11} + x_{12} + x_{13} + x_{14} = 0,\ x_2 = x_{21} + x_{22} = 2$

(An alternate solution is $x_1 = 2, x_2 = 1$ which also gives the optimal objective value $Z = 12$)

13.9-1. $\nabla f(x_1,x_2) = \left(\frac{1}{1+x_1+x_2}, \frac{1}{1+x_1+x_2}\right)$

$\nabla f(1,1) = (\frac{1}{3},\frac{1}{3})$

Solving: max $\frac{1}{3}x_1+\frac{1}{3}x_2$
s.t. $x_1+2x_2 \le 5$ $\Rightarrow$ $x_1=5, x_2=0$
$x_1\ge 0, x_2\ge 0$

$x^{(1)} = (1,1)+t(4,-1)$

$t^*=1$ $f(x)$ increases as t increases

New $X = (5,0)$ (which is the solution found in 13.6-5(b))

2nd iter: $\nabla f(5,0) = (\frac{1}{6},\frac{1}{6})$

Solving LP: max $\frac{1}{6}x_1+\frac{1}{6}x_2$
s.t. $x_1+2x_2\le 5$ $\Rightarrow$ $x_1=5, x_2=0$
$x_1\ge 0, x_2\ge 0$

$x^{(2)} = (5,0)+t(0,0)$

So $X=(5,0)$ is optimal

Since $\nabla f(x_1,x_2) = (c_1,c_2)$ $c_1=c_2\ge 0$, the LP to be solved is

max $c_1x_1+c_1x_2$
s.t. $x_1+2x_2\le 5$ $\Rightarrow$ $x_1=5, x_2=0$ is always the solution.
$x_1\ge 0, x_2\ge 0$

Thus, $x^{(1)} = (x_1^{(0)}, x_2^{(0)}) + t\left(5-x_1^{(0)}, 0-x_2^{(0)}\right)$ (at $t=1$) $= (5,0)$ after 1st iteration.

($t^*=1$ since $f(x)$ increases with t)

(There are no complications for $(x_1,x_2)=(0,0)$)

13.9-2. $\nabla f(x_1,x_2) = \left(\frac{1}{x_1+1}, -2x_2\right)$

$\nabla f(0,0) = (1,0)$

Solving: max x_1
s.t. $x_1+2x_2\le 3$ $\Rightarrow$ $x_1=3, x_2=0$
$x_1\ge 0, x_2\ge 0$

$x^{(1)} = (0,0)+t(3,0)$

$t^*=1$ ($f(x)$ increases with t)

$X^{(1)} = (3,0)$ (the solution found in 13.6-6(b))

2nd iter: $\nabla f(3,0) = (\frac{1}{4},0)$

Solving LP: max $\frac{1}{4}x_1$
s.t. $x_1+2x_2\le 3$ $\Rightarrow$ $x_1=3, x_2=0$
$x_1\ge 0, x_2\ge 0$

$x^{(2)} = (3,0)+t(0,0)$

So $X=(3,0)$ is optimal.

13.9-3.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0, 0)	-6	-3	(1, 0)	1	(1, 0)
2	(1, 0)	-4	-3	(1, 0)	1e-8	(1, 0)

Final solution: (1, 0).

$\nabla f(x_1, x_2) = (2x_1 - 6, 3x_2^2 - 3)$

but since $x_1 \le 1, x_2 \le 1$ (since $x_1 + x_2 \le 1, x_1 \ge 0, x_2 \ge 0$)

$2x_1 - 6 < 3x_2^2 - 3$ $\quad (2x_1 - 6 \le -4 < -3 \le 3x_2^2 - 3)$

So the resulting LP will be of the form

$$\text{Min } c_1x_1 + c_2x_2 \qquad c_1 < c_2$$
$$\text{s.t. } x_1 + x_2 \le 1$$
$$x_1 \ge 0, x_2 \ge 0$$

$\Rightarrow$ (1,0) will always be the optimal solution.

$$X^{(1)} = (x_1^{(0)}, x_2^{(0)}) + t(1 - x_1^{(0)}, -x_2^{(0)})$$

So at $t^* = 1$, $X^{(1)} = (1,0)$, the optimal

13.9-4.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	c_3	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0, 0, 0)	8	2	1	(12, 0, 0)	0.33	(4, 0, 0)
2	(4, 0, 0)	0	2	1	(0, 4, 0)	0.25	(3, 1, 0)

Final solution: (3, 1, 0).

13.9-5.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(5, 5)	15	10	(30, 0)	0.088	(7.196,4.561)
2	(7.196,4.561)	4.459	22.3	(0, 15)	0.119	(6.337,5.807)
3	(6.337,5.807)	12.88	8.89	(30, 0)	0.07	(7.996, 5.4)
4	(7.996, 5.4)	4.615	18.79	(0, 15)	0.089	(7.283,6.256)
5	(7.283,6.256)	10.89	9.082	(30, 0)	0.054	(8.514,5.917)
6	(8.514,5.917)	4.611	16.72	(0, 15)	0.072	(7.903,6.569)
7	(7.903,6.569)	9.666	9.056	(30, 0)	0.045	(8.887,6.277)

Final solution: (8.8866, 6.277).

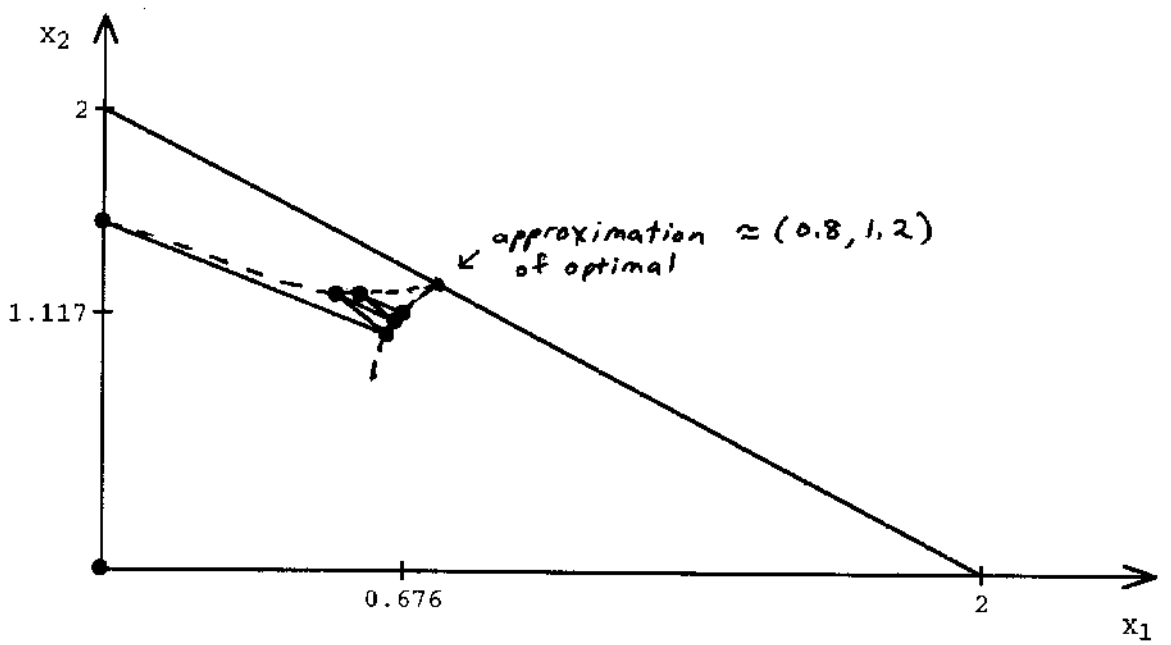
13.9-6.

a)

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0, 0)	2	3	(0, 2)	0.75	(0, 1.5)
2	(0, 1.5)	2	0	(2, 0)	0.32	(0.64, 1.02)
3	(0.64, 1.02)	0.72	0.96	(0, 2)	0.175	(0.528,1.192)
4	(0.528,1.192)	0.944	0.617	(2, 0)	0.092	(0.663,1.082)
5	(0.663,1.082)	0.674	0.835	(0, 2)	0.126	(0.579,1.198)
6	(0.579,1.198)	0.842	0.603	(2, 0)	0.068	(0.676,1.117)

Final solution: (0.676,1.1166).

b)



13.9-7.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0, 0)	126	182	(2, 6)	1	(2, 6)
2	(2, 6)	90	26	(4, 3)	0.333	(2.667, 5)
3	(2.667, 5)	78	52	(4, 3)	1e-8	(2.667, 5)

Final solution: (2.6667, 5).

This is optimal since iter. 3 gave same point as iter. 2

13.9-8.

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}(k)$	t^*	$\mathbf{x}(k)$
1	(0, 0)	32	50	(3, 2)	0.729	(2.188,1.458)
2	(2.188,1.458)	-9.87	14.81	(0, 3.2)	0.131	(1.902,1.686)
3	(1.902,1.686)	4.499	5.634	(3, 2)	0.111	(2.024,1.721)
4	(2.024,1.721)	-1.15	4.078	(0, 3.2)	0.028	(1.966,1.763)

Final solution: (1.9662,1.7629).

13.9-9.

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(0, 0)	40	30	(3, 0)	0.616	(1.847, 0)
2	(1.847, 0)	0.001	35.54	(0, 2)	0.406	(1.097,0.812)

13.9-10. (a)

k	X^(k-1)	c1	c2	X[LP]^k	t*	X^k
1	(0.25, 0.25)	2.813	3.5	(0, 1)	1	(0, 1)
2	(0, 1)	3	2	(1, 0)	0.333	(0.333,0.667)
3	(0.333,0.667)	2.667	2.667	(1, 0)	0.001	(0.334,0.666)

(b) The KKT conditions:

1a) $3 - 3x_1^2 - u \le 0$

2a) $x_1(3 - 3x_1^2 - u) = 0$

1b) $4 - 2x_2 - u \le 0$

2b) $x_2(4 - 2x_2 - u) = 0$

3) $x_1 + x_2 \le 1$

4) $u(x_1 + x_2 - 1) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

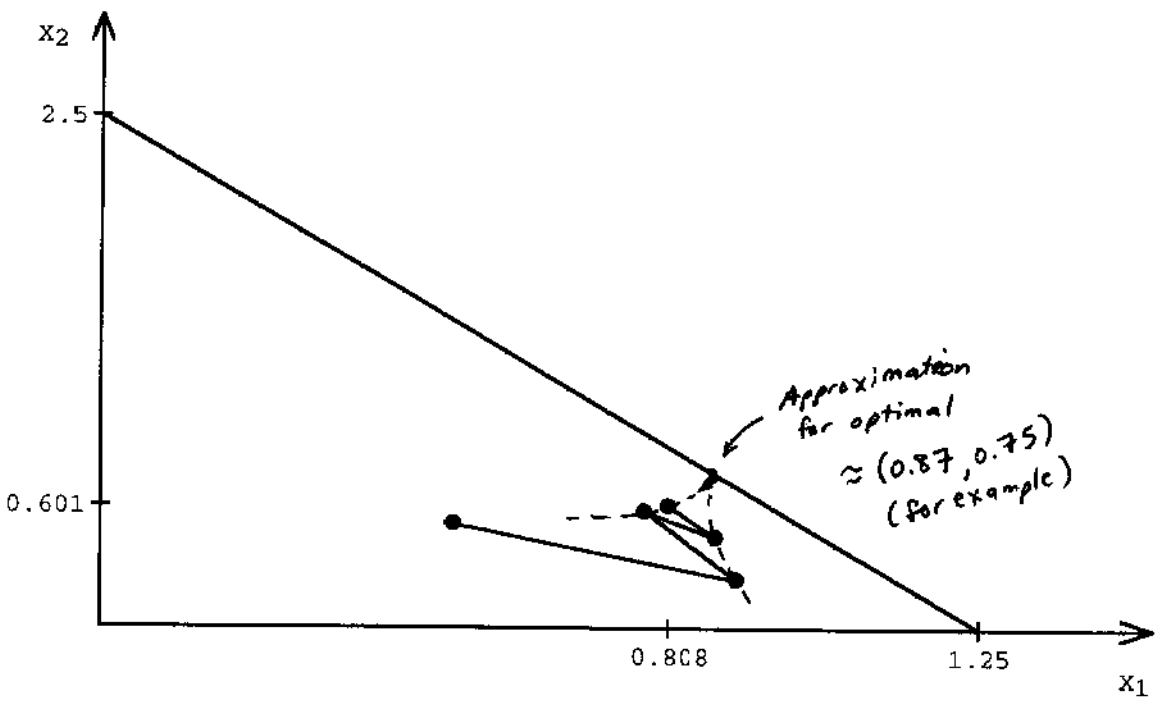
It is easy to verify that $(x_1, x_2, u) = (1/3, 2/3, 4/3)$ satisfies these conditions, so $(x_1, x_2) = (1/3, 2/3)$, the estimated solution from part (a), is optimal.

13.9-11. a)

k	$\mathbf{x}^{(k-1)}$	c_1	c_2	$\mathbf{x}_{LP}^{(k)}$	t^*	$\mathbf{x}^{(k)}$
1	(0.5, 0.5)	3.5	1	(1.25, 0)	0.541	(0.906,0.229)
2	(0.906,0.229)	1.027	1.541	(0, 2.5)	0.148	(0.771,0.566)
3	(0.771,0.566)	2.164	0.857	(1.25, 0)	0.216	(0.875,0.444)
4	(0.875,0.444)	1.323	1.112	(0, 2.5)	0.076	(0.808,0.601)

Final solution: (0.8079,0.6011).

b)



(c) The KKT conditions:

1a) $4 - 4x_1^3 - 4u \le 0$

2a) $x_1(4 - 4x_1^3 - 4u) = 0$

1b) $2 - 2x_2 - 2u \le 0$

2b) $x_2(2 - 2x_2 - 2u) = 0$

3) $4x_1 + 2x_2 \le 5$

4) $u(4x_1 + 2x_2 - 5) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

Are satisfied by $(x_1, x_2, u) = (.8934, .7131, .5737)$

So $(x_1, x_2) = (.8934, .7131)$ is optimal.

13.10-1. a) $P(\underline{x}; r) = 3x_1 + 4x_2 - x_1^3 - x_2^2 - r\left[\frac{1}{1-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

b) $\nabla P(\underline{x}; r) = \begin{bmatrix} 3 - 3x_1^2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_1^2}\right] \\ 4 - 2x_2 + r\left[\frac{-1}{(1-x_1-x_2)^2} + \frac{1}{x_2^2}\right] \end{bmatrix}$

$\Rightarrow \nabla P\left(\left(\frac{1}{4}, \frac{1}{4}\right); 1\right) = \begin{bmatrix} 14\frac{13}{16} \\ 15\frac{1}{2} \end{bmatrix}$

$x' + t\nabla P(x'; 1) = \left(\frac{1}{4} + 14\frac{13}{16}t, \frac{1}{4} + 15\frac{1}{2}t\right)$

$t^* = 0.006606$

New $x' = (0.3479, 0.3524)$

c)

k	r	x_1	x_2	f(x)
0		0.25	0.25	1.672
1	1	0.343	0.357	2.29
2	0.01	0.322	0.619	3.023
3	0.0001	0.331	0.663	3.169

d) The true solution is $(\frac{1}{3}, \frac{2}{3})$. The percentage error in:

x_1 is $\frac{|1/3 - .331|}{1/3} = 0.70\%$

x_2 is $\frac{|1/3 - .663|}{2/3} = 0.55\%$

$f(x)$ is $\frac{|3\frac{5}{27} - 3.169|}{3\frac{5}{27}} = 0.51\%$

13.10-2. a) $P(\underline{x}; r) = 4x_1 - x_1^4 + 2x_2 - x_2^2 - r\left[\frac{1}{5-4x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$

b) $\nabla P(\underline{x}; r) = \begin{bmatrix} 4 - 4x_1^3 + r\left[\frac{-4}{(5-4x_1-2x_2)^2} + \frac{1}{x_1^2}\right] \\ 2 - 2x_2 + r\left[\frac{-2}{(5-4x_1-2x_2)^2} + \frac{1}{x_2^2}\right] \end{bmatrix}$

$\Rightarrow \nabla P\left(\left(\frac{1}{2}, \frac{1}{2}\right); 1\right) = \begin{bmatrix} 6\frac{1}{2} \\ 4\frac{1}{2} \end{bmatrix}$

$x' + t\nabla P(x'; 1) = \left(\frac{1}{2} + 6\frac{1}{2}t, \frac{1}{2} + 4\frac{1}{2}t\right)$

$t^* = 0.03167$

New $x' = (0.7058, 0.6425)$

13.10-2. c)

k	r	X_1	X_2	f(x)
0		0.5	0.5	2.688
1	1	0.669	0.716	3.395
2	0.01	0.871	0.671	3.801
3	0.0001	0.891	0.708	3.849
4	0.000001	0.894	0.712	3.854

13.10-3. a) $P(\underset{\sim}{X}; r) = -X_1^4 - 2X_1^2 - 2X_1X_2 - 4X_2^2 - r\left[\frac{1}{2X_1+X_2-10} + \frac{1}{X_1+2X_2-10} + \frac{1}{X_1} + \frac{1}{X_2}\right]$

b) $\nabla P(\underset{\sim}{X}; 100) = \begin{bmatrix} -4X_1^3 - 4X_1 - 2X_2 + 100\left[\frac{2}{(2X_1+X_2-10)^2} + \frac{1}{(X_1+2X_2-10)^2} + \frac{1}{X_1^2}\right] \\ -2X_1 - 8X_2 + 100\left[\frac{1}{(2X_1+X_2-10)^2} + \frac{2}{(X_1+2X_2-10)^2} + \frac{1}{X_2^2}\right] \end{bmatrix}$

$\Rightarrow \nabla P((5,5); 100) = \begin{bmatrix} -514 \\ -34 \end{bmatrix}$

$X' + t\nabla P(X'; 100) = (5 - 514t, 5 - 34t)$

$t^* = 0.003529$

New $X' = (3.1862, 4.8802)$

c)

k	r	X_1	X_2	f(x)
0		5	5	-825
1	100	2.725	6.072	-251
2	1	2.587	4.976	-183
3	0.01	2.562	4.891	-177
4	0.0001	2.557	4.888	-176

Note that because of minimization, we converted

Min f(x) to Max $-f(x)$

Also, $g(x) \geq b$

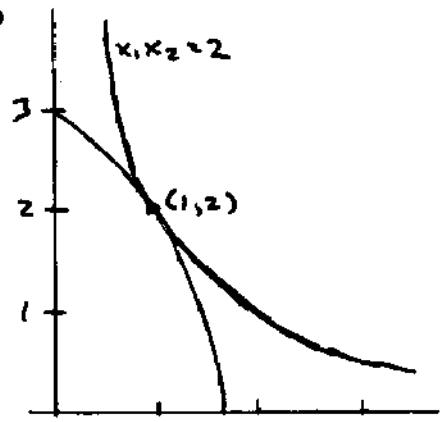
becomes $-g(x) \leq -b$.

13.10-4. (a) The KKT conditions:

1a) $x_2 - 2u_1 x_1 \le 0$
2a) $x_1(x_2 - 2u_1 x_1) = 0$
1b) $x_1 - u_1 \le 0$
2b) $x_2(x_1 - u_1) = 0$
3) $x_1^2 + x_2 \le 3$
4) $u_1(x_1^2 + x_2 - 3) = 0$
5) $x_1 \ge 0, x_2 \ge 0$
6) $u_1 \ge 0$

$(x_1, x_2) = (1,2)$ satisfies these with $u_1 = 1$.

(b)



13.10-5. a) $P(\underline{x}; r) = -2x_1 - (x_2-3)^2 - r\left(\frac{1}{x_1-3} + \frac{1}{x_2-3}\right)$

b) $\frac{\partial P(x;r)}{\partial x_1} = -2 + \frac{r}{(x_1-3)^2} = 0 \Rightarrow x_1 = \sqrt{\frac{r}{2}} + 3$

$\frac{\partial P(x;r)}{\partial x_2} = -2x_2 + 6 + \frac{r}{(x_2-3)^2} = 0 \Rightarrow x_2 = \sqrt[3]{\frac{r}{2}} + 3$

r	x_1	x_2
1	3.7071	3.7937
10^{-2}	3.0707	3.1710
10^{-4}	3.0071	3.0368
10^{-6}	3.0007	3.0079

Note $(x_1, x_2) \to (3,3)$ as $r \to 0$, so $(x_1, x_2) = (3,3)$ is optimal

c)

k	r	X1	X2	f(X)
0		4	4	-9
1	1	3.707	3.794	-8.044
2	0.01	3.07	3.179	-6.172
3	0.0001	3.007	3.056	-6.017
4	0.000001	3.001	3.011	-6.002

13.10-6. a) $P(x;r) = -\frac{(x_1+1)^3}{3} - x_2 - r\left(\frac{1}{x_1-1} + \frac{1}{x_2}\right)$

b) $\frac{\partial P(x;r)}{\partial x_1} = -(x_1+1)^2 + \frac{r}{(x_1-1)^2} = 0 \Rightarrow x_1 = \sqrt{1+\sqrt{r}}$

$\frac{\partial P(x;r)}{\partial x_2} = -1 + \frac{r}{x_2^2} = 0 \Rightarrow x_2 = \sqrt{r}$

r	x_1	x_2
1	1.4142	1
10^{-2}	1.0488	0.1
10^{-4}	1.0050	0.01
10^{-6}	1.0005	0.001

Note that $(x_1, x_2) \to (1,0)$ as $r \to 0$, so $(x_1, x_2) = (1,0)$ is optimal.

13.10-6. c)

k	r	X1	X2	f(X)
0		2	1	-8
1	1	1.459	1	-4.762
2	0.01	1.056	0.1	-2.273
3	0.0001	1.006	0.01	-2.027
4	0.000001	1.001	0.001	-2.003

13.10-7. $P(x;r) = x_1^2 - x_2^2 - x_1 - x_2 + x_1 x_2 - r/x_2$

k	r	X_1	X_2	f(X)
0		1	1	-3
1	1	-0.18	0.638	-1.01
2	0.01	-0.46	0.079	0.127
3	0.0001	-0.5	0.008	0.238

13.10-8. $P(x;r) = 2x_1 + 3x_2 - x_1^2 - x_2^2 - r\left(\frac{1}{2-x_1-x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right)$

k	r	X_1	X_2	f(X)
0		0.5	0.5	2
1	1	0.649	0.781	2.61
2	0.01	0.691	1.184	3.055
3	0.0001	0.743	1.243	3.118
4	0.000001	0.749	1.249	3.124

13.10-9. $P(x;r) = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2 - r\left(\frac{1}{4-x_1} + \frac{1}{12-2x_2} + \frac{1}{18-3x_1-2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right)$

k	r	X_1	X_2	f(X)
0		2	3	645
1	100	2.292	4.523	798.8
2	1	2.62	4.972	851.9
3	0.01	2.661	4.999	856.5
4	0.0001	2.665	5.002	856.9

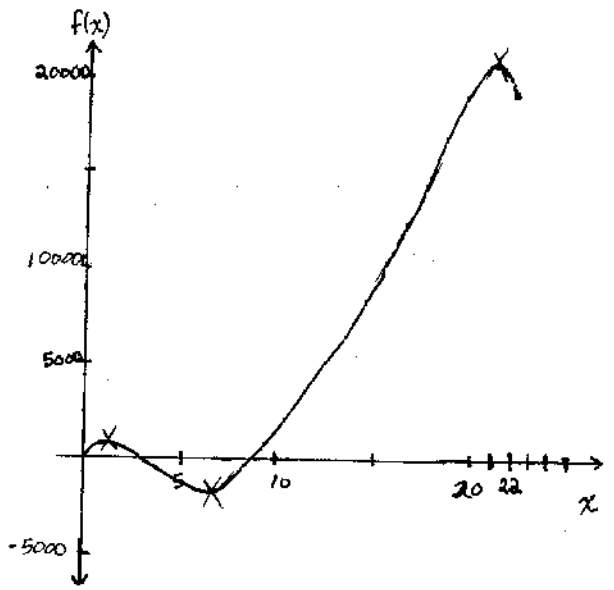
13.10-10. (a) Solving for the roots of $x^2+x-500=0$ we find x is feasible in the range $[0, \frac{-1+\sqrt{2001}}{2}]$ or $[0, 21.866]$

$f'(x) = 1000 - 800x + 120x^2 - 4x^3$

$f''(x) = -800 + 240x - 12x^2$

$f'''(x) = 240 - 24x$

A rough sketch of f(x):



X corresponds to a local maximum or minimum

(b)

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		0	5	2.5	585.94
1	-312.5	0	2.5	1.25	700.68
2	+179.7	1.25	2.5	1.875	720.06
3	-104.5	1.25	1.875	1.5625	732.56
4	+27.71	1.5625	1.875	1.7188	731.48
5	-40.82	1.5625	1.7188	1.6406	733.36
6	-7.166	1.5625	1.6406	1.6016	733.3
Stop					

Iteration	df(X)/dX	X(L)	X(U)	New X'	f(X')
0		18	21.866	19.933	19931
1	+ 1053	19.933	21.866	20.899	20546
2	+180.4	20.899	21.866	21.383	20509
3	-346.2	20.899	21.383	21.141	20559
4	- 75.1	20.899	21.141	21.02	20560
5	+54.58	21.02	21.141	21.081	20562
6	-9.778	21.02	21.081	21.051	20561
Stop					

There is a local maximum near 1.6016 and a global maximum near 21.051

(601

13.10-10. c)

k	r	X_1	f(x)	
0		3	399	Initial trial solution X = 3
1	1000	2.171	672.8	gives X = 1.625 (f(x) = 733.4)
2	100	1.704	732	as max
3	10	1.633	733.4	
4	1	1.625	733.4	
0		15	9375	Initial trial solution X = 15
1	1000	21.04	20561	gives X = 21.07 (f(x) = 20562)
2	100	21.07	20562	as max
3	10	21.07	20562	
4	1	21.07	20562	

The global maximum is $X^* = 21.07$

13.10-11. (a) $P(x;r) = 3x_1x_2 - 2x_1^2 - x_1^2 - r\left(\frac{1}{4 - x_1^2 - 2x_2^2} + \frac{1}{x_2 - 2x_2} + \frac{1}{x_1} + \frac{1}{x_2}\right) - \frac{(2 - x_1x_2^2 - x_1^2x_2)^2}{\sqrt{r}}$

(b)

k	r	X1	X2	f(X)
0		1	1	0
1	1	0.915	1.007	0.0758
2	0.01	0.848	1.169	0.1692
3	0.0001	0.843	1.175	0.1697

13.10-12. (a) $P(x;r) = x_1^3 + 4x_2^2 + 16x_3 + r\left(\frac{1}{x_1 - 1} + \frac{1}{x_2 - 1} + \frac{1}{x_3 - 1}\right) + \frac{(5 - x_1 - x_2 - x_3)^2}{\sqrt{r}}$

(b)

k	r	X1	X2	X3	f(X)
0		1.5	1.5	2	-44.38
1	0.01	1.95	1.434	1.047	-32.38
2	0.0001	2.179	1.743	1.007	-38.62
3	0.000001	2.208	1.784	1.001	-39.51
4	0.00000001	2.21	1.786	1.002	-39.6

13.10-13. (a) $P(x;r) = \sin 3x_1 + \cos 3x_2 + \sin(x_1 + x_2) + r\left(\frac{1}{1 + x_1^2 - 10x_2} + \frac{1}{100 - 10x_1 - x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right)$

(b) SUMT can be used to obtain the global minimum if "enough" different starting points are used and SUMT is run from each. If we choose a lattice of points over the feasible region so that the adjacent points don't change by more than $\frac{3}{5}\pi$ then this set of points will work for f(x). Since sin and cos have period 2π, choosing lattice points with grid size not greater than $\frac{2}{3}\pi$ will ensure that the arguments of the sin and cos terms in f will not change by more than 2π between adjacent lattice points. Since the second constraint ensures $X_1 \le 10$ and $X_2 \le 10$, we can see that at most $\left(\frac{10}{2\pi/3}\right)^2 \approx 23$ starting points are required if chosen well.

13.11-1. a) Yes. $f(\underline{x}) = f_1(x_1) + f_2(x_2)$

$f_1(x_1) = 4x_1 - x_1^2 \qquad \frac{\partial^2 f_1}{\partial x_1^2} = -2 < 0$

$f_2(x_2) = 10x_2 - x_2^2 \qquad \frac{\partial^2 f_2}{\partial x_2^2} = -2 < 0$

So $f(\underline{x})$ is concave.

$g(x) = x_1^2 + 4x_2^2$ is convex (again $g = g_1(x_1) + g_2(x_2)$ $g_1(x_1) = x_1^2$, $g_2(x_2) = 4x_2^2$ both convex)

So this is a convex programming problem.

b) No. This is not a quadratic programming problem since the constraints are not linear.

c) No. The Frank-Wolfe algorithm in 13.9 requires linear constraints so cannot be used for this problem.

d) KKT conditions are:

1a) $4 - 2x_1 - 2x_1 u \le 0$

1b) $x_1(4 - 2x_1 - 2x_1 u) = 0$

2a) $10 - 2x_2 - 8x_2 u \le 0$

2b) $x_2(10 - 2x_2 - 8x_2 u) = 0$

3) $x_1^2 + 4x_2^2 - 16 \le 0$

4) $u(x_1^2 + 4x_2^2 - 16) = 0$

5) $x_1 \ge 0, x_2 \ge 0$

6) $u \ge 0$

If $x_1 = 1$, (1b) $\Rightarrow u = 1$ but this violates (4) so $(x_1, x_2) = (1,1)$ cannot be optimal.

e) Let $x_1 = x_{11} + x_{12} + x_{13} + x_{14}$, $x_2 = x_{21} + x_{22}$

Obj: $f_1(x_1) = 4x_1 - x_1^2$, $f_2(x_2) = 10x_2 - x_2^2$

$f_1(0) = 0$, $f_1(1) = 3$, $f_1(2) = 4$, $f_1(3) = 3$, $f_1(4) = 0$

$f_2(0) = 0$, $f_2(1) = 9$, $f_2(2) = 16$

$s_{11} = \frac{3-0}{1-0} = 3$, $s_{12} = \frac{4-3}{2-1} = 1$, $s_{13} = -1$, $s_{14} = -3$

$s_{21} = 9$, $s_{22} = 7$

Constraints: $g_1(x_1) = x_1^2$, $g_2(x_2) = 4x_2^2$

$g_1(0) = 0$, $g_1(1) = 1$, $g_1(2) = 4$, $g_1(3) = 9$, $g_1(4) = 16$

$g_2(0) = 0$, $g_2(1) = 4$, $g_2(2) = 16$

$t_{11} = \frac{1-0}{1-0} = 1$, $t_{12} = \frac{4-1}{2-1} = 3$, $t_{13} = 5$, $t_{14} = 7$

$t_{21} = 4$, $t_{22} = 12$

13.11-1. e) (cont')

The approximate LP is:

$$\text{Maximize} \quad 3x_{11} + x_{12} - x_{13} - 3x_{14} + 9x_{21} + 7x_{22}$$

$$\text{subject to:} \quad x_{11} + 3x_{12} + 5x_{13} + 7x_{14} + 4x_{21} + 12x_{22} \leq 16$$

$$0 \leq x_{ij} \leq 1 \quad (\text{all } i, j)$$

f) The Simplex Method gives the solution:

Value of the
Objective Function: Z = 18.4166667

Variable		Value
X_1	(x_{11})	1
X_2	(x_{12})	0
X_3	(x_{13})	0
X_4	(x_{14})	0
X_5	(x_{21})	1
X_6	(x_{22})	0.91667

In terms of the original variables, we have

$$x_1 = x_{11} + x_{12} + x_{13} + x_{14} = 1 \quad , \quad x_2 = x_{21} + x_{22} = 1.91667$$

g) $$P(\underline{x}; r) = 4x_1 - x_1^2 + 10x_2 - x_2^2 - r\left[\frac{1}{16 - x_1^2 - 4x_2^2} + \frac{1}{x_1} + \frac{1}{x_2}\right]$$

h)

k	r	X_1	X_2	f(X)
0		2	1	13
1	1	1.504	1.754	18.22
2	0.01	1.409	1.862	18.8
3	0.0001	1.41	1.871	18.86
4	0.000001	1.411	1.871	18.86

13-1 a) If Lydia wants to ignore the risk of her investment she should invest all her money into the stock that promises the highest expected return. According to the predictions of the investment advisors the expected returns equal 0.2 (=20%) for Bigbell, 0.42 for Lotsofplace, 1 for Internetlife, 0.5 for Healthtomorrow, 0.46 for Quicky, and 0.3 for Automobile Alliance. Therefore, she should invest 100% of her money into Internetlife. The risk of this portfolio equals 0.333.

b) Lydia should put 40% of her money into the stock with the highest expected returns, 40% into the stock with the second highest expected returns, and 20% into the stock with the third highest expected returns.

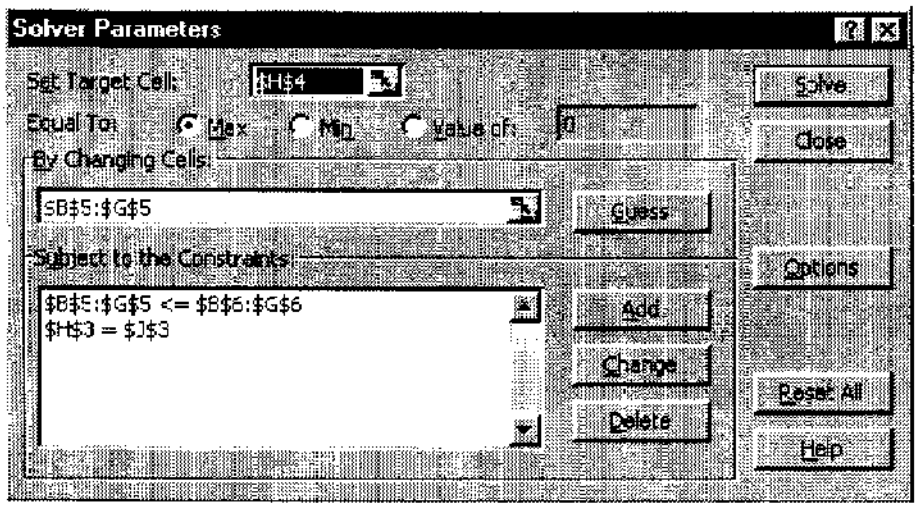
We can also find this intuitive solution by solving a simple linear programming problem. We define the variables as follows:

BB = proportion (in percentage points) of Lydia's money invested in the stock Bigbell.

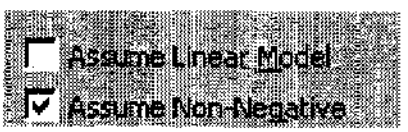
All other variables are defined correspondingly. The sum of all variables must equal 1 and no variable can exceed 0.4.

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	objective	0.2	0.42	1	0.5	0.46	0.3	0.692		
5	Solution	0	0	0.4	0.4	0.2	0			
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40	0.04548		risk
7										
8		Formula in cell H3:		"=SUMPRODUCT(B3:G3,B5:G5)"						
9		Formula in cell H4:		"=SUMPRODUCT(B4:G4,B5:G5)"						
10										
11		Formula in cell H6:		"=SUMPRODUCT(B22:G22,B5:G5)"						
12										
13										
14		BB	LOP	ILI	HEAL	QUI	AUA			
15	BB	0.032	0.005	0.03	-0.031	-0.027	0.01			
16	LOP	0.005	0.1	0.085	-0.07	-0.05	0.02			
17	ILI	0.03	0.085	0.333	-0.11	-0.02	0.042			
18	HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.06			
19	QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.02			
20	AUA	0.01	0.02	0.042	-0.06	-0.02	0.08			
21										
22		-0.0058	-0.004	0.0852	0.016	0.025	-0.0112			
23										
24		Formula in cell B22:			"=MMULT($B$5:$G$5,B$15:B$20)"					
25		Formula in cell C22:			"=MMULT($B$5:$G$5,C$15:C$20)"					
26		Formula in cell D22:			"=MMULT($B$5:$G$5,D$15:D$20)"					
27		Formula in cell E22:			"=MMULT($B$5:$G$5,E$15:E$20)"					
28		Formula in cell F22:			"=MMULT($B$5:$G$5,F$15:F$20)"					
29		Formula in cell G22:			"=MMULT($B$5:$G$5,G$15:G$20)"					

The solver dialogue box for this problem appears as follows:



Throughout this case we use the following solver options:

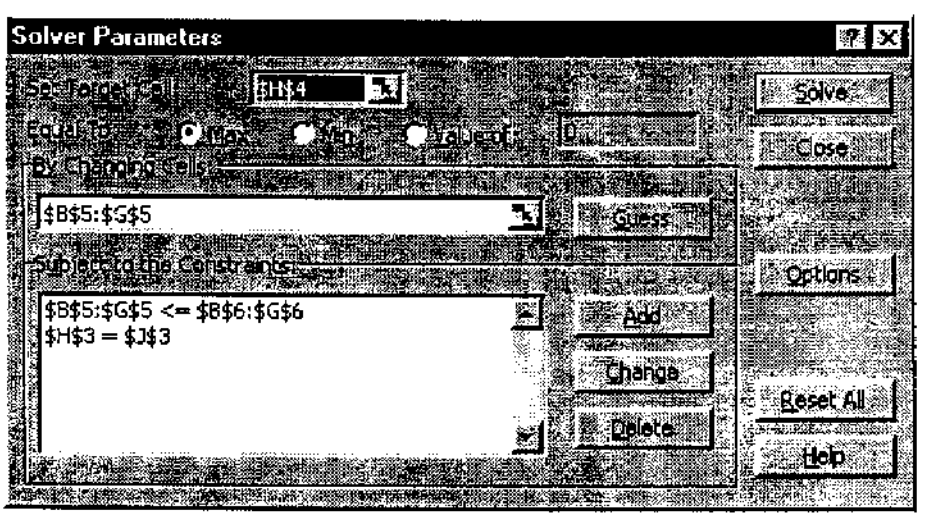


The total expected return of her portfolio is 69.2% with a total variance of 0.04548.

c) The risk of Lydia's portfolio is a quadratic function of her decision variables. We, therefore, apply quadratic programming to her decision problem.

d) Lydia's objective is now to maximize return – beta*risk. The constraints are the same as in part (b).

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	objective	0.2	0.42	1	0.5	0.46	0.3	0.68063		
5	Solution	0	0	0.4	0.4	0.2	0	0.692		Return
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40	0.04548		Risk
7								0.25		beta
8		Formula in cell H3:		"=SUMPRODUCT(B3:G3,B5:G5)"						
9		Formula in cell H4:		"=H5-H7*H6"						
10		Formula in cell H5:		"=SUMPRODUCT(B4:G4,B5:G5)"						
11		Formula in cell H6:		"=SUMPRODUCT(B22:G22,B5:G5)"						
12										
13										
14		BB	LOP	ILI	HEAL	QUI	AUA			
15	BB	0.032	0.005	0.03	-0.031	-0.027	0.01			
16	LOP	0.005	0.1	0.085	-0.07	-0.05	0.02			
17	ILI	0.03	0.085	0.333	-0.11	-0.02	0.042			
18	HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.06			
19	QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.02			
20	AUA	0.01	0.02	0.042	-0.06	-0.02	0.08			
21										
22		-0.0058	-0.004	0.0852	0.016	0.025	-0.0112			
23										
24		Formula in cell B22:			"=MMULT($B$5:$G$5,B$15:B$20)"					
25		Formula in cell C22:			"=MMULT($B$5:$G$5,C$15:C$20)"					
26		Formula in cell D22:			"=MMULT($B$5:$G$5,D$15:D$20)"					
27		Formula in cell E22:			"=MMULT($B$5:$G$5,E$15:E$20)"					
28		Formula in cell F22:			"=MMULT($B$5:$G$5,F$15:F$20)"					
29		Formula in cell G22:			"=MMULT($B$5:$G$5,G$15:G$20)"					



Due to the small emphasis (beta = 0.25) Lydia puts on risk her investment portfolio is the same as in part (b).

e) For the investment policy of part (d) Lydia invests \$6000 in ILI (1500 shares at \$4), \$6000 in HEAL (120 shares at \$50), and \$3000 in QUI (20 shares at \$150).

f) For beta = 0.5 the optimal portfolio does not change (we only display the relevant output of the spreadsheet):

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	objective	0.2	0.42	1	0.5	0.46	0.3	0.66926		
5	Solution	0	0	0.4	0.4	0.2	0	0.692		Return
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40	0.04548		Risk
7								0.5		beta

For beta = 1 we obtain the following result:

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	objective	0.2	0.42	1	0.5	0.46	0.3	0.6468257		
5	Solution	0	0.033962	0.4	0.4	0.16604	0	0.6906415		Return
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40	0.0438158		Risk
7								1		beta

Lydia invests 40% of her money in ILI and HEAL respectively, 16.604% in QUI, and 3.396% in LOP.

When beta = 2 the portfolio changes again:

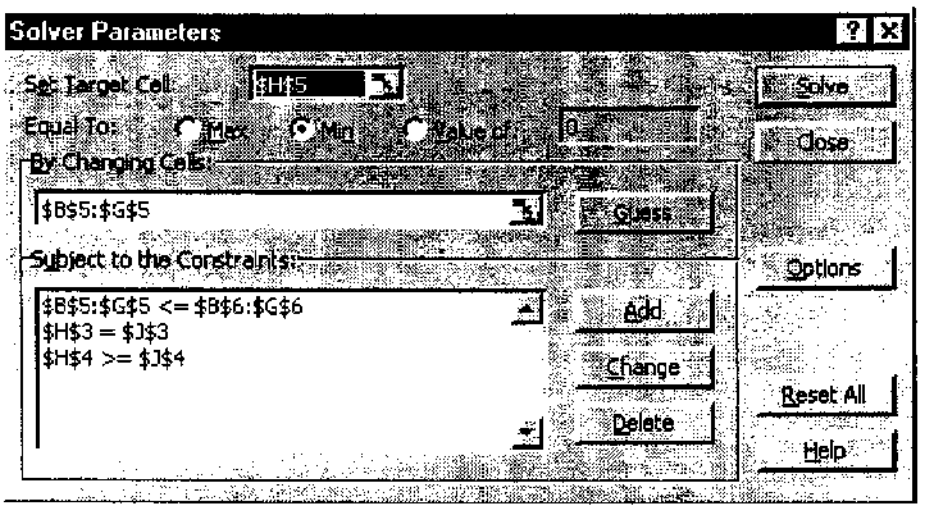
	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	objective	0.2	0.42	1	0.5	0.46	0.3	0.6037645		
5	Solution	0	0.071698	0.4	0.4	0.1283	0	0.6891321		Return
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40	0.0426838		Risk
7								2		beta

Lydia invests 40% of her money in ILI and HEAL respectively, 12.83% in QUI, and 7.17% in LOP.

g) The general rule is that as we include more and more different stocks into Lydia's portfolio, the total variance (risk) decreases - or gets "diversified away". LOP has a negative covariance with respect to QUI and HEAL. Including LOP into the portfolio thus reduces total variance. The benefit of this lower variance outweighs the fact that LOP offers a slightly lower expected return than QUI (42% vs. 46%). However, low variance becomes more and more important as beta increases. That explains why LOP is being substituted for QUI.

h) The return of Lydia's portfolio is no longer a part of the objective but now becomes part of a new constraint: return >= 0.35. The objective is now to minimize the risk.

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	return	0.2	0.42	1	0.5	0.46	0.3	0.3591996	>=	0.35
5	Solution	0.317717	0.198843	0	0.168351	0.209	0.10609	0.0013602		Risk
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40			
7										
8										
9		Formula in cell H3:		"=SUMPRODUCT(B3:G3,B5:G5)"						
10		Formula in cell H4:		"=SUMPRODUCT(B4:G4,B5:G5)"						
11		Formula in cell H5:		"=SUMPRODUCT(B22:G22,B5:G5)"						
12										
13										
14		BB	LOP	ILI	HEAL	QUI	AUA			
15	BB	0.032	0.005	0.03	-0.031	-0.027	0.01			
16	LOP	0.005	0.1	0.085	-0.07	-0.05	0.02			
17	ILI	0.03	0.085	0.333	-0.11	-0.02	0.042			
18	HEAL	-0.031	-0.07	-0.11	0.125	0.05	-0.06			
19	QUI	-0.027	-0.05	-0.02	0.05	0.065	-0.02			
20	AUA	0.01	0.02	0.042	-0.06	-0.02	0.08			
21										
22		0.00136	0.00136	0.00819	0.00136	0.00136	0.00136			
23										
24		Formula in cell B22:			"=MMULT($B$5:$G$5,B$15:B$20)"					
25		Formula in cell C22:			"=MMULT($B$5:$G$5,C$15:C$20)"					
26		Formula in cell D22:			"=MMULT($B$5:$G$5,D$15:D$20)"					
27		Formula in cell E22:			"=MMULT($B$5:$G$5,E$15:E$20)"					
28		Formula in cell F22:			"=MMULT($B$5:$G$5,F$15:F$20)"					
29		Formula in cell G22:			"=MMULT($B$5:$G$5,G$15:G$20)"					



Lydia's optimal portfolio consists of 31.77% BB, 19.88% LOP, 16.84% HEAL, 20.90% QUI, and 10.61% AUA. Her expected return equals 35.92% with a risk of 0.00136.

i) Since the return constraint in not binding in part (h), decreasing the right-hand-side will not change the optimal solution. The minimum risk for a minimum expected return of 25% is the same as for a minimum expected return of 35%.

For a minimum expected return of 40% we obtain a new portfolio:

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	BB	LOP	ILI	HEAL	QUI	AUA	Totals		
3	budget	1	1	1	1	1	1	1	=	1
4	return	0.2	0.42	1	0.5	0.46	0.3	0.4	>=	0.4
5	Solution	0.229309	0.21032	0.03392	0.219701	0.18757	0.11979	0.0023269		Risk
6	Maximum	0.40	0.40	0.40	0.40	0.40	0.40			

Lydia's optimal portfolio consists of 22.93% BB, 21.03% LOP, 3.39% ILI, 21.97% HEAL, 18.76% QUI, and 11.92% AUA. Her expected return equals 40% with a risk of 0.00233.

) Lydia's approach is very risky. She puts a lot of confidence in the advice of the two investment experts. Of course, Lydia cannot expect to find an optimal investment strategy with her model if the estimates she uses for the input parameters are wrong.

13.2 a) When Charles sells a portion of his B-Bonds in a given year, the first DM 6100 of interest are tax-free, but the interest earnings exceeding DM 6100 are levied a 30 percent tax. Therefore, Charles encounters decreasing marginal returns, and we can use separable programming to solve this problem.

Define the following variables:

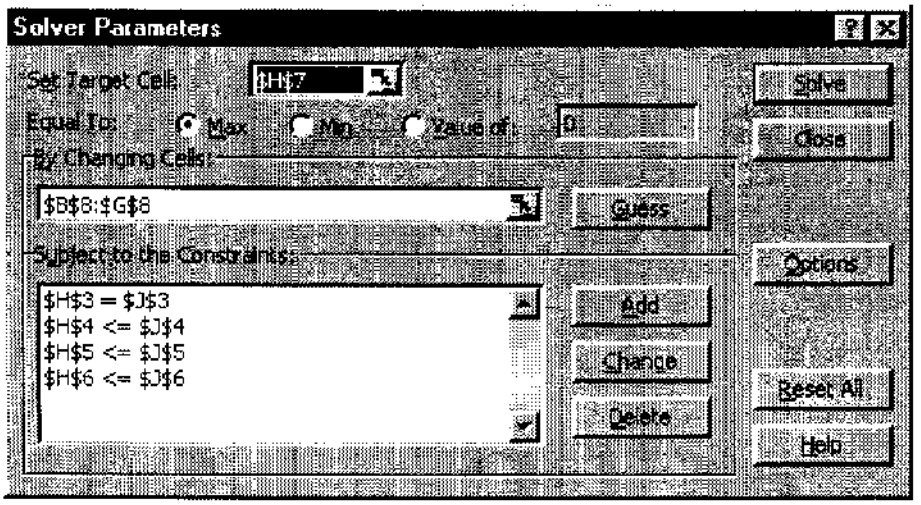
NoTax5 = The base amount of B-Bonds Charles sells in the fifth year that yield untaxed interest

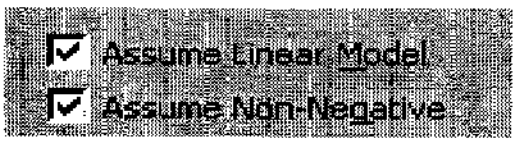
Tax5 = The base amount of B-Bonds Charles sells in the fifth year that yield taxed interest

Similarly, NoTax6, Tax6, NoTax7, and Tax7 are defined.

The sum of these six variables must equal the total of DM 30,000 that Charles invested at the beginning of year 1. When Charles sells B-Bonds with the base amount NoTax5, he earns 0.5001*NoTax5 in interest. In order for him not to pay any taxes on this amount, the interest must be less than or equal DM 6100. We have to include this constraint. Any additional base amount of B-bonds sold in year 5 yields Charles only 0.7*0.5001=0.35007. A similar reasoning applies to the variables for the other years. The objective is to maximize Charles' interest income.

	A	B	C	D	E	F	G	H	I	J
1										
2	Server	NoTax5	Tax5	NoTax6	Tax6	NoTax7	Tax7	Totals		
3	Selling	1	1	1	1	1	1	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	<=	6100
5	Untaxed6	0	0	0.6351	0	0	0	6100	<=	6100
6	Untaxed7	0	0	0	0	0.7823	0	6100	<=	6100
7	Interest	0.5001	0.35007	0.6351	0.44457	0.7823	0.54761	19098.62		
8	Solution	0	0	9604.79	0	7797.52	12597.69			
9										
10		Formula in cell H3:			"=SUMPRODUCT(B3:G3,B8:G8)"					
11		Formula in cell H4:			"=SUMPRODUCT(B4:G4,B8:G8)"					
12		Formula in cell H5:			"=SUMPRODUCT(B5:G5,B8:G8)"					
13		Formula in cell H6:			"=SUMPRODUCT(B6:G6,B8:G8)"					
14		Formula in cell H7:			"=SUMPRODUCT(B7:G7,B8:G8)"					





b) The optimal investment strategy for Charles is to sell a base amount of DM 9604.79 at the end of year 6 and the remaining DM 20395.21 at the end of year 7. His total after-tax interest income equals DM 19098.62.

c) When Charles sells all B-Bonds in the seventh year, then he must pay 30% of taxes on the amount of interest income exceeding DM 6100. This amount is earned interest not only from the last year, but it includes interest from all the previous years. So Charles does not pay 30% tax on the 9 percent interest he earned in the last year, but he effectively pays tax on the total interest of all the years. This tax payment decreases his after-tax interest so much that it pays for him to sell some of his bonds in the sixth year in order to take advantage of the yearly tax-free income of DM 6100. Compare the total amount of interest Charles earns if he sells tax-free after year 6 and taxed after year 7: In the former case his total interest equals 63.51% while in the latter case it is only 54.761%. Therefore, it is better to sell some bonds at the end of the sixth year than to keep them until the end of the last year.

d) The following observation greatly simplifies the analysis of this problem: The interest rate on the CD is much lower than the yearly interest rates on the B-Bonds. Therefore, it can never be optimal for Charles to sell B-Bonds in year 5 in order to buy a CD for year 6 if he does not take advantage of the maximal tax-free amount of selling B-Bonds in year 6. Put differently, Charles will only buy a CD for year 6 if he already plans to sell B-Bonds in year 6 to obtain at least the maximal tax-free amount of interest. The same argument applies to year 7. This observation implies that Charles will never earn untaxed interest on a CD. Therefore, his yearly interest on the CD will always be $0.7*0.04 = 0.028 = 2.8\%$.

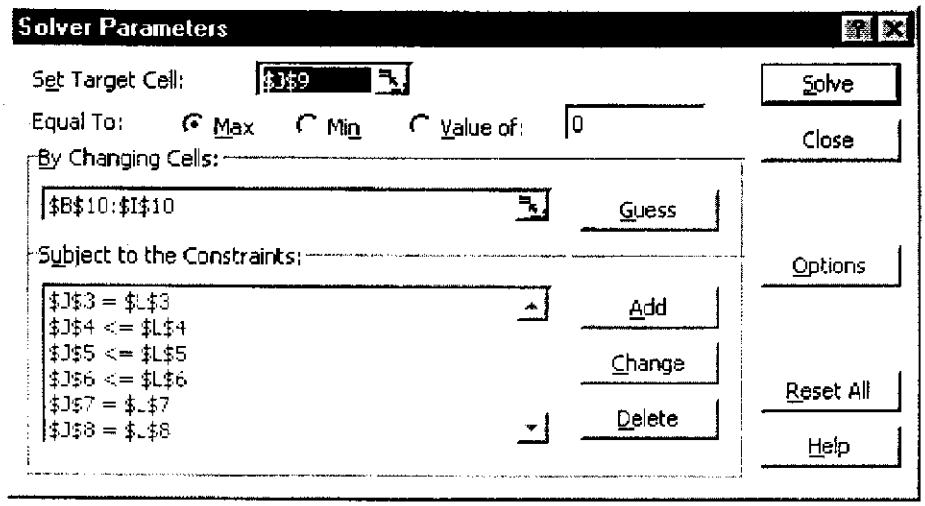
Define

CD6 = Amount invested in a CD in year 6

CD7 = Amount invested in a CD in year 7

The amount of money Charles can invest in a CD in year 6 equals the base amount of B-Bonds sold in year 5 plus the total after-tax interest earned on the base amount. This condition results in the constraint CD6 = 1.5001*NoTax5 + 1.35007*Tax5. The corresponding constraint for CD7 is CD7 = 1.6351*NoTax6 + 1.44457*Tax6 + 1.028*CD6.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	NoTax5	Tax5	CD6	NoTax6	Tax6	CD7	NoTax7	Tax7	Totals		
3	Selling	1	1	0	1	1	0	1	1	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	6100	<=	6100
5	Untaxed6	0	0	0	0.6351	0	0	0	0	6100	<=	6100
6	Untaxed7	0	0	0	0	0	0	0.7823	0	6100	<=	6100
7	CDinvest5	1.5001	1.35007	-1	0	0	0	0	0	0	=	0
8	CDinvest6	0	0	1.028	1.6351	1.44457	-1	0	0	0	=	0
9	Interest	0.5001	0.35007	0.028	0.6351	0.44457	0.028	0.7823	0.5476	19997.86		
10	Solution	12197.56	0	18297.56	9604.79	0	34514.68	7797.524	00.133			
11												
12		Formula in cell J3:			"=SUMPRODUCT(B3:I3,B10:I10)"							
13		Formula in cell J4:			"=SUMPRODUCT(B4:I4,B10:I10)"							
14		Formula in cell J5:			"=SUMPRODUCT(B5:I5,B10:I10)"							
15		Formula in cell J6:			"=SUMPRODUCT(B6:I6,B10:I10)"							
16		Formula in cell J7:			"=SUMPRODUCT(B7:I7,B10:I10)"							
17		Formula in cell J8:			"=SUMPRODUCT(B8:I8,B10:I10)"							
18		Formula in cell J9:			"=SUMPRODUCT(B9:I9,B10:I10)"							



The options for the solver are the same as before.

Charles should sell the maximal base amount of B-bonds in year 5 that yields tax-free interest and then invest this money (base amount + interest) into a one-year CD for year 6. In year 6 he should sell again the maximal base amount of B-bonds that yields tax-free interest and then invest this money (base amount + interest) and the money from his CD into a one-year CD for year 7. In year 7 he should sell the remainder of the base amount of B-bonds. He again takes advantage of the maximum tax-free amount, but he also sells a base amount of DM 400.13 for which he must pay taxes on the interest earnings.

e) The right-hand side of the selling constraint needs to be changed:

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	NoTax5	Tax5	CD6	NoTax6	Tax6	CD7	NoTax7	Tax7	Totals		
3	Selling	1	1	0	1	1	0	1	1	50000	=	50000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	6100	<=	6100
5	Untaxed6	0	0	0	0.6351	0	0	0	0	6100	<=	6100
6	Untaxed7	0	0	0	0	0	0	0.7823	0	6100	<=	6100
7	CDInvest5	1.5001	1.35007	-1	0	0	0	0	0	0	=	0
8	CDInvest6	0	0	1.028	1.6351	1.44457	-1	0	0	0	=	0
9	Interest	0.5001	0.35007	0.028	0.6351	0.44457	0.028	0.7823	0.54761	30950.06		
10	Solution	12197.56	0	18297.56	9604.79	0	34514.68	7797.52	20400.13			
11												
12		Formula in cell J3:			"=SUMPRODUCT(B3:I3,B10:I10)"							
13		Formula in cell J4:			"=SUMPRODUCT(B4:I4,B10:I10)"							
14		Formula in cell J5:			"=SUMPRODUCT(B5:I5,B10:I10)"							
15		Formula in cell J6:			"=SUMPRODUCT(B6:I6,B10:I10)"							
16		Formula in cell J7:			"=SUMPRODUCT(B7:I7,B10:I10)"							
17		Formula in cell J8:			"=SUMPRODUCT(B8:I8,B10:I10)"							
18		Formula in cell J9:			"=SUMPRODUCT(B9:I9,B10:I10)"							

The optimal investment strategy is similar to the previous one, except that Charles must now pay taxes on the interest earned from selling a base amount of DM 20400.13 in year 7.

f) The right-hand sides of the Untaxed5, Untaxed6, and Untaxed7 constraints need to be changed.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	Server	NoTax5	Tax5	CD6	NoTax6	Tax6	CD7	NoTax7	Tax7	Totals		
3	Selling	1	1	0	1	1	0	1	1	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	0	<=	12200
5	Untaxed6	0	0	0	0.6351	0	0	0	0	9148.59	<=	12200
6	Untaxed7	0	0	0	0	0	0	0.7823	0	12200	<=	12200
7	CDInvest 5	1.5001	1.35007	-1	0	0	0	0	0	0	=	0
8	CDInvest 6	0	0	1.028	1.6351	1.44457	-1	0	0	0	=	0
9	Interest	0.5001	0.35007	0.028	0.6351	0.44457	0.028	0.7823	0.54761	22008.09		
10	Solution	0	0	0	14404.96	0	23553.55	15595.04	0			
11												
12		Formula in cell J3:			"=SUMPRODUCT(B3:I3,B10:I10)"							
13		Formula in cell J4:			"=SUMPRODUCT(B4:I4,B10:I10)"							
14		Formula in cell J5:			"=SUMPRODUCT(B5:I5,B10:I10)"							
15		Formula in cell J6:			"=SUMPRODUCT(B6:I6,B10:I10)"							
16		Formula in cell J7:			"=SUMPRODUCT(B7:I7,B10:I10)"							
17		Formula in cell J8:			"=SUMPRODUCT(B8:I8,B10:I10)"							
18		Formula in cell J9:			"=SUMPRODUCT(B9:I9,B10:I10)"							

By getting married in the fifth year, Charles can increase his interest income by (22008.09 - 19997.86) = 2010.23 German marks. He should sell the maximal base amount of B-Bonds earning tax-free interest in year 7 (DM 15595.04). The remainder of DM 14404.96 he should sell at the end of year 6. His entire interest income on this base amount will be tax-free. He then should invest the total amount (base amount + interest) in a CD for the seventh year.

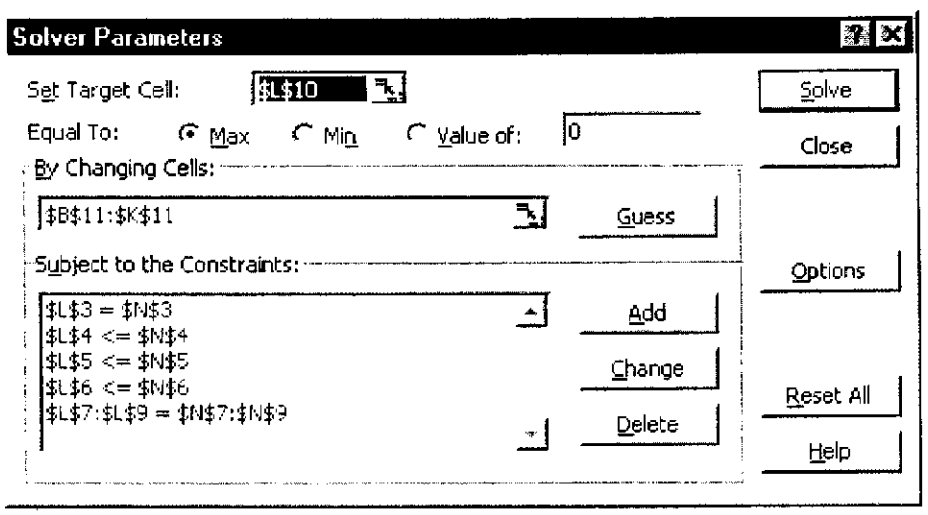
g) Instead of maximizing his interest income Charles now wants to maximize the expected dollar amount he will have at the end of the seventh year. He considers exchanging marks for dollars either at the end of year 5 or at the end of year 7. We define

CD-US = The amount of money (in dollars) Charles invests in a two-year CD at the end of year 5

US = The amount of dollars Charles converts at the end of year 7

The total amount of money in dollars Charles has at the end of year 7 equals (1.036)^2*CD-US + US; this function is the new objective. At the end of year 5, 1 US$ is assumed to equal DM 1.50, so Charles can exchange marks for dollars at this rate in year 5; this condition must be included as a constraint. Similarly, we include a constraint for the currency conversion at the end of the last year.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2	Server	NoTax5	Tax5	CD6	CD-US	NoTax6	Tax6	CD7	NoTax7	Tax7	US	Totals		
3	Selling	1	1	0	0	1	1	0	1	1	0	30000	=	30000
4	Untaxed5	0.5001	0	0	0	0	0	0	0	0	0	6100	<=	6100
5	Untaxed6	0	0	0	0	0.6351	0	0	0	0	0	0	<=	6100
6	Untaxed7	0	0	0	0	0	0	0	0.7823	0	0	6100	<=	6100
7	CDinvest5	1.5001	1.35007	-1	-1.5	0	0	0	0	0	0	0	=	0
8	CDinvest6	0	0	1.028	0	1.6351	1.44457	-1	0	0	0	0	=	0
9	Conversion	0	0	0	0	0	0	1.028	1.7823	1.5461	-1.8	0	=	0
10	Dollars	0	0	0	1.073296	0	0	0	0	0	1	30478.23		
11	Solution	12197.56	10004.92	0	21203.27	0	0	0	7797.52	0	7720.84			
12														
13														
14			Formula in cell L3:			"=SUMPRODUCT(B3:K3,B11:K11)"								
15			Formula in cell L4:			"=SUMPRODUCT(B4:K4,B11:K11)"								
16			Formula in cell L5:			"=SUMPRODUCT(B5:K5,B11:K11)"								
17			Formula in cell L6:			"=SUMPRODUCT(B6:K6,B11:K11)"								
18			Formula in cell L7:			"=SUMPRODUCT(B7:K7,B11:K11)"								
19			Formula in cell L8:			"=SUMPRODUCT(B8:K8,B11:K11)"								
20			Formula in cell L9:			"=SUMPRODUCT(B9:K9,B11:K11)"								
21			Formula in cell L10			"=SUMPRODUCT(B10:K10,B11:K11)"								



Charles converts DM (1.5001*12197.56 + 1.35007*10004.92) to dollars at the end of year 5. With the exchange rate of DM 1.50 for $1, he is able to invest $21203.27 in the American CD. At the end of the seventh year he converts the remaining DM (1.7823*7797.52) to dollars. The total amount of his investments at the end of year 7 will be $30478.23.

CHAPTER 14

14.1-1 Let player I be the labor union with strategy i being to decrease its wage demand by $10(i-1)¢$.
Let player II be the management with strategy i being to increase its offer by $10(i-1)¢$.

The payoff matrix is:

		II					
		1	2	3	4	5	6
I	1	1.35	1.2	1.3	1.4	1.5	1.6
	2	1.5	1.35	1.3	1.4	1.5	1.6
	3	1.4	1.4	1.35	1.4	1.5	1.6
	4	1.3	1.3	1.3	1.35	1.5	1.6
	5	1.2	1.2	1.2	1.2	1.35	1.6
	6	1.1	1.1	1.1	1.1	1.1	1.35

14.1-2 Label the products respectively A and B. Then the strategies for each manufacturer are:
1- Normal development of both products
2- Crash development of product A
3- Crash development of product B

Let $p_{ij} = \frac{(\%\text{ increase to I from A}) + (\%\text{ increase to I from B})}{2}$

The payoff matrix then becomes

I \ II	1	2	3	Row min
1	8	10	10	8 ← max
2	4	-4	13	-4
3	4	13	-4	-4
Column Maximum:	8	13	13	
	↑ min			

Hence, both manufacturers should use normal development and I will increase his share of the market by 8%.

14.1-3 Each player has the same strategy set. A strategy must specify the first chip chosen and, for each possible first choice by the opponent, choices of second and third chips. For example, a typical strategy is "Pick i first. If opponent chooses W, pick j_1 and k_1. If opponent chooses R, pick j_2 and k_2. If opponent chooses B, pick j_3 and k_3" where, for $\ell = 1,2,3$ $\{i, j_\ell, k_\ell\} = \{W, R, B\}$. There are 3 choices for i and, for each i, 8 choices of "conditional" strategies, forming 24 distinct strategies. Payoffs are determined from the table (all net payoffs are either \$120, 0 or -120, depending on whether player 1 wins 3 times, wins 1 time and ties 1 time or loses 3 times).

14.2-1

a) Strategies 4, 5 and 6 of player II are dominated by strategy 3.
Strategies 4, 5 and 6 of player I are dominated by strategy 3.
Strategy 1 of player II is dominated by strategy 3.
Strategy 1 of player I is dominated by strategy 3.
Strategy 2 of player II is dominated by strategy 3.
Strategy 2 of player I is dominated by strategy 3.
Therefore, the optimal strategy is for the labor union to decrease its demand by 20¢ and for management to increase its offer by 20¢.
A wage of \$1.35 will be decided.

b)

		II						
		1	2	3	4	5	6	Row Minimum
I	1	1.35	1.2	1.3	1.4	1.5	1.6	1.2
	2	1.5	1.35	1.3	1.4	1.5	1.6	1.3
	3	1.4	1.4	1.35	1.4	1.5	1.6	1.35 ← max
	4	1.3	1.3	1.3	1.35	1.5	1.6	1.3
	5	1.2	1.2	1.2	1.2	1.35	1.6	1.2
	6	1.1	1.1	1.1	1.1	1.1	1.35	1.1
Column Maximum		1.5	1.4	1.35 ↑ min	1.4	1.5	1.6	

14.2-2

(a) Strategy 3 of player I is dominated by strategy 2.
Strategy 3 of player II is dominated by strategy 1.
Strategy 1 of player I is dominated by strategy 2.
Strategy 2 of player II is dominated by strategy 1.
Therefore, the optimal strategy is for player I to choose strategy 2 and player II to choose strategy 1 resulting in a payoff of 1 to player I.

14.2-2 b) Strategy 1 of player II is dominated by strategy 3.
Strategy 2 of player I is dominated by both strategies 1 and 3.
Strategy 2 of player II is dominated by strategy 3.
Strategy 1 of player I is dominated by strategy 3.
Therefore, the optimal strategy is for player I to choose strategy 3 and player II to choose strategy 3 resulting in a payoff of 1 to player II.

14.2-3 Strategy 1 of player II is dominated by strategy 3.
Strategy 4 of player II is dominated by strategy 2.
Strategies 1 and 2 of player I are dominated by strategy 3.
Strategy 2 of player II is dominated by strategy 3.
Therefore, the optimal strategy is for player I to choose strategy 3 and player II to choose strategy 3 resulting in a payoff of 1 to player II.

14.2-4

	II: 1	2	3	Row Minimum
I: 1	1	-1	1	-1
2	-2	0	3	-2
3	3	1	2	1 ← max
Column Maximum	3	1 ↑ min	3	

v = 1, player I uses strategy 3 and player II uses strategy 2.

The game is stable with saddlepoint (3,2)

14.2-5

	II: 1	2	3	4	Row Minimum
I: 1	3	-3	-2	-4	-4
2	-4	-2	-1	1	-4
3	1	-1	2	0	-1 ← max
Column Maximum	3	-1 ↑ min	2	1	

v = -1, player I uses strategy 3 and player II uses strategy 2.

The game is stable with saddlepoint (3,2)

14.2-6 (a)

	II: 1	2	3	Row Minimum
I: 1	2	3	1	1 ← max
2	1	4	0	0
3	3	-2	-1	-2
Column maximum	3	4	1 ↑ min	

v = 1, player I uses strategy 1 and player II uses strategy 3.

14.2-6 (b) Strategy 1 of player II is dominated by strategy 3.
Strategy 3 of player I is dominated by strategies 1 and 2.
Strategy 2 of player II is dominated by strategy 3.
Strategy 2 of player I is dominated by strategy 1.
Therefore, the optimal strategy is for player I to choose strategy 1 and player II to choose strategy 3 resulting in a payoff of 1 to player I.

14.2-7 (a)

I \ II	1	2	3	Row Minimum
1	7	-1	3	-1
2	1	0	2	0 ← max
3	-5	-3	1	-5
Column Maximum:	7	0	3	
		↑ min		

Hence, $v = 0$ with Politician I using Issue 2 and Politician II using Issue 2.

(b) Let p_{ij} = P{winning or tying election for Politician I}
Then the payoff matrix becomes

I \ II	1	2	3
1	1	0	3/5
2	1/5	0	2/5
3	0	0	1/5

Strategy 3 of Politician I dominated by strategy 2.
Strategy 2 of Politician I dominated by strategy 1.
Strategies 1 and 3 of Politician II dominated by strategy 2.
Hence, eliminating dominated strategies gives $v = 0$ with Politician I using Issue 1 and Politician II using Issue 2. Therefore, Politician II can prevent Politician I from winning or tying.

(c) Let $p_{ij} = \begin{cases} 1 & \text{if Politician I will win or tie} \\ 0 & \text{if Politician II will win} \end{cases}$

Then the payoff matrix becomes

I \ II	1	2	3
1	1	0	0
2	0	0	0
3	0	0	0

(CONT)

14.2-7 (c) cont. Min [Column maxima] = 0 = Max [Row minima]. Hence the minimax criterion yields v=0 (Politician I cannot win). Politician I can use any issue and Politician II can use Issue 2 or 3. However, since Issue 1 offers Politician I his only chance of winning, he should use that and hope Politician II makes an error and also uses Issue 1.

14.2-8 Advantages: It provides the best possible guarantee on what the worst outcome can be, regardless of how skillfully the opponent plays the game. Therefore it reduces the risk of very undesirable outcomes to a minimum.

Disadvantages: It is a very conservative approach, and, therefore, it may yield far from the best attainable results if the opponent is not skillful.

14.3-1 (a)

Strategies for player I	Strategies for player II
1- Pass on heads or tails	1- If player I bets, call.
2- Bet on heads or tails	2- If player I bets, pass
3. Pass on heads, bet on tails	
4- Pass on tails, bet on heads	

b)

I \ II	1	2
1	-5	-5
2	0	5
3	$-15/2$	0
4	$5/2$	0

Strategies 1 and 3 of player I are dominated by strategy 2. Eliminating the dominated strategies we obtain the payoff table:

I \ II	1	2
2	0	5
4	$5/2$	0

14.3-1 (c) The payoff matrix is:

I \ II	1	2	Row minimum
1	-5	-5	-5
2	0	5	0
3	-15/2	0	-15/2
4	5/2	0	0
Column maximum	5/2 ↑ min	5	max = 0

Min[Column maxima] ≠ Max[Row minima], therefore there is no saddle point.

If either player chose a pure strategy, the other player could adjust his strategy in such a way as to cause the first player to want to change his strategy, too. Mixed strategies are needed.

d) expected payoff = $p_{11} x_2 y_1 + p_{12} x_2 y_2 + p_{41} x_4 y_1 + p_{42} x_4 y_2$,

where $x_2 + x_4 = 1$ are the probabilities of each player using
$y_1 + y_2 = 1$ each non-dominated strategy (from (b)).

case (i): $y_1 = 1, y_2 = 0$ $\quad \frac{5}{2} x_4 = \frac{5}{2}(1 - x_2)$

(ii): $y_1 = 0, y_2 = 1$ $\quad 5 x_2 = 5(1 - x_4)$

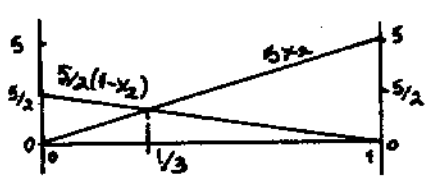
(iii): $y_1 = \frac{1}{2} = y_2$ $\quad 5 x_2 (\frac{1}{2}) + 5/2\, x_4 (\frac{1}{2})$

$= \frac{5}{2} x_2 + \frac{5}{4}(1 - x_2)$

$= \frac{5}{4} x_2 + \frac{5}{4}$

14.4-1

(y_1, y_2)	Expected Payoff
(1, 0)	$5/2(1 - x_2)$
(0, 1)	$5 x_2$



$5/2(1 - x_2) = 5x_2 \Rightarrow (x_1^*, x_2^*, x_3^*, x_4^*) = (0, 1/3, 0, 2/3)$ and $v = 5/3$

$y_1^*(5/2)(1 - x_2) + y_2^*(5 x_2) = 5/3$ for $0 \le x_2 \le 1 \Rightarrow \frac{5}{2} \cdot y_1^* = 5/3$

$5\, y_2^* = 5/3$

$\Rightarrow (y_1^*, y_2^*) = (2/3, 1/3)$

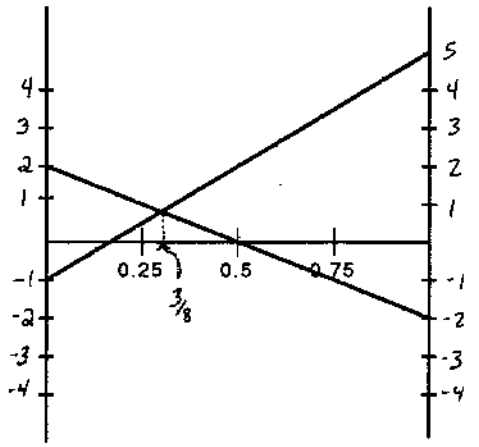
14.4-2

(y_1, y_2)	Expected Payoff
$(1,0)$	$3x_1 - 1(1-x_1) = 4x_1 - 1$
$(0,1)$	$-2x_1 + 2(1-x_1) = -4x_1 + 2$

$4x_1 - 1 = -4x_1 + 2 \Rightarrow x_1^* = 3/8,\ x_2^* = 5/8$

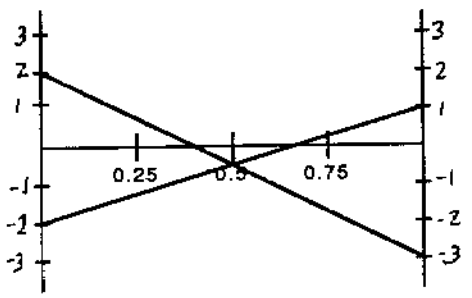
$v = 4(3/8) - 1 = 1/2$

$$\left.\begin{aligned} 3y_1^* - 2y_2^* &= 1/2 \\ -y_1^* + 2y_2^* &= 1/2 \end{aligned}\right\} \Rightarrow y_1^* = y_2^* = 1/2$$



The payoff matrix for player II is:

I \ II	1	2
1	-3	2
2	1	-2

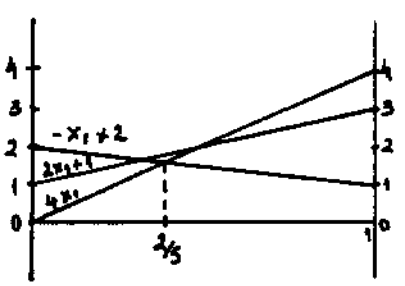


(x_1, x_2)	Expected Payoff
$(1,0)$	$-3y_1 + 2(1-y_1) = -5y_1 + 2$
$(0,1)$	$y_1 - 2(1-y_1) = 3y_1 - 2$

$-5y_1 + 2 = 3y_1 - 2 \Rightarrow y_1^* = 1/2,\ y_2^* = 1/2$

14.4-3 (a)

(y_1, y_2, y_3)	Expected payoff
$(1,0,0)$	$4x_1$
$(0,1,0)$	$3x_1 + (1-x_1) = 2x_1 + 1$
$(0,0,1)$	$x_1 + 2(1-x_1) = -x_1 + 2$



$4x_1 = -x_1 + 2 \Rightarrow (x_1^*, x_2^*) = (2/5, 3/5)$

and $v = 8/5$

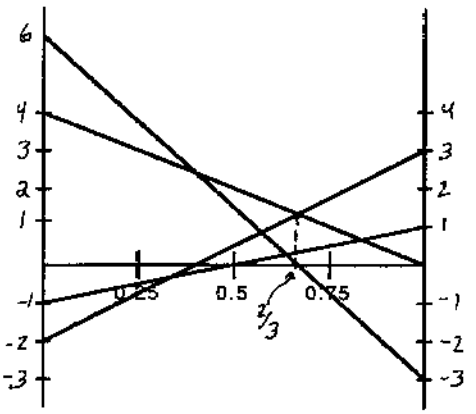
$y_1^*(4x_1) + y_3^*(-x_1+2) = 8/5$ for $0 \le x_1 \le 1 \Rightarrow 2y_3^* = 8/5$

$4y_1^* + y_3^* = 8/5$

$\Rightarrow (y_1^*, y_2^*, y_3^*) = (1/5, 0, 4/5)$

14.4-3(b) Strategy 3 of player II is dominated by strategy 1 reducing the table to:

		II	
		1	2
I	1	1	-1
	2	0	4
	3	3	-2
	4	-3	6



(x_1, x_2, x_3, x_4)	Expected Payoff
$(1, 0, 0, 0)$	$y_1 - (1-y_1) = 2y_1 - 1$
$(0, 1, 0, 0)$	$4(1-y_1) = -4y_1 + 4$
$(0, 0, 1, 0)$	$3y_1 - 2(1-y_1) = 5y_1 - 2$
$(0, 0, 0, 1)$	$-3y_1 + 6(1-y_1) = -9y_1 + 6$

$y_1^* = 2/3 \quad y_2^* = 1/3$

$v = 4/3$

14.4-4(a)

Strategies for A.J. Team	Strategies for G.N. Team
1. John does not swim butterfly	1. Mark does not swim butterfly
2. John does not swim backstroke	2. Mark does not swim backstroke
3. John does not swim breaststroke	3. Mark does not swim breaststroke

Let the payoff entries be the total points won in all three events when a given pair of strategies are used by the teams. Then the payoff matrix becomes:

AJ \ GN	1	2	3
1	14	13	12
2	13	12	12
3	12	12	13

Strategy 2 of A.J. Team dominated by strategy 1.
Strategy 1 of G.N. Team dominated by strategy 2

Final payoff table

A.J. \ G.N.	2	3
1	13	12
3	12	13

(y_2, y_3)	Expected Payoff
$(1, 0)$	$13x_1 + 12(1-x_1) = x_1 + 12$
$(0, 1)$	$12x_1 + 13(1-x_1) = -x_1 + 13$

$\Rightarrow x_1 + 12 = -x_1 + 13$

$\therefore x_1^* = 1/2, \; x_2^* = 0, \; x_3^* = 1/2$

and $V = 12\ 1/2$

144-4 (a) (cont.)

$y_2^*(x_1+12)+y_3^*(-x_1+13) = 12\frac{1}{2}$ for $0 \le x_1 \le 1 \Rightarrow 12y_2^* + 13y_3^* = 12\frac{1}{2}$

$13y_2^* + 12y_3^* = 12\frac{1}{2}$

$\therefore \quad y_1^* = 0, \quad y_2^* = \frac{1}{2}, \quad y_3^* = \frac{1}{2}$

That is, John should always swim the backstroke and should swim the butterfly or breaststroke each with probability $\frac{1}{2}$. Also, Mark should always swim the butterfly and should swim the backstroke or breaststroke each with probability $\frac{1}{2}$. And the A.J. Team can expect to get $12\frac{1}{2}$ points in the three events.

(b) The strategies for the two teams are as in part (a).

Let $p_{ij} = \begin{cases} \frac{1}{2} & \text{if } p_{ij} \ge 13 \text{ for part (a); that is, A.J. Team wins} \\ -\frac{1}{2} & \text{if } p_{ij} < 13 \text{ for part (a); that is, A.J. Team loses} \end{cases}$

The payoff matrix becomes

A.J. \ G.N.	1	2	3
1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
2	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
3	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Strategy 2 of A.J. Team dominated by strategy 1.
Strategy 1 of GN Team dominated by strategy 2.

After eliminating the dominated strategies the matrix is

AJ \ GN	2	3
1	$\frac{1}{2}$	$-\frac{1}{2}$
3	$-\frac{1}{2}$	$\frac{1}{2}$

If $12\frac{1}{2}$ is added to each entry, the optimal strategies are unchanged. Furthermore, the payoff matrix of part (a) is obtained. Hence the strategies given in part (a) are still optimal and $v = 12\frac{1}{2} - 12\frac{1}{2} = 0$

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14.4.4(c) Since John and Mark are the best swimmers on their respective teams, they will always swim in two events since the team can do no better if they swim in only one or no events. Hence, if either does not swim in the first event, the butterfly, he will surely swim the last two events.
Thus the strategies for the A.J. team are:

1. John enters the butterfly and then enters the backstroke regardless of whether Mark enters the butterfly.
2. John enters the butterfly and then swims the backstroke if Mark enters the butterfly, but swims breaststroke if Mark does not.
3. John enters the butterfly and then swims breaststroke if Mark enters the butterfly, but swims the backstroke if Mark does not.
4. John enters the butterfly and then swims the breaststroke regardless of whether Mark enters the butterfly.
5. John does not swim the butterfly and then enters both the breaststroke and the backstroke.

The strategies for the G.N. Team are as above but with the roles of John and Mark reversed.

The payoff matrix is

A.J. \ G.N.	1	2	3	4	5
1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
3	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
4	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
5	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Strategy 3 of G.N. Team dominates all others. Since the resulting payoff matrix is:

AJ \ GN	3
1	$-\frac{1}{2}$
2	$-\frac{1}{2}$
3	$-\frac{1}{2}$
4	$-\frac{1}{2}$
5	$-\frac{1}{2}$

if G.N. Team uses strategy 3, it will win, regardless of the strategy chosen by the A.J. Team.

(d) Strategy 2 of AJ Team dominates strategies 1, 3, 4. Thus, if the coach for the GN Team may choose any of his strategies at random, the coach for the AJ. Team should choose either strategy 2 or 5. The payoff matrix becomes (after eliminating the dominated strategies of AJ Team):

A.J. \ G.N.	1	2	3	4	5
2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
5	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The two rows are identical except for columns 1 and 4.

Thus, if the coach for the AJ. Team knows that the other coach has a tendency to enter Mark in butterfly and backstroke more often than breast-stroke, that means column 1 is more likely to be chosen than column 4. Therefore, the coach for the AJ. Team should choose strategy 2.

14.5-1 Adding 3 to the entries of table 12.6 we obtain the payoff table

I \ II	1	2	3
1	3	1	5
2	8	7	0

The new linear programming model for player I is:

Maximize x_3

subject to
$$3x_1 + 8x_2 - x_3 \geq 0$$
$$x_1 + 7x_2 - x_3 \geq 0$$
$$5x_1 - x_3 \geq 0$$
$$x_1 + x_2 = 1$$
$$x_1, x_2, x_3 \geq 0$$

The new linear programming model for player II is:

Minimize y_4

subject to
$$3y_1 + y_2 + 5y_3 - y_4 \leq 0$$
$$8y_1 + 7y_2 - y_4 \leq 0$$
$$y_1 + y_2 + y_3 = 1$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Based on the information given in Section 12.5, the optimal solutions for these new models are:

$(x_1^*, x_2^*, x_3^*) = (7/11, 4/11, 35/11)$ and $(y_1^*, y_2^*, y_3^*, y_4^*) = (0, 5/11, 6/11, 35/11)$

Note that $x_3^* = y_4^*$ and also $x_3^* = y_4^* = v + 3$ where v is the original game value.

14.5-2

a) Maximize x_4

subject to
$$5x_1 + 2x_2 + 3x_3 - x_4 \geq 0$$
$$4x_2 + 2x_3 - x_4 \geq 0$$
$$3x_1 + 3x_2 - x_4 \geq 0$$
$$x_1 + 2x_2 + 4x_3 - x_4 \geq 0$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

b) Solve Automatically by the Simplex Method:

Optimal Solution

Value of the
Objective Function: Z = 2.36842105

Variable	Value
X_1	0.05263
X_2	0.73684
X_3	0.21053
X_4	2.36842

14.5-3 To insure $x_4 \geq 0$ add 3 to each entry of the payoff table.

a)

Maximize x_4

subject to:
$$7x_1 + 2x_2 + 5x_3 - x_4 \geq 0$$
$$5x_1 + 3x_2 + 6x_3 - x_4 \geq 0$$
$$6x_2 + x_3 - x_4 \geq 0$$
$$x_1 + x_2 + x_3 = 1$$
$$x_i \geq 0 \text{ for } i = 1, 2, 3, 4.$$

Solve Automatically by the Simplex Method:

b) **Optimal Solution**

Value of the Objective Function: Z = 3.79166567

Variable	Value
X_1	0.33333
X_2	0.625
X_3	0.04167
X_4	3.79167

14.5-4

a) To insure $x_5 \geq 0$ add 4 to each entry of the payoff table.

Maximize

subject to:
$$5x_1 + 6x_2 + 4x_3 - x_5 \geq 0$$
$$x_1 + 7x_2 + 8x_3 + 4x_4 - x_5 \geq 0$$
$$6x_1 + 4x_2 + 3x_3 + 2x_4 - x_5 \geq 0$$
$$2x_1 + 7x_2 + x_3 + 6x_4 - x_5 \geq 0$$
$$5x_1 + 2x_2 + 6x_3 + 3x_4 - x_5 \geq 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$

$x_i \geq 0$ for $i = 1, 2, 3, 4, 5$

Solve Automatically by the Simplex Method:

b) **Optimal Solution**

Value of the Objective Function: Z = 3.98101266

Variable	Value
X_1	0.31013
X_2	0.26582
X_3	0.20886
X_4	0.21519
X_5	3.98101

14.5-5 Following Table 6.14, the dual of player I's problem is:

$$\min. \quad y_{m+1}$$

s.t.
$$p_{11} y_1' + p_{12} y_2' + \cdots + p_{1m} y_m' + y_{m+1} \geq 0$$
$$p_{21} y_1' + p_{22} y_2' + \cdots + p_{2m} y_m' + y_{m+1} \geq 0$$
$$\vdots \qquad\qquad \vdots$$
$$p_{m1} y_1' + p_{m2} y_2' + \cdots + p_{mm} y_m' + y_{m+1} \geq 0$$
$$-y_1' - y_2' - \cdots - y_m' = 1$$
$$y_i' \leq 0, \quad i = 1, 2, \ldots, m \quad (y_{m+1} \text{ free})$$

Now let $y_i = -y_i'$, $i = 1, 2, \ldots, m$

and we get the linear program given for player II.

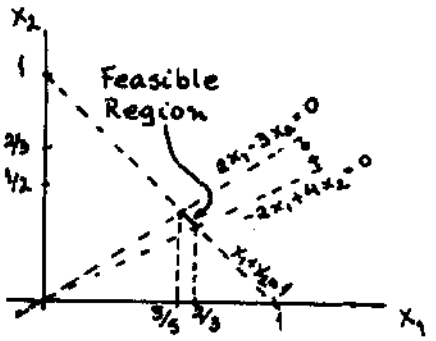
14.5-6 Taking the dual of the player I problem gives:

$$\begin{aligned} \text{m.w.} \quad & y_4 \\ \text{s.t.} \quad & -2y_1' + 2y_2' + y_4 \geq 0 \\ & 5y_1' + 4y_2' - 3y_3' + y_4 \geq 0 \\ & -y_1' - y_2' - y_3' = 1 \\ & y_1', y_2', y_3' \leq 0 \quad y_4 \text{ free} \end{aligned}$$

let $y_i = -y_i'$, $i = 1,2,3$

→ substitution produces the given player II problem.

14.5-7



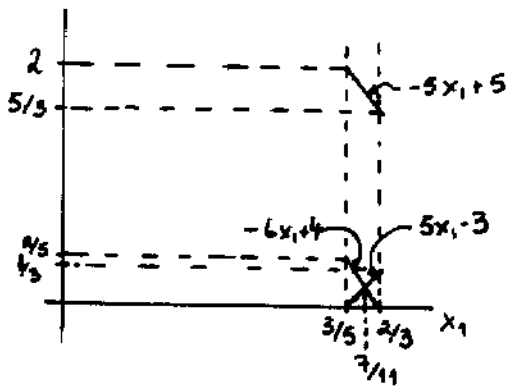
Therefore the feasible region may be algebraically described by: $x_2 = 1 - x_1 \quad 3/5 \leq x_1 \leq 2/3$

The restrictions may be rewritten as:

$x_3 \leq -5x_1 + 5 \quad 3/5 \leq x_1 \leq 2/3$

$x_3 \leq -6x_1 + 4 \quad 3/5 \leq x_1 \leq 2/3$

$x_3 \leq 5x_1 - 3 \quad 3/5 \leq x_1 \leq 2/3$



$-6x_1 + 4 = 5x_1 - 3$

$\Rightarrow x_1 = 7/11$

Therefore the algebraic expression for the maximizing value of x_3 for any point in the feasible region is:

$$x_3 = \begin{cases} 5x_1 - 3 & \text{for } 3/5 \leq x_1 \leq 7/11 \\ -6x_1 + 4 & \text{for } 7/11 \leq x_1 \leq 2/3 \end{cases}$$

Hence, the optimal solution is:

$x_1^* = 7/11$

$x_2^* = 1 - 7/11 = 4/11$

$x_3^* = 5(7/11) - 3 = 2/11$

14.5-8 AUTOMATIC SIMPLEX METHOD: FINAL TABLEAU

Bas Var	Eq No	Z	Coefficient of X1	X2	X3	X4	X5	X6	X7	X8	X9	X10	Right side
									1M	1M	1M	1M	
Z	0	1	0	0	0	0	0.455	0.545	0	-0.45	-0.55	0.182	0.182
X2	1	0	0	1	0	0	-0.09	0.091	0	0.091	-0.09	0.364	0.364
X4	2	0	0	0	0	1	-0.91	-0.09	-1	0.909	0.091	1.636	1.636
X1	3	0	1	0	0	0	0.091	-0.09	0	-0.09	0.091	0.636	0.636
X3	4	0	0	0	1	0	0.455	0.545	0	-0.45	-0.55	0.182	0.182

The optimal primal solution is $(x_1, x_2) = (0.636, 0.364)$ with a payoff = 0.182

The optimal dual solution is $(y_1, y_2, y_3) = (0, 0.455, 0.545)$

14.5-9 (a) Since saddlepoints can be found from the linear programming formulation of the game, part (a) follows from part (b).

(b) Consider the linear programming formulation of the problem for Player II. The i^{th} and k^{th} constraints are

$$p_{i1}y_1 + p_{i2}y_2 + \cdots + p_{in}y_n \leq y_{n+1}$$

$$p_{k1}y_1 + p_{k2}y_2 + \cdots + p_{kn}y_n \leq y_{n+1}$$

If row k weakly dominates row i, then

$$p_{i1}y_1 + \cdots + p_{in}y_n \leq p_{k1}y_1 + \cdots + p_{kn}y_n \text{ for all } y_1, \ldots y_n$$

That is, the i^{th} constraint is redundant since it is implied by the k^{th} constraint. Hence, eliminating weakly dominated pure strategies for Player I corresponds to eliminating redundant constraints in the linear program for Player II. Similarly, eliminating weakly dominated pure strategies for Player II corresponds to eliminating redundant constraints in the linear program for Player I.

Since this process cannot eliminate feasible solutions or create new ones, all optimal strategies cannot be eliminated and new ones cannot be created.

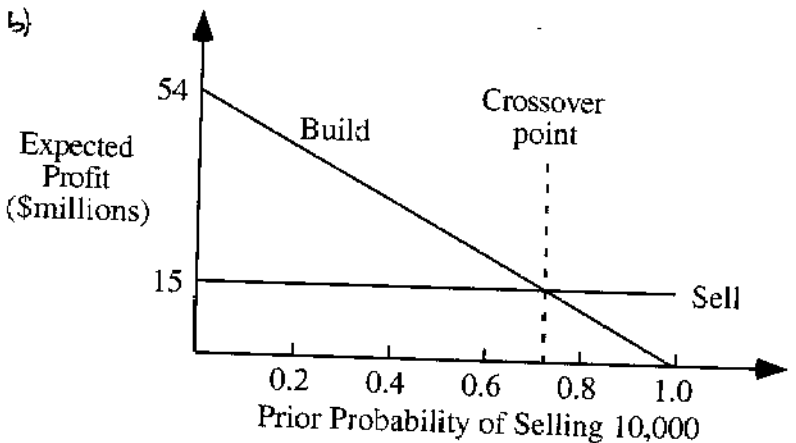
CHAPTER 15

15.2-1

a)

Alternative	State of Nature	
	Sell 10,000	Sell 100,000
Build Computers	0	54
Sell Rights	15	15

b)



c)

Let p = prior probability of selling 10,000.

For Build:

$$EP = p(0) + (1-p)(54) = -54p + 54$$

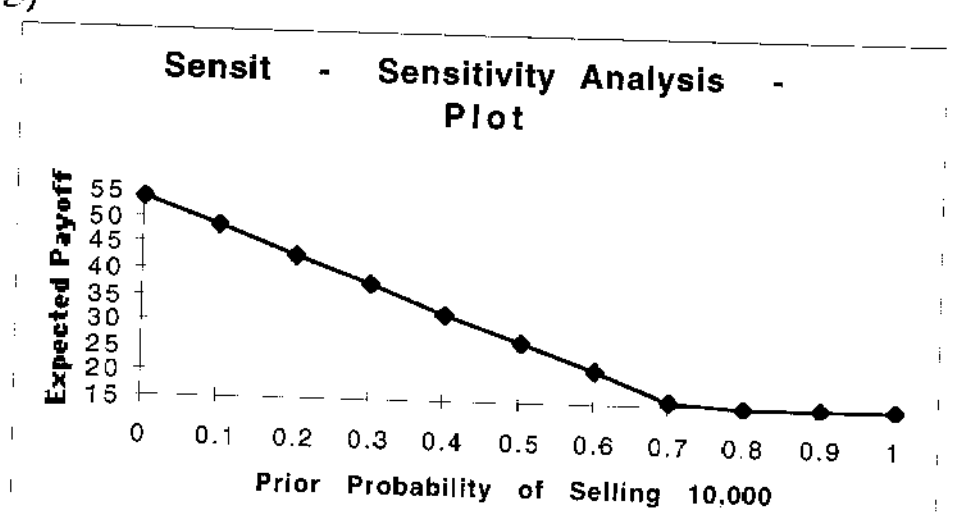
For Sell:

$$EP = p(15) + (1-p)(15) = 15$$

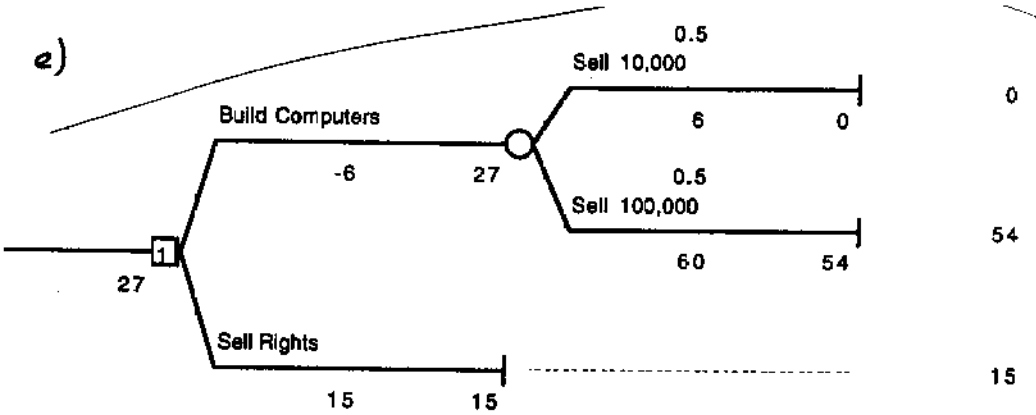
Build and Sell cross when $-54p + 54 = 15$ or $54p = 39$ or $p = 0.722$

They should build when $p \leq 0.722$, and sell when $p > 0.722$.

d)



e)



Choose to build computers (expected payoff is $27 million).

15.2-2

a)

Alternative	State of Nature			
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases
Buy 10 cases	50	50	50	50
Buy 11 cases	47	55	55	55
Buy 12 cases	44	52	60	60
Buy 13 cases	41	49	57	65
Prior Probability	0.2	0.4	0.3	0.1

b)

Alternative	State of Nature				Minimum	
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases		
Buy 10 cases	50	50	50	50	50	Maximum
Buy 11 cases	47	55	55	55	47	
Buy 12 cases	44	52	60	60	44	
Buy 13 cases	41	49	57	65	41	
Prior Probability	0.2	0.4	0.3	0.1		

c) Jean should purchase 10 cases

Alternative	State of Nature				
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases	
Buy 10 cases	50	50	50	50	Maximum
Buy 11 cases	47	55	55	55	
Buy 12 cases	44	52	60	60	
Buy 13 cases	41	49	57	65	
Prior Probability	0.2	0.4	0.3	0.1	
		Maximum			

Jean should purchase 11 cases.

d)

Alternative	State of Nature				Expected Payoff	
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases		
Buy 10 cases	50	50	50	50	50	
Buy 11 cases	47	55	55	55	53.4	
Buy 12 cases	44	52	60	60	53.6	Maximum
Buy 13 cases	41	49	57	65	51.4	
Prior Probability	0.2	0.4	0.3	0.1		

Jean should purchase 12 cases.

e)

0.2 and 0.5

Alternative	State of Nature				Expectec Payoff	
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases		
Buy 10 cases	50	50	50	50	50	
Buy 11 cases	47	55	55	55	53.4	
Buy 12 cases	44	52	60	60	55.2	Maximum
Buy 13 cases	41	49	57	65	53	
Prior Probability	0.2	0.2	0.5	0.1		

Jean should purchase 12 cases.

0.3 and 0.4

Alternative	State of Nature				Expected Payoff	
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases		
Buy 10 cases	50	50	50	50	50	
Buy 11 cases	47	55	55	55	53.4	
Buy 12 cases	44	52	60	60	54.4	Maximum
Buy 13 cases	41	49	57	65	52.2	
Prior Probability	0.2	0.3	0.4	0.1		

Jean should purchase 12 cases.

0.5 and 0.2

Alternative	State of Nature				Expected Payoff	
	Sell 10 cases	Sell 11 cases	Sell 12 cases	Sell 13 cases		
Buy 10 cases	50	50	50	50	50	
Buy 11 cases	47	55	55	55	53.4	Maximum
Buy 12 cases	44	52	60	60	52.8	
Buy 13 cases	41	49	57	65	50.6	
Prior Probability	0.2	0.5	0.2	0.1		

Jean should purchase 11 cases.

15.2-3

a) Optimal (maximin) action is a_1 or a_3 (both have values of -10).

b)

Alternative	State of Nature: Improving	Stable	Worsening	
Conservative	30	5	-10	
Speculative	40	10	-30	Maximum
Counter-cyclical	-10	0	15	
Prior Probability	0.1	0.5	0.4	
		Maximum		

Warren should make the speculative investment.

c)

Alternative	State of Nature: Improving	Stable	Worsening	Expected Payoff	
Conservative	30	5	-10	1.5	
Speculative	40	10	-30	-3	
Counter-cyclical	-10	0	15	5	Maximum
Prior Probability	0.1	0.5	0.4		

Warren should make the counter-cyclical investment.

15.2-4

a)

Alternative	State of Nature: Improving	Stable	Worsening	Expected Payoff	
Conservative	30	5	-10	-1.5	
Speculative	40	10	-30	-11	
Counter-cyclical	-10	0	15	8	Maximum
Prior Probability	0.1	0.3	0.6		

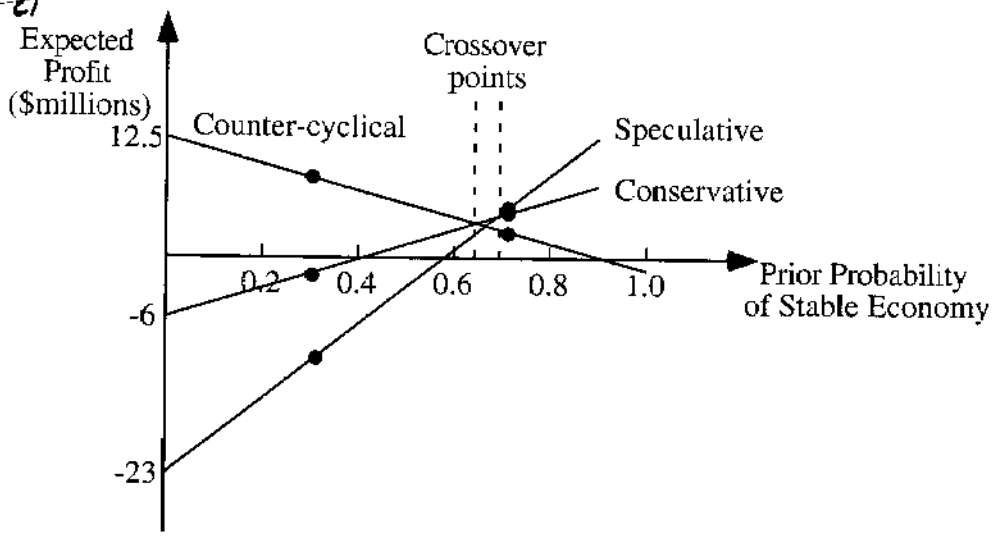
Warren should make the counter-cyclical investment.

b)

Alternative	State of Nature: Improving	Stable	Worsening	Expected Payoff	
Conservative	30	5	-10	4.5	
Speculative	40	10	-30	5	Maximum
Counter-cyclical	-10	0	15	2	
Prior Probability	0.1	0.7	0.2		

Warren should make the speculative investment.

c)



Counter-cyclical and conservative cross at approximately p=0.62.
Conservative and speculative cross at approximately p = 0.68.

d) Let p = prior probability of stable economy

For the conservative option:

$$\begin{aligned} EP &= (0.1)(30) + p(5) + (1\text{-}0.1\text{-}p)(\text{-}10) \\ &= 3 + 5p - 9 + 10p \\ &= 15p - 6 \end{aligned}$$

For the speculative option:

$$\begin{aligned} EP &= (0.1)(40) + p(10) + (1\text{-}0.1\text{-}p)(\text{-}30) \\ &= 4 + 10p - 27 + 30p \\ &= 40p - 23 \end{aligned}$$

For the counter-cyclical option:

$$\begin{aligned} EP &= (0.1)(\text{-}10) + p(0) + (1\text{-}0.1\text{-}p)(15) \\ &= \text{-}1 + 0 + 13.5 - 15p \\ &= \text{-}15p + 12.5 \end{aligned}$$

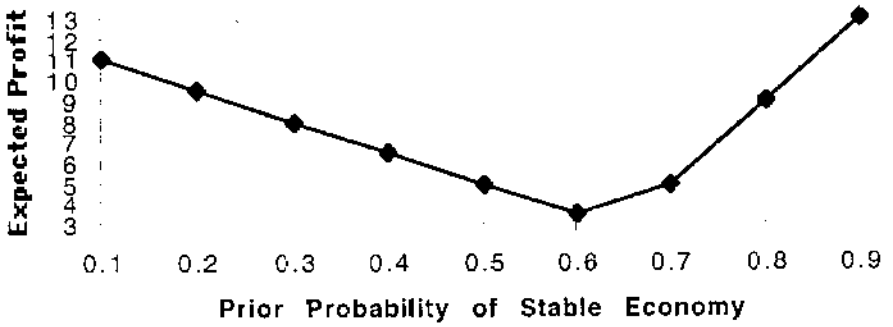
Counter-cyclical and conservative cross when
$-15p + 12.5 = 15p - 6$ or $30p = 18.5$ or p=0.617

Conservative and speculative cross when
$15p - 6 = 40p - 23$ or $25p = 17$ or $p = 0.68$

They should choose the counter-cyclical option when p<0.617, the conservative option when $0.617 \le p < 0.68$, and the speculative option when $p \ge 0.68$.

e)

Sensit - Sensitivity Analysis - Plot



15.2-5

a)

Alternative	State of Nature S1	S2	Minimum	
A1	80	25	25	
A2	30	50	30	
A3	60	40	40	Maximum
Prior Probability	0.4	0.6		

Alternative 3 should be chosen

b)

Alternative	State of Nature S1	S2	
A1	80	25	
A2	30	50	Maximum
A3	60	40	
Prior Probability	0.4	0.6	
		Maximum	

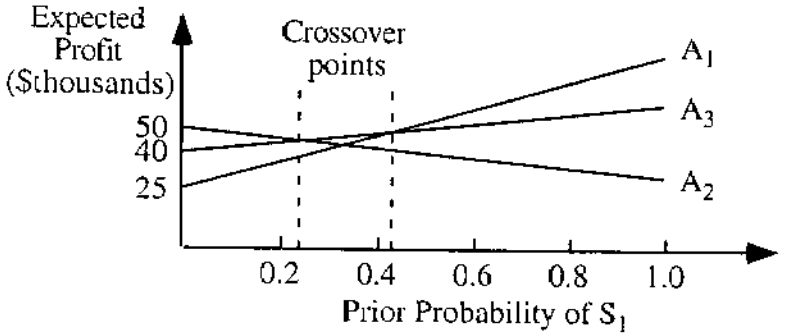
Alternative 2 should be chosen.

c)

Alternative	State of Nature S1	S2	Expected Payoff	
A1	80	25	47	
A2	30	50	42	
A3	60	40	48	Maximum
Prior Probability	0.4	0.6		

Alternative 3 should be chosen.

d)



A_2 and A_3 cross at approximately p=0.25.
A_1 and A_3 cross at approximately p = 0.43.

e) Let p = prior probability of S_1.

For A_1:

$$EP = p(80) + (1-p)(25)$$
$$= 80p + 25 - 25p$$
$$= 55p + 25$$

For A_2:

$$EP = p(30) + (1-p)(50)$$
$$= 30p + 50 - 50p$$
$$= -20p + 50$$

For A_3:

$$EP = p(60) + (1-p)(40)$$
$$= 60p + 40 - 40p$$
$$= 20p + 40$$

A_2 and A_3 cross when $-20p + 50 = 20p + 40$ or $40p = 10$ or p=0.250.
A_1 and A_3 cross when $55p + 25 = 20p + 40$ or $35p = 15$ or $p = 0.429$

They should choose A_2 when $p<0.250$, A_3 when $0.250 \le p < 0.429$, and A_1 when $p \ge 0.429$.

15.2-6

a)

Alternative	State of Nature S1	S2	S3	Minimum	
A1	220	170	110	110	
A2	200	180	150	150	Maximum
Prior Probability	0.6	0.3	0.1		

Alternative 2 should be chosen.

b)

Alternative	State of Nature S1	S2	S3	
A1	220	170	110	Maximum
A2	200	180	150	
Prior Probability	0.6	0.3	0.1	
	Maximum			

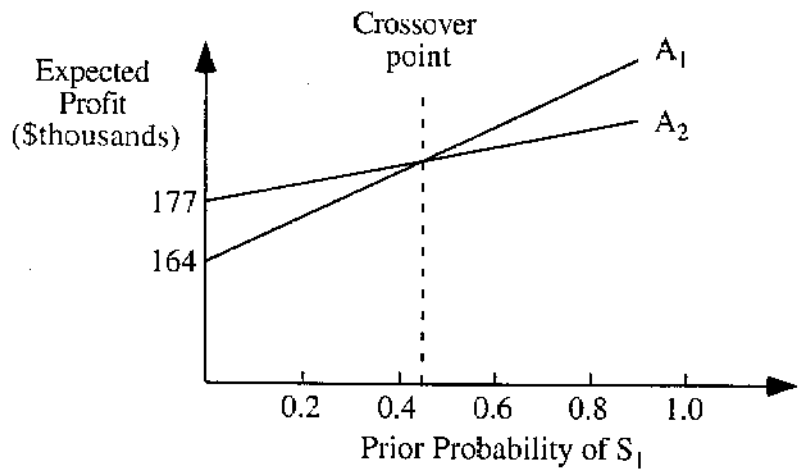
Alternative 1 should be chosen.

c)

Alternative	State of Nature S1	S2	S3	Expected Payoff	
A1	220	170	110	194	Maximum
A2	200	180	150	189	
Prior Probability	0.6	0.3	0.1		

Alternative 1 should be chosen.

d)



Let p = prior probability of S_1.
For A_1:

$$EP = p(220) + (1\text{-}0.1\text{-}p)(170) + (0.1)(110)$$
$$= 220p + 153 - 170p + 11$$
$$= 50p + 164$$

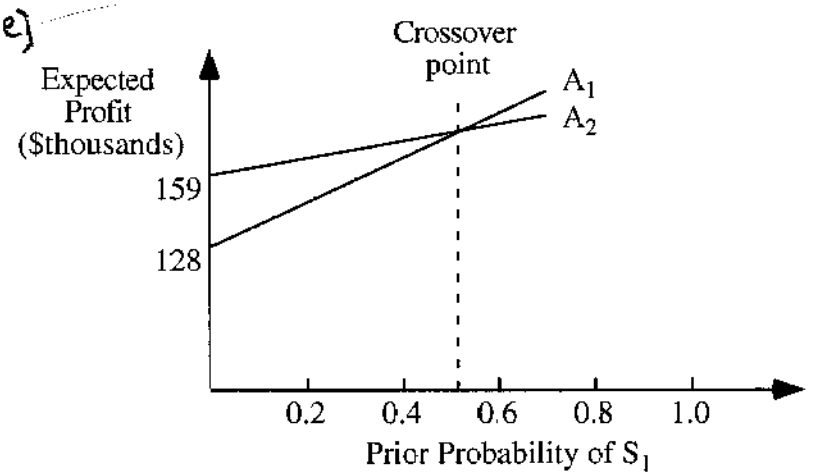
For A_2:

$$EP = p(200) + (1\text{-}0.1\text{-}p)(180) + (0.1)(150)$$
$$= 200p + 162 - 180p + 15$$
$$= 20p + 177$$

$A1$ and $A2$ cross when $50p + 164 = 20p + 177$ or $30p = 13$ or $p=0.433$.

They should choose $A2$ when $p \leq 0.433$, A_1 when $p > 0.433$.

e)



Let p = prior probability of S_1.

For A_1:

$$EP = p(220) + (0.3)(170) + (1\text{-}0.3\text{-}p)(110)$$
$$= 220p + 51 + 77 - 110p$$
$$= 110p + 128$$

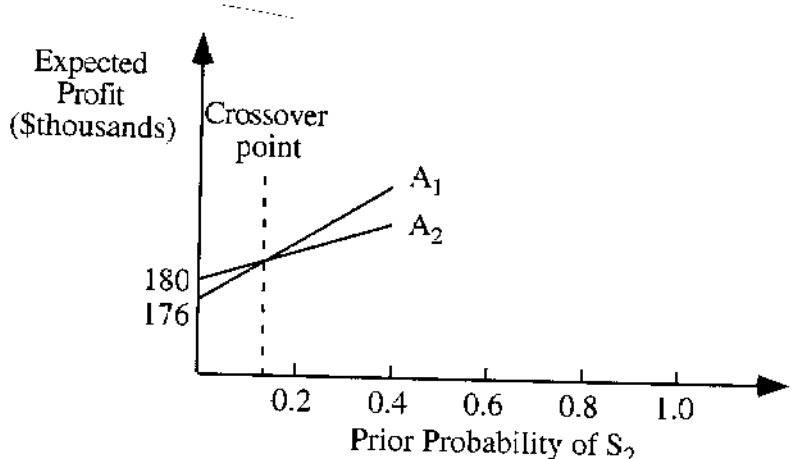
For A_2:

$$EP = p(200) + (0.3)(180) + (1\text{-}0.3\text{-}p)(150)$$
$$= 200p + 54 + 105 - 150p$$
$$= 50p + 159$$

A_1 and A_2 cross when $110p + 128 = 50p + 159$ or $60p = 31$ or p=0.517.

They should choose A_2 when $p \leq 0.517$, A_1 when $p > 0.517$.

F)



Let p = prior probability of S_2.

For A_1:

$$EP = (0.6)(220) + p(170) + (1\text{-}0.6\text{-}p)(110)$$
$$= 132 + 170p + 44 - 110p$$
$$= 60p + 176$$

(CONT'D)

(CONT'D) (F)

For A_2:

$$EP = (0.6)(200) + p(180) + (1\text{-}0.6\text{-}p)(150)$$
$$= 120 + 180p + 60 - 150p$$
$$= 30p + 180$$

A_1 and A_2 cross when $60p + 176 = 30p + 180$ or $30p = 4$ or $p=0.133$.

They should choose A_2 when $p \leq 0.133$, A_1 when $p > 0.133$.

g) Alternative A_1 should be chosen.

15.2-7

a)

Alternative	State of Nature Dry	Moderate	Damp
Crop 1	20	35	40
Crop 2	22.5	30	45
Crop 3	30	25	25
Crop 4	20	20	20
Prior Probability	0.3	0.5	0.2

b)

Alternative	State of Nature Dry	Moderate	Damp	Expected Payoff	
Crop 1	20	35	40	31.5	Maximum
Crop 2	22.5	30	45	30.75	
Crop 3	30	25	25	26.5	
Crop 4	20	20	20	20	
Prior Probability	0.3	0.5	0.2		

Grow crop 1.

c)

Prior Probability of moderate weather = 0.2

Alternative	State of Nature Dry	Moderate	Damp	Expected Payoff	
Crop 1	20	35	40	33	
Crop 2	22.5	30	45	35.25	Maximum
Crop 3	30	25	25	26.5	
Crop 4	20	20	20	20	
Prior Probability	0.3	0.2	0.5		

Grow crop 2.

(CONT'D)

(CONT'D) c)

Prior Probability of moderate weather = 0.3

Alternative	State of Nature Dry	Moderate	Damp	Expected Payoff	
Crop 1	20	35	40	32.5	
Crop 2	22.5	30	45	33.75	Maximum
Crop 3	30	25	25	26.5	
Crop 4	20	20	20	20	
Prior Probability	0.3	0.3	0.4		

Grow crop 2.

Prior Probability of moderate weather = 0.4

Alternative	State of Nature Dry	Moderate	Damp	Expected Payoff	
Crop 1	20	35	40	32	
Crop 2	22.5	30	45	32.25	Maximum
Crop 3	30	25	25	26.5	
Crop 4	20	20	20	20	
Prior Probability	0.3	0.4	0.3		

Grow crop 2.

Prior Probability of moderate weather = 0.6

Alternative	State of Nature Dry	Moderate	Damp	Expected Payoff	
Crop 1	20	35	40	31	Maximum
Crop 2	22.5	30	45	29.25	
Crop 3	30	25	25	26.5	
Crop 4	20	20	20	20	
Prior Probability	0.3	0.6	0.1		

Grow crop 1.

15.2-8 The prior distribution is $P\{\theta=\theta_1\}=\frac{2}{3}$, $P\{\theta=\theta_2\}=\frac{1}{3}$.

For a_1: $-E[p(a_1,\theta)] = \frac{2}{3}(1.155\times10^7)+\frac{1}{3}(1.414\times10^7) = 1.241\times10^7$

For a_2: $-E[p(a_2,\theta)] = \frac{2}{3}(1.012\times10^7)+\frac{1}{3}(1.207\times10^7) = 1.077\times10^7$

For a_3: $-E[p(a_3,\theta)] = \frac{2}{3}(1.047\times10^7)+\frac{1}{3}(1.135\times10^7) = 1.076\times10^7$

Since these are expected negative payoffs, the optimal (Bayes) action is a_3 (order 25).

15.2-9 For a_1: $E[p(a_1,\theta)] = .4(2x)+.2(50)+.4(10) = 14+.8x$

For a_2: $E[p(a_2,\theta)] = .4(25)+.2(40)+.4(90) = 54$

For a_3: $E[p(a_3,\theta)] = .4(35)+.2(3x)+.4(30) = 26+.6x$

At $x=50$, the optimal action is a_3 with expected payoff 56.

At $x=75$, the optimal action is a_1 with expected payoff 74.

The maximum amount that should be spent to increase x to 75 is 18.

15.3-1

a)

Alternative	State of Nature: Sell 10,000	Sell 100,000
Build Computers	0	54
Sell Rights	15	15
Prior Probability	0.5	0.5
Maximum Payoff	15	54

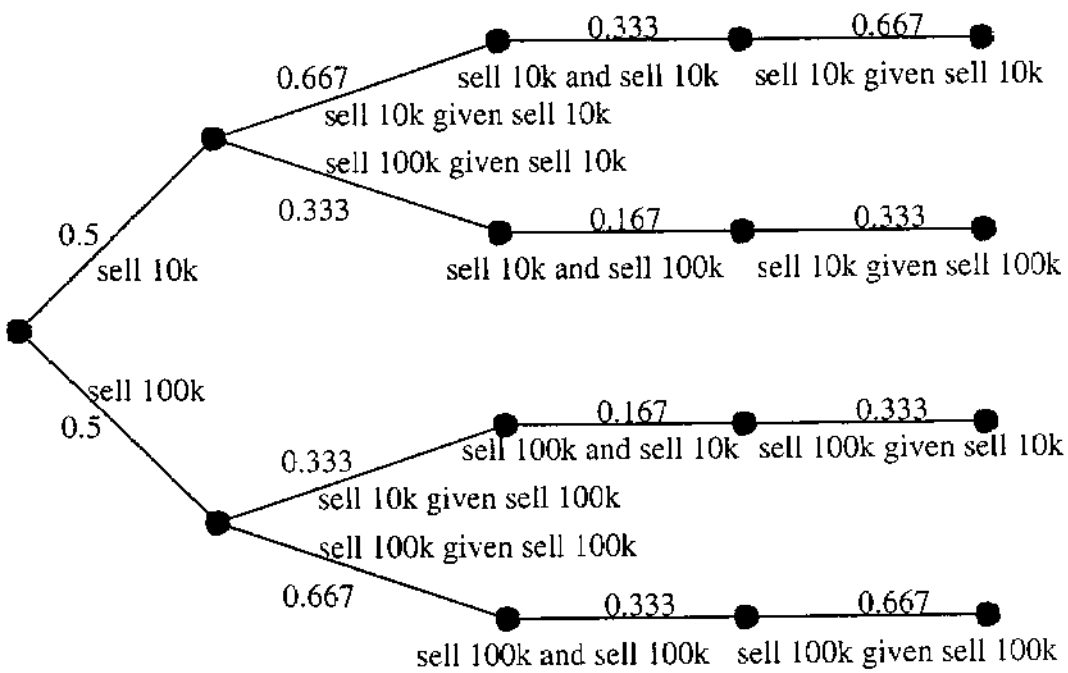
Expected Payoff with Perfect Information =	34.5

EVPI = EP (with perfect info) – EP (without more info) = 34.5 – 27 = S7.5 million

b) Since the market research will cost S1 million it might be worthwhile to perform it.

c)

Prior Probabilities P (state)	Conditional Probabilities P(finding | state)	Joint Probabilities P(state and finding)	Posterior Probabilities P(state | finding)



d)

Data:		P(Finding I State) Finding	
State of Nature	Prior Probability	Sell 10,000	Sell 100,000
Sell 10,000	0.5	0.666666667	0.333333333
Sell 100,000	0.5	0.333333333	0.666666667

Posterior Probabilities:		P(State I Finding) State of Nature	
Finding	P(Finding)	Sell 10,000	Sell 100,000
Sell 10,000	0.5	0.666666667	0.333333333
Sell 100,000	0.5	0.333333333	0.666666667

e)

$EVE = [.5(1800) + .5(3600)] - 2700 = 0$

So performing the market research is not worthwhile.

15.3-2

a)

Alternative	State of Nature S1	S2	S3	Expected Payoff	
A1	4	0	0	0.8	
A2	0	2	0	1	Maximum
A3	3	0	1	0.9	
Prior Probability	0.2	0.5	0.3		

Choose alternative 2 (expected payoff is \$1,000).

b)

Alternative	State of Nature S1	S2	S3
A1	4	0	0
A2	0	2	0
A3	3	0	1
Prior Probability	0.2	0.5	0.3
Maximum Payoff	4	2	1

Expected Payoff with Perfect Information =	2.1

EVPI = EP (with perfect info) – EP (without more info) = 2.1 – 1 = \$1,100

c) Since the information will cost \$1,000 and the value is \$1,100, it might be worthwhile to spend the money.

15.3-3

a)

Alternative	State of Nature S1	S2	S3	Expected Payoff	
A1	50	100	-100	35	Maximum
A2	0	10	-10	1	
A3	20	40	-40	14	
Prior Probability	0.5	0.3	0.2		

Choose alternative 1 (expected payoff is $35).

b)

Alternative	State of Nature S1	S2	S3
A1	50	100	-100
A2	0	10	-10
A3	20	40	-40
Prior Probability	0.5	0.3	0.2
Maximum Payoff	50	100	-10

Expected Payoff with Perfect Information =	53

EVPI = EP (with perfect info) – EP (without more info) = 53 – 35 = $18

c)

Betsy should consider spending up to $18 to obtain more information.

15.3-4 a)

Alternative	State of Nature S1	S2	S3	Expected Payoff	
A1	-100	10	100	33	
A2	-10	20	50	29	
A3	10	10	60	35	Maximum
Prior Probability	0.2	0.3	0.5		

Choose alternative 3 (expected payoff is $35,000).

b) If S1 occurs for certain then choose A3 (expected payoff is $10,000).

If S1 does not occur for certain then the chance of S2 occurring is 3/8 and the chance of S3 occurring is 5/8. So choose A1 (expected payoff is $66,250).

A1: (3/8)(10) + (5/8)(100) = 66.25
A2: (3/8)(20) + (5/8)(50) = 38.75
A3: (3/8)(10) + (5/8)(60) = 41.25

EVPI = EP (with perfect info) – EP (without more info)
= [(0.2)(10) + (0.8)(66.25)] – 35 = $20,000

The maximum amount you should pay for the information is $20,000. The decision with this information would be to choose A1 with an expected payoff of $66,250.

c) If S2 occurs for certain then choose A2 (expected payoff is $20,000).

If S2 does not occur for certain then the chance of S1 occurring is 2/7 and the chance of S3 occurring is 5/7. So choose A3 (expected payoff is $45,714).

A1: (2/7)(-100) + (5/7)(100) =42.857
A2: (2/7)(-10) + (5/7)(50) = 32.857
A3: (2/7)(10) + (5/7)(60) = 45.714

EVPI = EP (with perfect info) – EP (without more info)
= [(0.3)(20) + (0.7)(45.714)] – 35 = $3,000

The maximum amount you should pay for the information is $3,000. The decision with this information would be to choose A3 with an expected payoff of $45,714.

d) If S3 occurs for certain then choose A1 (expected payoff is $100,000).

If S3 does not occur for certain then the chance of S1 occurring is 2/5 and the chance of S2 occurring is 3/5. So choose A3 (expected payoff is $10,000).

A1: (2/5)(-100) + (3/5)(100) = -34
A2: (2/5)(-10) + (3/5)(20) = 8
A3: (2/5)(10) + (3/5)(10) = 10

EVPI = EP (with perfect info) – EP (without more info)
= [(0.5)(100) + (0.5)(10)] – 35 = 20

The maximum amount you should pay for the information is $20,000. The decision with this information would be to choose A1 with an expected payoff of $100,000.

e) EVPI = EP (with perfect info) – EP (without more info)
= [(0.2)(10) + (0.3)(20) + (0.5)(100)] – 35 = 23

A maximum of S23,000 should be paid for the information.

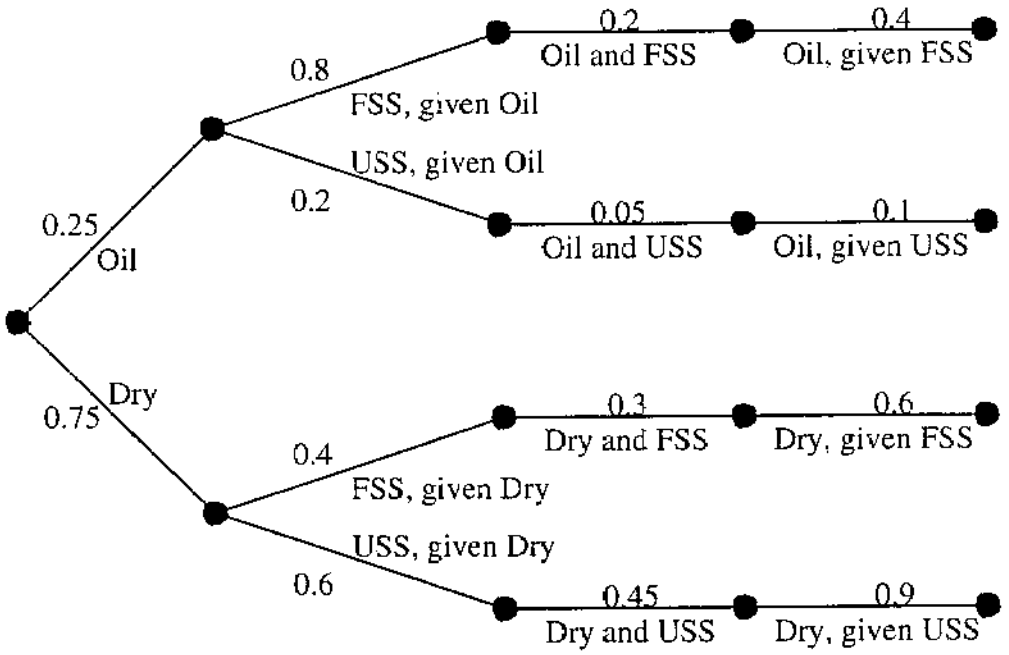
Choose the alternative that maximizes the payoff given the known state of nature. For S1 choose A3, for S2 choose A2, and for S3 choose A1.

The resulting expected payoff is $58,000
(0.2)(10) + (0.3)(20) + (0.5)(100) = 58

f) The maximum amount you should pay for testing is $23,000.

15.3-5 a)

Prior Probabilities P (state)	Conditional Probabilities P(finding | state)	Joint Probabilities P(state and finding)	Posterior Probabilities P(state | finding)



b)

Data:		P(Finding | State) Finding	
State of Nature	Prior Probability	FSS	USS
Oil	0.25	0.8	0.2
Dry	0.75	0.4	0.6

Posterior Probabilities:		P(State | Finding) State of Nature	
Finding	P(Finding)	Oil	Dry
FSS	0.5	0.4	0.6
USS	0.5	0.1	0.9

c) The optimal policy is to do a seismic survey and sell if it is unfavorable or drill if it is favorable.

15.3-6

a)

Alternative	State of Nature S1	S2	Expected Payoff	
A1	400	-100	100	Maximum
A2	0	100	60	
Prior Probability	0.4	0.6		

Choose A1 (expected payoff is \$100).

b)

EVPI = EP (with perfect info) – EP (without more info) = 220 – 100 = \$120

This indicates that it might be worthwhile to do the research.

c) P(state and finding) = P(state) P(finding/state)

i) P(S1 and S1) = (0.4)(0.6) = 0.24
ii) P(S1 and S2) = (0.4)(0.4) = 0.16
iii) P(S2 and S1) = (0.6)(0.2) = 0.12
iv) P(S2 and S2) = (0.6)(0.8) = 0.48

d) P(S1) = 0.24 + 0.12 = 0.36
P(S2) = 0.16 + 0.48 = 0.64

e)

$$P(\text{state} \mid \text{finding}) = \frac{P(\text{state and finding})}{P(\text{finding})}$$

$$P(S1|S1) = \frac{0.24}{0.36} = 0.667$$

$$P(S1|S2) = \frac{0.16}{0.64} = 0.25$$

$$P(S2|S1) = \frac{0.12}{0.36} = 0.333$$

$$P(S2|S2) = \frac{0.48}{0.64} = 0.75$$

f)

Data: State of Nature	Prior Probability	P(Finding I State) Finding S1	S2
S1	0.4	0.6	0.4
S2	0.6	0.2	0.8

Posterior Probabilities: Finding	P(Finding)	P(State I Finding) State of Nature S1	S2
S1	0.36	0.6667	0.3333
S2	0.64	0.25	0.75

g)

Alternative	State of Nature S1	S2	Expected Payoff	
A1	400	-100	233.5	Maximum
A2	0	100	33.3	
Prior Probability	0.667	0.333		

If S1 is predicted then the choose A1 (expected payoff is $233.33)

h)

Alternative	State of Nature S1	S2	Expected Payoff	
A1	400	-100	25	
A2	0	100	75	Maximum
Prior Probability	0.25	0.75		

If S2 is predicted then the choose A2 (expected payoff is $75)

i) Expected payoff given research is (0.36)(233.33) + (0.64)(75) – 100 = $32

j) The optimal policy is to do no research and choose A1.

15.3-7. a) Payoff Table:

Action \ State of Nature	θ_1 Fair coin	θ_2 2-headed coin
a_1 (guess fair)	5	-5
a_2 (guess 2-head)	-5	5
Prior probability	0.5	0.5

b) Neither option is better. Both give expected payoffs of $0.

c) $EVPI = [(0.5)5 + 0.5(5)] - 0 = \$5$

d) $P(\text{fair} \mid \text{flip tail}) = \dfrac{P(\text{flip tail} \mid \text{fair})P(\text{fair})}{P(\text{flip tail} \mid \text{fair})P(\text{fair}) + P(\text{flip tail} \mid \text{2-head})P(\text{2-head})}$

$= \dfrac{(0.5)(0.5)}{(0.5)(0.5) + 0(0.5)} = 1$

So $P(\text{2-head} \mid \text{flip tail}) = 0$

$P(\text{fair} \mid \text{flip head}) = \dfrac{P(\text{flip head} \mid \text{fair})P(\text{fair})}{P(\text{flip head} \mid \text{fair})P(\text{fair}) + P(\text{flip head} \mid \text{2-head})P(\text{2-head})}$

$= \dfrac{(0.5)(0.5)}{(0.5)(0.5) + 1(0.5)} = \dfrac{1}{3}$

$P(\text{2-head} \mid \text{flip head}) = \dfrac{2}{3}$

e)

Data:		P(Finding | State)	
State of Nature	Prior Probability	Finding Head	Tail
Fair	0.5	0.5	0.5
2-Head	0.5	1	0

Posterior Probabilities		P(State | Finding) State of Nature	
Finding	P(Finding)	Fair	2-Head
Head	0.75	0.333	0.667
Tail	0.25	1	0

f) If flip tail:

for a_1: $E[p(a_1,\theta \mid \text{flip tail})] = 1(5) + 0(-5) = 5$

for a_2: $E[p(a_2,\theta \mid \text{flip tail})] = 1(-5) + 0(5) = -5$

If flip head:

for a_1: $E[p(a_1,\theta \mid \text{flip head})] = \frac{1}{3}(5) + \frac{2}{3}(-5) = -\frac{5}{3}$

for a_2: $E[p(a_2,\theta \mid \text{flip head})] = \frac{1}{3}(-5) + \frac{2}{3}(5) = \frac{5}{3}$

So if tail occurs, guess fair coin, if head occurs, guess 2-headed coin.

g) $P(\text{flip tail}) = P(\text{flip tail} \mid \text{fair})P(\text{fair}) + P(\text{flip tail} \mid \text{2-head})P(\text{2-head})$

$= (0.5)(0.5) + 0(0.5) = 0.25$

$P(\text{flip head}) = 0.75$

So, $EVE = [(0.75)\frac{5}{3} + (0.25)5] - 0 = \$2.5$

The most you should be willing to pay would be $2.50.

15.3-8

a) $EVPI = [(\frac{2}{3})(-1.012\times10^7) + (\frac{1}{3})(-1.135\times10^7)] - (-1.076\times10^7) = 230{,}000$

b) $P(\theta=21 \mid \text{30 spares req.}) = \dfrac{P(\text{30 spares req.} \mid \theta=21)P(\theta=21)}{P(\text{30 spares req} \mid \theta=21)P(\theta=21) + P(\text{30 spares req} \mid \theta=24)P(\theta=24)}$

$= \dfrac{(0.013)(\frac{2}{3})}{(0.013)(\frac{2}{3}) + (0.036)(\frac{1}{3})} = .419$

$P(\theta=24 \mid \text{30 spares req}) = .581$

For a_1: $E[p(a_1,\theta \mid \text{30 spares req})] = .419(-1.155\times10^7) + .581(-1.414\times10^7) = -1.305\times10^7$

For a_2: $E[p(a_2,\theta \mid \text{30 spares req})] = .419(-1.012\times10^7) + .581(-1.207\times10^7) = -1.125\times10^7$

For a_3: $E[p(a_3,\theta \mid \text{30 spares req})] = .419(-1.047\times10^7) + .581(-1.135\times10^7) = -1.098\times10^7$

So optimal action is a_3 (order 25).

15.3-9

a)

Alternative	State of Nature: Poor Risk	Average Risk	Good Risk
Extend Credit	-15,000	10,000	20,000
Don't Extend Credit	0	0	0
Prior Probabilities	0.2	0.5	0.3

b)

Alternative	State of Nature: Poor Risk	Average Risk	Good Risk	Expected Payoff	
Extend Credit	-15000	10000	20000	8000	Maximum
Don't Extend Credit	0	0	0	0	
Prior Probability	0.2	0.5	0.3		

Choose to extend credit (expected payoff is $8,000).

c)

Alternative	State of Nature: Poor Risk	Average Risk	Good Risk
Extend Credit	-15000	10000	20000
Don't Extend Credit	0	0	0
Prior Probability	0.2	0.5	0.3
Maximum Payoff	0	10000	20000

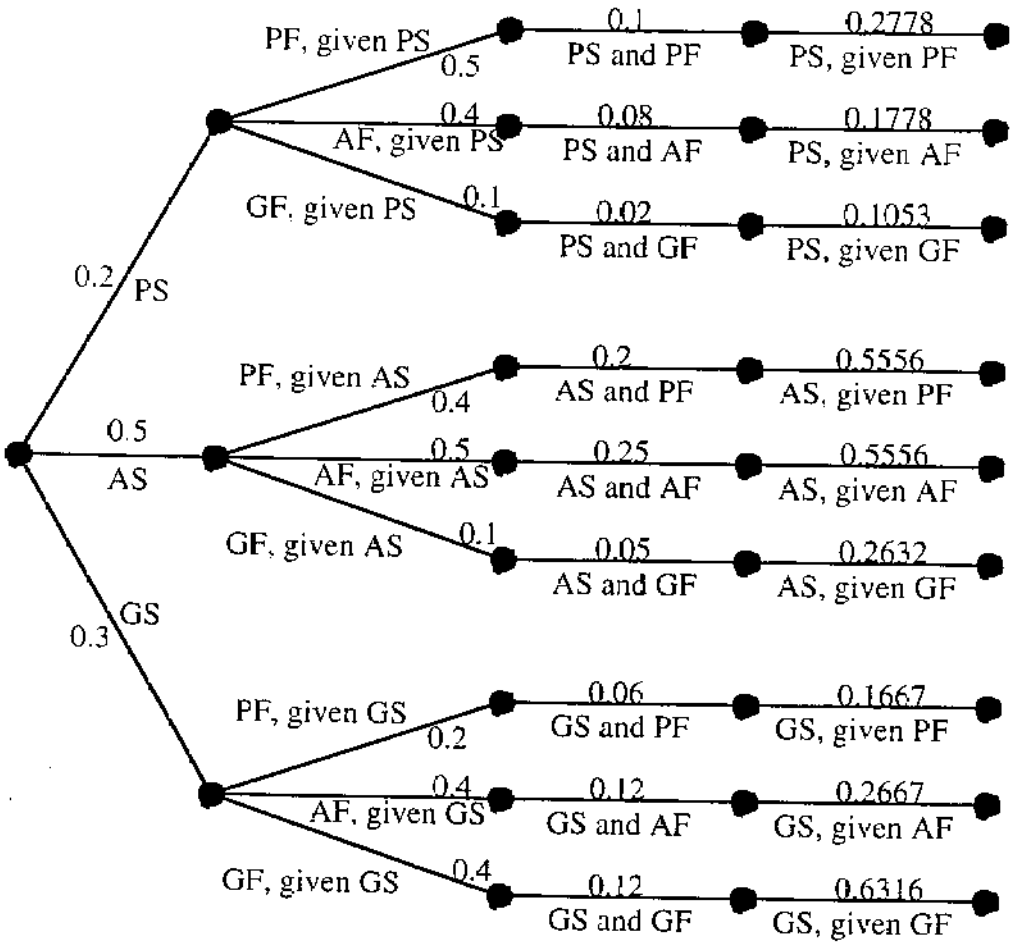
Expected Payoff with Perfect Information =	11000

EVPI = EP (with perfect info) – EP (without more info) = 11,000 – 8,000 = $3,000

This indicates that the credit-rating organization should not be used.

d) PF = Poor Finding AF = Average Finding GF = Good Finding
PS = Poor State AS = Average State GS = Good State

Prior Probabilities P (state)	Conditional Probabilities P(finding | state)	Joint Probabilities P(state and finding)	Posterior Probabilities P(state | finding)



e)

Data:		P(Finding | State) Finding		
State of Nature	Prior Probability	Poor	Average	Good
Poor	0.2	0.5	0.4	0.1
Average	0.5	0.4	0.5	0.1
Good	0.3	0.2	0.4	0.4

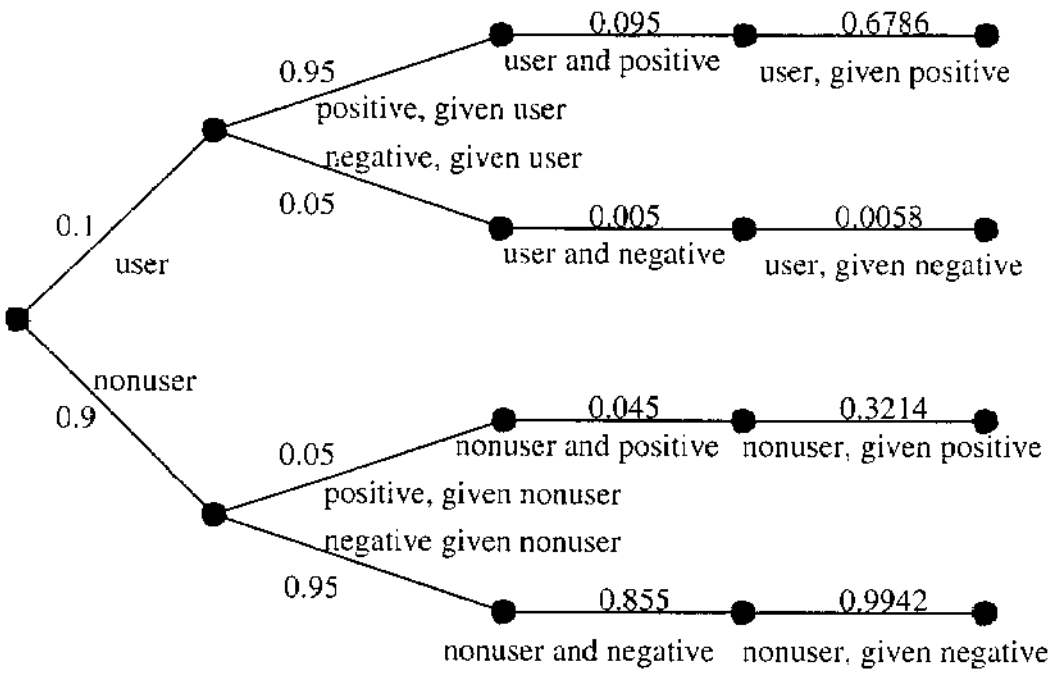
Posterior Probabilities:		P(State | Finding) State of Nature		
Finding	P(Finding)	Poor	Average	Good
Poor	0.36	0.2778	0.5556	0.1667
Average	0.45	0.1778	0.5556	0.2667
Good	0.19	0.1053	0.2632	0.6316

f) Vincent should not get the credit rating and extend credit.

15.3-10

a through d)

Prior Probabilities P (state)	Conditional Probabilities P(finding | state)	Joint Probabilities P(state and finding)	Posterior Probabilities P(state | finding)



e)

Data:		P(Finding | State) Finding	
State of Nature	Prior Probability	Positive	Negative
Positive	0.1	0.95	0.05
Negative	0.9	0.05	0.95

Posterior Probabilities:		P(State | Finding) State of Nature	
Finding	P(Finding)	Positive	Negative
Positive	0.14	0.67857	0.32143
Negative	0.86	0.00581	0.99419

15.3-11

a)

Alternative	State of Nature: Successful	Unsuccessful
Develop new product	1,500,000	-1,800,000
Don't develop new product	0	0
Prior Probabilities	0.667	0.333

b)

Alternative	State of Nature: Successful	Unsuccessful	Expected Payoff	
Develop product	1500000	-1800000	400000	Maximum
Don't develop product	0	0	0	
Prior Probability	0.666667	0.3333333		

Choose to develop new product (expected payoff is $400,000).

c)

Alternative	State of Nature: Successful	Unsuccessful
Develop product	1500000	-1800000
Don't develop product	0	0
Prior Probability	0.666667	0.333333
Maximum Payoff	1500000	0

Expected Payoff with Perfect Information =	1000000

EVPI = EP (with perfect info) – EP (without more info)
= 1,000,000 – 400,000 = $600,000

This indicates that consideration should be given to conducting the market survey.

d)

Data: State of Nature	Prior Probability	P(Finding | State) Finding: Successful	Unsuccessful
Successful	0.6666667	0.8	0.2
Unsuccessful	0.3333333	0.3	0.7

Posterior Probabilities: Finding	P(Finding)	P(State | Finding) State of Nature: Successful	Unsuccessful
Successful	0.6333333	0.842105	0.157894737
Unsuccessful	0.3666667	0.363636	0.636363636

e) $E[p(a_1, \theta \mid \text{pred. succ.})] = 0.8421(1{,}500{,}000) + 0.1579(-1{,}800{,}000) = \$979{,}000$

$E[p(a_2, \theta \mid \text{pred. succ.})] = 0$

$E[p(a_1, \theta \mid \text{pred. no succ.})] = .3636(1{,}500{,}000) + .6364(-1{,}800{,}000) = -\$600{,}000$

$E[p(a_2, \theta \mid \text{pred. no succ.})] = 0$

Optimal a_1 if pred. success, a_2 if not.

$P(\text{pred. success}) = P(\text{pred. succ.} \mid \theta_1)P(\theta_1) + P(\text{pred. succ.} \mid \theta_2)P(\theta_2)$

$= 0.8(2/3) + (0.2)\,1/3 = 0.6$

Expected payoff given info $= 0.6(979{,}000) + 0.4(0) = \$587{,}000$

So $EVE = 587{,}000 - 400{,}000 = \$187{,}000$

Cost of survey $= \$300{,}000 > \$187{,}000$

So optimal strategy is: Do not conduct survey, market product.

15.3-12 a)

Alternative	State of Nature p=0.05	p=0.25
Screen	-1500	-1500
Don't screen	-750	-3750
Prior Probabilities	0.8	0.2

b)

Alternative	State of Nature p=0.05	p=0.25	Expected Payoff	
Screen	-1500	-1500	-1500	
Don't screen	-750	-3750	-1350	Maximum
Prior Probability	0.8	0.2		

Choose to not screen (expected loss is $1,350).

c)

Alternative	State of Nature p=0.05	p=0.25
Screen	-1500	-1500
Don't screen	-750	-3750
Prior Probability	0.8	0.2
Maximum Payoff	-750	-1500

Expected Payoff with Perfect Information =	-900

EVPI = EP (with perfect info) – EP (without more info) = -900 + 1350 = $450

This indicates that consideration should be given to inspecting the single item.

d)

Data:		P(Finding | State) Finding	
State of Nature	Prior Probability	defective	nondefective
p=0.05	0.8	0.05	0.95
p=0.25	0.2	0.25	0.75

Posterior Probabilities:		P(State | Finding) State of Nature	
Finding	P(Finding)	p=0.05	p=0.25
defective	0.09	0.444444	0.555555556
nondefective	0.91	0.835165	0.164835165

e) $P(\text{defective}) = (0.05)(0.8) + (0.25)(0.2) = 0.09$

$P(\text{non-defective}) = 0.91$

$EVE = [(0.09)(-1500) + (0.91)(-1245)] - (-1350) = 82.05$

Since the cost of the inspection is \$125 > \$82.05, the single item is not worthwhile inspecting.

F) If defective found:

$E[p(\text{screen}, \theta \mid \text{defective})] = 0.444(1500) + 0.556(1500) = -1500$

$E[p(\text{non-screen}, \theta \mid \text{defect})] = 0.444(-750) + 0.556(-3750) = -2418$

If non-defective found:

$E[p(\text{screen}, \theta \mid \text{non-defective})] = -1500$

$E[p(\text{non-screen}, \theta \mid \text{non-defect})] = 0.835(-750) + 0.165(-3750) = -1245$

So optimal policy w/experimentation is to screen if defective is found, and not screen if non-defective is found.

15.3-13. a) $E[p(a_1, \theta)] = 0.6(0) + 0.4(-1) = -0.4$

$E[p(a_2, \theta)] = 0.6(-1) + 0.4(0) = -0.6$

Optimal action is a_1 (say coin 1 is tossed)

b) If outcome is heads (H):

$$P(\text{coin 1} \mid H) = \frac{P(H \mid \text{coin 1})P(\text{coin 1})}{P(H \mid \text{coin 1})P(\text{coin 1}) + P(H \mid \text{coin 2})P(\text{coin 2})}$$

$$= \frac{0.3(0.6)}{0.3(0.6) + 0.6(0.4)} = \frac{3}{7}$$

$P(\text{coin 2} \mid H) = \frac{4}{7}$

$E[p(a_1, \theta \mid H)] = \frac{3}{7}(0) + \frac{4}{7}(-1) = -\frac{4}{7}$

(CONT'D)

(CONT'D)

$E[p(a_2,\theta|H)] = \frac{3}{7}(-1) + \frac{4}{7}(0) = -\frac{3}{7}$

Optimal action is a_2.

If outcome is tails (T):

$P(\text{coin 1}|T) = \frac{0.7(0.6)}{0.7(0.6)+(0.4)(0.4)} = .7241$, $P(\text{coin 2}|T) = .2759$

$E[p(a_1,\theta|T)] = .7241(0) + .2759(-1) = -.2759$

$E[p(a_2,\theta|T)] = .7241(-1) + .2759(0) = -.7241$

Optimal action is a_1.

15.3-14 a)

a \ θ	0	1	2	3	4	5
a_1	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40
a_2	1.60	-0.40	-0.40	-0.40	-0.40	-0.40
a_3	0.60	0.40	0.20	0.00	-0.20	-0.40

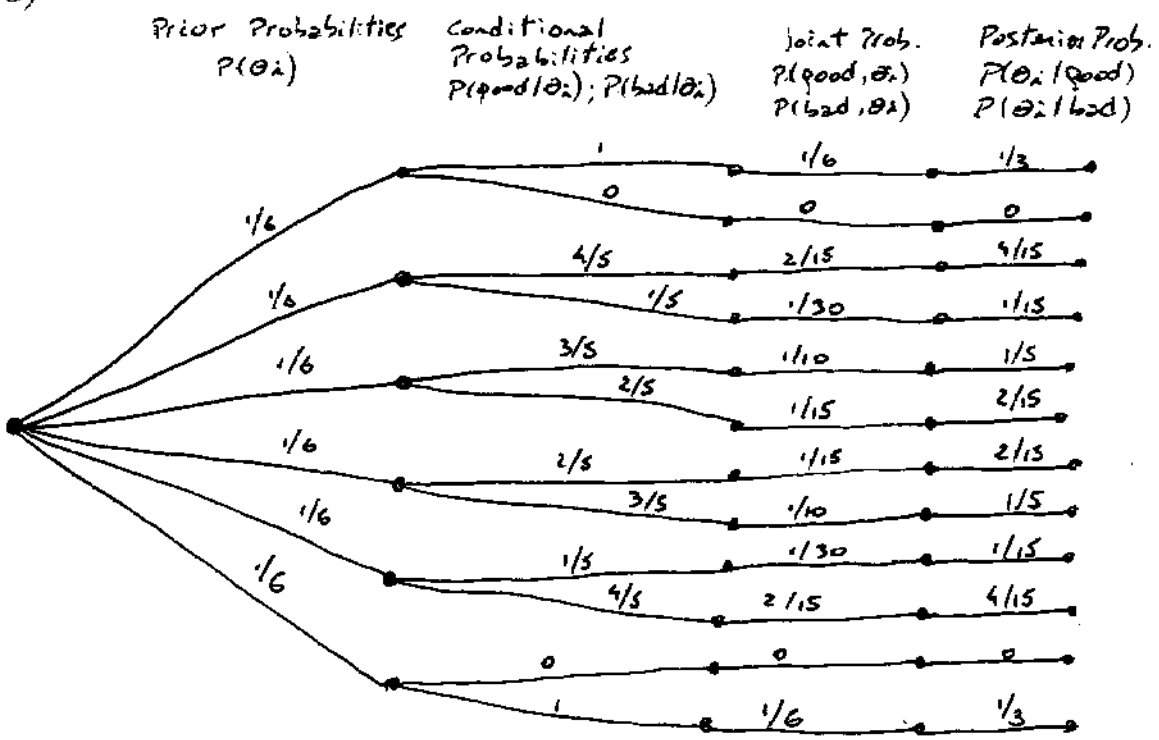
b) $E[p(a_1,\theta)] = -0.40$

$E[p(a_2,\theta)] = \frac{1}{6}(1.60) + \frac{5}{6}(-0.40) = -0.067$

$E[p(a_3,\theta)] = \frac{1}{6}(0.60) + \frac{1}{6}(0.40) + \ldots + \frac{1}{6}(-0.40) = 0.10$

Optimal action is a_3 (sell film for \$1).

c)



d) If test = good:

$E[\rho(a_1, \theta \mid good)] = -0.40$

$E[\rho(a_2, \theta \mid good)] = \frac{1}{3}(1.60) + \frac{2}{3}(-0.40) = 0.267$

$E[\rho(a_3, \theta \mid good)] = \frac{1}{3}(0.60) + \frac{4}{15}(0.40) + \frac{1}{5}(0.20) + 0 + \frac{1}{15}(-0.20) + 0 = 0.333$

If test = bad:

$E[\rho(a_1, \theta \mid bad)] = -0.40$

$E[\rho(a_2, \theta \mid bad)] = 0(1.60) + 1(-0.40) = -0.40$

$E[\rho(a_3, \theta \mid bad)] = 0 + \frac{1}{15}(0.40) + \frac{2}{15}(0.20) + 0 + \frac{4}{15}(-0.20) + \frac{1}{3}(-0.40) = -0.133$

In both cases, the optimal action is a_2 (sell film for \$1).

The expected payoff is $\frac{1}{2}(0.333) + \frac{1}{2}(-0.133) = 0.10$.

15.3-15 a) Payoff Table:

State of Nature / Action	θ_1 coin 1	θ_2 coin 2
a_1 (predict 0 H's)	4	36
a_2 (predict 1 H)	32	48
a_3 (predict 2 H's)	64	16
prior probabilities	0.5	0.5

$E[\rho(a_1, \theta)] = \frac{1}{2}(4) + \frac{1}{2}(36) = \$20$

$E[\rho(a_2, \theta)] = \frac{1}{2}(32) + \frac{1}{2}(48) = \$40$

$E[\rho(a_3, \theta)] = \frac{1}{2}(64) + \frac{1}{2}(16) = \$40$

Optimal action is a_2 or a_3 (both have expected payoff \$40)

b) Prior Distribution:

k	θ_1	θ_2
$P_\theta(k)$	0.5	0.5

$Q_{X \mid \theta=k}(x)$

x \ k	θ_1	θ_2
X_1	0.8	0.4
X_2	0.2	0.6

Posterior Distribution:

$h_{\theta \mid X=x}(k)$

x \ k	θ_1	θ_2
X_1	0.667	0.333
X_2	0.25	0.75

c) If toss heads (H):

$E[p(a_1,\theta|H)] = 0.667(4) + 0.333(36) = 14.67$

$E[p(a_2,\theta|H)] = 0.667(32) + 0.333(48) = 37.33$

$E[p(a_3,\theta|H)] = 0.667(64) + 0.333(16) = 48$

Optimal action is a_3 (predict 2 H's)

If toss tails (T):

$E[p(a_1,\theta|T)] = 0.25(4) + 0.75(36) = 28$

$E[p(a_2,\theta|T)] = 0.25(32) + 0.75(48) = 44$

$E[p(a_3,\theta|T)] = 0.25(64) + 0.75(16) = 28$

Optimal action is a_2 (predict 1 H)

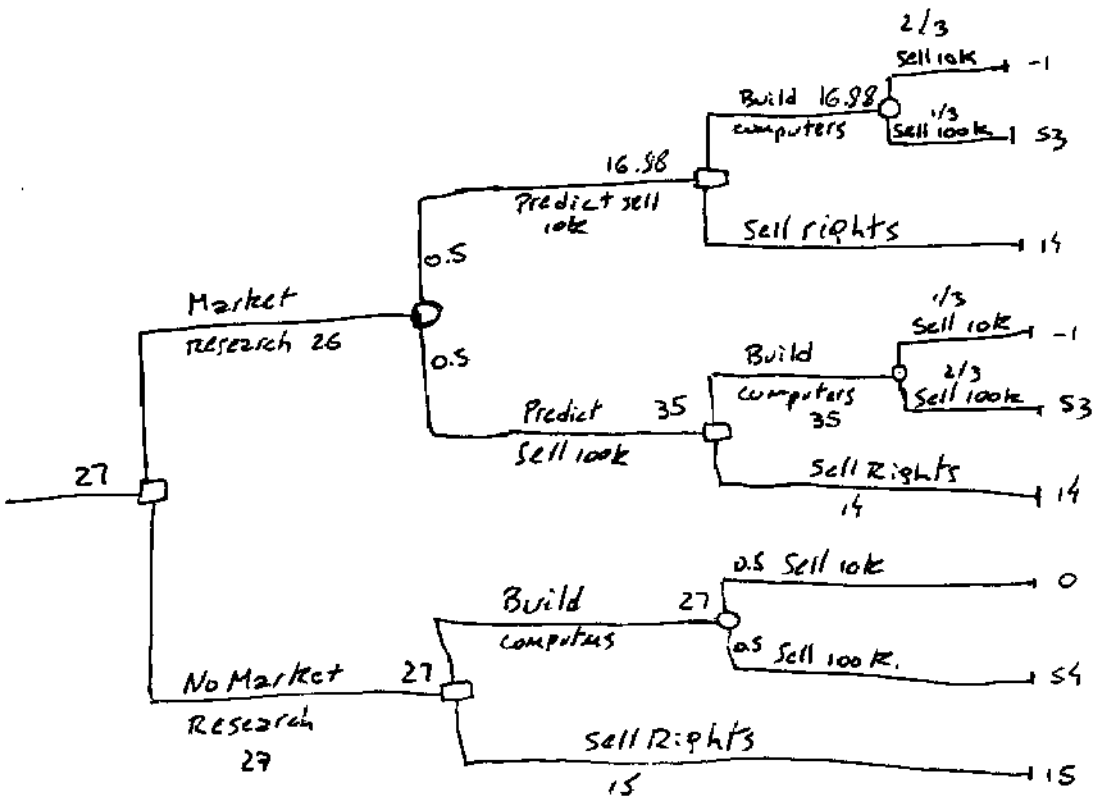
$P(H) = 0.6 \quad P(T) = 0.4$

Expected payoff $= 0.6(48) + 0.4(44) = \$46.40$

d) $EVE = 46.40 - 40 = \$6.40 < \$30$

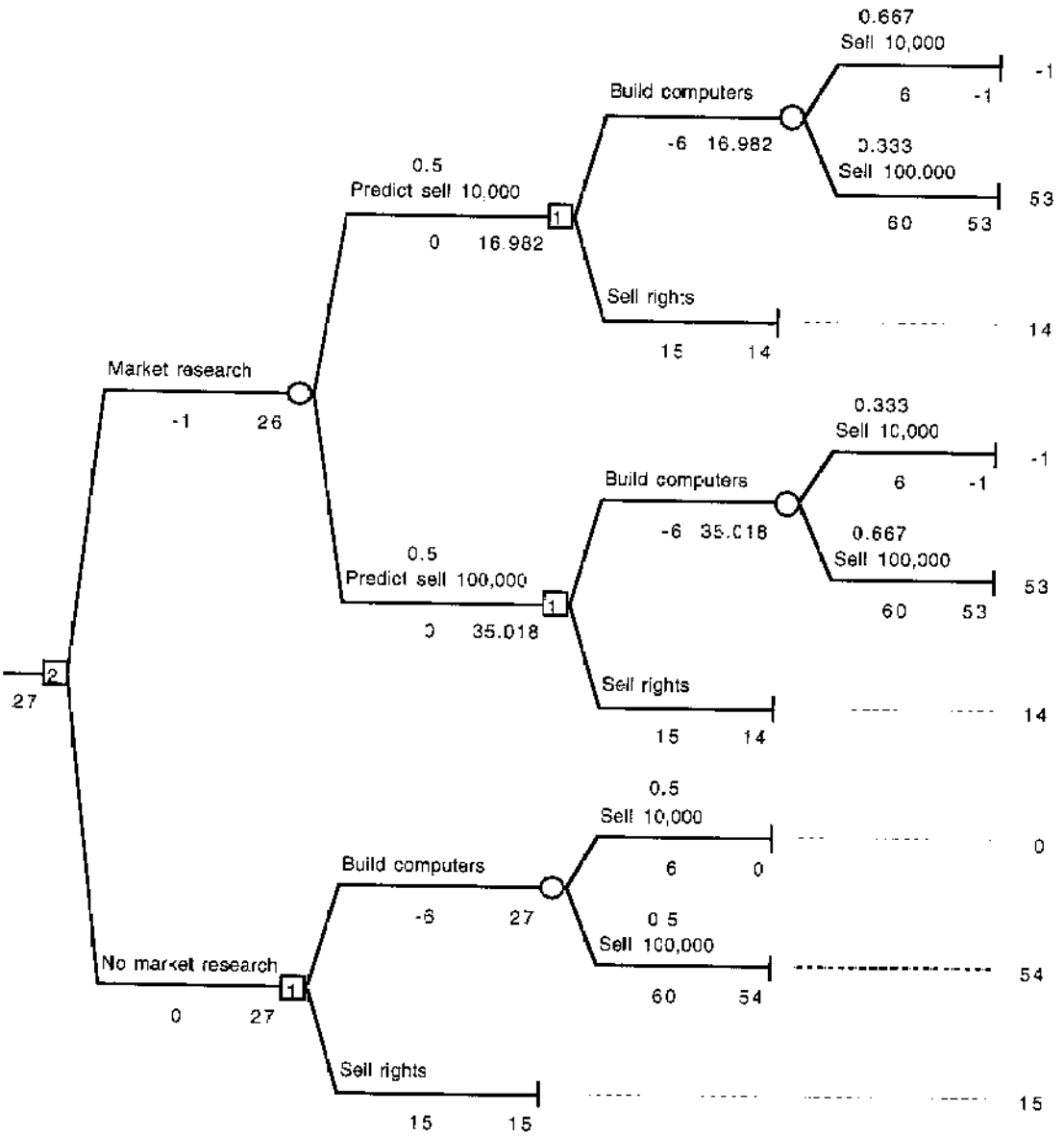
So the optimal policy is to not pay for the experiment and choose action 2 or 3.

15.5-1 a)



So the optimal policy is: No Market Research and build computers.

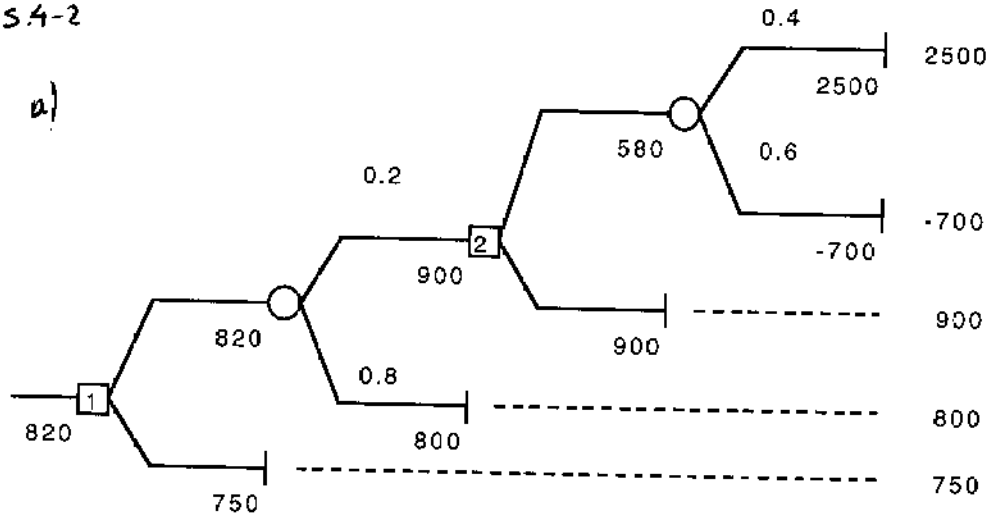
b)



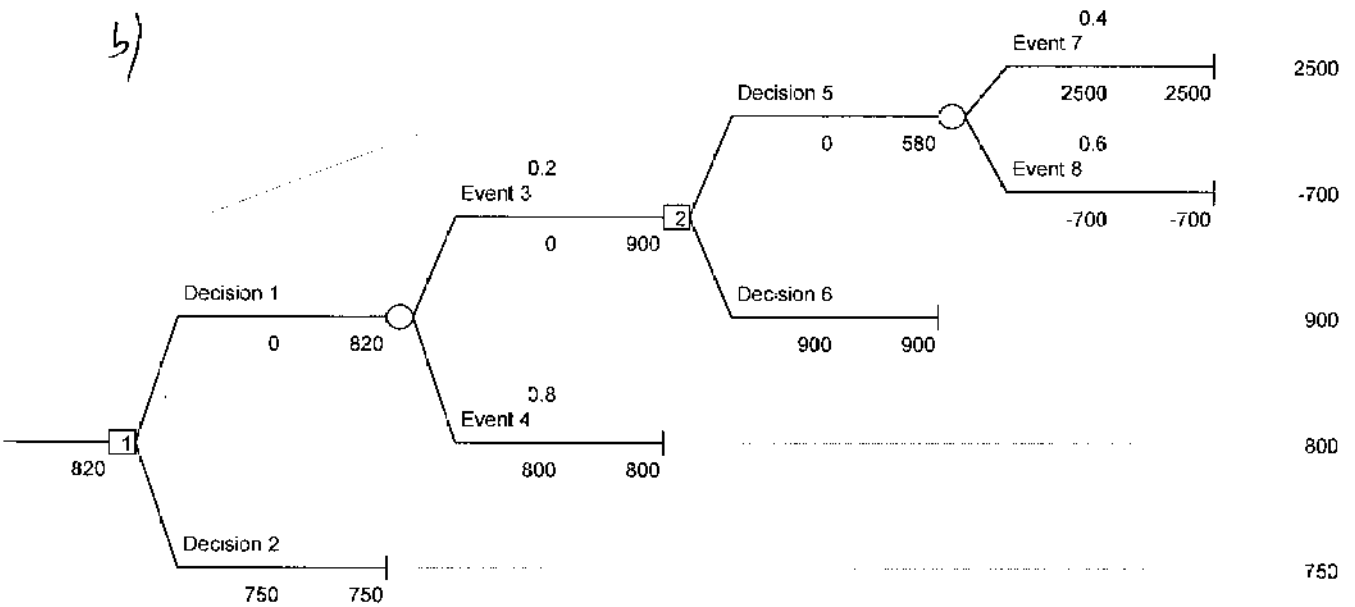
The optimal policy is to do no market research and build the computers.

15.4-2

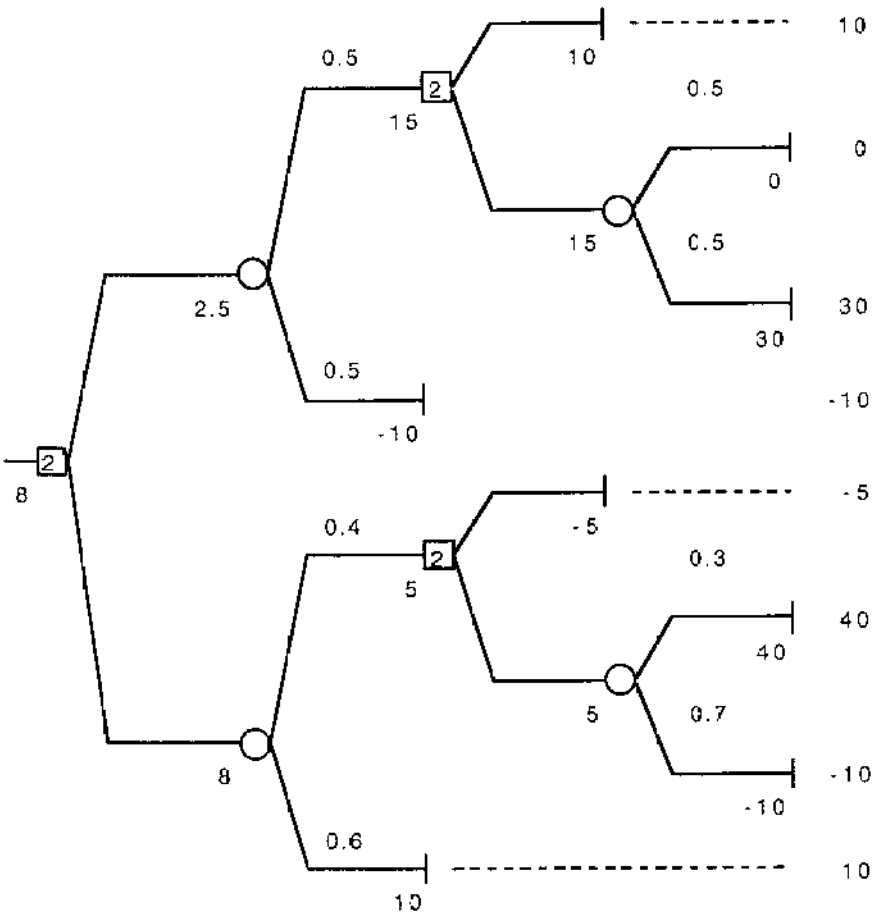
a)


0.4
2500
2500
580
0.6
-700
-700
0.2
2
900
900
900
820
0.8
800
800
1
820
750
750

b)


0.4
Event 7
2500
2500
2500
Decision 5
0
580
0.6
Event 8
-700
-700
-700
0.2
Event 3
0
900
2
Decision 1
0
820
Decision 6
900
900
900
0.8
Event 4
1
820
800
800
800
Decision 2
750
750
750

15.4-3



15.4-4

a)

Alternative	State of Nature W	L
Hold campaign	3	-2
Don't hold campaign	0	0
Prior Probabilities	0.6	0.4

b)

Alternat ve	State of Nature W	L	Expected Payoff	
Hold campaign	3	-2	1	Maximum
Don't hold campaign	0	0	0	
Prior Probability	0.6	0.4		

Choose to hold the campaign (expected payoff is $1 million).

c)

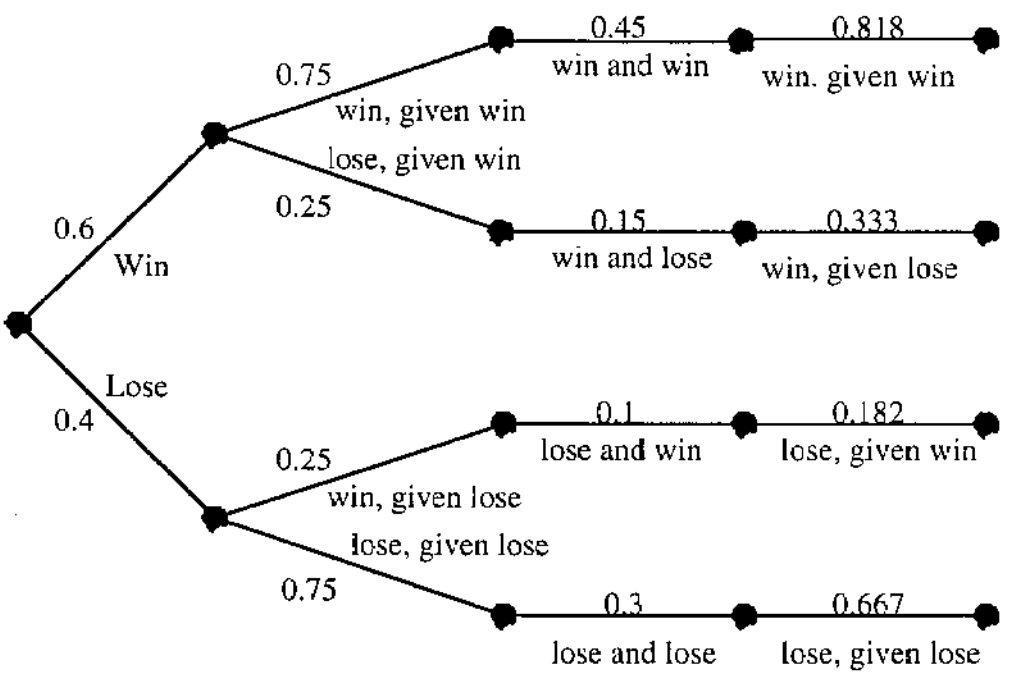
Alternative	State of Nature W	L
Hold campaign	3	-2
Don't hold campaign	0	0
Prior Probability	0.6	0.4
Maximum Payoff	3	0

Expected Payoff with Perfect Information =	1.8

EVPI = EP (with perfect info) – EP (without more info) = 1.8 - 1 = $800,000

d)

Prior Probabilities P (state)	Conditional Probabilities P(finding I state)	Joint Probabilities P(state and finding)	Posterior Probabilities P(state I finding)



e)

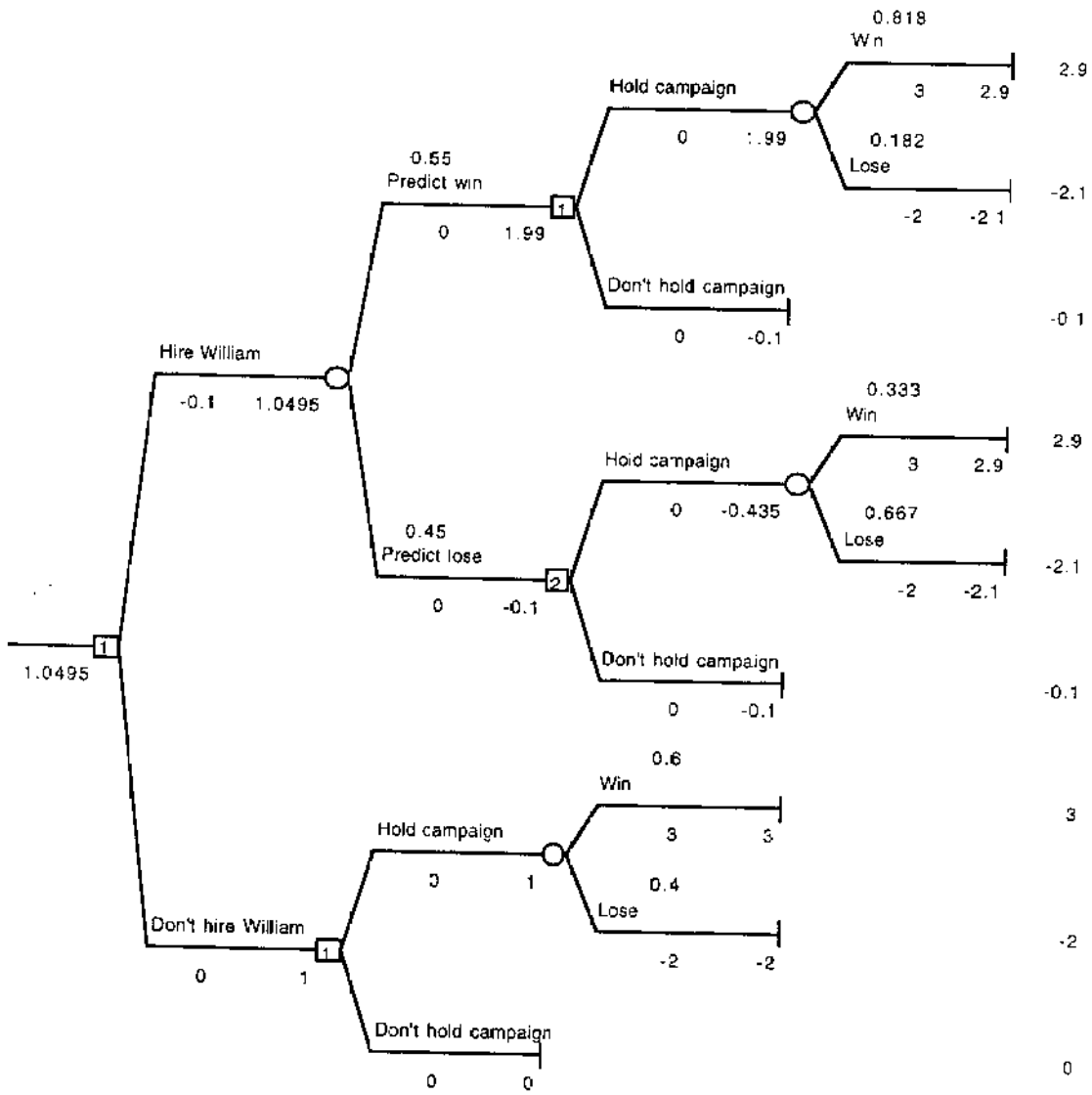
Data: State of Nature	Prior Probability	P(Finding I State) Finding Win	Lose
Win	0.6	0.75	0.25
Lose	0.4	0.25	0.75

(CONT'D)

(CONT'D)

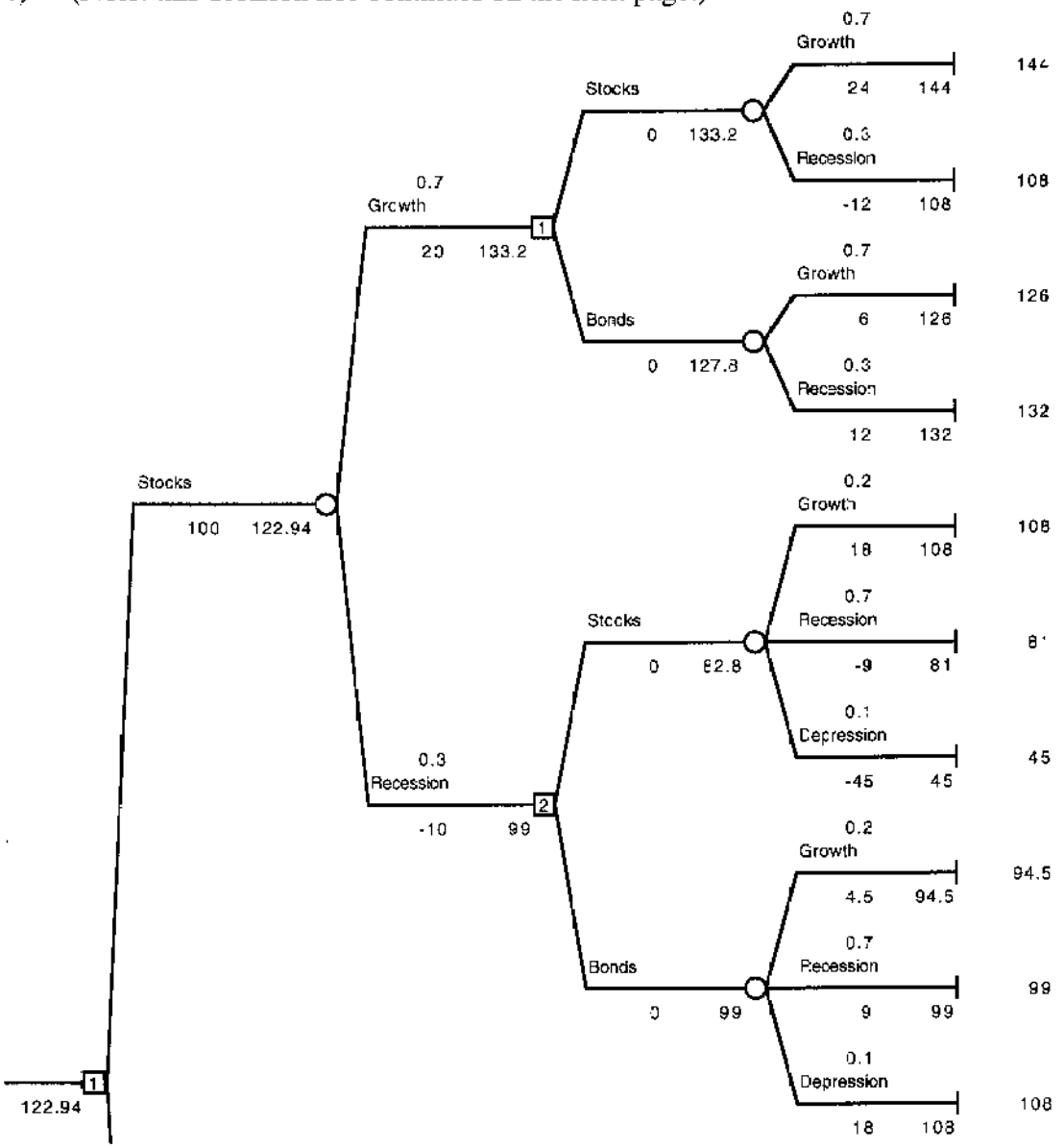
Posterior Probabilities:		P(State I Finding) State of Nature	
Finding	P(Finding)	Win	Lose
Win	0.55	0.8182	0.1818
Lose	0.45	0.3333	0.6667

f & g) Leland University should hire William. If he predicts a winning season then they should hold the campaign, if he predicts a loosing season then they should not hold the campaign.



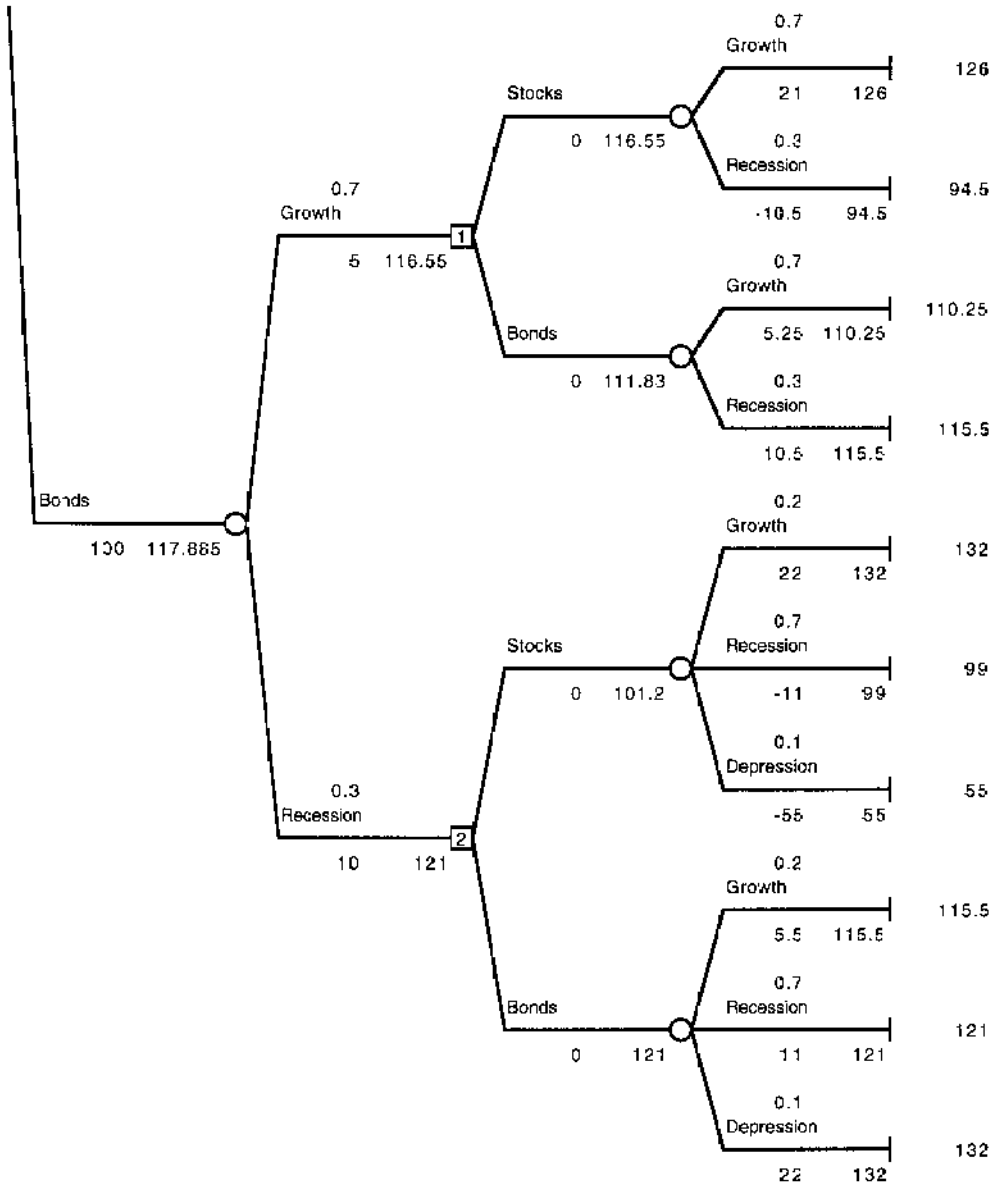
15.4-5

a & c) (Note: this decision tree continues on the next page.)



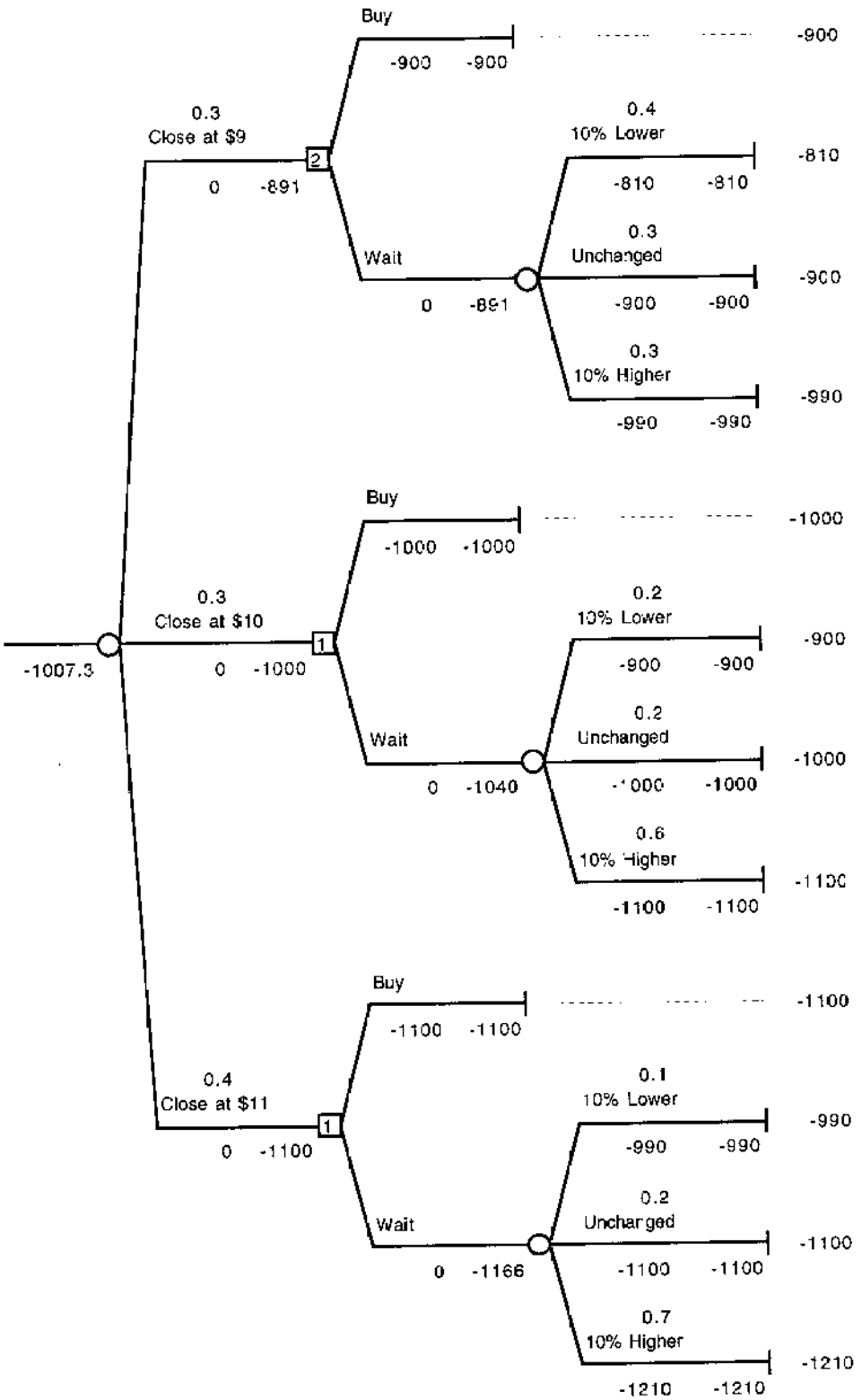
(CONT'D)

(CONT'D)

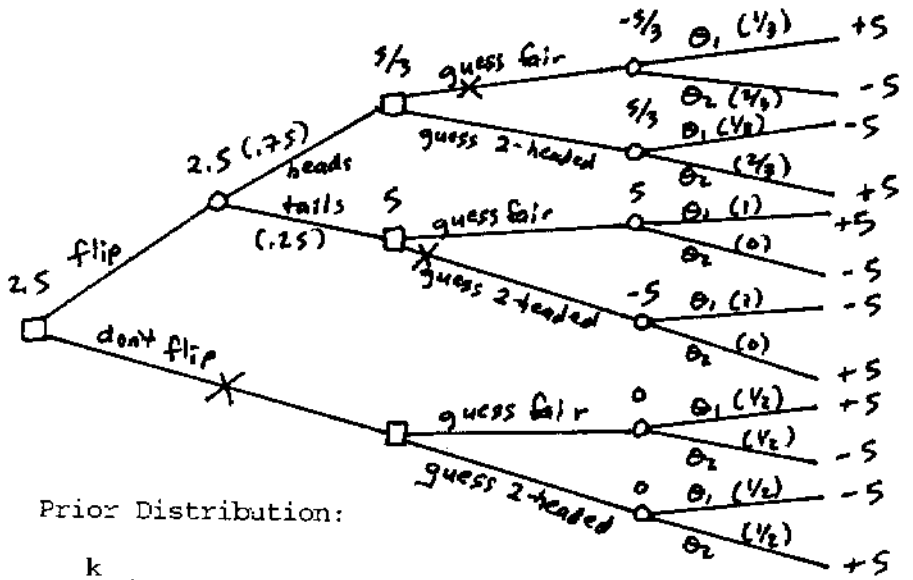


b) The comptroller should invest in stocks the first year. If there is growth during the first year then she should invest in stocks again the second year. If there is a recession during the first year then she should invest in bonds for the second year.

10.4-6 a & b) The optimal policy is to wait until Wednesday to buy if the price is \$9 on Tuesday. If the price is \$10 or \$11 on Tuesday then buy on Tuesday.



15.4-7. a)



b) Prior Distribution:

k	θ_1	θ_2
$P_\theta(k)$	0.5	0.5

$Q_{X|\theta=k}(x)$

x \ k	θ_1	θ_2
X_1	0.5	0
X_2	0.5	1

Posterior Distribution:

$h_{\theta|X=x}(k)$

x \ k	θ_1	θ_2
X_1	1	0
X_2	0.333	0.667

c) See (a). The optimal policy is to flip the coin and guess that it is fair if it comes up tails, and 2-headed if it comes up heads.

15.4-8

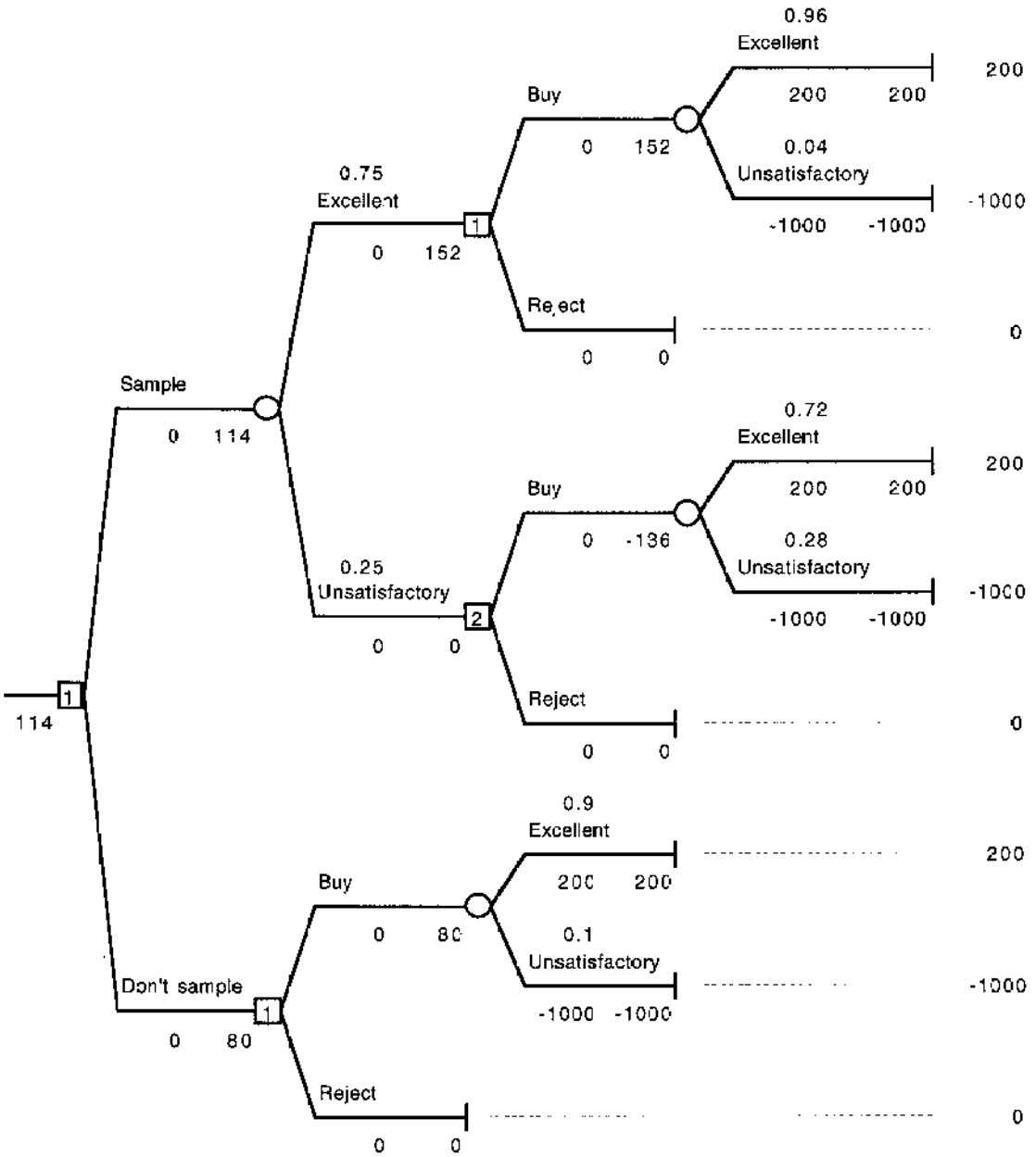
The optimal policy is to sample the fruit and buy if it is excellent and reject if it is unsatisfactory.

Data:		P(Finding I State) Finding	
State of Nature	Prior Probability	Excellent	Unsatisfactory
Excellent	0.9	0.8	0.2
Unsatisfactory	0.1	0.3	0.7

Posterior Probabilities:		P(State I Finding) State of Nature	
Finding	P(Finding)	Excellent	Unsatisfactory
Excellent	0.75	0.96	0.04
Unsatisfactory	0.25	0.72	0.28

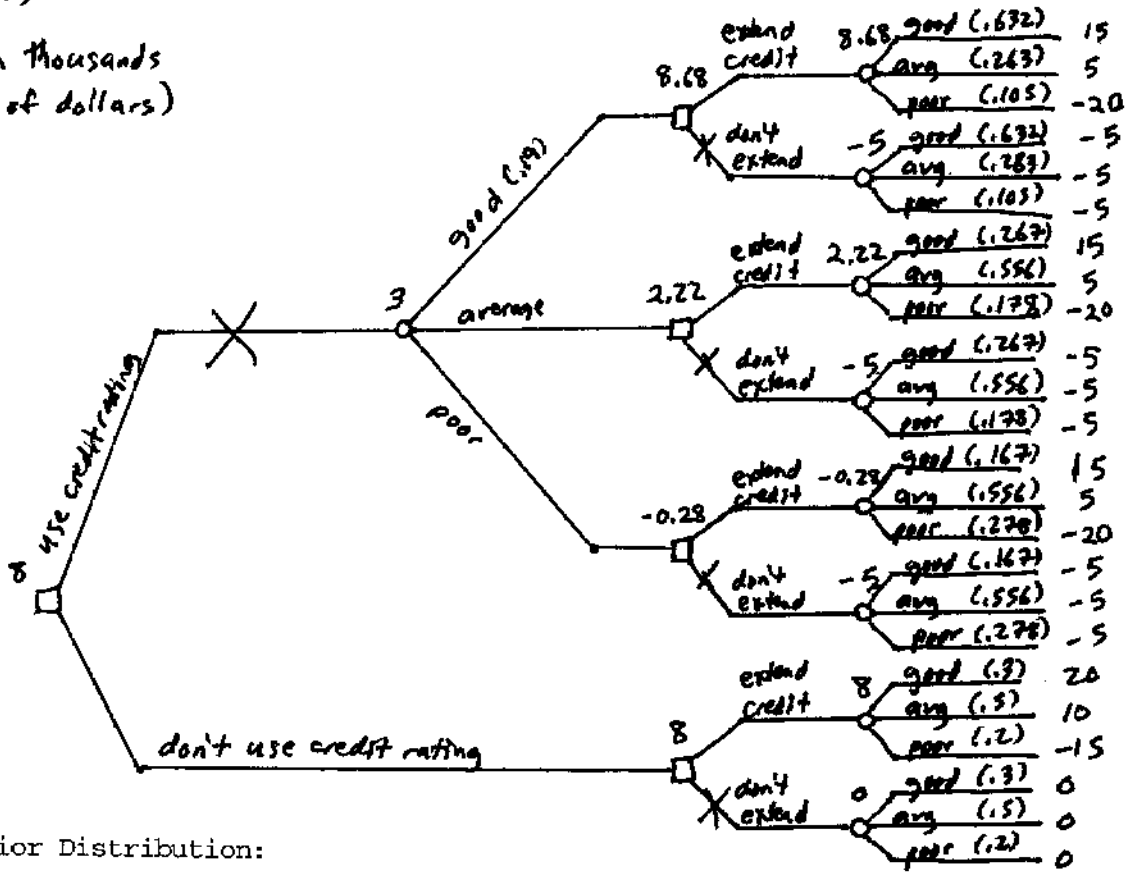
(CONT'D)

(CONT'D)


0.96
Excellent
200
Buy
200 200
0 152
0.04
Unsatisfactory
-1000
-1000 -1000
0.75
Excellent
1
0 152
Reject
0 0
0
Sample
0 114
0.72
Excellent
200
Buy
200 200
0 -136
0.28
Unsatisfactory
-1000
-1000 -1000
0.25
Unsatisfactory
2
0 0
1
114
Reject
0 0
0
0.9
Excellent
200
Buy
200 200
0 80
0.1
Unsatisfactory
-1000
-1000 -1000
Don't sample
1
0 80
Reject
0 0
0

15.4-9 .a)

(in thousands of dollars)



b) Prior Distribution:

k	θ_1	θ_2	θ_3
$P_\theta(k)$	0.2	0.5	0.3

$Q_{X|\theta=k}(x)$

x \ k	θ_1	θ_2	θ_3
X_1	0.5	0.4	0.2
X_2	0.4	0.5	0.4
X_3	0.1	0.1	0.4

Posterior Distribution:

$h_{\theta|X=x}(k)$

x \ k	θ_1	θ_2	θ_3
X_1	0.278	0.556	0.167
X_2	0.178	0.556	0.267
X_3	0.105	0.263	0.632

c) Optimal to not use credit rating, but extend credit. (See (a)).

15.4-10 a)

Alternative	State of Nature	
	Successful	Unsuccessful
Introduce new product	40	-15
Don't introduce new product	0	0
Prior Probabilities	0.5	0.5

Alternative	State of Nature		Expected Payoff	
	Successful	Unsuccessful		
Introduce product	40	-15	12.5	Maximum
Don't introduce product	0	0	0	
Prior Probability	0.5	0.5		

Choose to introduce the new product (expected payoff is $12.5 million).

b)

Alternative	State of Nature	
	Successfu	Unsuccessful
Introduce product	40	-15
Don't introduce product	0	0
Prior Probability	0.5	0.5
Maximum Payoff	40	0

Expected Payoff with Perfect Information =	20

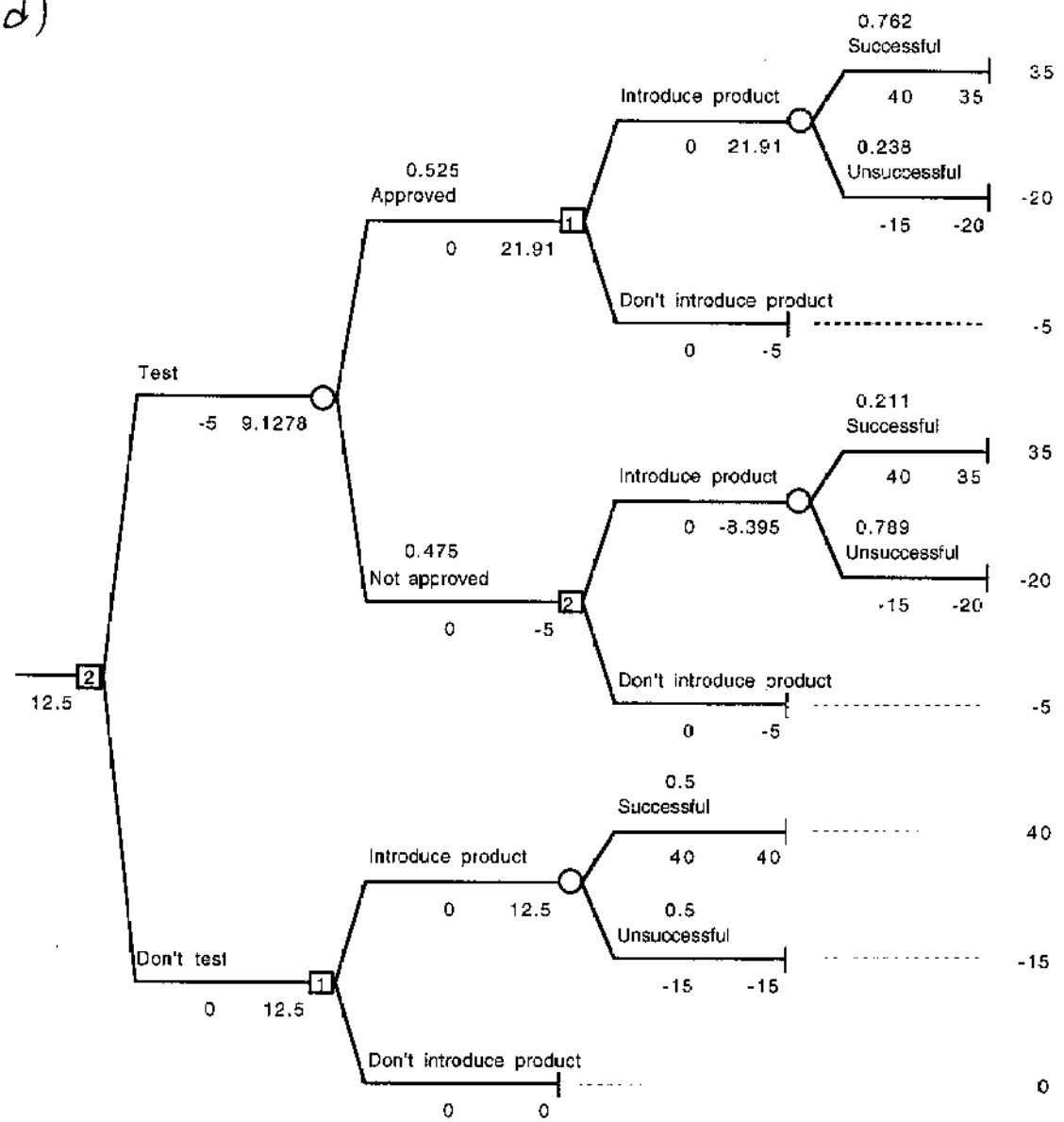
EVPI = EP (with perfect info) – EP (without more info) = 20 – 12.5 = $7.5 million

c) The optimal policy is not to test but to introduce the new product.

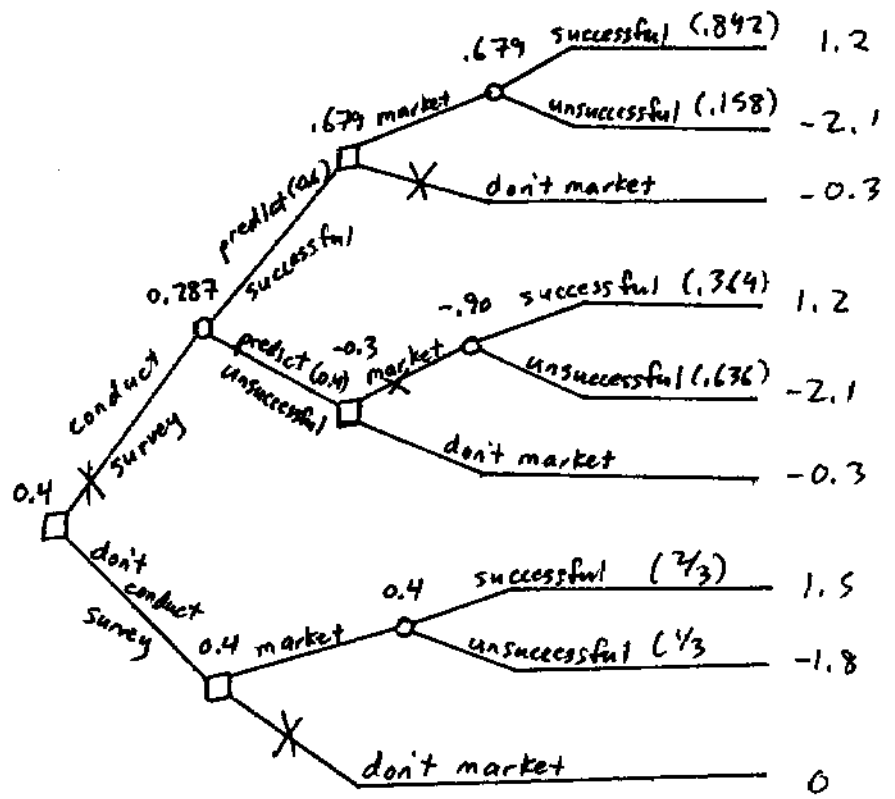
Data:		P(Finding I State)	
		Finding	
State of Nature	Prior Probability	FSS	USS
Successful	0.5	0.8	0.2
Unsuccessful	0.5	0.25	0.75

Posterior Probabilities:		P(State I Finding)	
		State of Nature	
Finding	P(Finding)	Successfunsuccessf	
FSS	0.525	0.7619	0.2381
USS	0.475	0.2105	0.7895

d)


0.762
Successful
35
Introduce product
40 35
0 21.91
0.238
Unsuccessful
-20
-15 -20
0.525
Approved
1
0 21.91
Don't introduce product
-5
0 -5
Test
-5 9.1278
0.211
Successful
35
Introduce product
40 35
0 -8.395
0.789
Unsuccessful
-20
-15 -20
0.475
Not approved
2
0 -5
Don't introduce product
-5
0 -5
2
12.5
0.5
Successful
40
Introduce product
40 40
0 12.5
0.5
Unsuccessful
-15
-15 -15
Don't test
1
0 12.5
Don't introduce product
0
0 0

15.4-11. a)



b)

Prior Distribution:

k	θ_1	θ_2
$P_\theta(k)$	0.667	0.333

$Q_{X|\theta=k}(x)$

x \ k	θ_1	θ_2
X_1	0.8	0.3
X_2	0.2	0.7

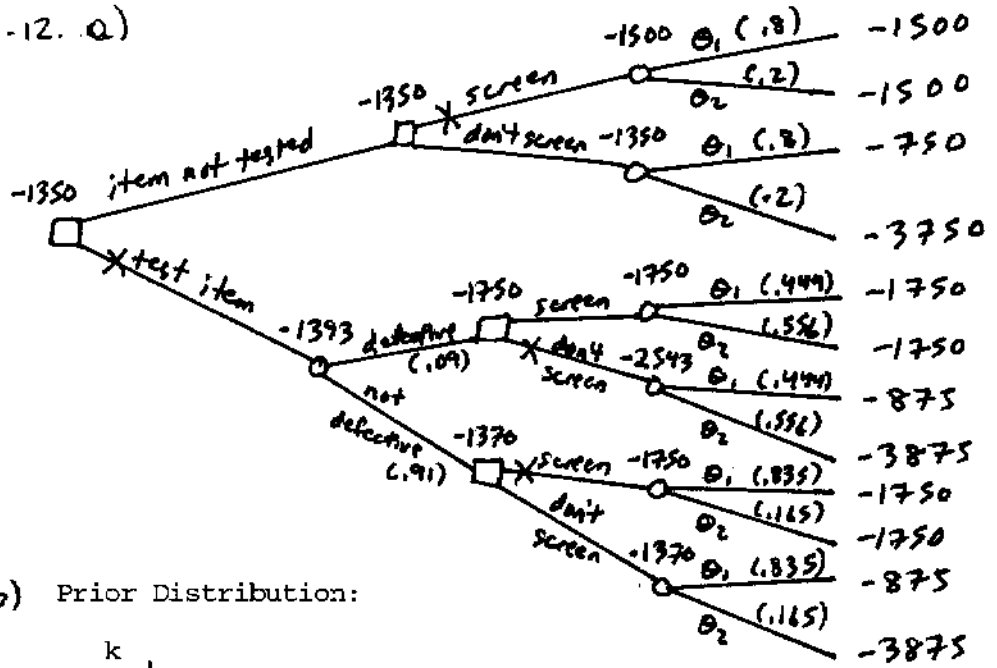
Posterior Distribution:

$h_{\theta|X=x}(k)$

x \ k	θ_1	θ_2
X_1	0.842	0.158
X_2	0.364	0.636

c) See (a). Optimal Policy is to not conduct a survey, but to market the new product.

15.4-12. a)



b) Prior Distribution:

k	θ_1	θ_2
$P_\theta(k)$	0.8	0.2

$Q_{X|\theta=k}(x)$

x \ k	θ_1	θ_2
X_1	0.95	0.75
X_2	0.05	0.25

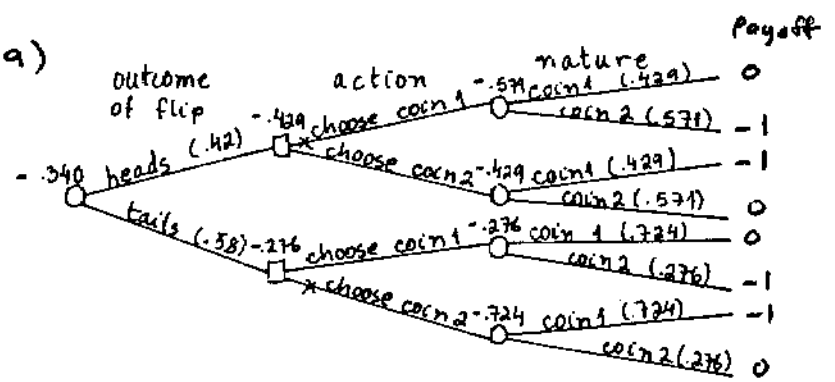
Posterior Distribution:

$h_{\theta|X=x}(k)$

x \ k	θ_1	θ_2
y_1	0.835	0.165
y_2	0.444	0.556

c) See (a). Optimal is not to test item and not to screen.

15.4-13. a)



b) Prior Distribution:

k	θ_1	θ_2
$P_\theta(k)$	0.6	0.4

$Q_{X|\theta=k}(x)$

x \ k	θ_1	θ_2
X_1	0.3	0.6
X_2	0.7	0.4

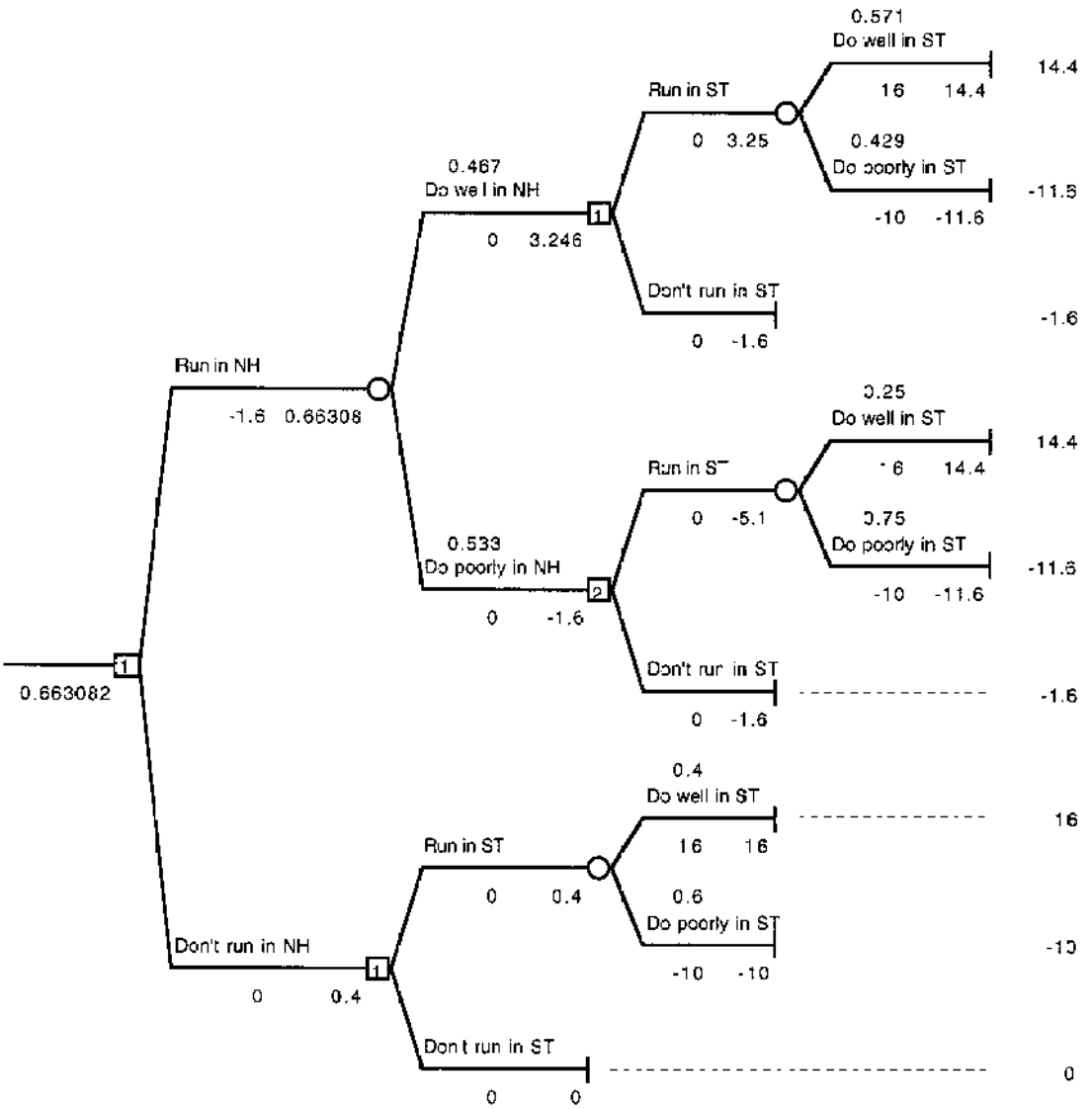
Posterior Distribution:

$h_{\theta|X=x}(k)$

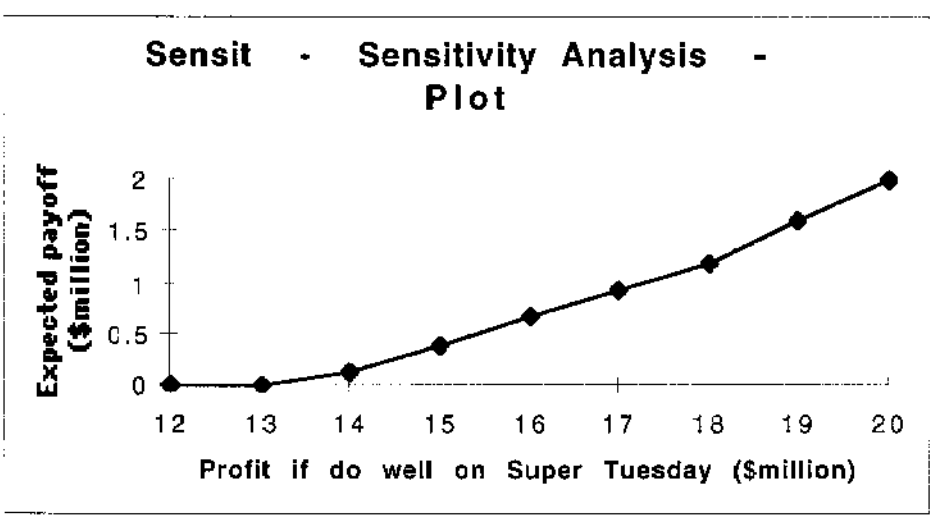
x \ k	θ_1	θ_2
X_1	0.429	0.571
X_2	0.724	0.276

c) See (a). Optimal to choose coin 1 if tails is flipped and coin 2 if heads is flipped

15.4-14. a) Chelsea should run in the NH primary. If she does well then she should run in the ST primaries. If she does poorly then she should not run in the ST primaries.



b)



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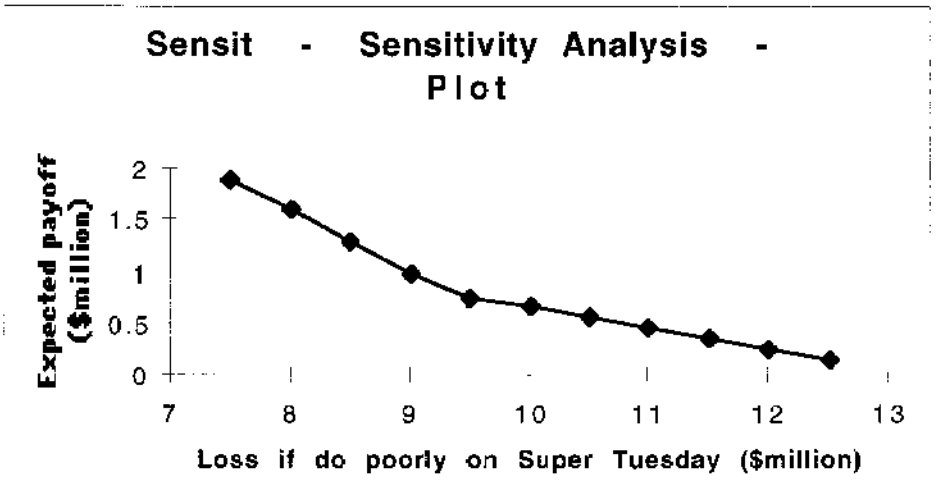
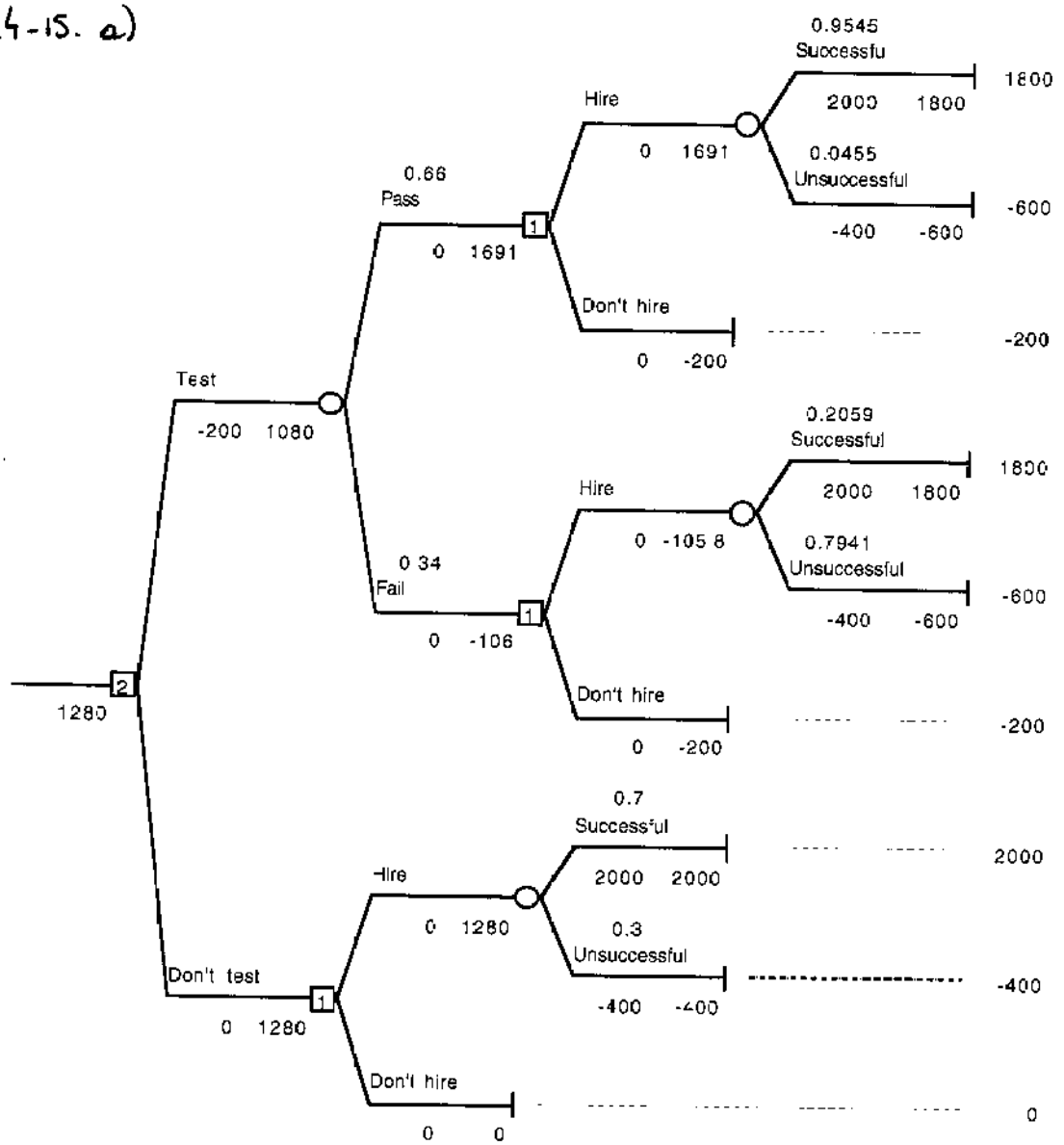


15.4-15. a)



b)

Data:		P(Finding I State) Finding	
State of Nature	Prior Probability	Pass	Fail
Successful	0.7	0.9	0.1
Unsuccessful	0.3	0.1	0.9

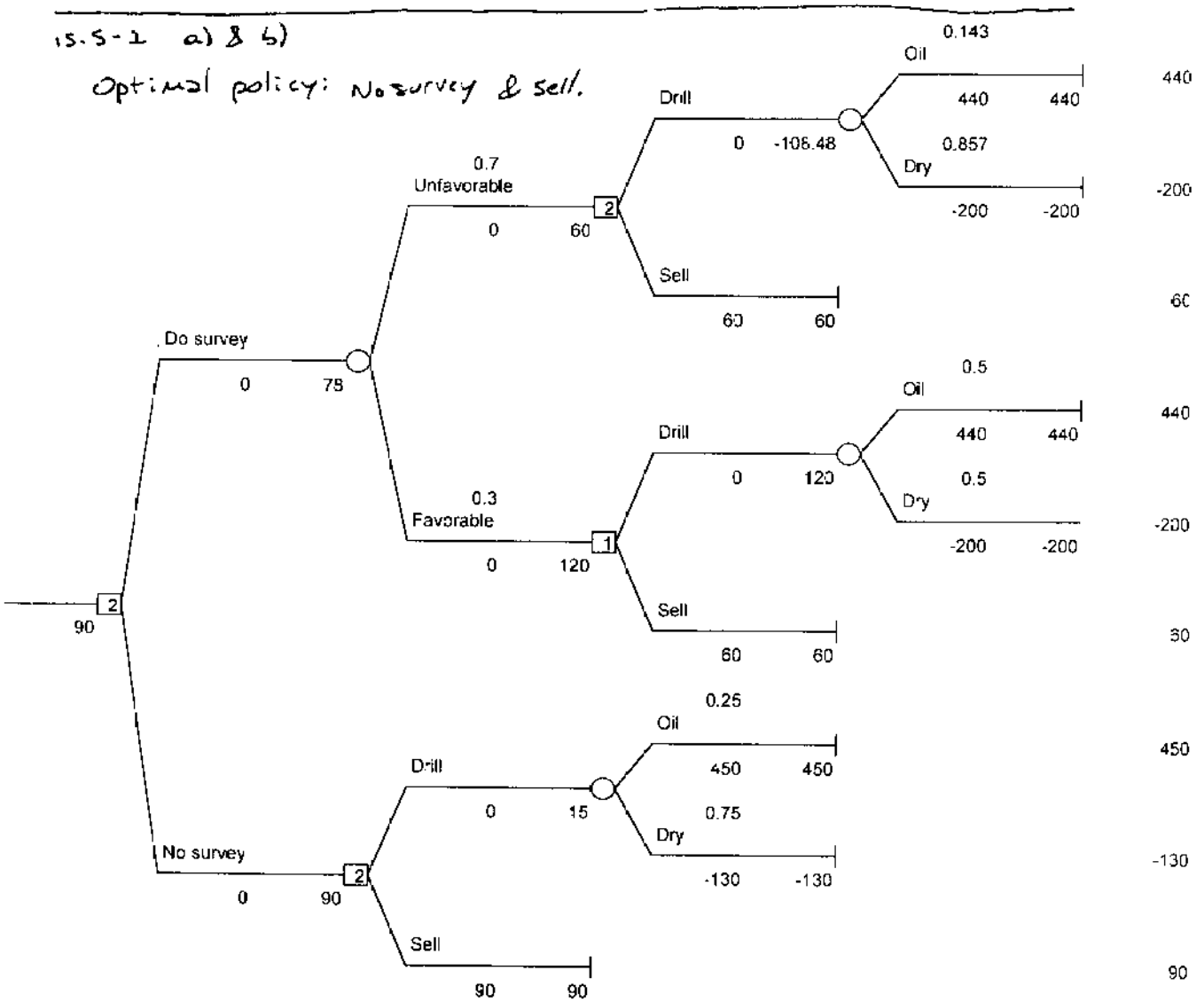
Posterior Probabilities:		P(State I Finding) State of Nature	
Finding	P(Finding)	Successful	Unsuccessful
Pass	0.66	0.9545	0.0455
Fail	0.34	0.2059	0.7941

c) The optimal policy is not to pay for testing and to hire Matthew.

d) If the fee is 0 then the company is still has the same policy so they should not be willing to pay anything for the detailed report.

15.5-1 a) & b)

Optimal policy: No survey & sell.



15-47

15.5-2 a)

Alternative	State of Nature: Earthquake	State of Nature: No earthquake	Expected Payoff	
Buy insurance	249820	249820	249820	
Don't buy insurance	90000	250000	249840	Maximum
Prior Probability	0.001	0.999		

Choose not to buy insurance (expected payoff is \$249,840).

b) U(insurance) = U(250,000-180) = $\sqrt{249,820}$ = 499.82
U(no insurance) = (0.999)U(250,000) + (0.001)U(90,000) = 499.8

Optimal policy is to buy insurance.

15.5-3 E(utility) of \$19,000 = U(19) = $\sqrt{25}$ = 5
E(utility) of investment = (0.3)U(10) + (0.7)U(30) = (0.3)$\sqrt{16}$ + (0.7)$\sqrt{36}$ = 5.4

Choose the investment to maximize expected utility.

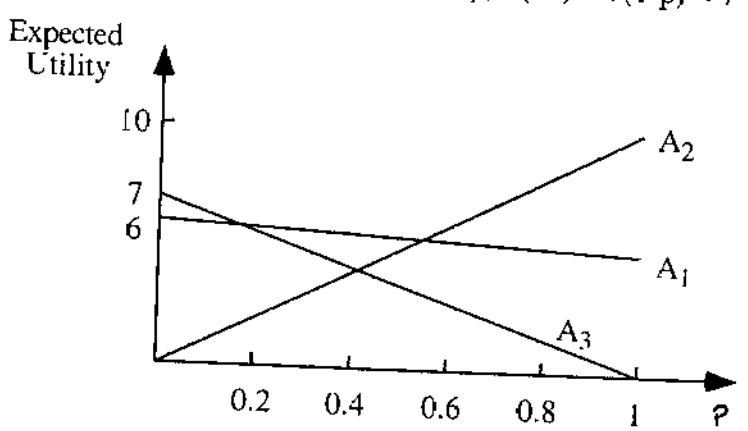
15.5-4 E(utility for A_1) = E(utility for A_2)
pU(10) + (1-p)U(30) = U(19)
(0.3)U(10) + (0.7)(20) = 16.7
U(10) = 9

15.5-5 a) E(utility for A_1) = E(utility for A_2)
pU(10) + (1-p)U(0) = U(1)
(0.125)U(10) + (1-0.125)(0) = 1
U(10) = 8

b) E(utility for A_1) = E(utility for A_2)
pU(10) + (1-p)U(0) = U(5)
(0.5625)(8) + (1-0.5625)(0) = U(5)
U(5) = 4.5

c) Answers will vary.

15.5-6 a) Expected utility of A_1 = pU(25) + (1-p)(36) = 5p + 6(1-p) = 6 – p.
Expected utility of A_2 = pU(100) + (1-p)U(0) = 10p + 0 = 10p.
Expected utility of A_3 = pU(0) + (1-p)U(49) = 7(1-p) = 7 – 7p.



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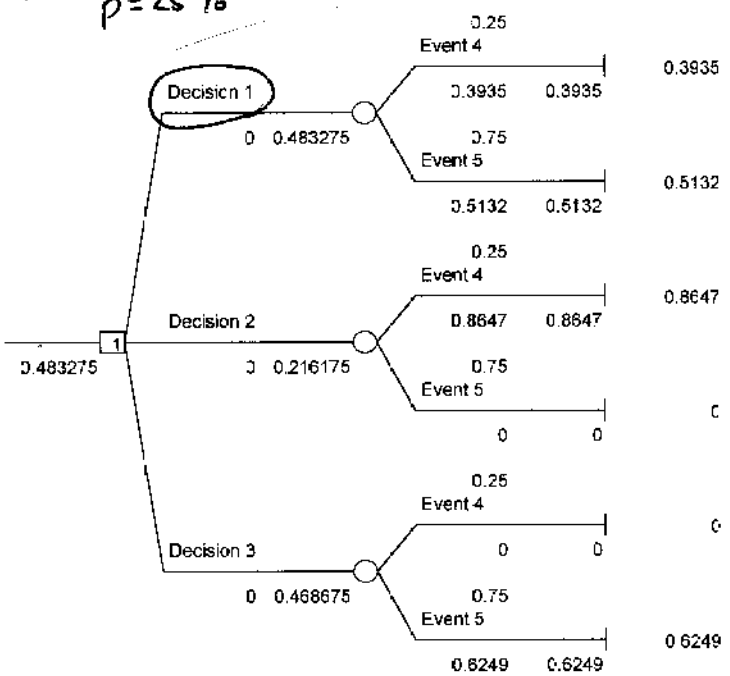
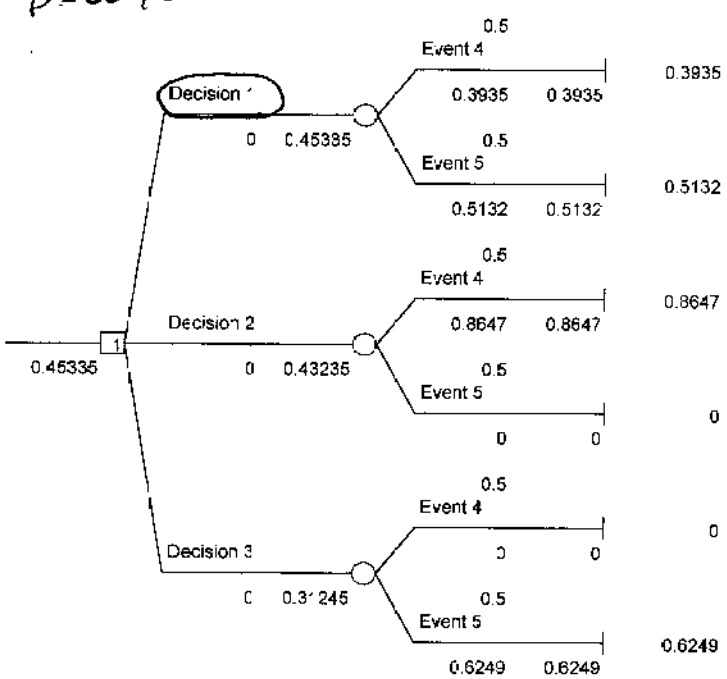
A_1 and A_3 cross when 6-p = 7-7p, or p=1/6.
A_1 and A_2 cross when 6-p=10p, or p=2/3.
Thus, A_3 is best when $p \leq 1/6$, A_1 is best when $1/6 < p \leq 2/3$, and A_2 is best when $p > 2/3$.

b)

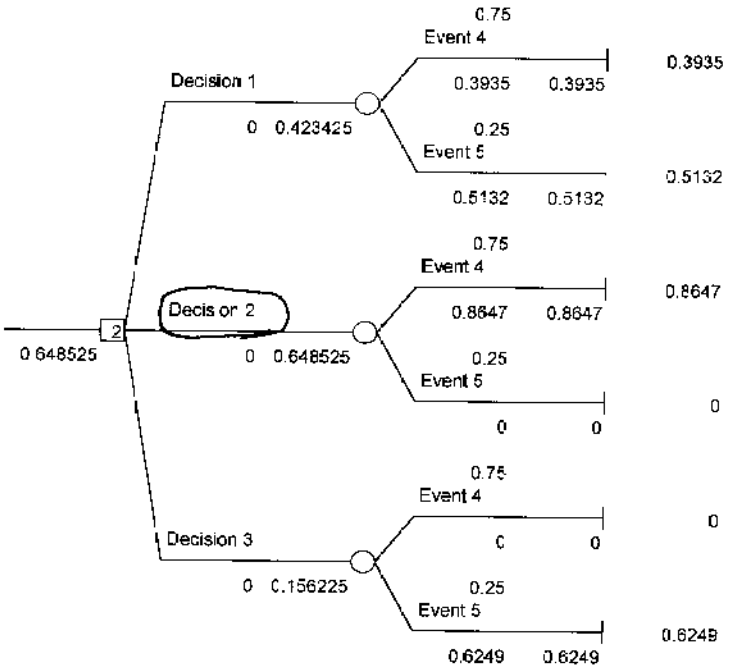
p = 25%



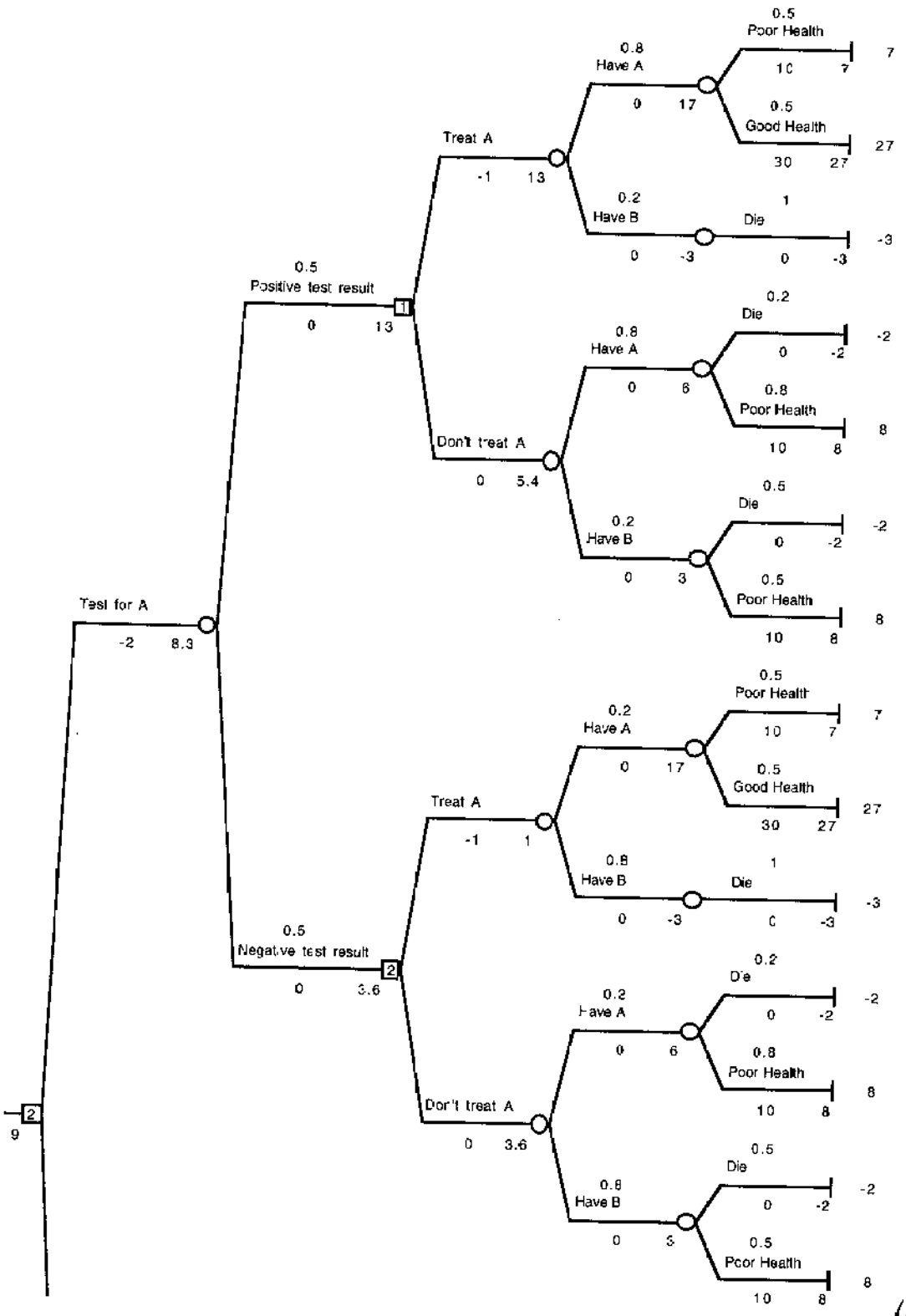
p = 50%



P = 75%


0.75
Event 4
0.3935
Decision 1
0.3935
0.3935
0 0.423425
0.25
Event 5
0.5132
0.5132
0.5132
0.75
Event 4
0.8647
Decis or 2
0.8647
0.8647
2
0.648525
0 0.648525
0.25
Event 5
0
0
0
0.75
Event 4
0
Decision 3
0
0
0 0.156225
0.25
Event 5
0.6249
0.6249
0.6249

The optimal policy is not to test for disease A but to treat disease A. (Note: this decision tree is continued on the next page.)



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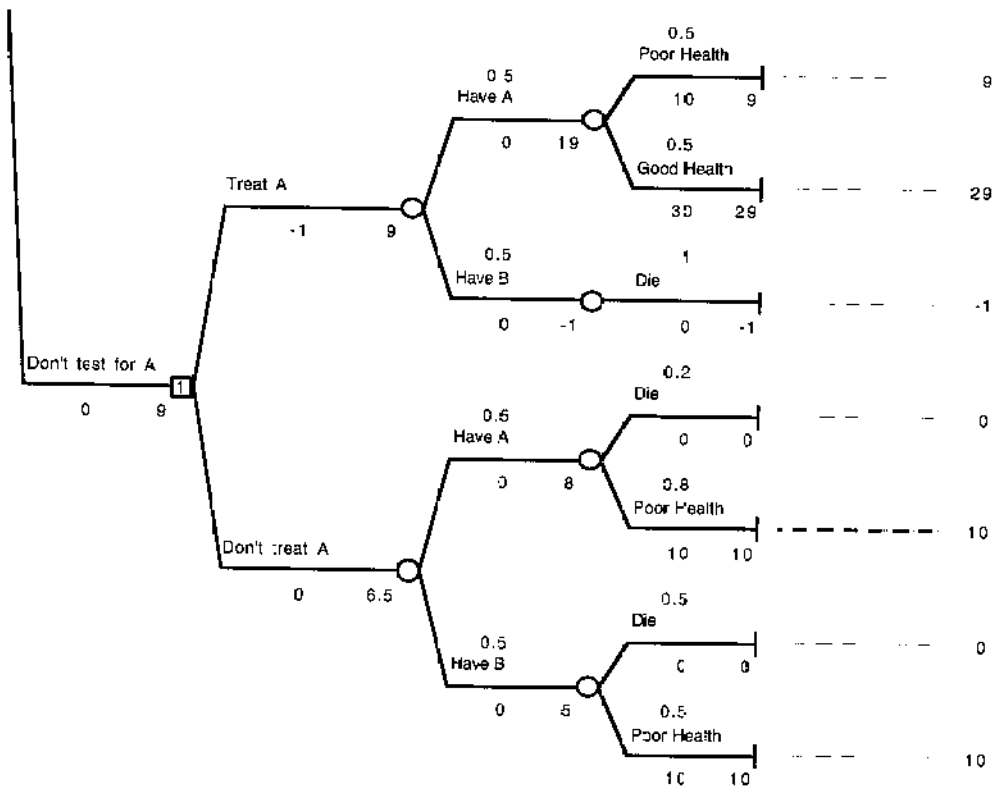
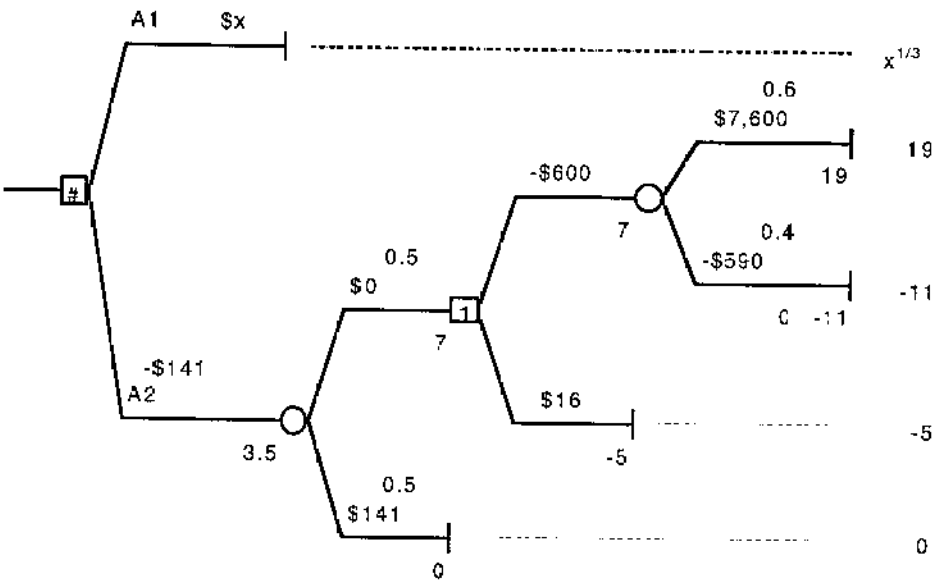


15.5-8

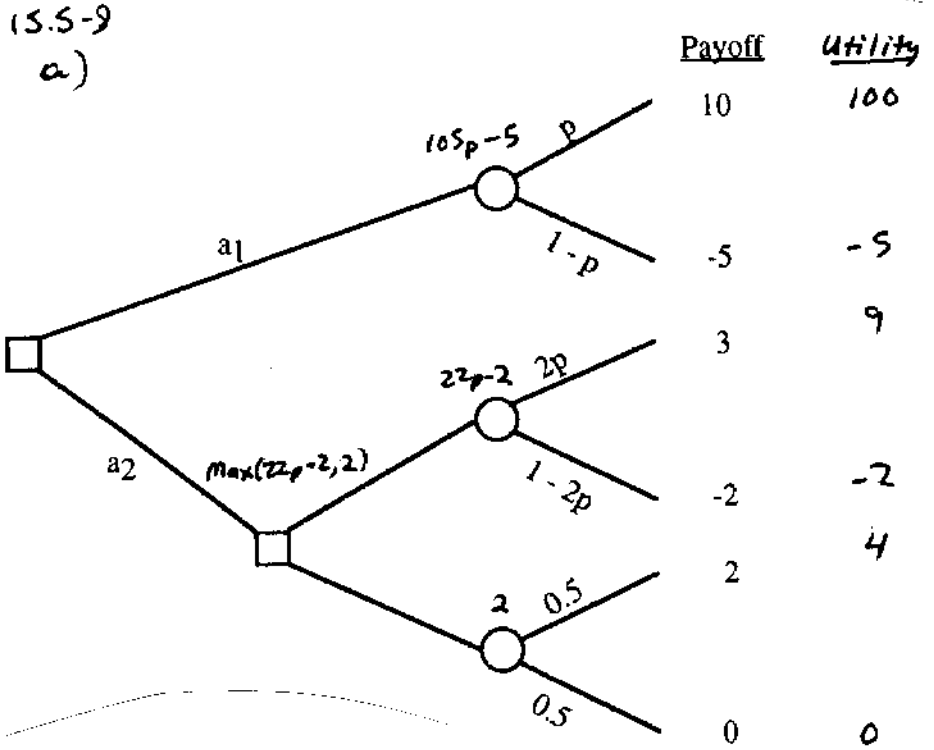
For the decision-maker to be indifferent between A1 and A2, U(x) must equal 3.5.

So, $x^{\frac{1}{3}} = 3.5$ or x = \$42.88.



15.5-9

a)



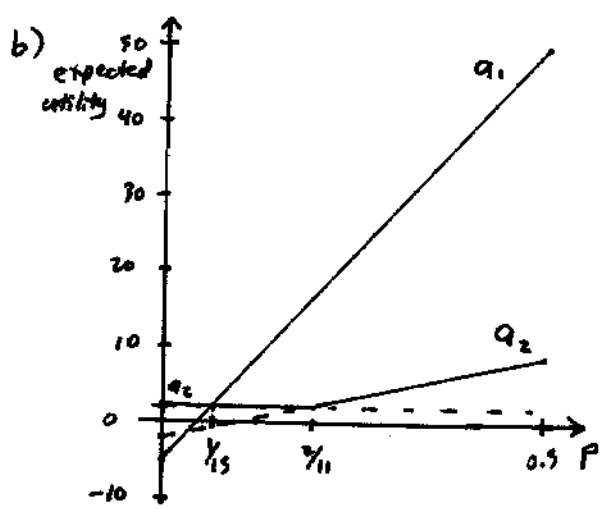
At $p = 0.25$

$105p - 5 = 21.25$

and $\max(22p-2, 2) = \max(3.5, 2) = 3.5$

So a_1 is the optimal action.

b)



As can be seen from the graph,

a_1 stays optimal for $\frac{1}{15} \le p \le 0.5$.

Cases

15.1 a) The relevant data are summarized in the following spreadsheet.

Data:

Price:

High	Medium	Low
50.00	40.00	30.00

Sales:

High	Medium	Low
50,000	30,000	20,000

Probability tables:

Prior probabilities:

Competition:	Severe	Moderate	Weak
p(competition)	0.20	0.70	0.10

Conditional Probabilities:

Price:	50.00		
Competition:	Severe	Moderate	Weak
50k Units	0.20	0.25	0.30
30k Units	0.25	0.30	0.35
20k Units	0.55	0.45	0.35

Price:	40.00		
Competition:	Severe	Moderate	Weak
50k Units	0.25	0.30	0.40
30k Units	0.35	0.40	0.50
20k Units	0.40	0.30	0.10

Price:	30.00		
Competition:	Severe	Moderate	Weak
50k Units	0.35	0.40	0.50
30k Units	0.40	0.50	0.45
20k Units	0.25	0.10	0.05

b) The scenario "moderate competition, sales of 30,000 units at a unit price of $30" has the largest total probability. Therefore, under the Maximum Likelihood Criterion, Charlotte should price the product at $30.

For the maximin criterion we conclude that for a price of:

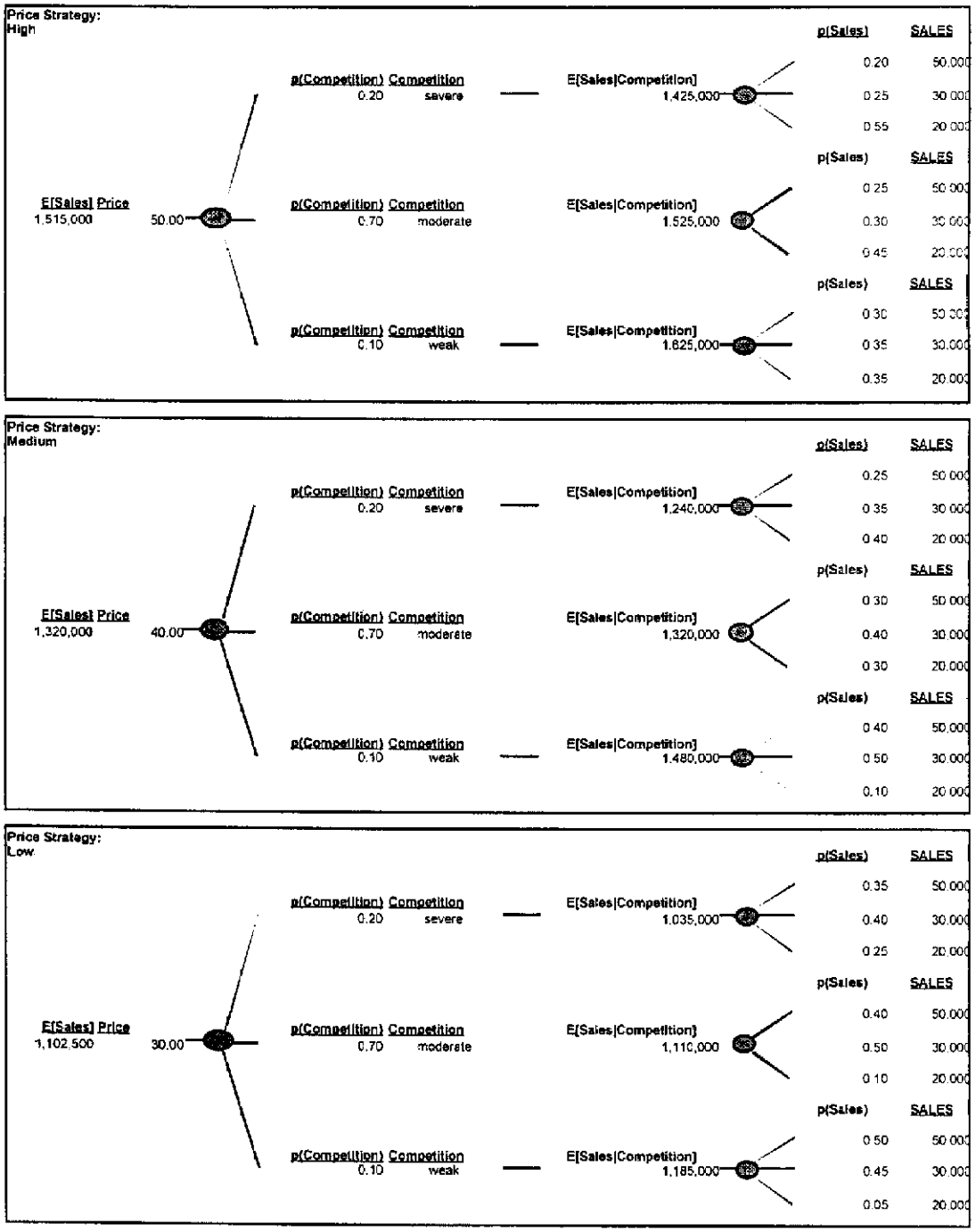
$30: 20k units sales @ $30 is the worst case.

$40: " " " " 40 " " " " .

$50: " " " " 50 " " " " .

The maximum of the three is: price @ $50, and that should be the policy under the maximin criterion.

c) The three branches of the decision tree for the decision problem without additional information follow.



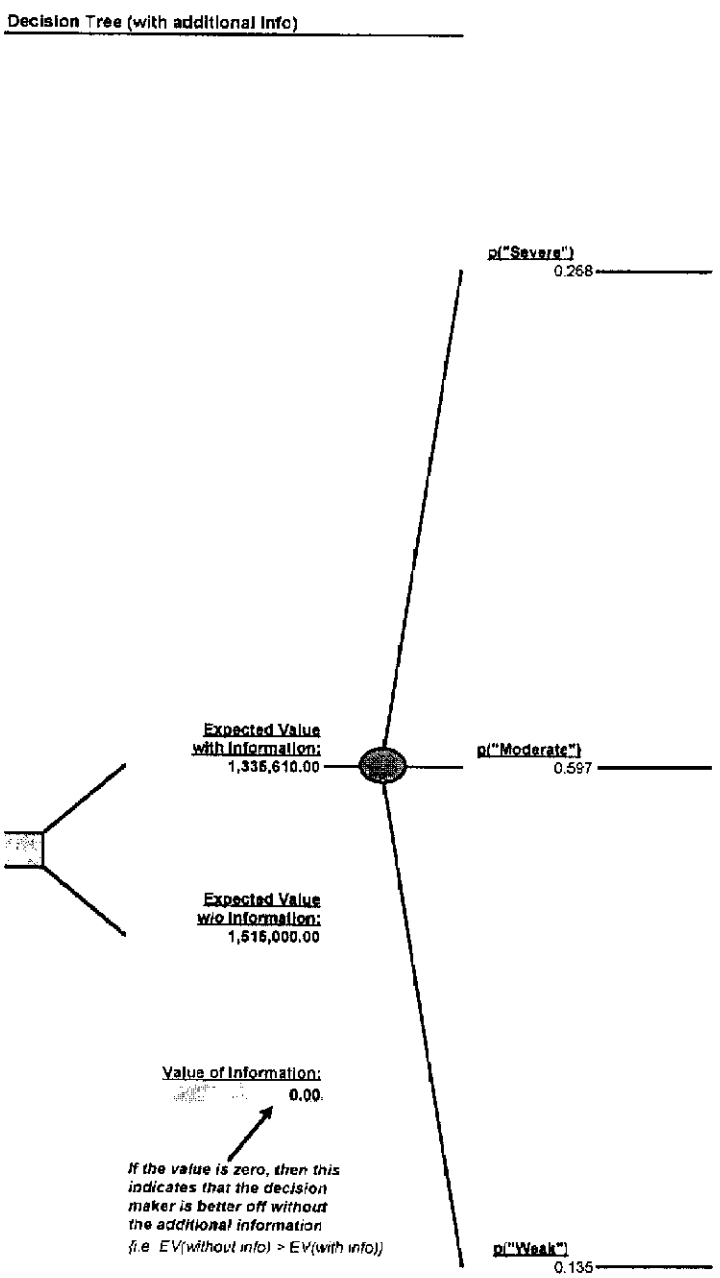
The calculation of the expected revenue for a high price strategy is as follows (please refer to the decision tree). Each final outcome ("Sales") is multiplied by its probability. For example, the expected sales given severe competition (E[Sales|Competition]) equals:

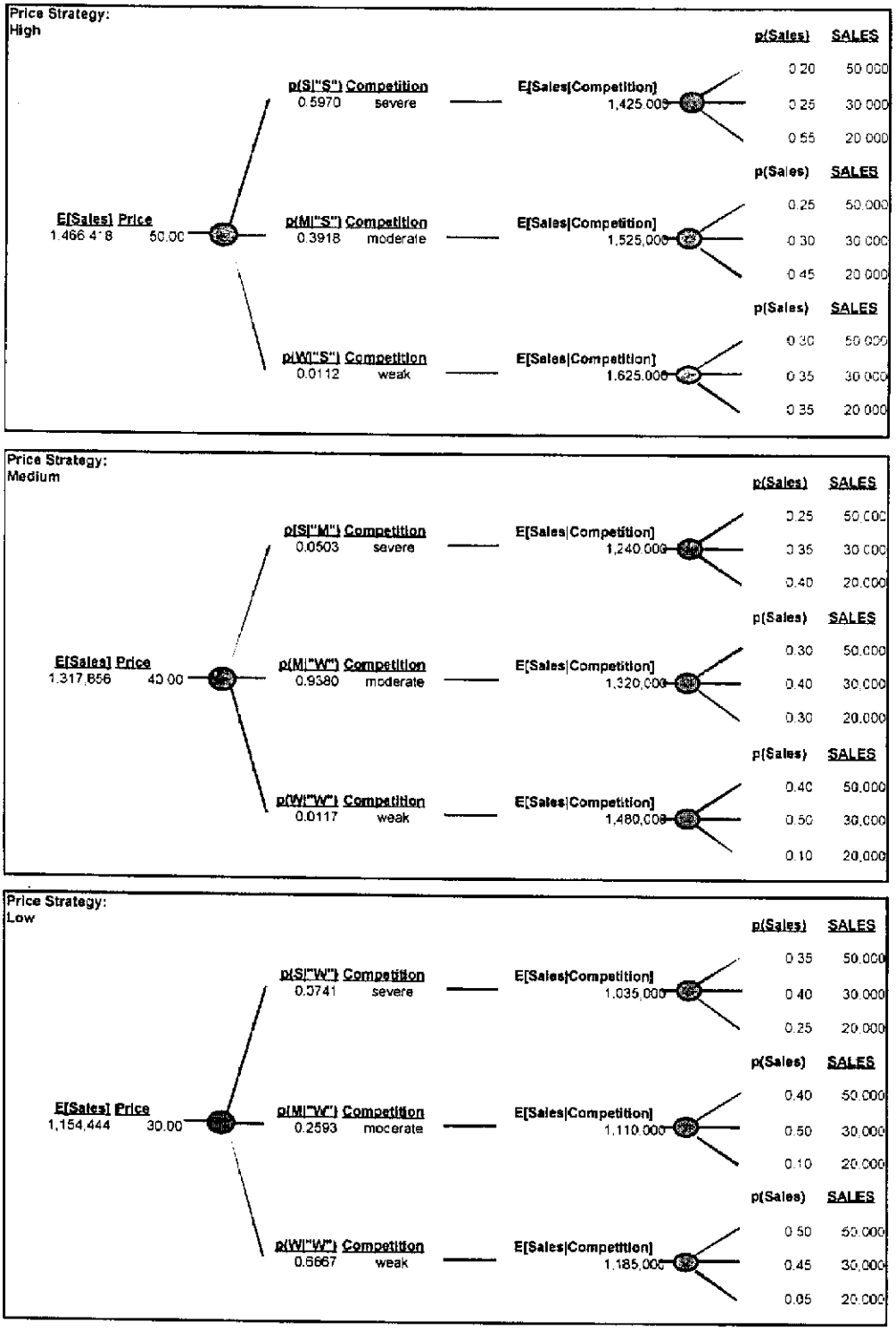
0.2*50,000+0.25*30,000+0.55*20,000 = 1,425,000

The remaining expected values are calculated accordingly.

The decision tree indicates that the alternative with the maximum expected value is the high price strategy. Therefore, Charlotte should price her product at $50.

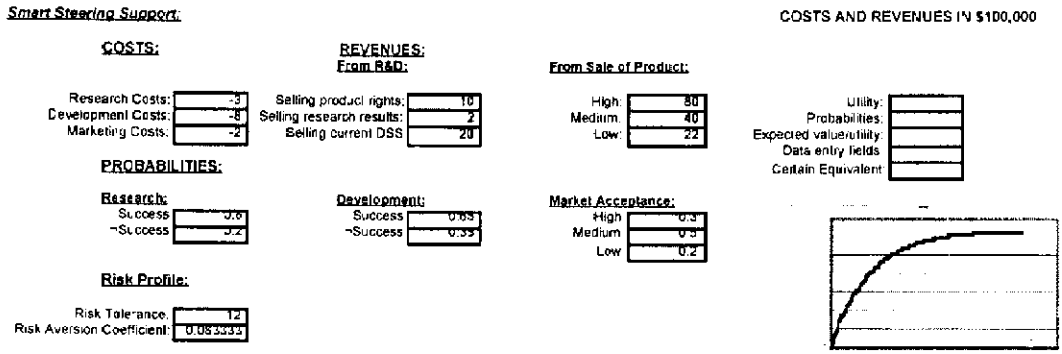
d) The decision tree for the decision problem with additional information follows in two sections. The branches of the decisions are on the next page. Please note that the decision alternative "Expected Value w/o Information" represents the entire decision tree of part (c). The calculations in this decision tree are performed in the same manner as in part (c).



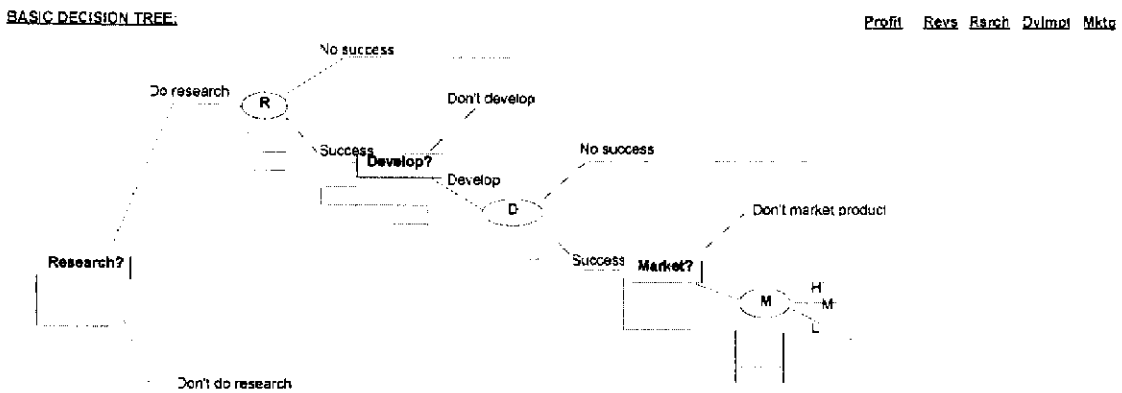


Since the expected value for the decision tree with additional (imperfect) information is less than that without information, Charlotte should not purchase the services of the market research company.

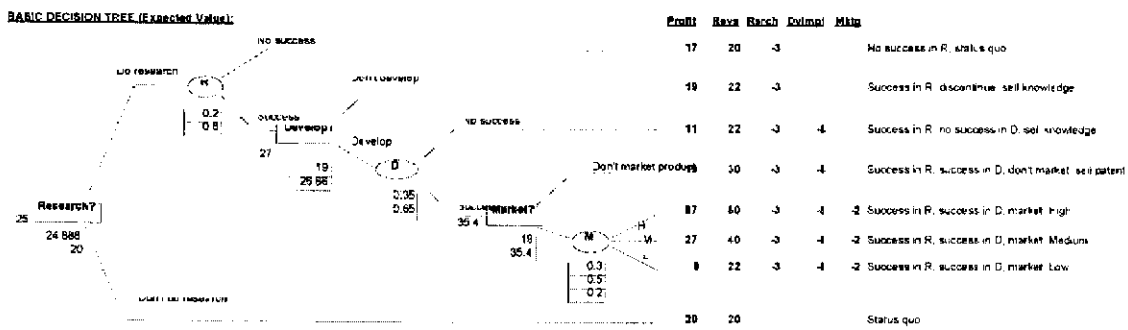
15.2 a) The available data are summarized in the table.



b) The basic decision tree is shown. Rectangular nodes represent decision forks and oval nodes represent chance forks.

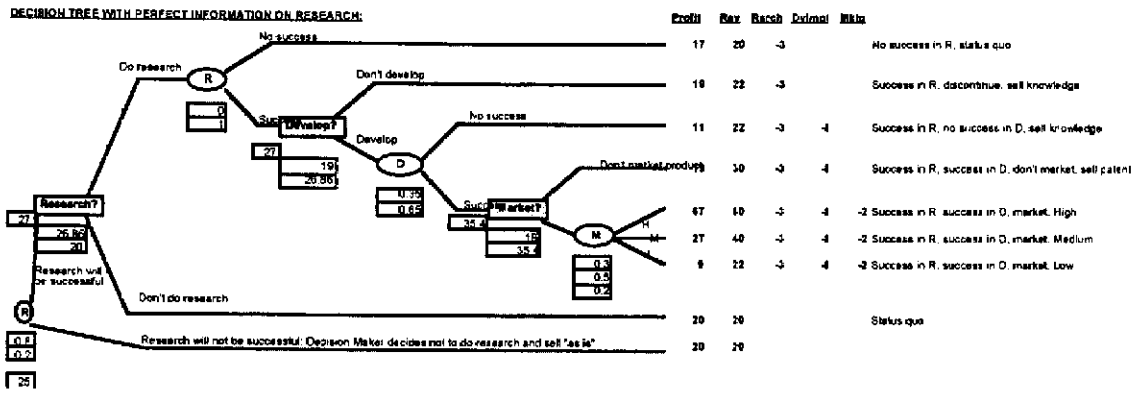


c) The decision tree displays all the expected payoffs.

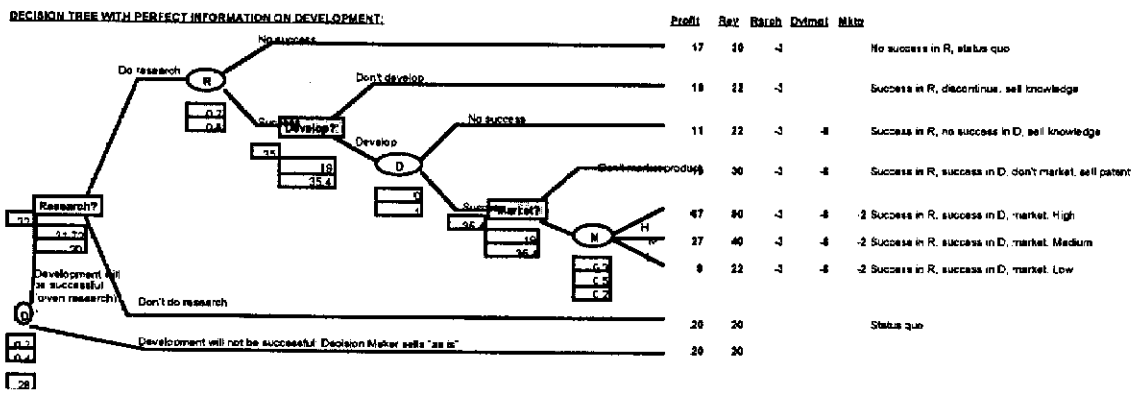


d) The best course of action is to do the research project. The expected payoff equals $2.4888 million.

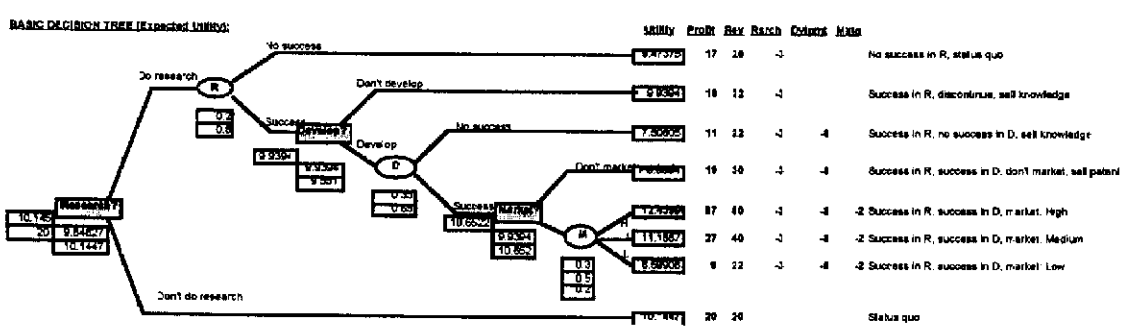
e) The decision tree with perfect information on research is displayed. The expected value in this case equals $2.5488 million. The difference between the expected values with and without information equals $60,000, which is the value of perfect information on research.



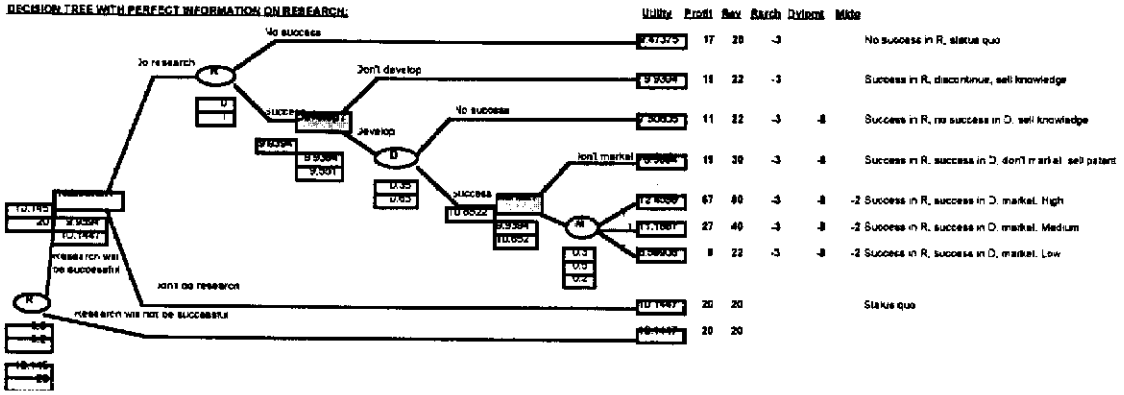
f) The decision tree with perfect information on development is displayed. The expected value in this case equals $2.7618 million. The difference between the expected values with and without information equals $273,000, which is the value of perfect information on development.



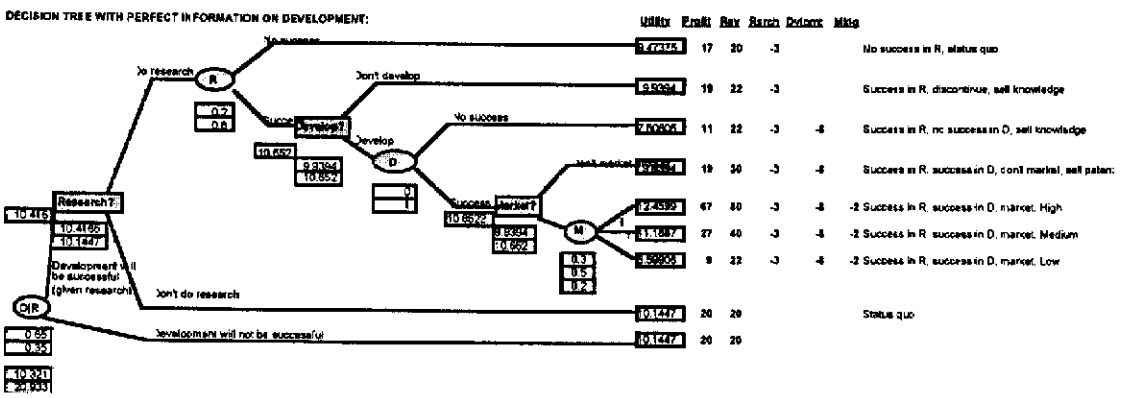
g-i) The decision tree with expected utilities is displayed. The expected utilities are calculated in the following way: for each of the outcome branches of the decision tree (e.g. profit of $6,700,000) the corresponding utility is calculated (e.g. 12.45992). Once this is done, the expected utilities are calculated. The best course of action is not to do the research (expected utility of 10.14469 vs. 9.846267 in the case of "Do research").



j) The expected utility for perfect information on research equals 9.939397 which is still less than the expected utility of the alternative "Don't do research" (10.14469). Therefore, the best course of action is not to do the research implying a value of zero for perfect information on the outcome of the research effort.



k) The expected utility for perfect information on development equals 10.321347 which is more than the expected utility for the case with no information (10.14469). The value of perfect information on development is calculated as the difference between the inverses of these two utility values ($U^{-1}[10.321347] - U^{-1}[10.14469] = 20.93274 - 20 = 0.93274$). The value of perfect information on the outcome of the development effort equals $93,274.



CHAPTER 16

16.2-1 a) Since the probability of weather tomorrow is only dependent on today, Markovian property holds for evolution of the weather.

b) Let 0 = Rain, 1 = Clear,

Then $P = P^{(1)} = \begin{matrix} \\ 0 \\ 1 \end{matrix} \begin{matrix} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{bmatrix} .5 & .5 \\ .1 & .9 \end{bmatrix} \end{matrix}$

16.2-2 a) Let 1 = increased today and yesterday
2 = increased today but decreased yesterday
3 = decreased today but increased yesterday
4 = decreased today and yesterday

Then $P = P^{(1)} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} \alpha_1 & 0 & 1-\alpha_1 & 0 \\ \alpha_2 & 0 & 1-\alpha_2 & 0 \\ 0 & \alpha_3 & 0 & 1-\alpha_3 \\ 0 & \alpha_4 & 0 & 1-\alpha_4 \end{bmatrix} \end{matrix}$

b) Because the state space is properly defined as (change today, change yesterday), and tomorrow's state will have its second component equal to today's first component.

16.2-3

Yes, it can be formulated as a Markov Chain with the following 8 ($= 2^3$) states

State	today	1 day ago	2 days ago
1	inc	inc	inc
2	inc	inc	dec
3	inc	dec	inc
4	inc	dec	dec
5	dec	inc	inc
6	dec	inc	dec
7	dec	dec	inc
8	dec	dec	dec

These states have included all the information needed to predict tomorrow's stockmarket whereas states in Prob. 14.2-2 doesn't consider the day before yesterday, which

16.3-1

a) $P^2 = \begin{bmatrix} 0.3 & 0.7 \\ 0.14 & 0.86 \end{bmatrix}$ $P^5 = \begin{bmatrix} 0.175 & 0.825 \\ 0.165 & 0.835 \end{bmatrix}$

$P^{10} = \begin{bmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{bmatrix}$ $P^{20} = \begin{bmatrix} 0.167 & 0.833 \\ 0.167 & 0.833 \end{bmatrix}$

16.3-1 (b) $P(\text{rain } n \text{ days from now} \mid \text{rain today}) = P_{11}^n$

If the probability it will rain today is 0.5,

$P(\text{rain } n \text{ days from now}) = p_n = 0.5 \cdot P_{11}^n$

Thus, $P_2 = 0.15$, $P_5 = 0.0875$, $P_{10} = 0.0835$, $P_{20} = 0.0835$

(c) We find $\pi_1 = 0.167$

$\pi_2 = 0.833$

Obviously when n grows large, every row of $P^{(n)}$ approaches to the stationary probabilies.

16.3-2 (a) Let states 0 and 1 denote that a 0 and a 1 have been recorded, respectively.

$$P = \begin{bmatrix} 1-q & q \\ q & 1-q \end{bmatrix} \quad \text{where } q = 0.01$$

(b) $$P^{(10)} = \begin{bmatrix} 0.909 & 0.091 \\ 0.091 & 0.909 \end{bmatrix}$$

The probability that a digit would recorded accurately is 0.909

(c) $$P^{(10)} = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}$$

The probability is 0.99.

16.3-3 a) Transition Matrix, $P = \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix}$

(b) $$p^5 = \begin{bmatrix} 0.062 & 0.312 & 0.156 & 0.156 & 0.312 \\ 0.312 & 0.062 & 0.312 & 0.156 & 0.156 \\ 0.156 & 0.312 & 0.062 & 0.312 & 0.156 \\ 0.156 & 0.156 & 0.312 & 0.062 & 0.312 \\ 0.312 & 0.156 & 0.156 & 0.312 & 0.062 \end{bmatrix}$$

$$p^{10} = \begin{bmatrix} 0.248 & 0.161 & 0.215 & 0.215 & 0.161 \\ 0.161 & 0.248 & 0.161 & 0.215 & 0.215 \\ 0.215 & 0.161 & 0.248 & 0.161 & 0.215 \\ 0.215 & 0.215 & 0.161 & 0.248 & 0.161 \\ 0.161 & 0.215 & 0.215 & 0.161 & 0.248 \end{bmatrix}$$

16.3-3 b) (cont'd)

$$P^{20} = \begin{bmatrix} 0.206 & 0.195 & 0.202 & 0.202 & 0.195 \\ 0.195 & 0.206 & 0.195 & 0.202 & 0.202 \\ 0.202 & 0.195 & 0.206 & 0.195 & 0.202 \\ 0.202 & 0.202 & 0.195 & 0.206 & 0.195 \\ 0.195 & 0.202 & 0.202 & 0.195 & 0.206 \end{bmatrix}$$

$$P^{40} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

$$P^{80} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}$$

(c) We can easily find out $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0.2$

16.4-1

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$

(b) P has three communicating classes:
$\{0\}$ is absorbing and, therefore, recurrent
$\{1, 2\}$ is recurrent
$\{3\}$ is transient

16.4-2

(a) P has one recurrent communicating class: $\{0, 1, 2, 3\}$

(b) P has one recurrent communicating class: $\{0, 1, 2\}$

16.4-3

This matrix has three communicating classes:
$\{0, 1\}$ is recurrent
$\{2\}$ is transient
$\{3, 4\}$ is recurrent

16.4-4

There is one class, so every state has the same period. From the definition, by inspection, the period is 4.

16.4-5 a) & b)

There are two classes:
$\{0, 1, 2, 4\}$ and $\{3\}$

From the definition, by inspection, the period for the states in $\{0, 1, 2, 4\}$ is 2 and the period for $\{3\}$ is 1. Class $\{0, 1, 2, 4\}$ is transient and $\{3\}$ is recurrent.

16.5-1 $P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$

From $\pi P = P \Rightarrow \alpha\pi_1 + (1-\beta)\pi_2 = \pi_1$

$\pi_1 + \pi_2 = 1$

$\Rightarrow \pi = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta}\right)$

16.5-2

Must show that $\pi_j = \frac{1}{M+1}$ for $j = 0, 1, \ldots, M$ satisfies the steady state equations: $\pi_j = \sum_{i=0}^{M} p_{ij}\pi_j$, for $j = 0, \ldots, M$ and $\sum_{j=0}^{M} \pi_j = 1$.

Since $\sum_{i=0}^{M} p_{ij} = 1$ for all j, this is easily verified.

Since the chain is irreducible, aperiodic and positive recurrent, this is the unique solution.

16.5-3 Since $M=5$, $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = \frac{1}{5} = 0.2$

The steady-state probabilities don't change if the probability for moving steps changes.

16.5-4 Run the OR courseware and we can find

$\pi_1 = 0.346$

$\pi_2 = 0.385$

$\pi_3 = 0.269$

which means the steady-state market share for A and B are 0.346 and 0.385 respectively.

16.5-5

(a) Assume demand occurs after the delivery:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} .6 & .4 & 0 & 0 & 0 & 0 & 0 \\ .3 & .3 & .4 & 0 & 0 & 0 & 0 \\ .1 & .2 & .3 & .4 & 0 & 0 & 0 \\ 0 & .1 & .2 & .3 & .4 & 0 & 0 \\ 0 & 0 & .1 & .2 & .3 & .4 & 0 \\ 0 & 0 & 0 & .1 & .2 & .3 & .4 \\ 0 & 0 & 0 & 0 & .1 & .2 & .7 \end{bmatrix} \end{array}$$

16.5-5 (CONT'D)

(b) $\pi P = \pi \Rightarrow (.139, .139, .139, .138, .141, .130, .174)$

(c) The steady probability that a pint of blood is to be discarded

$= P\{D=0\} \cdot P\{\text{state is } 7\} = 0.4 \times 0.174 = 0.0696$

(d) $P\{\text{emergency deliveries}\}$

$= \sum_{i=1}^{2} P\{\text{state } i\} \cdot P\{D > i\}$

$= 0.139 \times (0.2+0.1) + 0.139 \times 0.1 = 0.0556$

16.5-6. (a) Let 0 = Low
1 = High

Never advertise $\Rightarrow P = \begin{matrix} & 0 & 1 \\ 0 & 3/4 & 1/4 \\ 1 & 1/2 & 1/2 \end{matrix}$

Always advertise $\Rightarrow P = \begin{matrix} & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1 & 1/4 & 3/4 \end{matrix}$

Marketing Manager's Proposal $\Rightarrow P = \begin{matrix} & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \end{matrix}$

(b) Never advertise $\Rightarrow \pi = (2/3, 1/3)$
Always advertise $\Rightarrow \pi = (1/3, 2/3)$
Marketing Manager's Proposal $\Rightarrow \pi = (1/2, 1/2)$

(c) Never advertise $\Rightarrow$ profit $= \frac{2}{3} \times 2 + \frac{1}{3} \times 4 = 2\frac{2}{3}$ million

Always advertise $\Rightarrow$ profit $= \frac{1}{3} \times 2 + \frac{2}{3} \times 4 - 1 = 2\frac{1}{3}$ million

Proposal $\Rightarrow$ profit $= \frac{1}{2} \times (2-1) + \frac{1}{2} \times 4 = 2.5$ million

So the best is to never advertise

16.5-7

For a (s,S) policy with s=2 and S=3,

$$C(X_{t-1}, D_t) = \begin{cases} 10 + (25)(3 - X_{t-1}) + 50 \max\{D_t - 3, 0\} & \text{for } X_{t-1} < 2 \\ 50 \max\{D_t - X_{t-1}, 0\} & \text{for } X_{t-1} \geq 2 \end{cases}$$

Thus
$$K(0) = E[C(0, D_t)] = 85 + 50\left[\sum_{j=4}^{\infty} (j-3) P\{D_t = j\}\right] \approx 86.2$$
$$K(1) = E[C(1, D_t)] = 60 + 50\left[\sum_{j=3}^{\infty} (j-3) P\{D_t = j\}\right] \approx 61.2$$
$$K(2) = E[C(2, D_t)] = 0 + 50\left[\sum_{j=2}^{\infty} (j-2) P\{D_t = j\}\right] \approx 5.2$$
$$K(3) = E[C(3, D_t)] = 0 + 50\left[\sum_{j=1}^{\infty} (j-3) P\{D_t = j\}\right] \approx 1.2$$

Again for a (2,3) policy,

$$X_{t+1} = \begin{cases} \max\{3 - D_{t+1}, 0\} & \text{for } X_t < 2 \\ \max\{X_t - D_{t+1}, 0\} & \text{for } X_t \geq 2 \end{cases}$$

So

$$P = \begin{bmatrix} .080 & .184 & .368 & .368 \\ .080 & .184 & .368 & .368 \\ .264 & .368 & .368 & 0 \\ .080 & .184 & .368 & .368 \end{bmatrix}$$

Solving the steady state equations gives $\pi_0 = .148, \pi_1 = .252, \pi_2 = .368, \pi_3 = .232$

Hence, the long-run average cost per week is

$$\sum_{j=0}^{3} K(j)\pi_j = 30.37$$

16.5-8

$$X_{t+1} = \begin{cases} \max\{X_t + 2 - D_{t+1}, 0\} & \text{for } X_t \leq 1 \\ \max\{X_t - D_{t+1}, 0\} & \text{for } X_t \geq 2 \end{cases}$$

$$\text{So } P = \begin{bmatrix} .264 & .368 & .368 & 0 \\ .080 & .184 & .368 & .368 \\ .264 & .368 & .368 & 0 \\ .080 & .184 & .368 & .368 \end{bmatrix}$$

Solving the steady state equations gives $\pi_0 = .182, \pi_1 = .285, \pi_2 = .368, \pi_3 = .165$

b) $$\lim_{n \to \infty} E\left(\frac{1}{n} \sum_{t=1}^{n} C(X_t)\right) = 0\pi_0 + 2\pi_1 + 8\pi_2 + 18\pi_3 = 6.48$$

16.5-9

a) $P_{11} = P\{D_{n+1}=0\} + P\{D_{n+1}=2\} + P\{D_{n+1}=4\} = 3/5$

$P_{12} = P\{D_{n+1}=1\} + P\{D_{n+1}=3\} = 2/5$

$P_{21} = P\{D_{n+1}=1\} + P\{D_{n+1}=3\} = 2/5$

$P_{22} = P\{D_{n+1}=0\} + P\{D_{n+1}=2\} + P\{D_{n+1}=4\} = 3/5$

So $P = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}$

(e)

(b) Since P is doubly stochastic, $\pi_0 = \pi_1 = 1/2$.

(d) $K(1) = E[c(1, D_n)] = (2/5)[3+2(1)] + (2/5)[3+2(2)] + (1/5)(1) +$
$+ (4/5)[1+2+3] = 9.8$

$K(2) = E[c(2, D_n)] = (2/5)[3+2(1)] + (1/5)[3+2(2)] + 1/5[2+1]$
$+ (4/5)[1+2] = 6.4$

So long-run average cost per unit time is
$9.8(1/2) + 6.4(1/2) = 8.10$.

16.5-10

(a) $P\{\text{waiting on line in Period } n\} = P_{02}^{(n)}$

and

$$P^2 = \begin{bmatrix} 0.64 & 0.16 & 0.04 & 0.16 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0.64 & 0.16 & 0.04 & 0.16 \end{bmatrix} \qquad P^5 = \begin{bmatrix} 0.62 & 0.195 & 0.037 & 0.148 \\ 0.594 & 0.174 & 0.046 & 0.186 \\ 0.64 & 0.232 & 0.026 & 0.102 \\ 0.62 & 0.195 & 0.037 & 0.148 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.615 & 0.192 & 0.039 & 0.154 \\ 0.616 & 0.193 & 0.038 & 0.153 \\ 0.614 & 0.191 & 0.039 & 0.156 \\ 0.615 & 0.192 & 0.039 & 0.154 \end{bmatrix} \qquad P^{20} = \begin{bmatrix} 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \\ 0.615 & 0.192 & 0.038 & 0.154 \end{bmatrix}$$

So, $n=2$, $p=0.04$; $n=5$, $p=0.037$

$n=10$, $p=0.039$; $n=20$, $p=0.038$

(b) $\pi_1 = 0.615$, $\pi_2 = 0.192$, $\pi_3 = 0.038$, $\pi_4 = 0.154$

(c) Long-run average cost per period = $\$30,000\,\pi_3 = \$1140$

16.6-1

(a) The process satisfies the four points listed in Sec. 15.3, and is thus Markovian.

Let 0 = Working
1 = Failed

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} .9 & .1 \\ .65 & .35 \end{bmatrix} \end{array}$$

b) $\mu_{00} = 1 + .1\mu_{10} \rightarrow \mu_{00} = 1.15$
$\mu_{01} = 1 + .9\mu_{01} \rightarrow \mu_{01} = 10$
$\mu_{10} = 1 + .35\mu_{10} \rightarrow \mu_{10} = 1.54$
$\mu_{11} = 1 + .65\mu_{01} \rightarrow \mu_{11} = 7.5$

16.6-2

(a) Let 0 = operational, 1 = down, 2 = repaired

Then
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0 & 1 \\ 0.9 & 0.1 & 0 \end{bmatrix} \end{array}$$

(b) From sec 14.5, we have $\mu_{ij} = 1 + \sum_{k \neq j} P_{ik}\mu_{kj}$

So, $\begin{cases} \mu_{00} = 1 + 0.1\,\mu_{10} \\ \mu_{10} = 1 + \mu_{20} \\ \mu_{20} = 1 + 0.1\,\mu_{10} \end{cases}$ such that $\begin{cases} \mu_{00} = \frac{11}{9} \\ \mu_{10} = \frac{20}{9} \\ \mu_{20} = \frac{11}{9} \end{cases}$

$\begin{cases} \mu_{01} = 1 + 0.9\,\mu_{01} \\ \mu_{11} = 1 + \mu_{21} \\ \mu_{21} = 1 + 0.9\,\mu_{01} \end{cases}$ such that $\begin{cases} \mu_{01} = 10 \\ \mu_{11} = 11 \\ \mu_{21} = 10 \end{cases}$

$\begin{cases} \mu_{02} = 1 + 0.9\,\mu_{02} + 0.1\,\mu_{12} \\ \mu_{12} = 1 + 0 \\ \mu_{22} = 1 + 0.9\,\mu_{02} + 0.1\mu_{12} \end{cases}$ such that $\begin{cases} \mu_{02} = 11 \\ \mu_{12} = 1 \\ \mu_{22} = 2.09 \end{cases}$

The expected number of days that machine will remain operational is 10.

(C) It would be the same due to the Markovian Property.

16.6-3

(a)
$$P = \begin{array}{c} \\ (1,1) \\ (0,1) \\ (1,0) \end{array} \begin{array}{c} \begin{array}{ccc} (1,1) & (0,1) & (1,0) \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix} \end{array}$$

b) $\mu_{33} = \frac{1}{\pi_3}$. So solving $\pi P = \pi$ and $\pi \cdot 1 = 1$ we get $\pi_3 = \frac{.009}{.99}$

Thus $\mu_{33} = \frac{.99}{.009} = 110$, which is the expected recurrence time of state (1,0).

16.6-4

a) Transition Matrix, $P = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$

b) $P^2 = \begin{bmatrix} 0.5 & 0.375 & 0.125 \\ 0.375 & 0.438 & 0.188 \\ 0.5 & 0.375 & 0.125 \end{bmatrix}$

$P^5 = \begin{bmatrix} 0.449 & 0.4 & 0.15 \\ 0.451 & 0.399 & 0.149 \\ 0.449 & 0.4 & 0.15 \end{bmatrix}$ $P^{10} = \begin{bmatrix} 0.45 & 0.4 & 0.15 \\ 0.45 & 0.4 & 0.15 \\ 0.45 & 0.4 & 0.15 \end{bmatrix}$

(c) $\begin{cases} \mu_{00} = 1 + 0.5\mu_{10} + 0.25\mu_{20} \\ \mu_{10} = 1 + 0.25\mu_{10} \\ \mu_{20} = 1 + 0.5\mu_{10} + 0.25\mu_{20} \end{cases} \Rightarrow \begin{cases} \mu_{00} = 20/9 \\ \mu_{10} = 4/3 \\ \mu_{20} = 20/9 \end{cases}$

$\begin{cases} \mu_{01} = 1 + 0.25\mu_{01} + 0.25\mu_{21} \\ \mu_{11} = 1 + 0.75\mu_{01} \\ \mu_{21} = 1 + 0.25\mu_{01} + 0.25\mu_{21} \end{cases} \Rightarrow \begin{cases} \mu_{01} = 2 \\ \mu_{11} = 2\frac{1}{2} \\ \mu_{21} = 2 \end{cases}$

$\begin{cases} \mu_{02} = 1 + 0.25\mu_{02} + 0.5\mu_{12} \\ \mu_{12} = 1 + 0.75\mu_{02} + 0.25\mu_{12} \\ \mu_{22} = 1 + 0.25\mu_{02} + 0.5\mu_{12} \end{cases} \Rightarrow \begin{cases} \mu_{02} = 20/3 \\ \mu_{12} = 8 \\ \mu_{22} = 20/3 \end{cases}$

(d) The steady-state probabilities is (0.45, 0.4, 0.15)

(e) $\pi \cdot C = 0(0.45) + 2(0.4) + 8(0.15) = \$2/\text{week}$.

16.6-5

(a) Transition Matrix, $P = \begin{bmatrix} 0 & 0.875 & 0.062 & 0.062 \\ 0 & 0.75 & 0.125 & 0.125 \\ 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Solution:

$\pi_1 = 0.154$
$\pi_2 = 0.538$
$\pi_3 = 0.154$
$\pi_4 = 0.154$

(b) $\pi \cdot C = 1(0.538) + 3(0.154) + 6(0.154) = \$1923.08$

(c) $\begin{cases} \mu_{00} = 1 + 0.875\mu_{10} + 0.0625\mu_{20} + 0.0625\mu_{30} \\ \mu_{10} = 1 + 0.75\mu_{10} + 0.125\mu_{20} + 0.125\mu_{30} \\ \mu_{20} = 1 + 0.5\mu_{20} + 0.5\mu_{30} \\ \mu_{30} = 0 + 1 \end{cases}$

$\Rightarrow \mu_{00} = 6.5$, which is expected recurren time for state 0.

16.7-1 a)

For the transition matrix, $P_{00} = P_{TT} = 1$; $P_{i,i-1} = q$, $P_{i,i+1} = p$, $P_{ik} = 0$ else. And,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & & \\ q & 0 & p & 0 & \cdots & \\ & \ddots & & \ddots & & \\ & & q & 0 & p & 0 \\ & & 0 & q & 0 & p \\ & & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Class 1: $\{0\}$ which is absorbing
Class 2: $\{T\}$ which is absorbing
Class 3: $\{1, 2, \ldots, T-1\}$ which is transient

(c) Let $f_{ik} = P\{\text{absorbtion at } k \text{ given start at } i\}$

$f_{00}=1$ $f_{03}=0$ $f_{30}=0$ $f_{33}=1$

since $P_{ij}=0$ for $|i-j|\neq 1$, and $P_{i,i+1}=p$; $P_{i,i-1}=q$

we get:

$f_{10} = q + p f_{20} \rightarrow f_{10} = q + pq f_{10} \rightarrow f_{10} = \frac{q}{1-pq} = .886$

$f_{13} = 1 - f_{10} \rightarrow f_{13} = .114$

$f_{20} = q f_{10} \rightarrow f_{20} = .62$

$f_{23} = 1 - f_{20} \rightarrow f_{23} = .38$

d) From the same formulas as in part c) we obtain:

$f_{10}=.38$ $f_{13}=.62$ $f_{20}=.114$ $f_{23}=.886$

It appears that $p > 1/2 \Rightarrow$ drift to T
$p < 1/2 \Rightarrow$ drift to 0

16.7-2. (a) 0 = Have to honor warranty
1 = Reorder in 1st year
2 = Reorder in 2nd year
3 = Reorder 3 years or older

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .01 & 0 & .99 & 0 \\ .05 & 0 & 0 & .95 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b) Probability manufacturer has to honor warranty $= f_{10}$

$f_{10} = .01 f_{00} + 0 f_{10} + .99 f_{20} + 0 f_{30}$

$f_{20} = .05 f_{00} + 0 f_{10} + 0 f_{20} + .95 f_{30}$

but $f_{00}=1$ and $f_{30}=0$

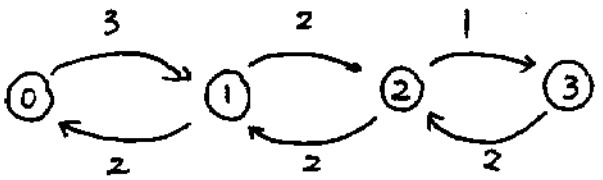
$\Rightarrow f_{10} = .01 + .99 f_{20}$

$f_{20} = .05$

$\Rightarrow f_{10} = .0595 = 5.95\%$

16.8-1. (a)

state:

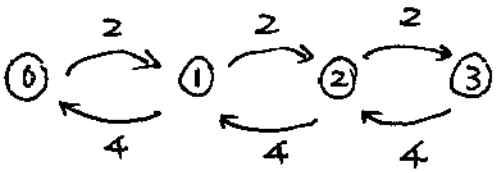


(b) Balance equations:

$$3\pi_0 = 2\pi_1$$
$$4\pi_1 = 3\pi_0 + 2\pi_2$$
$$3\pi_2 = 2\pi_1 + 2\pi_3$$
$$2\pi_3 = \pi_2$$
$$\pi_1 + \pi_2 + \pi_3 + \pi_0 = 1$$

(c) Solve them, we get $\pi_0 = \frac{4}{19}$, $\pi_1 = \frac{6}{19}$, $\pi_2 = \frac{6}{19}$, $\pi_3 = \frac{3}{19}$

16.8-2 (a) State: number of jobs at the work center



(b) Steady-state equations:

$$2\pi_0 = 4\pi_1$$
$$6\pi_1 = 2\pi_0 + 4\pi_2$$
$$6\pi_2 = 2\pi_1 + 4\pi_3$$
$$2\pi_2 = 4\pi_3 \quad , \quad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

(c) We can easily get

$$\pi = \left(\frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15}\right)$$

16.3

CHAPTER 17

17.2-1 input source: population having hair
calling units: customers wanting haircuts
queue: customers waiting for a barber
service discipline: usually first in, first out.
service mechanism: barbers and equipment.

17.2-2

a) $L = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = 2$ which represents the average number of customers in the shop, including those getting their hair cut.

b)

n	# in queue	probability	product
0	0		
1	0		
2	0		
3	1	0.25	0.25
4	2	0.0625	0.125

Lq = 0.375 which represents the average number of customers in the shop waiting to get a haircut.

c) $E(\#\text{ customers being served}) = 1 \cdot P_1 + 2(P_2 + P_3 + P_4) =$
$= \frac{4}{16} + 2\left(\frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right) = \frac{13}{8}$

d)

$W = \frac{L}{\lambda} = \frac{2}{4} = 0.5 = 30$ minutes

$W_q = \frac{L_q}{\lambda} = \frac{0.375}{4} = 0.094 = 5.625$ minutes

These quantities mean that customers will be in the shop an average of half an hour, including the time to get a haircut, and will have to wait an average of 5.625 minutes before their haircut will begin.

e) W - Wq = 0.5 – 0.094 = 0.406 hours = 24.36 minutes

17.2-3

a) A parking lot is a queueing system for providing parking with cars as the customers, and parking spaces as the servers. The service time is the amount of time a car spends in a space. The queue capacity is 0.

b)

$$L = 0(P_0) + 1(P_1) + 2(P_2) + 3(P_3) = 0(0.2) + 1(0.3) + 2(0.3) + 3(0.2) = 1.5 \text{ cars}$$

$$L_q = 0 \text{ cars}$$

$$W = \left(\frac{L}{\lambda}\right) = \left(\frac{1.5}{2}\right) = 0.75 \text{ hours}$$

$$W_q = \left(\frac{L_q}{\lambda}\right) = \left(\frac{0}{2}\right) = 0 \text{ hours}$$

c) A car spends an average of 45 minutes in a parking space.

17.2-4

a) False. The queue is where customers wait before being served.

b) False. Queueing models conventionally assume that the queue is an infinite queue.

c) True. The most common is first come first served.

17.2-5

a) A bank is a queueing system with people as the customers, and tellers as the servers.

b)

$$W_q = 1 \text{ minute}$$

$$W = W_q + \frac{1}{\mu} = 1 + 2 = 3 \text{ minutes}$$

$$L_q = \lambda W_q = \frac{40}{60}(1) = 0.667 \text{ customers}$$

$$L = \lambda W = \frac{40}{60}(3) = 2 \text{ customers}$$

17.2-6

The utilization factor ρ represents the fraction of time that the server is busy. The server is busy except when there are zero people in the system. P_0 is the probability of having 0 customers in the system. Hence, $\rho = 1 - P_0$.

17.2-7

$$\lambda_2 = 2\lambda_1, \quad \mu_2 = 2\mu_1, \quad L_2 = 2L_1$$

$$\frac{W_2}{W_1} = \frac{L_2/\lambda_2}{L_1/\lambda_1} = 1$$

17.2-8

(a) $L = \begin{cases} L_q & \text{when no one is in the system} \\ L_q + 1 & \text{otherwise} \end{cases}$

So $L = P_0 L_q + (1 - P_0)(L_q + 1) = L_q + (1 - P_0)$

(b) $L = \lambda W = \lambda (W_q + 1/\mu) = \lambda W_q + \lambda/\mu = L_q + \rho$

(c) $L = L_q + \rho = L_q + (1 - P_0)$ from (a) and (b). So $\rho = 1 - P_0$.

17.2-9

$$L = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{s-1} n P_n + \sum_{n=s}^{\infty} n P_n = \sum_{n=0}^{s-1} n P_n + \sum_{n=s}^{\infty} (n-s) P_n + \sum_{n=s}^{\infty} s P_n =$$

$$= \sum_{n=0}^{s-1} n P_n + L_q + s \sum_{n=s}^{\infty} P_n =$$

$$= \sum_{n=0}^{s-1} n P_n + L_q + s \left(1 - \sum_{n=0}^{s-1} P_n\right)$$

17.3-1

Part	Customers	Servers
(a)	customers waiting checkout	checkers
(b)	fires	fire fighting units
(c)	cars	toll collectors
(d)	broken bicycles	bicycle repairpersons
(e)	ships to be loaded or unloaded	Longshoremen + equipment
(f)	machines needing operator	operator
(g)	materials to be handled	handling equipment
(h)	calls for plumbers	plumbers
(i)	custom orders	customized process
(j)	typing requests	typists

17.4-1

$\lambda_n = 1/2$ for $n > 0$ and $\mu_n = \begin{cases} 1/2 & \text{for } n = 1 \\ 1 & \text{for } n \geq 2 \end{cases}$

(a) $P\{\text{next arrival before 1:00}\} = 1 - e^{-1/2} = .393$

$P\{\text{next arrival between 1:00 and 2:00}\} = (1 - e^{-1/2 \cdot 2}) - (1 - e^{-1/2}) = .239$

$P\{\text{next arrival after 2:00}\} = e^{-2 \cdot 1/2} = .368$

(b) $P\{\text{next arrival between 1:00 and 2:00} \mid \text{no arrivals between 12:00 and 1:00}\} = 1 - e^{-1/2} = .393$

(c) $P\{\text{no arrivals between 1:00 and 2:00}\} = \dfrac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-1/2} = .607$

$P\{\text{one arrival between 1:00 and 2:00}\} = \dfrac{(\lambda t)^1 e^{-\lambda t}}{1!} = 1/2 \cdot e^{-1/2} = .303$

$P\{\text{two or more arrivals between 1:00 and 2:00}\} = 1 - e^{-1/2} - \frac{1}{2} e^{-1/2}$

17.4-1

(d) $P\{\text{none served by 2:00}\} = e^{-1} = .368$

$P\{\text{none served by 1:10}\} = e^{-1(1/6)} = .846$

$P\{\text{none served by 1:01}\} = e^{-1(1/60)} = .983$

17.4-2

$\lambda_n = 2$ for $n \geq 0 \Rightarrow P\{n \text{ arrivals in an hour}\} = \frac{2^n e^{-2}}{n!}$

(a) $P\{0 \text{ arrivals in an hour}\} = e^{-2} = .135$

b) $P\{2 \text{ arrivals in an hour}\} = \frac{2^2 e^{-2}}{2!} = 2e^{-2} = .270$

c) $P\{5 \text{ or more arrivals in an hour}\} = 1 - \sum_{n=0}^{4} P\{n \text{ arrivals in an hour}\}$

$= 1 - e^{-2} - 2e^{-2} - 2e^{-2} - (4/3)\cdot e^{-2} - (2/3)e^{-2}$

$= 1 - 7e^{-2} = .0527$

17.4-3

$Pay = 100 \cdot P\{T<2\} + 80 \cdot P\{T>2\} = 100 - 20 \cdot P\{T>2\}$

$P\{T_{old} > 2\} = e^{-\frac{1}{4}\cdot 2} = e^{-\frac{1}{2}} = 0.607$

$P\{T_{special} > 2\} = e^{-\frac{1}{2}\cdot 2} = e^{-1}$

$\text{Increase} = Pay_{special} - Pay_{old} = 20(P\{T_{old} > 2\} - P\{T_{special} > 2\})$

$= 20(e^{-\frac{1}{2}} - e^{-1})$

17.4-4

Given the memoryless property, the system turns into a two-server queue after first completion occurs.

T = amount of time after 1 and before next service completion

$P\{T < t\} = P\{\min(T_2, T_3) < t\}$

So, T satisfies exponential distribution with mean $0.5/2 = 0.25$ (property 3)

17.4-5 By memoryless property, $U = \min(T_1, T_2, T_3)$

$T_1 \sim \exp(1/20)$, $T_2 \sim \exp(1/15)$, $T_3 \sim \exp(1/10)$

$U \sim \exp(\frac{1}{20} + \frac{1}{15} + \frac{1}{10}) = \exp(\frac{13}{60})$

So, expected waiting time $= \frac{60}{13} = 4\frac{8}{13}$ minutes

17.4-6 a) From aggregation property of poisson process, the arrival process is still poisson with mean rate 10 per hour. So, distribution of time between consecutive arrivals is exponential with mean of 6 minutes.

17.4-6

b) The probability distribution is the minimum of two exponential random variables. By property 3, it is exponential with mean of 5 minutes.

17.4-7

a) Exponential with mean of 5 minutes.

b) $W = W_q + T_s$, W_q and T_s are independent

$EW = EW_q + ET_s = 5 + 10 = 15$ minutes $= \frac{1}{4}$ hour

$\text{Var}W = \text{var}W_q + \text{var}T_s = (\frac{1}{12})^2 + (\frac{1}{6})^2 = \frac{5}{144} = 0.0347$

d) $\tilde{W} = 5 + W_s$

$\tilde{W}$ has mean of 20 minutes, but holds the same variance as W_s

17.4-8

a) $P\{T_X < T_Y\} = \frac{\lambda_X}{\lambda_X + \lambda_Y} = \frac{1}{2} = P\{X \text{ completes before } Y\}$

b)

$P\{Z \text{ completes before } X\} = P\{Y \text{ completes before } X,\ Z \text{ completes before } X\}$

$= P\{T_Y < T_X\} \cdot P\{T_Z < T_X'\} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

T_X starts from the end of 15-minute interval

T_X' starts from the moment Y leaves the system

c) Same as b), $\frac{1}{4}$

d) $T_X = \frac{1}{4} + \exp(4)$

So, $F_X(t) = \begin{cases} 0 & t < \frac{1}{4} \\ 1 - e^{-4t} & t \geq \frac{1}{4} \end{cases}$

$ET_X = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ hour. $\text{var}T_X = (\frac{1}{4})^2 = \frac{1}{16} = 0.0625$

e) $T_Y = \frac{1}{6} + \exp(4)$

So, $F_Y(t) = \begin{cases} 0 & t < \frac{1}{6} \\ 1 - e^{-4t} & t \geq \frac{1}{6} \end{cases}$

$ET_Y = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$, $\text{var}T_Y = (\frac{1}{4})^2 = \frac{1}{16} = 0.0625$

17.4-8

f) $W_2 = W_q + T_2$, $W_q \sim \exp(8)$, $T_2 \sim \exp(4)$

So, $EW_2 = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$, $\text{var}\, W_2 = \frac{1}{16} + \frac{1}{64} = \frac{5}{64} = 0.0781$

g) $P\{2 \text{ arrivals}\} = \frac{(\lambda t)^2 e^{-\lambda t}}{2!} = \frac{e^{-1}}{2!} = \frac{1}{2e}$ $(\lambda = 1)$

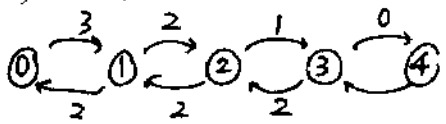
$= 0.1839$

17.4-9 a) False b) False c) False

17.4-10. Let $U = \min\{T_1, \dots, T_n\}$. So $P(T_j = U) = \int_0^\infty P\{T_i > T_j \text{ For all } i \neq j \mid T_j = t\} \alpha_j e^{-\alpha_j t} dt\} =$

$= \int_0^\infty e^{-t \sum_{i=1}^n \alpha_i} \alpha_j \cdot e^{-\alpha_j t} dt = \alpha_j \int_0^\infty e^{-t \sum_{i=1}^n \alpha_i} dt = \alpha_j / \sum_{i=1}^n \alpha_i$. #

17.5-1 a)



b) $P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{3}{2} P_0$, $P_2 = C_2 P_0 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \cdot P_0 = \frac{3}{2} P_0$

$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \cdot P_0 = \frac{3}{4} P_0$, $P_4 = P_5 = \dots = 0$

$P_0 + P_1 + P_2 + P_3 = (1 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4}) P_0 = 1$, $P_0 = \frac{4}{19}$

$P_1 = \frac{12}{38}$, $P_2 = \frac{12}{38}$, $P_3 = \frac{6}{38}$

c) $L = \sum_{n=0}^{\infty} n P_n = P_1 + 2P_2 + 3P_3 = \frac{12}{38} + \frac{24}{38} + \frac{9}{19} = \frac{27}{19} = 1.421$

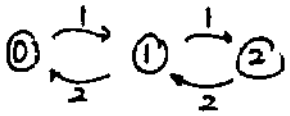
$L_q = P_2 + 2P_3 = \frac{12}{19} = 0.632$

$\bar{\lambda} = \sum_{n=0}^{\infty} \lambda_n P_n = 3P_0 + 2P_1 + P_2 = \frac{30}{19} = 1.579$

$W = L / \bar{\lambda} = \frac{27}{19} \times \frac{19}{30} = \frac{9}{10} = 0.9$

$W_q = L_q / \bar{\lambda} = \frac{12}{19} \cdot \frac{19}{30} = \frac{2}{5} = 0.4$

17.5-2 a)



b) $\begin{cases} 2P_1 = P_0 & P_0 + 2P_2 = 3P_1 \\ P_1 = 2P_2 & P_0 + P_1 + P_2 = 1 \end{cases}$

c) $P_0 = \frac{4}{7}$ $P_1 = \frac{2}{7}$ $P_2 = \frac{1}{7}$

17.5-2

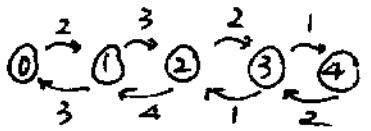
d) $p_1 = \frac{\lambda_0}{\mu_1} p_0 = \frac{1}{2} p_0 \quad p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 = \frac{1}{4} p_0$

$(1 + \frac{1}{2} + \frac{1}{4}) p_0 = p_1 + p_2 + p_3 = 1$, we have $p_0 = \frac{4}{7}$, $p_1 = \frac{2}{7}$, $p_2 = \frac{1}{7}$

$L = p_1 + 2p_2 = \frac{4}{7} \quad L_q = p_2 = \frac{1}{7} \quad \bar{\lambda} = \lambda_0 p_0 + \lambda_1 p_1 = \frac{6}{7}$

$W = L/\bar{\lambda} = \frac{2}{3} \quad W_q = L_q/\lambda = \frac{1}{6}$

17.5-3 a)



b)
$$\begin{cases} 2p_0 = 3p_1 & ① \\ 2p_0 + 4p_2 = 6p_1 & ② \\ 3p_1 + p_3 = 6p_2 & ③ \\ 2p_2 + 2p_4 = 2p_3 & ④ \\ p_3 = 2p_4 & ⑤ \\ p_1 + p_2 + p_3 + p_4 = 1 & ⑥ \end{cases}$$

c) ① $\Rightarrow p_1 = \frac{2}{3} p_0$

② $\Rightarrow p_2 = (6 \cdot \frac{2}{3} p_0 - 2p_0)/4 = \frac{1}{2} p_0$

③ $\Rightarrow p_3 = \frac{1}{2} \cdot 6p_0 - 3 \cdot \frac{2}{3} p_0 = p_0$

④ $\Rightarrow p_4 = (2p_0 - 2 \cdot \frac{1}{2} p_0)/2 = \frac{1}{2} p_0$

⑤ $p_0 + \frac{2}{3} p_0 + \frac{1}{2} p_0 + p_0 + \frac{1}{2} p_0 = 1$

$\Rightarrow p_0 = \frac{3}{11} \quad p_1 = \frac{2}{11} \quad p_2 = \frac{3}{22} \quad p_3 = \frac{3}{11} \quad p_4 = \frac{3}{22}$

d) $p_1 = \frac{\lambda_0}{\mu_1} \cdot p_0 = \frac{2}{3} p_0$,

$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \cdot p_0 = \frac{1}{2} p_0$

$p_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \cdot p_0 = p_0$

$p_4 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3 \mu_4} \cdot p_0 = \frac{1}{2} p_0$

From $p_0 + p_1 + p_2 + p_3 + p_4 = 1 \Rightarrow p_0 = \frac{3}{11}$.

17.5-3 (CONT'D)

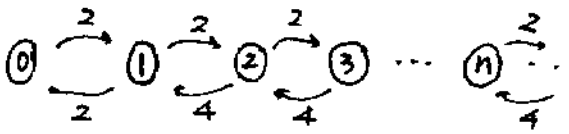
$L = p_1 + 2p_2 + 3p_3 = \frac{20}{11}$

$L_q = p_2 + 2p_3 = \frac{12}{11}$

$\bar{\lambda} = \lambda_0 p_0 + \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = \frac{18}{11}$

$W = L/\bar{\lambda} = \frac{10}{9} \qquad W_q = L_q/\bar{\lambda} = \frac{2}{3}$

17.5-4 a)



b) $p_1 = \frac{2}{2} p_0$

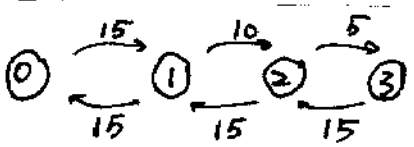
$p_2 = \frac{1}{2} p_0 \quad \cdots \quad p_n = \left(\frac{1}{2}\right)^{n-1} p_0$

$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n p_0 + p_0 = 3p_0 = 1 \quad , \quad p_0 = \frac{1}{3}$

$p_n = \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{3}$

c) arrival rate 1,
service rate 2.

17.5-5 a)



b)
$$\begin{cases} 15p_0 = 15p_1 & ① \\ 15p_0 + 15p_2 = 25p_1 & ② \\ 10p_0 + 15p_3 = 20p_2 & ③ \\ 5p_2 = 15p_3 & ④ \end{cases}$$

c) $p_0 = p_1 \quad , \quad p_2 = \frac{5p_1 - 3p_0}{3} = \frac{2}{3} p_0 \quad , \quad p_3 = \frac{1}{3} p_2 = \frac{2}{9} p_0$

$p_0 + p_1 + p_2 + p_3 = 1 = \frac{26}{9} p_0 \Rightarrow p_0 = \frac{9}{26}$

$p_1 = \frac{9}{26} \qquad p_2 = \frac{3}{13} \qquad p_3 = \frac{1}{13}$

Or $p_1 = \frac{\lambda_0}{\mu_1} p_0 = p_0$

$p_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 = \frac{2}{3} p_0$

$p_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} p_0 = \frac{2}{9} p_0$

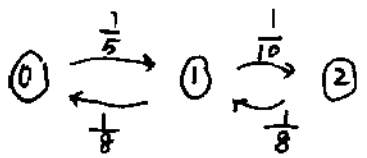
M11

17.5-5

d) $L = P_1 + 2P_2 + 3P_3 = \frac{27}{26} = 1.04$, $\bar{\lambda} = \lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 = \frac{255}{26} = 9.81$

$W = L/\bar{\lambda} = \frac{9}{85}$ (hour) $= 0.106$

17.5-6 a)



State = # of machines in breakdown state.

b) $P_1 = \frac{8}{5}P_0$, $P_2 = \frac{8}{5}\cdot\frac{8}{10}P_0 = \frac{32}{25}P_0$,

$P_0 + P_1 + P_2 = 1 \Rightarrow P_0 = \frac{25}{97}$, $P_1 = \frac{40}{97}$, $P_2 = \frac{32}{97}$

c) $\bar{\lambda} = P_0\cdot\lambda_0 + P_1\lambda_1 = \frac{1}{5}\cdot\frac{25}{97} + \frac{1}{10}\cdot\frac{40}{97} = \frac{9}{97} = 0.093$

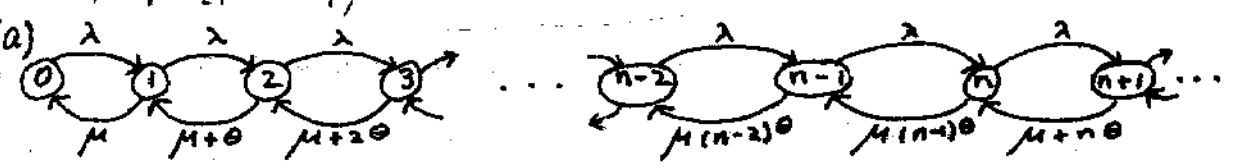
$L = P_1 + 2P_2 = \frac{104}{97} = 1.072$, $L_q = \frac{32}{97} = 0.330$

$W = L/\bar{\lambda} = \frac{104}{9}$, $W_q = \frac{32}{9} = 3.556$

d) $P_1 + P_2 = \frac{72}{97} = 0.742$

e) $P_0 + \frac{1}{2}P_1 = \frac{45}{97} = 0.464$

17.5-7 a)



(b) $\mu P_1 = \lambda P_0$

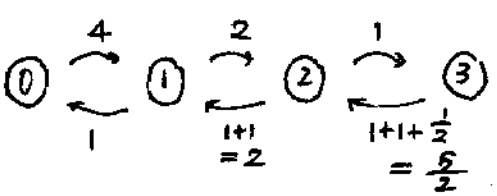
$\lambda P_0 + (\mu+\theta)P_2 = (\mu+\lambda)P_1$

$\vdots$

$\lambda P_{n-1} + (\mu+n\theta)P_{n+1} = (\lambda+\mu+(n-1)\theta)P_n$

$\vdots$

17.5-8 a)



b) $P_1 = 4P_0$, $P_2 = 4P_0$, $P_3 = \frac{2}{5}P_0$

$P_0 + P_1 + P_2 + P_3 = (1+4+4+\frac{2}{5})P_0 = 1 \Rightarrow P_0 = \frac{5}{47}$

$P_1 = \frac{20}{47}$, $P_2 = \frac{20}{47}$, $P_3 = \frac{2}{47}$

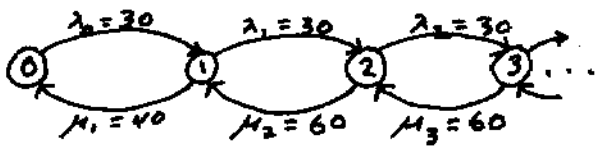
M12

17.5-8 c) Fraction of lost customers $= P_1 \cdot \frac{1}{2} + P_2 \cdot \frac{3}{4} + P_3 \cdot 1 = \frac{27}{47}$

d) $L = P_1 + 2P_2 + 3P_3 = \frac{66}{47} = 1.404$

$L_q = P_2 + 2 \cdot P_3 = \frac{24}{47} = 0.511$

17.5-9 a)



(b) $P_0 = \left[1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu_1 \mu_2^{n-1}}\right]^{-1} = \left[1 + \frac{\lambda}{\mu_1} \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu_2}\right)^{n-1}\right]^{-1} = \left[1 + \frac{\lambda}{\mu_1}\left(\frac{1}{1-\frac{\lambda}{\mu_2}}\right)\right]^{-1}$

$= \left[1 + \frac{3}{4}\left(\frac{1}{1-\frac{1}{2}}\right)\right]^{-1} = 2/5 = .4$

$P_n = P_0 \frac{\lambda^n}{\mu_1 \mu_2^{n-1}} = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right)^n$ for $n \geq 1$

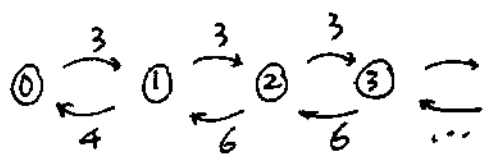
c) $L = \sum_{n=0}^{\infty} n P_n = \frac{3}{5} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{3}{5} \cdot \frac{1}{2} \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{(1-\frac{1}{2})^2} = \frac{6}{5}$

$L_q = L - (1 - P_0) = 6/5 - (1 - 2/5) = 3/5$

$W = L/\lambda = 1/25$

$W_q = L_q/\lambda = 1/50$

17.5-10 a)



b) $P_1 = \frac{3}{4} P_0$, $P_2 = \frac{3}{4} \cdot \frac{1}{2} P_0$, ..., $P_n = \frac{3}{4} \cdot \left(\frac{1}{2}\right)^{n-1} P_0$

$\sum_{n=0}^{\infty} P_n = P_0 + \frac{3}{4} P_0 + \frac{3}{4} \cdot \frac{1}{2} P_0 + \cdots$

$= \frac{5}{2} P_0 = 1$

So, $P_0 = \frac{2}{5}$, $P_1 = \frac{3}{10}$, $P_n = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)^{n-1}\left(\frac{2}{5}\right) = \frac{3}{10} \cdot \left(\frac{1}{2}\right)^{n-1}$

Let P_i' = P{ in steady-state i documents have been received but not yet completed }

17.5-10 (CONT'D)

Then $P_0' = P_0 + P_1 = \frac{7}{10}$

$P_n' = P_{n+1} = \frac{3}{10}\cdot(\frac{1}{2})^n \quad , n \geq 1$

c) $L = \sum_{n=1}^{\infty} n P_n = \frac{3}{10}\left(1 + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2^2} + \cdots\right)$

$= \frac{3}{10}\cdot 4\cdot\left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots\right)$

$= -\frac{6}{5}\,\frac{d}{d\rho}\left(\frac{1}{\rho-1}\right)\Big|_{\rho=2} = \frac{6}{5}$

$W = L/\lambda = \frac{2}{3}$

$L_q = \sum_{n=1}^{\infty}(n-1)P_n = L - (1-P_0) = \frac{3}{5}$

$W_q = L_q/\lambda = \frac{1}{5}$

17.5-11

(a) $L_q = W_q = 0$

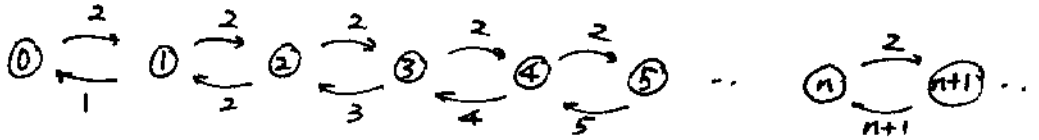
(b)

(c) $P_n = \frac{\lambda^n}{n!\mu^n}P_0 = \frac{\rho^n}{n!}P_0$

(d) $P_0 = \frac{1}{\sum_{i=0}^{\infty}\frac{\rho^n}{n!}} = \frac{1}{e^{\rho}} = e^{-\rho}$ notice that ρ need not be ≤ 1 in this case.

(e) $W = 1/\mu$, $L = \lambda W = \lambda/\mu = \rho$

17.5-12



Then, $P_1 = \frac{\lambda_0}{\mu_1}P_0 = 2P_0$,

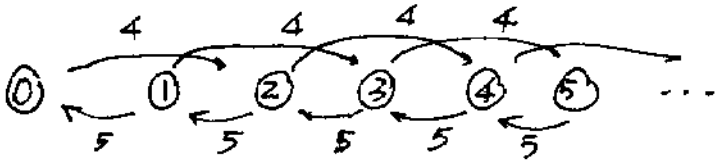
$P_2 = 2\cdot 1\cdot P_0$

$\vdots$

$P_n = \frac{2^n}{n!}\cdot P_0$

$\sum_{n=0}^{\infty} P_n = e^2\cdot P_0 = 1$, So $P_0 = e^{-2}$, $P_1 = 2e^{-2}$

17.5-13 a)



b) $5p_1 = 4p_0$

$5p_2 = 9p_1$

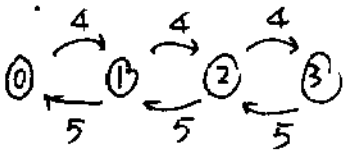
$5p_3 + 4p_0 = 9p_2$

$\vdots$

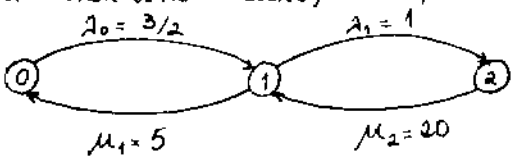
$5p_{n+1} + 4p_{n-2} = 9p_n$

$\vdots$

c)



17.5-14 (a) State n = # machines broken down.
Since the 3rd machine is shut off when the 2nd machine breaks, $n = 0, 1$ or 2.



(b) The balance equations are:

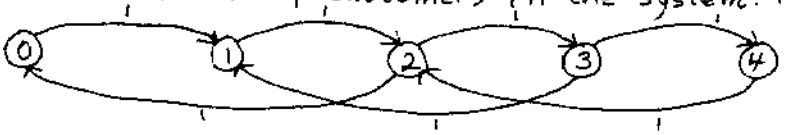
$$5P_1 = (3/2)P_0$$
$$(3/2)P_0 + 20P_2 = 6P_1$$
$$P_1 = 20P_2$$

Solving the balance equations we obtain

$$(P_0, P_1, P_2) = (200/263, 60/263, 3/263) \approx (.760, .228, .011)$$

(c) E[# operators available] $= 2P_0 + 2P_1 + 0P_2 =$
$= 520/263 = 1.977$

17.5-15 (a) Let n = number of customers in the system. Then the rate diagram is:



17.5-15. (CONT'D)

The balance equations are:

$$P_0 = P_2$$
$$P_1 = P_0 + P_3$$
$$2P_2 = P_1 + P_4$$
$$2P_3 = P_2$$
$$P_4 = P_3$$

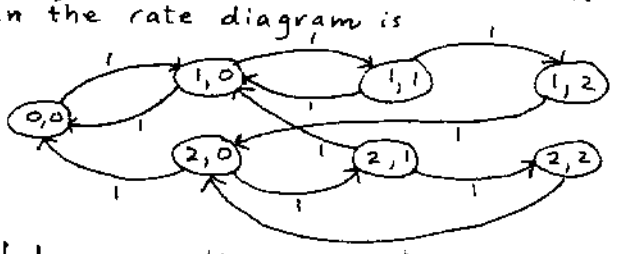
(b) The state space has to be more complex in this case because you need to know how many customers are being worked on by the server.

Let the state be (s, q) where

s = number of customers being served

q = number of customers in the queue

Then the rate diagram is



The balance equations are:

$$P_{00} = P_{10} + P_{20}$$
$$2P_{10} = P_{00} + P_{11} + P_{21}$$
$$2P_{11} = P_{10}$$
$$P_{12} = P_{11}$$
$$2P_{20} = P_{12} + P_{22}$$
$$2P_{21} = P_{20}$$
$$P_{22} = P_{21}$$

17.5-16

(a) Let the state be (n_1, n_2) where

n_1 = number of type 1 customers in the system

n_2 = number of type 2 customers in the system

Then the rate diagram is:

(b) The balance equations are:

(c)
$$12P_{01} = 5P_{00}$$
$$15P_{00} = 12(P_{01} + P_{10})$$
$$22P_{10} = 10P_{00} + 24P_{20}$$
$$24P_{20} = 10P_{10}$$
$$(P_{00} + P_{10} + P_{01} + P_{20} = 1)$$
$$\Rightarrow P_{00} = \frac{72}{187},\ P_{10} = \frac{60}{187},\ P_{01} = \frac{30}{187},\ P_{20} = \frac{25}{187}$$

(d) Type 1 customers are blocked when the system is in state (2,0) or (0,1) which means that the fraction unable to enter the system is $P_{20} + P_{01} = 55/187$.

Type 2 customers are blocked when the system is in state (2,0),(1,0) or (0,1) which means that the fraction unable to enter the system is $P_{20} + P_{10} + P_{01} = 115/187$.

17.6-1 a) $\lambda = 2$, $\mu = 4$, $s = 1$, $\rho = 1/2$

For M/M/1 queue, $P_0 = 1 - \lambda/\mu = 1/2$ and $P_n = (1-\rho)\rho^n = (1/2)^{n+1}$

desired proportion of time $= \sum_{i=0}^{4} P_i = 31/32$

b)

P_0 =	0.5
P_1 =	0.25
P_2 =	0.125
P_3 =	0.0625
P_4 =	0.03125

Total = 97%

17.6-2 $\lambda = 10$, $\mu = 15$, $P_0 = (1 - \frac{\lambda}{\mu}) = \frac{1}{3}$ = propotion of time no one is waiting

17.6-3 (a) $W \sim \exp(\mu - \lambda)$, $W = \frac{1}{\mu - \lambda}$

$$P\{W > W\} = (\mu-\lambda) e^{-(\mu-\lambda)\frac{1}{\mu-\lambda}} = (\mu-\lambda)/e$$

(b) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$, $W_q(t) = \begin{cases} 1-\rho & t=0 \\ 1-\rho e^{-\mu(1-\rho)t} & t>0 \end{cases}$

So, $P\{W_q > W_q\} = 1 - W_q(W_q) = \rho e^{-\mu(1-\rho)\frac{\lambda}{\mu(\mu-\lambda)}}$

$$= \frac{\lambda}{\mu} \cdot e^{-\frac{\lambda}{\mu}}$$

17.6-4 $P_0 = 1-\rho$, $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

$$\frac{(1-P_0)^2}{W_q P_0} = \frac{\rho^2}{\frac{\lambda}{\mu(\mu-\lambda)} \cdot (1-\frac{\lambda}{\mu})} = \frac{\lambda^2/\mu}{\lambda/\mu^2} = \lambda$$

$$\frac{1-P_0}{W_q P_0} = \frac{\rho}{\frac{\lambda}{\mu(\mu-\lambda)} \cdot \frac{\mu-\lambda}{\mu}} = \frac{\lambda/\mu}{\lambda/\mu^2} = \mu$$

17.6-5 $\lambda = 3$, $\mu = 4$, $s = 1$, $\rho = 3/4$

The system without the storage restriction is a M/M/1 queue. If n square feet of floor space were available for waiting, the proportion of time this would be sufficient is $\sum_{i=0}^{n+1} P_i$. Thus we want to find n_ℓ such that $\sum_{i=0}^{n_\ell+1} P_i \geq q_\ell$ for $\ell = 1,2,3$, where $q_1 = .5, q_2 = .9, q_3 = .99$

Now $\sum_{i=0}^{n_\ell+1} P_i \geq q_\ell \Leftrightarrow \sum_{i=0}^{n_\ell+1} (1-\rho)\rho^i \geq q_\ell \Leftrightarrow (1-\rho)\frac{(1-\rho^{n_\ell+2})}{(1-\rho)} \geq q_\ell \Leftrightarrow$

$\Leftrightarrow 1 - \rho^{n_\ell+2} \geq q_\ell \Leftrightarrow \rho^{n_\ell+2} \leq 1 - q_\ell \Leftrightarrow (n_\ell+2)\ln\rho \leq \ln(1-q_\ell)$

$\Leftrightarrow (n_\ell+2) \geq \frac{\ln(1-q_\ell)}{\ln\rho} \Leftrightarrow n_\ell \geq \frac{\ln(1-q_\ell)}{\ln\rho} - 2$

part	q_ℓ	$\frac{\ln(1-q_\ell)}{\ln\rho} - 2$	floor space required
(a)	.50	.409	1
(b)	.90	6.004	7
(c)	.99	14.008	15

17.6-6. a) True

b) False. $L = \lambda W = \frac{\rho}{1-\rho}$

c) False. $L = \frac{\rho}{(1-\rho)}$, $L = 9$ when $\rho = 0.9$, but $L = 99$ when $\rho = 0.99$

17.6-7 a) False

b) True. when $\rho > 1$, $L \to \infty$

c) TRUE

17.6-8 a) True

b) False

c) True

17.6-9

a)

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{40 - 30} = 3 \text{ customers}$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{40 - 30} = 0.1 \text{ hours}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{40(40 - 30)} = 0.075 \text{ hours}$$

$$L_q = \lambda W_q = 30(0.075) = 2.25 \text{ customers}$$

$$P_0 = 1 - \rho = 1 - 0.75 = 0.25$$

$$P_1 = (1 - \rho)\rho = (1 - 0.75)0.75 = 0.188$$

$$P_2 = (1 - \rho)\rho^2 = (1 - 0.75)0.75^2 = 0.141$$

There is a 42% chance of having more than 2 customers at the checkout stand.

b)

Data		
λ =	30	(mean arrival rate)
μ =	40	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	3.975E-31
when t =	7

Prob(ω_q>t) =	1.447E-22
when t =	5

P0+P1+P2= 0.578125

Results	
L =	3
L_q =	2.25
W =	0.1
W_q =	0.075
ρ =	0.75
P_0 =	0.25
P_1 =	0.1875
P_2 =	0.140625

17.6-9 c)

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{60 - 30} = 1 \text{ customer}$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{60 - 30} = 0.033 \text{ hours}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{60(60 - 30)} = 0.017 \text{ hours}$$

$$L_q = \lambda W_q = 30(0.075) = 0.5 \text{ customer}$$

$$P_0 = 1 - \rho = 1 - 0.5 = 0.5$$

$$P_1 = (1 - \rho)\rho = (1 - 0.5)0.5 = 0.25$$

$$P_2 = (1 - \rho)\rho^2 = (1 - 0.5)0.5^2 = 0.125$$

There is a 12.5% chance of having more than 2 customers at the checkout stand.

d)

Data		
λ =	30	(mean arrival rate)
μ =	60	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	6.283E-92
when t =	7

Prob(ω$_q$>t) =	3.588E-66
when t =	5

P0+P1+P2= 0.875

Results	
L =	1
L_q =	0.5
W =	0.03333333
W_q =	0.01666667
ρ =	0.5
P_0 =	0.5
P_1 =	0.25
P_2 =	0.125

e) The manager should adopt the new approach of adding another person to bag the groceries.

17.6-10

a)

Data		
λ =	10	(mean arrival rate)
μ =	20	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	0.0067379
when t =	0.5

Prob(ω_q>t) =	0.003369
when t =	0.5

P0+P1+P2+P3+P4+P5= 0.984375

Results	
L =	1
L_q =	0.5
W =	0.1
W_q =	0.05
ρ =	0.5
P_0 =	0.5
P_1 =	0.25
P_2 =	0.125
P_3 =	0.0625
P_4 =	0.03125
P_5 =	0.015625

All the criteria are currently being satisfied.

b)

Data		
λ =	15	(mean arrival rate)
μ =	20	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	0.082085
when t =	0.5

Prob(ω_q>t) =	0.0615637
when t =	0.5

P0+P1+P2+P3+P4+P5= 0.822021484

Results	
L =	3
L_q =	2.25
W =	0.2
W_q =	0.15
ρ =	0.75
P_0 =	0.25
P_1 =	0.1875
P_2 =	0.140625
P_3 =	0.10546875
P_4 =	0.07910156
P_5 =	0.05932617

None of the criteria are now satisfied.

17.6-10

c)

Data		
λ =	25	(mean arrival rate)
μ =	20	(mean service rate)
s =	2	(# servers)

Pr(ω>t) =	0.0010217
when t =	0.5

Prob(ω_q>t) =	0.0002659
when t =	0.5

P0+P1+P2+P3+P4+P5= 0.926640437

Results	
L =	2.05128205
L_q =	0.80128205
W =	0.08205128
W_q =	0.03205128
ρ =	0.625
P_0 =	0.23076923
P_1 =	0.28846154
P_2 =	0.18028846
P_3 =	0.11268029
P_4 =	0.07042518
P_5 =	0.04401574

In this case, the first and third criteria are satisfied but the second is not.

17.6-11

a)

Data		
λ =	2	(mean arrival rate)
μ =	1	(mean service rate)
s =	4	(# servers)

Pr(ω>t) =	0.0079019
when t =	5

Prob(ω_q>t) =	7.896E-06
when t =	5

P0+P1+P2+P3+P4+P5+P6+P7+P8+P9= 0.997282609

Results	
L =	2.17391304
L_q =	0.17391304
W =	1.08695652
W_q =	0.08695652
ρ =	0.5
P_0 =	0.13043478
P_1 =	0.26086957
P_2 =	0.26086957
P_3 =	0.17391304
P_4 =	0.08695652
P_5 =	0.04347826
P_6 =	0.02173913
P_7 =	0.01086957
P_8 =	0.00543478
P_9 =	0.00271739

All the guidelines are currently being met.

17.6-11

b)

Data		
λ =	3	(mean arrival rate)
μ =	1	(mean service rate)
s =	4	(# servers)

Results	
L =	4.52830189
L_q =	1.52830189
W =	1.50943396
W_q =	0.50943396
ρ =	0.75
P_0 =	0.03773585
P_1 =	0.11320755
P_2 =	0.16981132
P_3 =	0.16981132
P_4 =	0.12735849
P_5 =	0.09551887
P_6 =	0.07163915
P_7 =	0.05372936
P_8 =	0.04029702
P_9 =	0.03022277

Pr(ω>t) =	0.0239006
when t =	5

Prob(ω_q>t) =	0.0034325
when t =	5

P0+P1+P2+P3+P4+P5+P6+P7+P8+P9=

0.9093317

The first two guidelines will not be satisfied in one year but the third will be.

c) Five tellers will be needed in a year,

17.6-12

(a)

λ	L	L_q	W	W_q	P{W > 5}
5	1	.50	2	1	.082
.9	9	8.10	10	9	.409
.99	99	98.01	100	99	.951

(b)

λ	λ/μ	ρ	P_0	L	L_q	W	W_q	P{W>5}
.5	1	.5	.3333	1.333	.333	2.667	.667	.150
.9	1.8	.9	.0526	9.474	7.674	10.526	8.526	.641
.99	1.98	.99	.0050	99.49	97.517	100.503	98.503	.956

17.63

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	2.061E-09
when t =	10

Prob(ω_q>t) =	0.8333333
when t =	0

P0+P1= 0.305555556

Results

L =	5
L_q =	4.16666667
W =	0.5
W_q =	0.41666667
ρ =	0.83333333
P_0 =	0.16666667
P_1 =	0.13888889
P_2 =	0.11574074
P_3 =	0.09645062

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	2	(# servers)

Pr(ω>t) =	1.894E-52
when t =	10

Prob(ω_q>t) =	0.245098
when t =	0

P0+P1+P2= 0.897875817

Results

L =	1.00840336
L_q =	0.17507003
W =	0.10084034
W_q =	0.017507
ρ =	0.41666667
P_0 =	0.41176471
P_1 =	0.34313725
P_2 =	0.14297386
P_3 =	0.05957244

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	3	(# servers)

Pr(ω>t) =	8.047E-53
when t =	10

Prob(ω_q>t) =	0.0577101
when t =	0

P0+P1+P2+P3= 0.983969426

Results

L =	0.85552951
L_q =	0.02219618
W =	0.08555295
W_q =	0.00221962
ρ =	0.27777778
P_0 =	0.43213296
P_1 =	0.3601108
P_2 =	0.15004617
P_3 =	0.04167949

Data

λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	4	(# servers)

Pr(ω>t) =	7.707E-53
when t =	10

Prob(ω_q>t) =	0.0110241
when t =	0

P0+P1+P2+P3+P4= 0.997703314

Results

L =	0.83623441
L_q =	0.00290108
W =	0.08362344
W_q =	0.00029011
ρ =	0.20833333
P_0 =	0.43433168
P_1 =	0.36194306
P_2 =	0.15080961
P_3 =	0.04189156
P_4 =	0.00872741

(CONT'D)

Data		
λ =	10	(mean arrival rate)
μ =	12	(mean service rate)
s =	5	(# servers)

Pr(ω>t) =	7.672E-53
when t =	10

Prob(ω_q>t) =	0.0017464
when t =	0

P0+P1+P2+P3+P4+P5= 0.999708926

Results	
L =	0.83368262
L_q =	0.00034929
W =	0.08336826
W_q =	3.4929E-05
ρ =	0.16666667
P_0 =	0.43457121
P_1 =	0.36214268
P_2 =	0.15089278
P_3 =	0.04191466
P_4 =	0.00873222
P_5 =	0.00145537

a) 2 servers
b) 3 servers
c) 2 servers
d) 1 server
e) 5 servers
f) 1 server
g) 3 servers

17.6-14 $\lambda=20$, $\mu=30$, This is M/M/1 Queue.

E(number of airplanes at the end of the thundestorm)

= E(# airplanes before thunderstone) + E(# arrived planes during the thunderstorm)

$= L_q + \lambda t = \lambda(W_q + t) = 20\left(\frac{20}{30(30-20)} + 0.5\right) = 11.33$

17.6-15 $\lambda=15$, $\mu=20$, this is a M/M/1 Queue

P{customer does not have to wait} $= P_0 = 1 - \frac{\lambda}{\mu} = \frac{1}{3}$

price/per gallon $= 1.2 \times \frac{1}{3} + 1 \times \frac{2}{3} = 1.067$

17.6-16 Expected cost $= \sum_{n=1}^{\infty} n \cdot p_n = \sum_{n=1}^{\infty} n \cdot \rho^n(1-\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$

17.6-17 Let $P\{W \le t\} = G(t)$ and let $\frac{dG(t)}{dt} = g(t)$

Then $P\{W > t\} = 1 - P(t)$

So $[1 - G(t)] = \sum_{n=0}^{\infty} P_n P\{S_{n+1} > t\} =$

$= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\int_t^{\infty} \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx\right] =$

$= \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^{n+1} x^n e^{-\mu x}}{n!} dx\right]$

Differentiating both sides we have:

$g(t) = \sum_{n=0}^{\infty} (1-\rho)\rho^n \left[\frac{\mu^{n+1} t^n e^{-\mu t}}{n!}\right] = (1-\rho)\mu e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}$

$= (1-\rho)\mu e^{-\mu t} e^{\lambda t} = \mu(1-\rho)e^{-\mu(1-\rho)t}$

Hence, by integration, $P(W > t) = 1 - \int_0^t g(x)dx = e^{-\mu(1-\rho)t}$

17.6-18 (a) Let $P\{W_q \le t\} = G(t)$ and let $\frac{dG(t)}{dt} = g(t)$

Then $P\{W_q > t\} = 1 - G(t)$

So $[1-G(t)] = \sum_{n=1}^{\infty} P_n P\{S_n > t\} = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[1 - \int_0^t \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!} dx\right]$

Differentiating both sides gives:

$$g(t) = \sum_{n=1}^{\infty} (1-\rho)\rho^n \left[\frac{\mu^n t^{n-1} e^{-\mu t}}{(n-1)!}\right]$$

$$= (1-\rho)\lambda e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= (1-\rho)\lambda e^{-\mu t} e^{\lambda t}$$

$$= \left(\frac{\lambda}{\mu}\right)(\mu-\lambda) e^{-(\mu-\lambda)t}$$

Then $W_q = \left(\frac{\lambda}{\mu}\right)\int_0^{\infty} t\,(\mu-\lambda)e^{-(\mu-\lambda)t} dt = \frac{\lambda}{\mu(\mu-\lambda)}$

(b) Let $P\{W_q \le t\} = G(t)$ and let $\frac{dG(t)}{dt} = g(t)$

Then $P\{W_q > t\} = 1 - G(t)$

So $[1-G(t)] = \sum_{n=s}^{\infty} P_n P\{S_{n-s+1} > t\}$

$$= \sum_{n=s}^{\infty} P_n \left[1 - \int_0^t \frac{(s\mu)^{n-s+1} x^{n-s} e^{-(s\mu)x}}{(n-s)!} dx\right]$$

Now $P_n = \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0$ for $n \ge s$

So differentiating both sides gives

$$g(t) = \sum_{n=s}^{\infty} \left[\frac{(\lambda/\mu)^n P_0}{s!\, s^{n-s}}\right]\left[\frac{(s\mu)^{n-s+1} t^{n-s} e^{-(s\mu)t}}{(n-s)!}\right]$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} \sum_{n=s}^{\infty} \frac{(\lambda t)^{n-s}}{(n-s)!}$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-s\mu t} e^{\lambda t}$$

$$= \frac{P_0 (s\mu)(\lambda/\mu)^s}{s!} e^{-(s\mu)(1-\rho)t}$$

So $W_q = \frac{P_0(\lambda/\mu)^s}{s!} \int_0^{\infty} t\,(s\mu) e^{-(s\mu)(1-\rho)t} dt$

$$= \frac{P_0(\lambda/\mu)^s}{s!(1-\rho)} \int_0^{\infty} t\,(s\mu)(1-\rho) e^{-(s\mu)(1-\rho)t} dt$$

$$= \frac{P_0 (\lambda/\mu)^s}{s!(1-\rho)^2 (s\mu)}$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!(1-\rho)^2 \lambda}$$

$$= L_q / \lambda$$

17.6-19

$\lambda = 4 \quad \mu = 3 \quad s = 2$

we have: $P_0 = 0.2$; $P_1 = 0.267$; $P_2 = 0.178$

the mean rate is:

$$\frac{\mu_0 P_0 + \mu_1 P_1 + \mu_2 P_2}{P_0 + P_1 + P_2} = \frac{0 \cdot P_0 + 3P_1 + 6P_2}{P_0 + P_1 + P_2} = 2.90$$

17.6-20

$\lambda = 4$

$\mu = 6$

$s = 2$

$L = 0.75$

$L_q = 0.083$

$W = 0.188$

$W_q = 0.021$

$P(W > t) = 1$, where $t = 0$

$P(W_q > t) = 0.003$, where $t = 0.5$

$P_0 = 0.5$
$P_1 = 0.33333$
$P_2 = 0.11111$
$P_3 = 0.03704$
$P_4 = 0.01235$
$P_5 = 0.00412$
$P_6 = 0.00137$
$P_7 = 0.00046$
$P_8 = 0.00015$
$P_9 = 0.00005$
$P_{10} = 0.00002$
$P_{11} = 0.00001$
$P_{12} = 0$
$P_{13} = 0$
$P_{14} = 0$
$P_{15} = 0$
$P_{16} = 0$
$P_{17} = 0$
$P_{18} = 0$
$P_{19} = 0$
$P_{20} = 0$

$$P\{W_q > 0.5 \mid \#\text{customers} \geq 2\} = \frac{P\{W_q > 0.5, \#\text{customers} \geq 2\}}{P\{\#\text{customers} \geq 2\}}$$

$$= \frac{P\{W_q > 0.5\}}{1 - P_0 - P_1} = \frac{0.003}{1 - (0.5 + 0.33333)} = 0.018$$

17.6-21

(a) $W = \frac{1}{\mu - \lambda}$

$W_{\text{clara}} = \frac{1}{20-16} = \frac{1}{4}$ hour $= 15$ minutes

$W_{\text{clarence}} = \frac{1}{20-14} = \frac{1}{6}$ hour $= 10$ minutes

$W_{\text{total}} = P\{\text{clara}\} \cdot W_{\text{clara}} + P\{\text{clarence}\} \cdot W_{\text{clarence}}$

$= \frac{16}{30} \cdot 15 + \frac{14}{30} \cdot 10 = 12.67$ minutes $= 0.211$ hour

(b) It is M/M/2 queue, $\lambda = 16+14 = 30$, $\mu = 20$, $S = 2$

Run OR courseware, $W = 0.114$ hour

(c) $\mu = 60/3.5$, $W = 0.249$

$\mu = 60/3.4$, $W = 0.204$

$\mu = 60/3.45$, $W = 0.225$

$\mu = 60/3.425$, $W = 0.214$

$\mu = 60/3.419$, $W = 0.212$

$\mu = 60/3.4185$, $W = 0.211$

⋮

So, if the expected processing time is 3.485 minutes, it causes the same expected waiting time.

17.6-22 a) When $\lambda = 10$, $\mu = 7.5$, $s = 2$, (current system)

$L = 2.4$, $L_q = 1.067$, $W = 0.24$, $W_q = 0.107$

When $\lambda = 5$, $\mu = 7.5$, $s = 1$, (next year)

$L = 2$, $L_q = 1.333$, $W = 0.4$, $W_q = 0.267$

So, next year yields smaller L but larger L_q, W and W_q

b) $W = \frac{1}{\mu - \lambda} \Rightarrow \mu = \frac{1}{W} + \lambda = \frac{1}{0.24} + 5 = 9.17$

c) $W_q = \frac{\lambda}{\mu(\mu - \lambda)} \Rightarrow \mu = \frac{\lambda W_q \pm \sqrt{\lambda^2 W_q^2 + 4\lambda W_q}}{2 W_q} \Rightarrow \mu = 9.78$

17.6-23 $L = 8$, $W = 2$, $\lambda = L/W = 4$

$W = \frac{1}{\mu - \lambda} = 2 \Rightarrow \mu = \frac{1}{2} + \lambda = 4\frac{1}{2}$

$P\{T_s > \frac{1}{3}\} = e^{-\frac{9}{2} \cdot \frac{1}{3}} = e^{-\frac{3}{4}}$

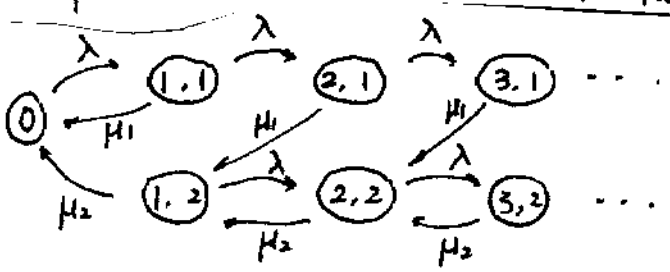
17.6-24 (a) The future evolution of the queueing system is affected by whether the parameter of the service time distribution for the customer currently in service is μ_1 or μ_2. So the current state of the system needs to include this information from the history of the process.

The states are (n, s) where

n = number of customers in the system ($n = 1, 2, \cdots$)

$s = \begin{cases} 1 & \text{if the current parameter is } \mu_1 \\ 2 & \text{if the current parameter is } \mu_2 \end{cases}$

except that $n = 0$ does not need s. Then we have:



(b) $\lambda p_0 = \mu_1 p_{1,1} + \mu_2 p_{1,2}$

$(\lambda + \mu_1) p_{1,1} = \lambda p_0$

$\vdots$

$(\lambda + \mu_1) p_{n,1} = \lambda p_{n-1,1}$

$\vdots$

$(\lambda + \mu_2) p_{1,2} = \mu_1 p_{2,1} + \mu_2 p_{2,2}$

$\vdots$

$(\lambda + \mu_2) p_{n,2} = \lambda p_{n-1,2} + \mu_1 p_{n+1,1} + \mu_2 p_{n+1,2} \quad (n \geq 2)$

$\vdots$

17.6-24

(c) Truncate (cut off) the balance equations at a very large n and then solve the resulting finite system of equations numerically. The resulting approximation of the stationary distribution should be essentially exact if the probability of exceeding the truncating value of n (in the exact model) is negligible.

(d) $L = \sum_{n=1}^{\infty} n(P_{n,1} + P_{n,2})$, $W = \frac{L}{\lambda}$

$L_q = \sum_{n=1}^{\infty} (n-1)(P_{n,1} + P_{n,2})$, $W_q = \frac{L_q}{\lambda}$

(e) Because the input is Poisson, the distribution of the state of the system is the same just before an arrival and at an arbitrary point in time

$$P\{W \le t\} = P\{W \le t \mid \text{arrival finds state } 0\} P_0 + \sum_{n=1}^{\infty} P\{W \le t \mid \text{arrival finds state } (n,1)\} P_{n,1} + \sum_{n=1}^{\infty} P\{W \le t \mid \text{arrival finds state } (n,2)\} P_{n,2}$$

These 3 conditional distributions of W are, respectively (1) $\exp(\mu_1)$ (2) a convolution of $\exp(\mu_1)$ and Erlang$(n/\mu_2, n)$ and (3) Erlang$((n+1)\mu_2, n+1)$

Then, $P\{W \le t\} = (1 - e^{-\mu_1 t}) P_0 + \sum_{n=1}^{\infty} \left[\int_0^t [1 - e^{-\mu_1 (t-t_1)}] \cdot \frac{\mu_2^n t_1^{n-1} e^{-\mu_2 t}}{(n-1)!} dt_1 \right] P_{n,1} + \sum_{n=1}^{\infty} \left[\int_0^t \frac{\mu_2^{n+1} x^n e^{-\mu_2 x}}{n!} dx \right] P_{n,2}$

17.6-25

(a) $\lambda P_0 = \mu P_1$ . . . . . (0)

$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_n$ . . . (1)

$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu) P_n$ . . . (n)

The solution given in Sec. 16.6 is:

$P_n = (1-\rho)\rho^n$ for $n = 0, 1, 2 \ldots$

Verifying that the above satisfy the balance equations:

equation (0): $\lambda \cdot (1-\rho) = \mu (1-\rho) \cdot \rho \Leftrightarrow \lambda = \mu \cdot \rho = \mu \frac{\lambda}{\mu}$ OK

equation (n): $\lambda (1-\rho)\rho^{n-1} + \mu (1-\rho)\rho^{n+1} = (\lambda+\mu)(1-\rho)\rho^n$

$\Leftrightarrow \lambda + \mu\rho^2 = (\lambda + \mu) \cdot \rho$

$\lambda + \frac{\lambda^2}{\mu}$ $\quad$ $\frac{\lambda^2}{\mu} + \lambda$ OK

17.6-25 (b) $\lambda P_0 = \mu P_1$
$\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$
$\lambda P_1 = \mu P_2$
The solution given in Sec. 16.6 is:

$$P_n = \left(\frac{1-\rho}{1-\rho^3}\right)\rho^n \quad \text{for } n = 0, 1, 2$$

Verifying:

- $\lambda \cdot \frac{1-\rho}{1-\rho^3} = \mu \cdot \frac{1-\rho}{1-\rho^3} \cdot \rho \Leftrightarrow \lambda = \mu\cdot\rho = \mu \cdot \frac{\lambda}{\mu}$ . . . ok
- $\lambda \cdot \frac{1-\rho}{1-\rho^3} + \mu \frac{1-\rho}{1-\rho^3}\cdot\rho^2 = (\lambda+\mu)\frac{1-\rho}{1-\rho^3}\rho \Leftrightarrow \lambda + \mu\cdot\rho^2 = (\lambda+\mu)\cdot\rho$
  $\lambda + \frac{\lambda^2}{\mu} \qquad \frac{\lambda^2}{\mu} + \lambda$
- $\lambda \cdot \frac{1-\rho}{1-\rho^3}\cdot\rho = \mu\frac{1-\rho}{1-\rho^3}\cdot\rho^2 \Leftrightarrow \lambda\cdot\rho = \mu\rho^2$ . . . ok
  $\frac{\lambda^2}{\mu} \qquad \frac{\lambda^2}{\mu}$

(c) $2\lambda P_0 = \mu P_1$
$2\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$
$2 P_1 = \mu P_2$
The solution given in Sec. 16.6 is:

$$P_0 = \left[\sum_{n=0}^{2} \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} = \left[1 + 2\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2\right]^{-1}$$

$$P_n = \frac{2!}{(2-n)!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad \text{for } n = 1, 2$$

Verifying:

- $2\lambda / (1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2) = \mu \cdot 2(\frac{\lambda}{\mu}) / (1 + 2(\frac{\lambda}{\mu}) + 2(\frac{\lambda}{\mu})^2)$
  $\Leftrightarrow 2\lambda = \mu \cdot 2 \cdot \frac{\lambda}{\mu}$ . . . ok

---

- $2\lambda/(1+2(\frac{\lambda}{\mu})+2(\frac{\lambda}{\mu})^2) + \mu\cdot 2(\frac{\lambda}{\mu})^2/(1+2(\frac{\lambda}{\mu})+2(\frac{\lambda}{\mu})^2) =$
  $= (\lambda+\mu)2(\lambda/\mu)/(1+2(\lambda/\mu)+2(\lambda/\mu)^2) \Leftrightarrow 2\lambda + 2\mu(\frac{\lambda}{\mu})^2 = 2(\lambda+\mu)(\frac{\lambda}{\mu})$
  $2(\lambda + \frac{\lambda^2}{\mu}) = 2(\frac{\lambda^2}{\mu} + \lambda)$ . . .
- $\lambda \cdot 2(\frac{\lambda}{\mu})/(1+2(\lambda/\mu)+2(\lambda/\mu)^2) = \mu\cdot 2\cdot(\lambda/\mu)^2/(1+2(\lambda/\mu)+2(\lambda/\mu)^2)$
  $\Leftrightarrow 2\frac{\lambda^2}{\mu} = 2\frac{\lambda^2}{\mu}$ . . . ok

17.6-26 (a) The M/M/s Model:

$\lambda = 6$

$\mu = 4$

$s = 3$

$L = 1.737$

$L_q = 0.237$

$W = 0.289$

$W_q = 0.039$

$P(W > t) = 0.026$, where $t = 1$

$P(W_q > t) = 0.237$, where $t = 0$

$P_0 = 0.21053$
$P_1 = 0.31579$
$P_2 = 0.23684$
$P_3 = 0.11842$
$P_4 = 0.05921$
$P_5 = 0.02961$
$P_6 = 0.0148$
$P_7 = 0.0074$
$P_8 = 0.0037$
$P_9 = 0.00185$
$P_{10} = 0.00093$
$P_{11} = 0.00046$
$P_{12} = 0.00023$
$P_{13} = 0.00012$
$P_{14} = 0.00006$
$P_{15} = 0.00003$
$P_{16} = 0.00001$
$P_{17} = 0.00001$
$P_{18} = 0$
$P_{19} = 0$
$P_{20} = 0$

(b) $P\{\text{a phone is answered immediately}\} = 1 - P\{W_q > 0\} = 0.763$

Or, $= P\{\text{at least one server is free}\} = P_0 + P_1 + P_2$

$= 0.21053 + 0.31579 + 0.23684 = 0.763$

(c) $P\{n \text{ calls on hold}\} = P'_n = P_{n+3} \quad (n \geq 1)$

$P'_0 = P_0 + P_1 + P_2 + P_3 = 0.88158$

(d) The printed measures are in the next page.

$P\{\text{arriving call is lost}\}$

$= P\{\text{all three servers are busy}\}$

$= P_3 = 0.13433$

17.6-26 (continued)

Finite Queue Variation of the M/M/s Model:

$\lambda = 6$

$\mu = 4$

$s = 3$

$K = 3$

$L = 1.299$

$L_q = 0$

$W = 0.25$

$W_q = 0$

$P_0 = 0.23881$
$P_1 = 0.35821$
$P_2 = 0.26866$
$P_3 = 0.13433$
$P_4 = 0$
$P_5 = 0$
$P_6 = 0$
$P_7 = 0$
$P_8 = 0$
$P_9 = 0$
$P_{10} = 0$
$P_{11} = 0$
$P_{12} = 0$
$P_{13} = 0$
$P_{14} = 0$
$P_{15} = 0$
$P_{16} = 0$
$P_{17} = 0$
$P_{18} = 0$
$P_{19} = 0$
$P_{20} = 0$

17.6-27 (a) This is a finite Calling Population of M/M/s Model.
Here $\lambda = 1$, $\mu = 2$, $S = 1$, $N = 3$

(b) Finite Calling Population of the M/M/s Model:

$\lambda = 1$

$\mu = 2$

$s = 1$

$N = 3$

$L = 1.421$

$L_q = 0.632$

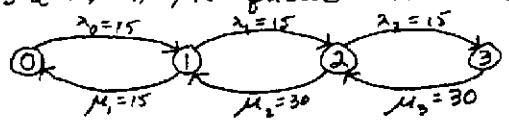
$W = 0.9$

$W_q = 0.4$

$P_0 = 0.21053$
$P_1 = 0.31579$
$P_2 = 0.31579$
$P_3 = 0.15789$
$P_4 = 0$
$P_5 = 0$
$P_6 = 0$
$P_7 = 0$
$P_8 = 0$
$P_9 = 0$
$P_{10} = 0$
$P_{11} = 0$
$P_{12} = 0$
$P_{13} = 0$
$P_{14} = 0$
$P_{15} = 0$
$P_{16} = 0$
$P_{17} = 0$
$P_{18} = 0$
$P_{19} = 0$
$P_{20} = 0$

17.6-28 This is a M/M/2/K queue with K=3, λ=15, μ=15.

(a)



(b)

λ = 15

μ = 15

s = 2

K = 3

L = 1

L_q= 0.091

W = 0.073

W_q= 0.007

P_0	= 0.36364
P_1	= 0.36364
P_2	= 0.18182
P_3	= 0.09091
P_4	= 0
P_5	= 0
P_6	= 0
P_7	= 0
P_8	= 0
P_9	= 0
P_{10}	= 0
P_{11}	= 0
P_{12}	= 0
P_{13}	= 0
P_{14}	= 0
P_{15}	= 0
P_{16}	= 0

So, $P\{0\} = P_0 + P_1 = 0.727$

$P\{1\} = P_2 = 0.182$

$P\{2\} = P_3 = 0.091$

17.6-29 This is a M/M/1/K queue with K=1, 3 and 5, respectively. Also, $\lambda = 1/4$ and $\mu = 1/3$ so that $\rho = 3/4$.
The fraction of customers lost = $P_K = \frac{(1-\rho)}{(1-\rho^{K+1})} \cdot \rho^K$

(a) zero spaces: $P_1 = \frac{(1-3/4)}{(1-(3/4)^2)} \cdot (3/4) = \frac{3}{7} = .429$

(b) two spaces: $P_3 = \frac{(1-3/4)}{(1-(3/4)^4)} \cdot (3/4)^3 = \frac{27}{175} = .154$

(c) four spaces: $P_5 = \frac{(1-3/4)}{(1-(3/4)^6)} \cdot (3/4)^5 = \frac{243}{3367} = .072$

17.6-30 M/M/s/K model

$$L_q = \sum_{n=s}^{\infty} (n-s) P_n =$$

$$= \sum_{n=s}^{K} (n-s) \frac{(\lambda/\mu)^n}{s!\, s^{n-s}} P_0$$

$$= \frac{P_0 (\lambda/\mu)^{s+1}}{s!\, s} \sum_{n=s}^{K} (n-s) \left(\frac{\lambda}{s\mu}\right)^{n-s-1} =$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \sum_{j=0}^{K-s} j\, \rho^{j-1} =$$

$$= \frac{P_0 (\lambda/\mu)^s \rho}{s!} \sum_{j=0}^{K-s} \frac{d(\rho^j)}{d\rho} =$$

17.6-30 (continued)

$$= \frac{P_0(\lambda/\mu)^s\rho}{s!}\frac{d}{d\rho}\left[\sum_{j=0}^{K-s}\rho^j\right] =$$

$$= \frac{P_0(\lambda/\mu)^s\rho}{s!}\frac{d}{d\rho}\left(\frac{1-\rho^{K-s+1}}{1-\rho}\right) =$$

$$= \frac{P_0(\lambda/\mu)^s\rho}{s!}\left[\frac{1-\rho^{K-s}-(K-s)\rho^{K-s}(1-\rho)}{(1-\rho)^2}\right]$$

17.6-31

W and W_q represent the waiting times of arriving customers who enter the system. The probability that such a customer finds n customers already there is:

$$P\{n \text{ customers in system} | \text{system not full}\} = \begin{cases} \frac{P_n}{1-P_K} & 0 \le n \le K-1 \\ 0 & n = K \end{cases}$$

And so:

(a) $$P\{W > t\} = \frac{1}{1-P_K}\sum_{n=0}^{K-1} P_n P\{S_{n+1} > t\}$$

(b) $$P\{W_q > t\} = \frac{1}{1-P_K}\sum_{n=0}^{K-1} P_n P\{S_n > t\}$$

17.6-32

a) The M/M/s model with a finite calling population fits this queueing system.

b)

Data		
λ =	0.333333	(max arrival rate)
μ =	0.5	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

Results	
L =	0.71805274
L_q =	0.21095335
W =	2.832
W_q =	0.832
ρ =	0.66666667
λ-bar =	0.2535497
P_0 =	0.49290061
P_1 =	0.32860041
P_2 =	0.14604462
P_3 =	0.03245436

The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively as shown in the spreadsheet above. The mean of this distribution is L=0.718 as shown above.

c) $W = \frac{L}{\bar{\lambda}} = \frac{0.718}{0.253} = 2.832$ hours.

d) The expected fraction of time that the repair technician will be busy is the system utilization, which is 0.667.

e) M/M/s model:

Data		
λ =	0.3333333	(mean arrival rate)
μ =	0.5	(mean service rate)
s =	1	(# servers)

Results	
L =	2
L_q =	1.33333333

Finite queue variation of the M/M/s model with K=3:

Data		
λ =	0.3333333	(mean arrival rate)
μ =	0.5	(mean service rate)
s =	1	(# servers)
K =	3	(max customers)

Results	
L =	1.01538462
L_q =	0.43076923
W =	3.47366421

F7

Data		
λ =	0.333333	(max arrival rate)
μ =	0.5	(mean service rate)
s =	2	(# servers)
N =	3	(size of population)

Results	
L =	0.55280899
L_q =	0.00898876
W =	2.03305785
W_q =	0.03305785
ρ =	0.33333333
λ-bar =	0.27191011
P_0 =	0.54606742
P_1 =	0.36404494
P_2 =	0.08089888
P_3 =	0.00898876

The probabilities that there are 0, 1, 2, or 3 machines not running are P_0, P_1, P_2, and P_3 respectively as shown in the spreadsheet above. The mean of this distribution is L=0.553 as shown above.

The expected fraction of time that the repair technician will be busy is the system utilization, which is 0.333.

17.6-33

a) Alternative 1:

Data		
λ =	1.2	(max arrival rate)
μ =	4	(mean service rate)
s =	1	(# servers)
N =	3	(size of population)

Results	
L =	0.32064422
L_q =	0.05270864
W =	0.29918033
W_q =	0.04918033

Three machines are the maximum that can be assigned to an operator while still achieving the required production rate. The average number not running is L=0.32. Thus, 1 – (0.32/3) = 89.7% of machines are running on the average.

$$\text{Utilization of servers} = \frac{\bar{\lambda}}{s\mu} = \frac{1.072}{(1)(4)} = 0.268.$$

b) Alternative 2:

Data		
λ =	4.8	(max arrival rate)
μ =	4	(mean service rate)
s =	3	(# servers)
N =	12	(size of population)

Results	
L =	1.12461693
L_q =	0.03708214
W =	0.25852352
W_q =	0.00852433

Three operators are required to achieve the required production rate. The average number not running is L=1.125. Thus, 1 – (1.125/12) = 90.6% of machines are running on the average.

$$\text{Utilization of servers} = \frac{\overline{\lambda}}{s\mu} = \frac{4.350}{(3)(4)} = 0.363.$$

c) Alternative 3:

Data		
λ =	4.8	(max arrival rate)
μ =	8	(mean service rate)
s =	1	(# servers)
N =	12	(size of population)

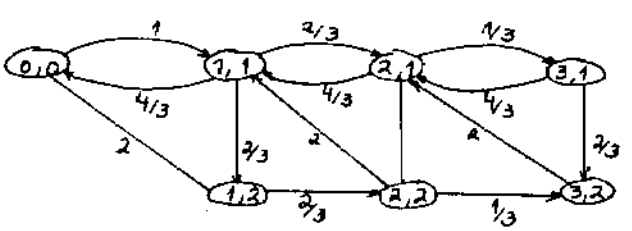
Results	
L =	1.03519555
L_q =	0.48698409
W =	0.23602691
W_q =	0.11103346

Two operators are required to achieve the required production rate. The average number not running is L=1.035. Thus, 1 – (1.035/12) = 91.4% of machines are running on the average.

$$\text{Utilization of servers} = \frac{\overline{\lambda}}{s\mu} = \frac{4.386}{(1)(8)} = 0.548.$$

17.6-34 (a) state is (n,i) where n is the number of failed machines ($n=0,1,2,3$) and i is the stage of service (which operation) for the machine under repair ($i=0$ if no machines are failed), 1, 2).

(b)



(c)

State	Rate in = Rate out
(0,0)	$4/3\ P_{1,1} + 2\ P_{1,2} = P_{0,0}$
(1,1)	$P_{0,0} + 4/3\ P_{2,1} + 2\ P_{2,2} = (4/3 + 2/3 + 2)\ P_{1,1}$
(2,1)	$2/3\ P_{1,1} + 4/3\ P_{3,1} + 2\ P_{3,2} = (4/3 + 2/3 + 1/3)\ P_{2,1}$
(3,1)	$1/3\ P_{2,1} = (4/3 + 2/3)\ P_{3,1}$
(1,2)	$2/3\ P_{1,1} = (2 + 2/3)\ P_{1,2}$
(2,2)	$2/3\ (P_{1,2} + P_{2,1}) = (2 + 1/3)\ P_{2,2}$
(3,2)	$1/3\ P_{2,2} + 2/3\ P_{3,1} + 2\ P_{3,2}$

17.6-35

$\mu_1 = 1/8$

(a) $\mu_4 = 1/4 < 4^c(1/8) \Rightarrow 2 = 4^c \Rightarrow c = 1/2$

(b) $\mu_4 = 1/5 = 4^c(1/8) \Rightarrow 8/5 = 4^c \Rightarrow c = \frac{\ln(8/5)}{\ln 4} = .339$

17.6-36 Using the L's of figure 16.10 and those calculated in problem 9,

$\lambda_0/s\mu_1$	c	.20	.40	.60
.50	s = 1	.7800	.6500	.6000
	s = 2	.9000	.8475	.7875
.90	s = 1	.2444	.1667	.1333
	s = 2	.3464	.2624	.2257
.99	s = 1	.0288	.0177	.0139
	s = 2	.0454	.0303	.252

17.7-1 (a) (i) exponential : $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

(ii) constant : $W_q = \frac{1}{2} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$

(iii) Erlang : $\sigma = \frac{1}{2}\left(0 + \frac{1}{\mu}\right) = \frac{1}{2\mu} \Rightarrow \sigma^2 = \frac{1}{4\mu^2} \Rightarrow K = 4$

$$W_q = \frac{1+4}{8} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{8} \frac{\lambda}{\mu(\mu-\lambda)}$$

So $W_q^{exp} = 2\, W_q^{c} = (8/5)\, W_q^{Erlang}$

(b) Let B = 1, (1/2) and (5/8) when the distribution is exponential, constant or Erlang, respectively.

Now $\lambda^{(2)} = 2\lambda^{(1)}$ and $\mu^{(2)} = 2\mu^{(1)}$

$$W_q^{(2)} = B\left[\frac{2\lambda^{(1)}}{2\mu^{(1)}(2\mu^{(1)} - 2\lambda^{(1)})}\right] = \frac{W_q^{(1)}}{2}$$

$$L_q^{(2)} = \lambda^{(2)} W_q^{(2)} = 2\lambda^{(1)} W_q^{(1)}/2 = \lambda^{(1)} W_q^{(1)} = L_q^{(1)}$$

So the waiting time is cut in half while the queue lenght is unchanged.

17.7-2

Data		
λ =	0.2	(mean arrival rate)
1/μ =	4	(expected service time)
σ =	4	(standard deviation)
s =	1	(# servers)

Results	
L =	4
L_q =	3.2
W =	20
W_q =	16

Data		
λ =	0.2	(mean arrival rate)
1/μ =	4	(expected service time)
σ =	3	(standard deviation)
s =	1	(# servers)

Results	
L =	3.3
L_q =	2.5
W =	16.5
W_q =	12.5

17.7-2 (CONT'D)

Data		
λ =	0.2	(mean arrival rate)
$1/\mu$ =	4	(expected service time)
σ =	2	(standard deviation)
s =	1	(# servers)

Results	
L =	2.8
L_q =	2
W =	14
W_q =	10

Data		
λ =	0.2	(mean arrival rate)
$1/\mu$ =	4	(expected service time)
σ =	1	(standard deviation)
s =	1	(# servers)

Results	
L =	2.5
L_q =	1.7
W =	12.5
W_q =	8.5

Data		
λ =	0.2	(mean arrival rate)
$1/\mu$ =	4	(expected service time)
σ =	0	(standard deviation)
s =	1	(# servers)

Results	
L =	2.4
L_q =	1.6
W =	12
W_q =	8

b) L_q is half with $\sigma = 0$ therefore it is quite important to reduce the variability of the service times.

c)

σ	L_q	Change	
4	3.2		
3	2.5	0.7	largest reduction
2	2	0.5	
1	1.7	0.3	
0	1.6	0.1	smallest reduction

d) μ needs to be increased 0.05 to achieve the same L_q.

17.7-3 For M/G/1, $L = \rho + \frac{\rho^2+\lambda^2\sigma_s^2}{2(1-\rho)}$, $L_q = \frac{\rho^2+\lambda^2\sigma_s^2}{2(1-\rho)}$ $W = L/\lambda$, $W_q = \frac{L_q}{\lambda}$

(a) False. when L increases, W also increases.

(b) False. when μ and σ^2 are small, L_q is not necessarily small.

(c) True. For exponential service time, $L_q = \frac{2\rho^2}{2(1-\rho)}$ since $\sigma_s^2 = 1/\mu^2$

For constant service time, $L_q = \frac{\rho^2}{2(1-\rho)}$ since $\sigma_s^2 = 0$

(d) False. We can easily find distribution with $\sigma_s^2 > \frac{1}{\mu^2}$

17.7-4 a)

Data		
λ =	30	(mean arrival rate)
1/μ =	0.0208333	(expectec service time)
σ =	0.0208333	(standard deviation)
s =	1	(# servers)

Results	
L =	1.66666667
L_q =	1.04166667
W =	0.05555556
W_q =	0.03472222

b)

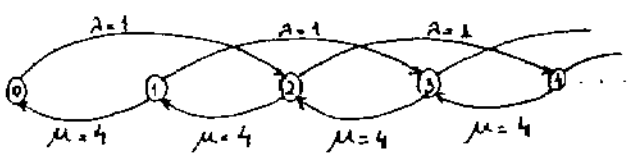
Data		
λ =	30	(mean arrival rate)
1/μ =	0.0208333	(expected service time)
σ =	0	(standard deviation)
s =	1	(# servers)

Results	
L =	1.14583333
L_q =	0.52083333
W =	0.03819444
W_q =	0.01736111

c) L_q in part b) is half of L_q in part a).

d) Marsha needs to reduce her service time to approximately 61 seconds.

17.7-5 (a)



$\mu P_1 = \lambda P_0$

$\mu P_2 = (\lambda+\mu) P_1$

$\lambda P_0 + \mu P_3 = (\lambda+\mu) P_2$

$\lambda P_{n-2} + \mu P_{n+1} = (\lambda+\mu) P_n$

(b) Poisson input with $\lambda = 1$ and Erlang service with $\mu = 4/2 = 2$ and $k = 2$.

c) $$L = L_q + \rho = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = \frac{(1)^2(0.354)^2 + 0.5^2}{2(1-0.5)} + 0.5 = 0.875$$

d) $$W = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{0.375}{1} + \frac{1}{2} = 0.875$$

e)

Data		
λ =	1	(mean arrival rate)
μ =	2	(mean service rate)
k =	2	(shape parameter)
s =	1	(# servers)

Results	
L =	0.875
L_q =	0.375
W =	0.875
W_q =	0.375

17.7-6 a & b) Current policy:

Data		
λ =	1	(mean arrival rate)
μ =	2	(mean service rate)
s =	1	(# servers)

Results	
L =	1
L_q =	0.5
W =	1
W_q =	0.5

Proposal:

Data		
λ =	0.25	(mean arrival rate)
μ =	0.5	(mean service rate)
k =	4	(shape parameter)
s =	1	(# servers)

Results	
L =	0.8125
L_q =	0.3125
W =	3.25
W_q =	1.25

Under the current policy an airplane looses 1 day of flying time as opposed to 3.25 days under the proposed policy.

Under the current policy 1 airplane is loosing flying time per day as opposed to 0.8125 airplanes.

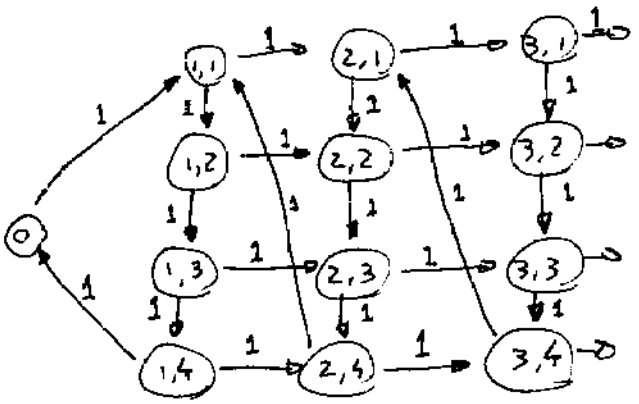
c) The comparison in part b) is the appropriate one for making the decision since it takes into account that airplanes will not have to come in for service as often.

17.7-7 a) Let the state be (n,s) where:

n = # of airplanes at the base

s = stage of service of the airplane being overhauled

b) The rate diagram is:



17.7-8

(a) Poisson Input

Erlang service with $\mu = 1/12$ and $k = 4$

(b) Poisson Input

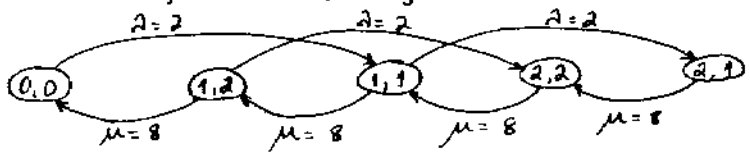
General service with mean $3+9 = 12$ and

variance $\frac{1}{(k\mu_1^2)} + \frac{1}{(k\mu_2^2)} = 30$

17.7-9 For the current arrangement, $\lambda = 24$ and $\mu = 30 \Rightarrow \rho = .8$
For the proposal, $\lambda = 48$, $\mu = 30$ and $s = 2 \Rightarrow \rho = .8$

Model	Current			Proposal	
	L at each crib	Total L	W = L/λ	L	W = L/λ
Figure 17.7	4.0	8.0	0.167	4.444	0.093
Figure 17.11	2.4	4.8	0.098	3.1	0.064
Figure 17.13	3.2	6.4	0.133	3.7	0.078
Figure 17.14	2.2	4.4	0.091	2.8	0.058

17.7-10 (a) Let state (i,j) denote i calling units in the system, with the calling unit being served at the j^{th} stage of his service. Then the state space is: $\{(0,0), (1,2), (1,1), (2,2), (2,1)\}$. The rate diagram is:



Note this analysis is possible because an Erlang distribution with $1/\mu = 1/4$ and $k = 2$ is equivalent to the sum of two independent exponentials with parameter $1/\mu = 1/8$.
Hence, the steady state equations are:

$8 P_{1,2} = 2 P_{0,0}$

$8 P_{1,1} = 10 P_{1,2}$

$2 P_{0,0} + 8 P_{2,2} = 10 P_{1,1}$

$2 P_{1,2} + 8 P_{2,1} = 8 P_{2,2}$

$2 P_{1,1} = 8 P_{2,1}$

(b) The solution to these equations is:
$(P_{0,0}; P_{1,2}; P_{1,1}; P_{2,2}; P_{2,1}) = (64/114, 16/114, 20/114, 9/114, 5/114)$
Hence $P_0 = \frac{64}{114} = .561$

$P_1 = \frac{16+20}{114} = .316$

$P_2 = \frac{9+5}{114} = .123$

$L = \frac{18+14}{52} = .561$

(c) If the service time is exponential, then the system is an M/M/1 queue limited to K = 2 and with $\lambda = 2$ and $\mu = 4$. So,

$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{(1/2)}{(1-1/8)} = \frac{4}{7} = .571$

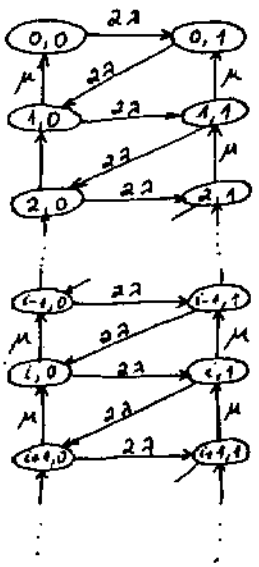
$P_1 = \left(\frac{1}{2}\right) P_0 = \frac{2}{7} = .286$

17.7-10 (CONT'D)

$$P_2 = \left(\frac{1}{2}\right)^2 P_0 = \frac{1}{7} = .143$$

$$L = \frac{2+2}{7} = \frac{4}{7} = .571$$

17.7-11. state is (n, i) where n is the number of customers in the system $(n \geq 1)$ and i is the number of completed arrival stages for currently arriving customer $(i = 0, 1)$.



17.7-12 (a) Let T be the repair time.

$$E(T) = E(T \mid \text{minor repair needed}).(0.9) + E(T \mid \text{major repair needed}).(0.1) = \frac{1}{2}(0.9) + 5(0.1) =$$

$$= .95 \text{ hours}$$

Now let X be a binary random variable with $P(X=1) = p = 0.9$ and $P(X=0) = q = 0.1$, Y_i be an exponential random variable with mean $1/\lambda_i$ $(i = 1, 2)$, with $\frac{1}{\lambda_1} = \frac{1}{2}$ and $\frac{1}{\lambda_2} = 5$. Then we may express T as follows:

$T = Y_1 \cdot X + Y_2(1-X)$ where X, Y_1, Y_2 are independent

To calculate $\sigma^2 = \text{Var}(T)$ we use the formula:

$$\text{Var}(T) = E(\text{Var}(T|X)) + \text{Var}(E(T|X))$$

$$\text{Var}(T|X) = \text{Var}(Y_1).X + \text{Var}(Y_2)(1-X) = \left(\frac{1}{\lambda_1^2}\right)X + \left(\frac{1}{\lambda_2^2}\right)(1-X)$$

$$\therefore E(\text{Var}(T|X)) = p/\lambda_1^2 + q/\lambda_2^2$$

17.7-18 (CONT'D)

$$E(T|X) = E(Y_1)\cdot X + E(Y_2)\cdot(1-X) = \frac{1}{\lambda_1}\cdot X + \frac{1}{\lambda_2}\cdot(1-X)$$

$$= \frac{1}{\lambda_2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)X$$

$$\therefore \mathrm{Var}(E(T|X)) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 \cdot \mathrm{Var}\,X = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq$$

Therefore,

$$\mathrm{Var}(T) = \frac{p}{\lambda_1^2} + \frac{q}{\lambda_2^2} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 pq = 4.5475$$

Now we can see that T has a variance much bigger than that of an exponential random variable with same mean, which would be $(.95)^2 = .9025$

(b) $\left.\begin{matrix}\mu = \frac{1}{.95} \\ \lambda = 1\end{matrix}\right\} \Rightarrow \rho = .95$

Since this is an M/G/1 queue we can apply the following formulas:

$P_0 = 1 - \rho = 1. - .95 = .05$

$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} = \frac{(4.5475)^2 + (.95)^2}{2 \times .05} = 215.82$

$L = \rho + L_q = 216.77$

$W_q = \frac{L_q}{\lambda} = 215.82$

$W = W_q + \frac{1}{\mu} = 216.77$

(c) $W|\text{major repair needed} = W_q + 5 = 220.82$

$W|\text{minor repair needed} = W_q + .5 = 216.32$

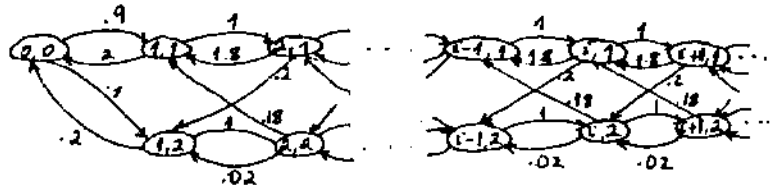
$L_{\text{major repair machines}} = (1)(0.1)(220.82) = 22.082$

$L_{\text{minor repair machines}} = (1)(0.1)(216.32) = 194.69$

(d) state is (n,i) where n is the number of failed machines and i is the type of repair being done on machine under repair ($i=1$ denotes minor repair and $i=2$ denotes major repair).

17.7-12 (CONT'D)

(e)



17.7-13 $\{X_n\}$ is an imbedded Markov chain with states .

And $X_{n+1} = \begin{cases} X_n - 1 + A_{n+1} & X_n \geq 1 \\ A_{n+1} & X_n = 0 \end{cases}$ and $X_{n+1} \leq 3$

A_{n+1} is the number of arrivals in the 10 minutes.

$Pr(A=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = a_n$, and $\lambda t = \frac{60}{50} \cdot \frac{10}{60} = 0.2$

So, the transition matrix is

$$P = \begin{bmatrix} a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ a_0 & a_1 & a_2 & 1-a_0-a_1-a_2 \\ 0 & a_0 & a_1 & 1-a_0-a_1 \\ 0 & 0 & a_0 & 1-a_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.819 & 0.164 & 0.016 & 0.001 \\ 0.819 & 0.164 & 0.016 & 0.001 \\ 0 & 0.819 & 0.164 & 0.017 \\ 0 & 0 & 0.819 & 0.181 \end{bmatrix}$$

b) Run OR courseware, we get

$P_0 = 0.801$, $P_1 = 0.177$, $P_2 = 0.02$, $P_3 = 0.002$

c) $L = P_1 + 2P_2 + 3P_3 = 0.223$

In M/D/1 model, $L^{\infty} = \rho + \frac{\rho^2}{2(1-\rho)} = 0.2 + \frac{0.04}{2(1-0.2)} = 0.225$

So, $L^{\infty} > L$

17.8-1

a) This system is an example of a nonpreemptive priority queueing system.

b)

n =	2	(# of priority classes)
μ =	20	(mean service rate)
s =	1	(# servers)

17.8-1 (CONT'D)

	λ_i	L	Lq	W	Wq
Priority Class 1	2	0.1666667	0.0666667	0.0833333	0.0333333
Priority Class 2	10	1.3333333	0.8333333	0.1333333	0.0833333
$\lambda =$	12				
$\rho =$	0.6				

c) $\frac{W_{q1}}{W_{q2}} = \frac{0.033}{0.083} = 0.4$

d) $\rho(12) = 0.6 = 7.2$ hours

17.8-2

	W_{q1}	L_{q1}	W_1	L_1	W_{q2}	L_{q2}	W_2	L_2
$s=1,\ \mu=10$	.133	.533	.233	.933	.667	2.667	.767	3.067
$s=2,\ \mu=5$	.119	.474	.319	1.274	.593	2.370	.795	3.170

If W_1 is the primary concern, one should choose the first alternative (one fast server). On the other hand, if W_{q1} is the primary concern, one should choose the second alternative (two slow servers)

17.8-3

a)

	u	a	b	W
0			1	
1	2.5	0.16	0.6	0.67
2	3.33	0.25	0.3	1.69
3	5	0.29	0.1	9.87

b)

			B	W
u	3.33	0	1	
r	0.30	1	0.7	0.62
A	4.44	2	0.4	1.10
		3	0.1	5.93

17.8-4 Conclusion: The approximation is not good for W_2 and W_3

$\lambda = 8,\ \lambda_1 = 2,\ \lambda_2 = 4,\ \lambda_3 = 2,\ \mu = 10$

(a) First come, first served: $W = \frac{1}{\mu - \lambda} = \frac{1}{2}$ days

(b) Nonpreemptive:

$A = \frac{\mu^2}{\lambda} = \frac{25}{2}$

$B_1 = 1 - (\lambda_1/\mu) = 4/5$

$B_2 = 1 - (\lambda_1+\lambda_2)/\mu = 2/5$

$B_3 = 1 - \lambda/\mu = 1/5$

So $W_1 = \frac{1}{AB_1} + \frac{1}{\mu} = \frac{1}{5} = .20$ days

$W_2 = \frac{1}{AB_1B_2} + \frac{1}{\mu} = \frac{7}{20} = .35$ days

$W_3 = \frac{1}{AB_2B_3} + \frac{1}{\mu} = \frac{11}{10} = 1.1$ days

(c) Preemptive:

$W_1 = \frac{1/\mu}{B_1} = \frac{1}{8} = .125$ days

$W_2 = \frac{1/\mu}{B_1B_2} = \frac{5}{16} = .3125$ days

$W_3 = \frac{1/\mu}{B_2B_3} = \frac{5}{4} = 1.25$ days

17.8-5

$\lambda_1 = 0.1,\ \lambda_2 = 0.4,\ \lambda_3 = 1.5,\ \lambda = \sum_{i=1}^{3} \lambda_i = 2,\ \mu = 3$

	Preemptive Priorities		Nonpreemptive Priorities	
	s=1	s=2	s=1	s=2
A	...	...	4.5	36
B_1	.967	...	.967	.983
B_2	.833	...	.833	.917
B_3	.333	...	.333	.667
$W_1 - \frac{1}{\lambda}$	.011	.00009	.230	.028
$W_2 - \frac{1}{\lambda}$	.080	.00289	.276	.031
$W_3 - \frac{1}{\mu}$	.867	.05493	.800	.045

17.8-6

a) The expected number of customers wouldn't change since customers of both types have exactly same arrival pattern and service times. The change of the prority wouldn't affect the total service rate from the server's view and thus the total queue size stays the same

b) Run OR courseware, for Nonpreemptive Prority-Disciplie Queueing model, we have

$\mu = 6$ $\quad \lambda_1 = 5$ $\quad \lambda_2 = 5$ $\quad L_1 = 1.37446$ $\quad L_2 = 4.08009$

$s = 2$

$N = 2$ $\quad L_p = L_1 + L_2 = 5.45455$

$W_1 = 0.27489$ $\quad (W_q)_1 = 0.10823$
$W_2 = 0.81602$ $\quad (W_q)_2 = 0.64935$

for M/M/2 queueing system,

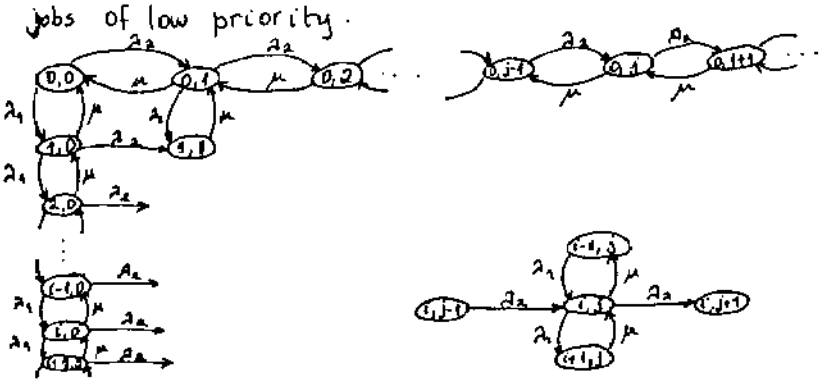
$\lambda = 10$ $\quad L = 5.455$

$\mu = 6$ $\quad L_q = 3.788$ $\quad$ Thus, $L_p = L$.

$s = 2$ $\quad W = 0.545$

17.8-7

Let state (i,j) denote i jobs of high priority and j jobs of low priority.



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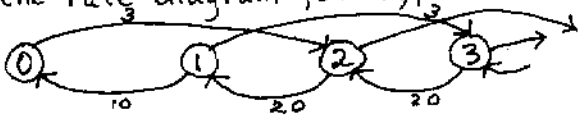
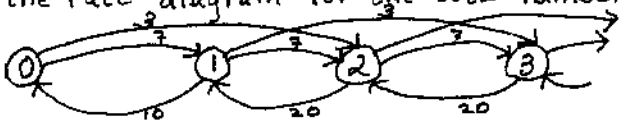
17.8-7 (CONT'D)

State	Rate in = Rate out
$(0,0)$	$\mu(P_{10} + P_{01}) = (\lambda_1 + \lambda_2) P_{00}$
$(i,0)$ for $i \geq 1$	$\mu P_{i+1,0} + \lambda_1 P_{i-1,0} = (\mu + \lambda_1 + \lambda_2) P_{i,0}$
$(0,j)$ for $j \geq 1$	$\mu(P_{1j} + P_{0,j+1}) + \lambda_2 P_{0,j-1} = (\mu + \lambda_1 + \lambda_2) P_{0,j}$
(i,j) for $i,j \geq 1$	$\mu P_{i+1,j} + \lambda_1 P_{i-1,j} + \lambda_2 P_{i,j-1} = (\mu + \lambda_1 + \lambda_2) P_{i,j}$

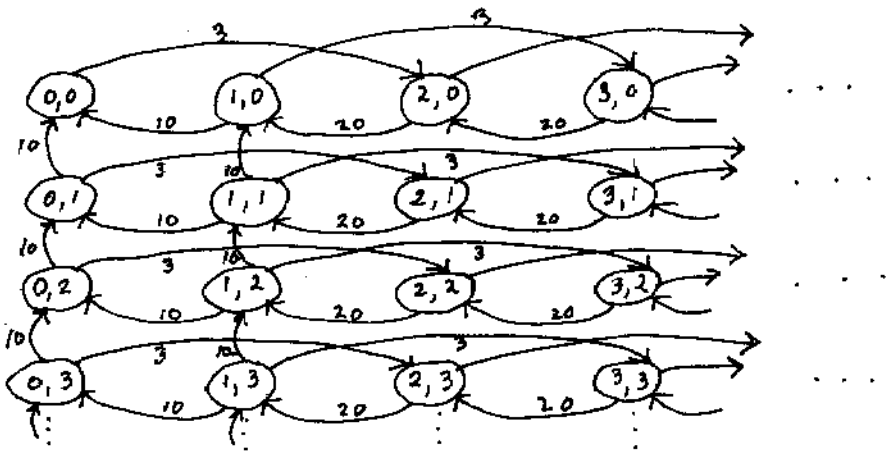
17.9-1. (a) Let the state be n_1 = number of type 1 customers in the system
Then the rate diagram for type 1 customers is:



(b) Let the state be n = number of customers in the system
Then the rate diagram for the total number of customers is:



(c) Let the state be (n_1, n_2) where
n_1 = number of type 1 customers in the system
n_2 = number of type 2 customers in the system
Then the rate diagram is:



17.9-2

(a) $P_{n_1} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1}$

$P_{n_2} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n_2}$

$P\{(N_1, N_2) = (n_1, n_2)\} = P_{n_1} P_{n_2} = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^{n_1}\left(\frac{2}{3}\right)^{n_2}$

(b) $P\{(N_1, N_2) = (0,0)\} = \frac{1}{6}$

(c) $L = L_1 + L_2 = 1 + 2 = 3$

$W = W_1 + W_2 = \frac{1}{10} + \frac{2}{10} = .3$ hour $= 18$ minutes

17.9-3

Facility j	a_j	P_{ij}: $j=1$	$j=2$	$j=3$	$\cdots$	$j=m-1$	$j=m$
$j=1$	λ	0	1	0	$\cdots$	0	0
$j=2$	0	0	0	1	$\cdots$	0	0
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$j=m-1$	0	0	0	0	$\cdots$	0	1
$j=m$	0	0	0	0	$\cdots$	0	0

17.9-4

(a)

Facility j	aj	Pij: i = 1	i = 2	i = 3
j = 1	10	0	0.3	0.4
j = 2	15	0.5	0	0.5
j = 3	3	0.3	0.2	0

Results

$\lambda_1 = 30$
$\lambda_2 = 40$
$\lambda_3 = 20$

(b) $p_{n_1} = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{n_1}$

$p_{n_2} = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n_2}$

$p_{n_3} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n_3}$

$P\{(N_1, N_2, N_3) = (n_1, n_2, n_3)\} = \left(\frac{1}{60}\right)\left(\frac{3}{4}\right)^{n_1}\left(\frac{4}{5}\right)^{n_2}\left(\frac{2}{3}\right)^{n_3}$

(c) $p = (p_0 + p_1)(p_0 + p_1)(p_0 + p_1)$
(facility 1) (facility 2) (facility 3)

$= \left(\frac{7}{16}\right)\left(\frac{9}{25}\right)\left(\frac{5}{9}\right) = \frac{7}{80} = 0.088$

(d) $L = L_1 + L_2 + L_3 = 3 + 4 + 2 = 9$

(e) $W = \dfrac{L}{a_1 + a_2 + a_3} = \dfrac{9}{28} = 0.321$

CASES

17-1 a) Status quo at the presses – 7.517 sheets of in-process inventory.

Data			Results	
λ =	7	(mean arrival rate)	L =	7.51737284
μ =	1	(mean service rate)	L_q =	0.51737284
s =	10	(# servers)		

Status quo at the inspection station – 7 wing sections of in-process inventory.

Data			Results	
λ =	7	(mean arrival rate)	L =	7
μ =	8	(mean service rate)	L_q =	6.125
s =	1	(# servers)		

Inventory cost = (7.517 + 7)(8) = $116.14 / hour
Machine cost = (10)(7) = $70 / hour
Inspector cost = $17 / hour

Total cost = $203.14 / hour

b) Proposal 1 will increase the in-process inventory at the presses to 11.147 sheets since the mean service rate has decreased.

Data			Results	
λ =	7	(mean arrival rate)	L =	11.0474066
μ =	0.6333333	(mean service rate)	L_q =	2.64740664
s =	10	(# servers)		

The in-process inventory at the inspection station will not change.

Inventory cost = (11.147 + 7)(8) = $145.18 / hour
Machine cost = (10)(6.5) = $65 / hour
Inspector cost = $17 / hour

Total cost = $227.18 / hour

This total cost is higher than for the status quo so should not be adopted. The main reason for the higher cost is that slowing down the machines won't change in-process inventory for the inspection station.

c) Proposal 2 will decrease the in-process inventory at the inspection station to 4.148 wing sections since the mean service rate has increased.

Data			Results	
λ =	7	(mean arrival rate)	L =	4.1475
μ =	8.3333333	(mean service rate)	L_q =	3.3075
k =	2	(shape parameter)		
s =	1	(# servers)	W =	0.5925
			W_q =	0.4725

The in-process inventory at the presses will not change.

Inventory cost = (7.517 + 4.148)(8) = $90.44 / hour
Machine cost = (10)(7) = $70 / hour
Inspector cost = $19 / hour

Total cost = $179.44 / hour

(CONT'D)

This total cost is lower than for the status quo so should be adopted. The main reason for the lower cost is the increase in the service rate and the resulting decrease in the in-process inventory.

d) As seen above, proposal 2 should be adopted to reduce the average level of in-process inventory at the inspection station from 7 to 4.148 wing sections. Inventory cost will be reduced by $25.70 / hour while inspector costs only increase $2 / hour for a total savings of $23.70 / hour.

18.2-1

	Service Costs	Waiting Costs
(a)	Salaries of checkers, cost of cash registers	Lost profit from lost business
(b)	Salaries of firemen, cost of fire trucks	Expected $ destruction due to waiting
(c)	Salaries of toll-takers, cost of constructing toll lane	Social cost of waiting by commuters
(d)	Salaries of repairpersons, cost of tools	Lost profit from lost business
(e)	Salaries of Longshoremen, cost of equipment	Lost profit from ships not loaded or unloaded
(f)	Salary of an operator as a function of their experience	Lost profit from lost productivity of unused machines
(g)	Salaries of operators, cost of equipment	Lost profit due to lost productivity of materials not handled
(h)	Salaries of plumbers, cost of tools	Lost profit from lost business
(i)	Salaries of employees, cost of equipment	Lost profit from lost business
(j)	Salaries of typists, cost of typewriters	Lost profit due to typing jobs being unfinished

18-3-1.

$s = 1, \lambda = 2, \mu = 4 \Rightarrow \rho = 1/2 \Rightarrow P_n = (1/2)^{n+1}$ and $f_n(t) = 2e^{-2t}$

The answers to (a) through (d) are based on the following identities:

(i) $\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$ if $|x|<1$

(ii) $\sum_{n=0}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x|<1$

(iii) $\sum_{n=0}^{\infty} n^3 x^n = \frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$ if $|x|<1$

(iv) $\int_0^b x e^{-\alpha x} dx = \frac{1}{\alpha^2}(1 - e^{-\alpha b} - \alpha b e^{-\alpha b})$

(v) $\int_0^{\infty} x e^{-\alpha x} dx = \frac{1}{\alpha^2}$

(vi) $\int_0^{\infty} x^2 e^{-\alpha x} dx = \frac{b^2}{\alpha} e^{-\alpha b} + \frac{2b}{\alpha^2} e^{-\alpha b} + \frac{2}{\alpha^3} e^{-\alpha b}$

(vii) $\int_0^{\infty} x^3 e^{-\alpha x} dx = \frac{6}{\alpha^4}$

(a) $E[WC] = \sum_{n=0}^{\infty} (10n + 2n^2) P_n = 10 \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^{n+1} + 2 \sum_{n=0}^{\infty} n^2 \left(\frac{1}{2}\right)^{n+1}$

$= 5 \sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} n^2 \left(\frac{1}{2}\right)^n$

18.3-1. (continued)

$$= 5\left(\frac{\frac{1}{2}}{(1-\frac{1}{2})^2}\right) + \left(\frac{2(\frac{1}{2})^2}{(1-\frac{1}{2})^3} + \frac{\frac{1}{2}}{(1-\frac{1}{2})^2}\right) \qquad \text{by (i) and (ii)}$$

$$= 16$$

(b)

$$E[WC] = \lambda E[h(w)] = 2\int_0^\infty (25w + w^3)(2e^{-2w})\,dw$$

$$= 100\int_0^\infty w e^{-2w}\,dw + 4\int_0^\infty w^3 e^{-2w}\,dw \quad = \text{by (v) and (vii)}$$

$$= 100 \cdot \frac{1}{2^2} + 4 \cdot \frac{6}{2^4} = 26.5$$

18.3-2 (a)

$$E[WC] = \sum_{n=0}^{2} 10n\left(\frac{1}{2}\right)^{n+1} + \sum_{n=3}^{5} 6n^2\left(\frac{1}{2}\right)^{n+1} + \sum_{n=6}^{\infty} n^3\left(\frac{1}{2}\right)^{n+1}$$

$$= 10\left(\frac{1}{4}\right) + 20\left(\frac{1}{8}\right) + 54\left(\frac{1}{16}\right) + 96\left(\frac{1}{32}\right) + 150\left(\frac{1}{64}\right) + \sum_{n=6}^{\infty} n^3\left(\frac{1}{2}\right)^{n+1}$$

$$= 20 + \frac{419}{128} = 23.273$$

(b)

$$E[WC] = 2\int_0^1 w(2e^{-2w}\,dw + 2\int_1^\infty w^2(2e^{-2w})\,dw$$

$$= 4\left[\frac{1}{2^2}(1 - e^{-2} - 2e^{-2})\right] + 4\left[\frac{e^{-2}}{2} + \frac{2e^{-2}}{2^2} + \frac{2e^{-2}}{2^3}\right]$$

$$= 1 - 3e^{-2} + 5e^{-2} = 1 + 2e^{-2} = 1.271$$

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18.4-1

a) 1 server is optimal.

Data

λ =	8	(mean arrival rate)
μ =	10	(mean service rate)
s =	1	(# servers)

$Pr(\omega>t)$ =	0.1353353
when t =	1

$Prob(\omega_q>t)$ =	0.1082682
when t =	1

C_s =	100	(cost/server/unit time)
C_w =	10	(waiting cost/unit time)
Cost of Service =	100	
Cost of Waiting =	40	
Total Cost =	140	

Results

L =	4
L_q =	3.2
W =	0.5
W_q =	0.4
ρ =	0.8
P_0 =	0.2
P_1 =	0.16
P_2 =	0.128
P_3 =	0.1024
P_4 =	0.08192
P_5 =	0.065536
P_6 =	0.0524288

b) 2 servers are optimal.

Data

λ =	8	(mean arrival rate)
μ =	10	(mean service rate)
s =	2	(# servers)

$Pr(\omega>t)$ =	0.0024788
when t =	1

$Prob(\omega_q>t)$ =	0.0009915
when t =	1

C_s =	100	(cost/server/unit time)
C_w =	100	(waiting cost/unit time)
Cost of Service =	200	
Cost of Waiting =	95.238095	
Total Cost =	295.2381	

Results

L =	0.95238095
L_q =	0.15238095
W =	0.11904762
W_q =	0.01904762
ρ =	0.4
P_0 =	0.42857143
P_1 =	0.34285714
P_2 =	0.13714286
P_3 =	0.05485714
P_4 =	0.02194286
P_5 =	0.00877714
P_6 =	0.00351086

c) 3 servers are optimal.

Data		
λ =	8	(mean arrival rate)
μ =	10	(mean service rate)
s =	3	(# servers)

Pr(ω>t) =	0.0006534
when t =	1

Prob(ω_q>t) =	0.0001742
when t =	1

C_s =	10	(cost/server/unit time)
C_w =	100	(waiting cost/unit time)
Cost of Service =	30	
Cost of Waiting =	81.892092	
Total Cost =	111.89209	

Results	
L =	0.81892092
L_q =	0.01892092
W =	0.10236511
W_q =	0.00236511
ρ =	0.26666667
P_0 =	0.44715447
P_1 =	0.35772358
P_2 =	0.14308943
P_3 =	0.03815718
P_4 =	0.01017525
P_5 =	0.0027134
P_6 =	0.00072357

18.4-2

Jim should operate 4 cash registers during the lunch hour.

Data		
λ =	66	(mean arrival rate)
μ =	30	(mean service rate)
s =	4	(# servers)

Pr(ω>t) =	1.371E-06
when t =	1

Prob(ω_q>t) =	7.54E-07
when t =	1

C_s =	9	(cost/server/unit time)
C_w =	18	(waiting cost/unit time)
Cost of Service =	36	
Cost of Waiting =	44.589575	
Total Cost =	80.589575	

Results	
L =	2.4771986
L_q =	0.2771986
W =	0.03753331
W_q =	0.00419998
ρ =	0.55
P_0 =	0.104562
P_1 =	0.2300364
P_2 =	0.25304004
P_3 =	0.1855627
P_4 =	0.10205948
P_5 =	0.05613272
P_6 =	0.03087299

18.4-3

Garrett-Tompkins should have 6 copiers.

Data

λ =	30	(mean arrival rate)
μ =	12	(mean service rate)
s =	6	(# servers)

Pr(ω>t) =	0.0009119
when t =	1

Prob(ω_q>t) =	0.00038
when t =	1

C_s =	1.5	(cost/server/unit time)
C_w =	25	(waiting cost/unit time)
Cost of Service =	9	
Cost of Waiting =	63.347229	
Total Cost =	72.347229	

Results

L =	2.53388915
L_q =	0.03388915
W =	0.08446297
W_q =	0.00112964
ρ =	0.41666667
P_0 =	0.08162026
P_1 =	0.20405065
P_2 =	0.25506331
P_3 =	0.21255276
P_4 =	0.13284547
P_5 =	0.06642274
P_6 =	0.02767614

18.4-4

$\lambda = 4,\ \mu = 5,\ c_s = 20$ and $g(n) = \begin{cases} 0 & \text{for } N = 0 \\ 120 & \text{for } N = 1 \\ 120 + 180(N-1) & \text{for } N \geq 2 \end{cases}$

$$E[WC] = \sum_{n=0}^{\infty} g(n) P_n = 120 \sum_{n=1}^{\infty} P_n + 180 \sum_{n=2}^{\infty} N \cdot P_n - 180 \sum_{n=2}^{\infty} P_n$$

$$= 120(1 - P_0) + 180[L - P_1] - 180[1 - P_0 - P_1] = 60 P_0 + 180 L - 60$$

s	$\rho = 4/55$	P_0	L	E[WC]	E[SC]	E[TC]
1	.8	.2	4.0	672.00	20.0	692.00
2	.4	.43	.95	136.80	40.0	176.80
3	.267	.45	.82	114.60	60.0	174.60
4	.2	.44	.80	110.40	80.0	190.40

Hence, $s^* = 3$ and E[TC] = \$ 174.60 per hour.

18.4-5

(a) Model 2, with s fixed (equal to 1), A = {40, 60},

$f(\mu) = \begin{cases} 10 & \text{for } \mu = 40 \\ 15 & \text{for } \mu = 60 \end{cases}, \lambda = 30$

That is, we must choose between a slow server (cashier) and a fast server (cashier + box boy)

(b) $E[WC] = \lambda E[h(W)] = \lambda E[(.05) W] = \lambda (.05) W = (.05) L = (.05) \cdot \dfrac{\lambda}{\mu - \lambda}$

Therefore we have

μ	sf(μ)	E[WC]	E[TC]	
40	10	.15	10.15	← minimum
60	15	.05	15.05	

Hence the status quo should be maintained.

18.4-6

$\lambda = 30$, $\mu_x = 50$, $\mu_y = 40$. Thus, $\rho_x = 0.6$, $\rho_y = 0.75$

$EW = \frac{1}{\mu - \lambda}$, $EW_x = \frac{1}{20}$ hour = 3 minutes, $EW_y = \frac{1}{10}$ hour = 6 minutes

Then, compensation difference $\leq 6000\left(\frac{1}{3} - \frac{1}{6}\right) = 1000$

18.4-7

Data

λ =	4	(mean arrival rate)
μ =	5	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	0.3678794
when t =	1

Prob(ω_q>t) =	0.2943036
when t =	1

P0+P1+P2+P3+P4= 0.67232

Results

L =	4
L_q =	3.2
W =	1
W_q =	0.8
ρ =	0.8
P_0 =	0.2
P_1 =	0.16
P_2 =	0.128
P_3 =	0.1024
P_4 =	0.08192

The train does not meet any of the criteria.

b)

Data

λ =	4	(mean arrival rate)
μ =	6.67	(mean service rate)
s =	1	(# servers)

Pr(ω>t) =	0.0692522
when t =	1

Prob(ω_q>t) =	0.0415306
when t =	1

P0+P1+P2+P3+P4= 0.922434109

Results

L =	1.49812734
L_q =	0.89842719
W =	0.37453184
W_q =	0.2246068
ρ =	0.59970015
P_0 =	0.40029985
P_1 =	0.24005988
P_2 =	0.14396395
P_3 =	0.0863352
P_4 =	0.05177523

The forklift truck meets all the criteria.

c) Tractor-trailer train: L($20)+$50 = $130/hour
Forklift truck: L($20)+$150 = $180/hour

d) The forklift truck should be chosen.

18.4-8

a) L = 1.5

$$W = \frac{L}{\lambda} = \frac{1.5}{0.2} = 7.5$$

$$W_q = W - \frac{1}{\mu} = 7.5 - \frac{1}{0.167} = 1.5$$

$$L_q = \lambda W_q = 0.2(1.5) = 0.3$$

18.4-8 (cont'd)

b)

Data		
λ =	0.2	(mean arrival rate)
μ =	0.3333333	(mean service rate)
s =	1	(# servers)

Results	
L =	1.05
L_q =	0.45
W =	5.25
W_q =	2.25

c) TC(Alternative 1) = \$70 + (\$100)(L) = \$220
TC(Alternative 2) = \$100 + (\$100)(L) = \$205

Alternative 2 should be adopted.

18.4-9 For alternative A, the system has Poisson input with λ = 1/45, Erlang service with k = 2 and μ = 1/36 and s = 2. For Alternative B, the system has Poisson input with λ = 1 / 45, Erlang service with k = 2 and μ = 1 / 18 and s = 1. Planes arrive at the rate of one every 45 hours or, equivalently, 8760 / 45 ≈ 195 arrivals per year. Since each plane goes to the maintenance shop five times a year, there are approximately 195/5 = 39 planes.
So, for Alternative A, E [SC] = \$100,000 x 39 / 8760 = \$1780.82 per hour
and for Alternative B, E [SC] = \$550,000 x 39 / 8760 = \$2448.63 per hour.

Also for Alternative A, $\rho = \dfrac{\left(\frac{1}{45}\right)}{(2)\left(\frac{1}{36}\right)} = \dfrac{2}{5} \Rightarrow L = .91$ from Fig. 18.13

and for Alternative B, $L = \left(\dfrac{1+k}{2k}\right)\left(\dfrac{\lambda^2}{\mu(\mu-\lambda)}\right) + \dfrac{\lambda}{\mu} = \dfrac{3}{4}\left[\dfrac{\left(\frac{1}{45}\right)^2}{\left(\frac{1}{18}\right)\left(\frac{1}{18}-\frac{1}{45}\right)}\right] + \dfrac{\left(\frac{1}{45}\right)}{\left(\frac{1}{18}\right)} = \dfrac{3}{5}$

Alternative	E[SC]	E[WC] = 3000L	E[TC]
A	1780.82	2730.00	4510.82
B	2448.63	1800.00	4248.63

So choose Alternative B with E [TC] = \$4248.63 per hour.

18.4-10

a)

Data		
λ =	0.05	(mean arrival rate)
$1/\mu$ =	15	(expected service time)
σ =	15	(standard deviation)
s =	1	(# servers)

Results	
L =	3
L_q =	2.25
W =	60
W_q =	45

b)

Data		
λ =	0.05	(mean arrival rate)
$1/\mu$ =	16	(expected service time)
σ =	9.49	(standard deviation)
s =	1	(# servers)

Results	
L =	2.96287563
L_q =	2.16287563
W =	59.2575125
W_q =	43.2575125

c) The new proposal shows that they will be slightly better off if they switch to the new queueing system.

d) TC (status quo) = \$40 + (L_q)(\$20) = \$85/hour
TC (proposal) = \$40 + (L_q)(\$20) = \$83/hour

18.4-11

a) $L = 2$

$$W = \frac{L}{\lambda} = \frac{2}{0.3} = 6.67$$

$$W_q = W - \frac{1}{\mu} = 6.67 - 5 = 1.67$$

$$L_q = \lambda W_q = 0.3(1.67) = 0.5$$

b)

Data		
λ =	0.3	(mean arrival rate)
1/μ =	3	(expected service time)
σ =	1.19	(standard deviation)
s =	1	(# servers)

Results	
L =	5.587245
L_q =	4.687245
W =	18.62415
W_q =	15.62415

c) TC(Alternative 1) = \$3000 + (\$150)(L) = \$3,300
TC(Alternative 2) = \$2750 + (\$150)(L) = \$3,589

Alternative 1 should be adopted.

18.4-12 For the status quo, the system has Poisson input with $\lambda = 15$, exponential service with $\mu = 15$, $s = 1$ and limited waiting room with $k = 4$. There is a waiting cost of 6 W_q for each customer due to loss of good will and also a waiting cost of \$45 per hour when the system is full (4 cars in system) due to loss of potential customers. So

$E[TC] = E[WC] = \lambda 6 W_q + 45 P_4 = 6 L_q + 45 P_4$.

In this case, $\frac{\lambda}{\mu} = 1$, so

$$P_0 = \frac{1}{k+1}, \quad P_n = \frac{1}{k+1} \quad n = 1, 2, \ldots, k \quad \therefore P_4 = \frac{1}{5}$$

$$L = \sum_{n=1}^{k} n P_n = \frac{1}{k+1} \sum_{n=1}^{k} n = \frac{k(k+1)}{2(k+1)} = \frac{k}{2}$$

$$L_q = L - (1 - P_0) = \frac{k}{2} - 1 + \frac{1}{k+1} = \frac{4}{2} - 1 + \frac{1}{5} = \frac{6}{5}$$

Thus, $E[TC] = 6(6/5) + 45(1/5) = \$16.20$ per hour.

For Proposal 1, the system has Poisson input with $\lambda = 15$, exponential service with $\mu = 20$ and $s = 1$. In addition to the waiting cost of 6 L_q due to los of good will, there is an expected waiting cost of \$2 per customer that waits longer than 1/2 hour before his car is ready. The expected value of this additional waiting cost is given by

$$2\lambda P\{W > 1/2\} = 2\lambda e^{-\mu(1-\rho)/2} = 30 e^{-2.5} = 2.46.$$

Also, $6 L_q = \frac{6\lambda^2}{\mu(\mu - \lambda)} = \frac{6 \times 225}{20 \times 5} = 13.50$

Hence, $E[TC] = 3 + 13.50 + 2.46 = 18.96$ per hour (where 3 is the capitalized cost of the new equipment.)

For Proposal 2, the system has a Poisson input with $\lambda = 15$, Erlang service with $\mu = 30$ and $k = 2$, and $s = 1$. The only waiting cost is 6 L_q due to loss of good will. In this case,

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu - \lambda)}\right) = \left(\frac{3}{4}\right)\left(\frac{225}{30 \times 15}\right) = \frac{3}{8} = 0.375.$$ Hence, $E[TC] = 10 + 2.25 = 12.25$.

Summarizing, Status Quo: $E[TC] = 0 + 6L_q + 45 P_4 = 16.20$

Proposal 1: $E[TC] = 3 + 6 L_q + 30 P\{W \geq 1/2\} = 18.96$

Proposal 2: $E[TC] = 10 + 6 L_q = 10 + 2.25 = 12.25$.

Hence, Proposal 2 should be adopted.

18.4-13

a) The customers are trucks to be loaded or unloaded and the servers are crews. The system currently has 1 server.

b)

Data		
λ =	1	(mean arrival rate)
μ =	4	(mean service rate)
s =	1	(# servers)

Results	
L =	0.33333333
L_q =	0.08333333
W =	0.33333333
W_q =	0.08333333
ρ =	0.25
P_0 =	0.75

Pr(ω>t) =	0.0497871
when t =	1

Prob(ω_q>t) =	0.0124468
when t =	1

c)

Data		
λ =	1	(mean arrival rate)
μ =	3	(mean service rate)
s =	1	(# servers)

Results	
L =	0.5
L_q =	0.16666667
W =	0.5
W_q =	0.16666667
ρ =	0.33333333
P_0 =	0.66666667

Pr(ω>t) =	0.1353353
when t =	1

Prob(ω_q>t) =	0.0451118
when t =	1

d)

Data		
λ =	1	(mean arrival rate)
μ =	2	(mean service rate)
s =	1	(# servers)

Results	
L =	1
L_q =	0.5
W =	1
W_q =	0.5
ρ =	0.5
P_0 =	0.5

Pr(ω>t) =	0.3678794
when t =	1

Prob(ω_q>t) =	0.1839397
when t =	1

e) A one person team should not be considered since that would lead to a utilization factor of ρ=1 which is not permitted in this model.

f &g) Total cost = (\$20)(# on crew)+(\$30)(L_q)
TC(4 members) = (20)(4)+(30)(0.0833) = \$82.50/hour
TC(3 members) = (20)(3)+(30)(0.167) = \$65/hour
TC(2 members) = (20)(2)+(30)(0.5) = \$55/hour

A crew of 2 people will minimize the expected total cost per hour.

18.4-13

h)

s	$\mu_s=\sqrt{s}$	$L=\frac{\lambda}{\mu_s-\lambda}$	E[WC]=15L	E[SC]=10s	E[TC]
1	1.0	∞	∞	10	∞
2	1.414	2.414	36.21	20	56.21
3	1.732	1.366	20.49	30	50.49
4	2.0	1.000	15.00	40	55.00
5	2.236	.809	13.75	50	63.75

Since clearly E[SC] > 50.49 for s > 6, it follows that $s^* = 3$.

18.4-14

$\lambda = 4$, $\mu = 6n$, $EN = \frac{\lambda}{\mu-\lambda} = \frac{4}{6n-4}$

Hourly cost = $C(n) = 18n + 20EN = 18n + \frac{80}{6n-4}$

Easy to see $C(n)$ is convex. Enumerate $C(n)$ at $n=1, 2, 3, \cdots$

We can easily find that $C(n)$ attain its minimum (when n is integer) at $n=2$. So, two loaders would minimize the expected hourly cost.

18.4-15

$\lambda = 3$, $ET = \frac{1}{\mu-3}$

Expect cost = $C(\mu) = 5\mu + 60ET\lambda = 5\mu + \frac{60\cdot 3}{\mu-3}$

$C'(\mu) = 5 - \frac{180}{(\mu-3)^2}$ So when $\mu = 9$, $C(\mu)$ attains its minimum

Then, the hourly wage of 45$ per hour would minimize the total cost

18.4-16

a) For $\mu=2$:

$\lambda=0.5$ $\mu=2$ $\rho=0.25$ $s=1$ $P(W>5)=.000553$ $P(W_Q>5)=.000138$

$L=1/3$ $L_Q=.083$ $W=.67$ $W_Q=.17$ $P_0=0.75$

$\mu=1$:

$\lambda=0.5$ $\mu=1$ $s=1$ $\rho=0.5$ $P(W>5)=.082$ $P(W_Q>5)=.041$

$L=1$ $L_Q=0.5$ $W=2$ $W_Q=1$ $P_0=0.5$

$\mu=2/3$:

$\lambda=0.5$ $\mu=2/3$ $s=1$ $\rho=0.75$ $P(W>5)=.435$ $P(W_Q>5)=.326$

$L=3$ $L_Q=2.25$ $W=6$ $W_Q=4.5$ $P_0=0.25$

(CONT'D)

18.4-16

b) $TC(\text{mean}=0.5) = 1.60 + 0.8 \times (1/3) = 1.87$

$TC(\text{mean}=1) = 0.40 + 0.8 \times 1 = 1.20$

$TC(\text{mean}=1.5) = 0.20 + 0.8 \times 3 = 2.60$

} So $\mu^* = 1$

18.4-17

Given that $s=1$ (by the Optimality of the Single Server result), $E[TC] = C_r \mu + C_w L = C_r \mu + C_w \left(\frac{\lambda}{\mu - \lambda}\right)$

$$\frac{d\,E[TC]}{d\,\mu} = C_r - \frac{C_w \lambda}{(\mu-\lambda)^2} = 0 \Rightarrow (\mu-\lambda)^2 = \frac{\lambda C_w}{C_r}$$

$$\Rightarrow \mu = \lambda + \sqrt{\lambda C_w / C_r}$$

$$\frac{d^2 E[TC]}{d\,\mu^2} = \frac{2\lambda C_w}{(\mu-\lambda)^3} > 0 \text{ for all } \mu > \lambda.$$

Hence, $\mu = \lambda + \sqrt{\lambda C_w / C_r}$ is the minimizing value.

18.4-18 a) Option 1:

Data		
$\lambda =$	150	(mean arrival rate)
$\mu =$	60	(mean service rate)
$s =$	3	(# servers)

Results	
$L =$	6.01123596
$L_q =$	3.51123596
$W =$	0.04007491
$W_q =$	0.02340824

Option 2:

Data		
$\lambda =$	150	(mean arrival rate)
$\mu =$	180	(mean service rate)
$s =$	1	(# servers)

Results	
$L =$	5
$L_q =$	4.16666667
$W =$	0.03333333
$W_q =$	0.02777778

b) W and L are smaller for Option 2 because it is a more efficient system. This is true because when there are only 1 or 2 customers in the system Option 2 is operating at full efficiency, while Option 1 will have idle servers. W_q and L_q are larger for Option 2 because there are fewer people in service (only 1 register) and therefore more people in line.

c) W should be the most important measure to customers since they should be most concerned with the total time spent in the system. Given this, Option 2 is better.

18.4-19 $E[TC] = D\mu + \lambda C/(\mu-\lambda)^2$

$$\frac{d\,E[TC]}{d\,\mu} = D - \frac{2\lambda C}{(\mu-\lambda)^3} = 0 \Rightarrow (\mu-\lambda)^3 = \frac{2\lambda C}{D} \Rightarrow \mu = \lambda + \left(\frac{2\lambda C}{D}\right)^{1/3}$$

Also, $\frac{d^2 E[TC]}{d\,\mu^2} = \frac{6\lambda C}{(\mu-\lambda)^4} > 0$ when $C > 0$

Hence, $\mu = \lambda + \left(\frac{2\lambda C}{D}\right)^{1/3}$ is the minimizing value.

18.4-20

(a) Original design would give a smaller number (expected) of customers in the system. This is due to the pooling effect of multiple servers.

(b) Original design is a $M/M/2$ queue where $\lambda = 5$, $\mu = 6$. Run ProModel, we find $L = 1.1$ from Figure 17.7. Alternative design is two $M/M/1$ queue, under which $L = \frac{2\lambda}{\mu - \lambda} = 10$. This has varified what we claimed in (a).

18.4-21

(a) Part (a) of Problem 17.6-33 is a special case of Model 3, in which we want to determine λ (since determining the number of machines assigned to one operator is equivalent to determining λ, the mean arrival rate) and s is fixed (equal to 1 in this case).

(b) (i) the resulting system would be a M/M/s with finite calling population (size of population is equal to the total number of machines) and the associated decision problem fits Model 1 (unknown s).

(ii) the resulting system is a collection of independent M/M/1 with finite calling populations and the appropriate decision model is a combination of Models 2 and 3 since in this case we have to determine μ (the service rate of a crew depends on how many operators are assigned to the crew) and λ (the mean arrival rate depends on the number of machines assigned to a crew) and s is fixed equal to 1.

(iii) the resulting system doesn't fit any of the models described in Chapter 15

Each of the proposed alternatives allows, in varied degrees, the sharing of resources (the operators) in contrast to the original proposal.

(CONT'D)

(CONT'D)

Now, since in the original proposal the operators would be idle most of the time, it is reasonable to expect that if we allow interaction, the production rate obtained with the same number of operators will increase, and thus, the required number of operators that are needed to achieve the desired production rate will decrease.
What might prevent this from happening?
In alternatives (i) and (iii), the travel time (not taken into account in the above argument) may pose a problem (instead of turning idle time into service time we may be turning idle time into travel time). Also, in alternative (iii) the service rate of a group of workers may be smaller than the individual service rate, since they won't be working together regularly. And finally, although this is not the case in Alternative (ii) (the components of a crew do work together regularly) even then we may have that $\mu' < n\mu$ (where μ' is the service rate of crew of n operators and μ is the individual service rate).

18.4-22 a) & b)

Data		
$\lambda =$	20	(mean arrival rate)
$\mu =$	30	(mean service rate)
s =	1	(# servers)
K =	2	(max customers)

Results	
L =	0.73684211
L_q =	0.21052632
W =	0.04666667
W_q =	0.01333333
ρ =	0.66666667
P_0 =	0.47368421
P_1 =	0.31578947
P_2 =	0.21052632

(CONT'D)

a) & b) (cont'd)

Data		
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	3	(max customers)

Results	
L =	1.01538462
L_q =	0.43076923
W =	0.05789474
W_q =	0.0245614
ρ =	0.66666667
P_0 =	0.41538462
P_1 =	0.27692308
P_2 =	0.18461538
P_3 =	0.12307692

Data		
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	4	(max customers)

Results	
L =	1.24170616
L_q =	0.62559242
W =	0.06717949
W_q =	0.03384615
ρ =	0.66666667
P_0 =	0.38388626
P_1 =	0.25592417
P_2 =	0.17061611
P_3 =	0.11374408
P_4 =	0.07582938

Data		
λ =	20	(mean arrival rate)
μ =	30	(mean service rate)
s =	1	(# servers)
K =	5	(max customers)

Results	
L =	1.42255639
L_q =	0.78796992
W =	0.07472354
W_q =	0.04139021
ρ =	0.66666667
P_0 =	0.36541353
P_1 =	0.24360902
P_2 =	0.16240602
P_3 =	0.10827068
P_4 =	0.07218045
P_5 =	0.0481203

c)

spaces	rate customers are lost (P_k)	change in P_k	profit/hour $(\$4)(\lambda)(1-P_k)$	change in profit/hour
2	0.21		\$63.20	
3	0.12	0.09	\$70.40	\$7.20
4	0.08	0.04	\$73.60	\$3.20
5	0.05	0.03	\$76.00	\$2.40

d) Since it cost \$200 per month per car length rented, each additional space must bring at least S200 per month (or \$1 per hour) in additional profit. Five spaces still bring more than that so 5 should be provided.

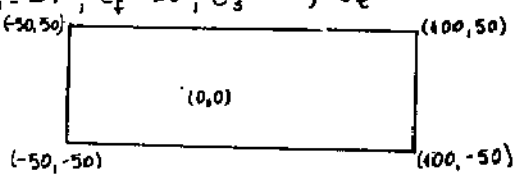
18.4-23 $a = b = c = d = 300$ and $v = 3$ miles/hour $= 264$ feet/min

So $E[T] = \frac{1}{264}\left[\frac{(300)^2+(300)^2}{(300+300)} + \frac{(300)^2+(300)^2}{(300+300)}\right] =$

$= 600/264$ minutes $= 2.27$ minutes.

18.4-24 $\mu = 30$, $s = 1$, $\lambda_p = 24$, $C_f = 20$, $C_s = 15$, $C_t = 25$.

$n = 1$: $\lambda = \frac{\lambda_p}{n} = 24$



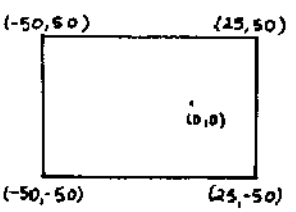
So $a = 50$, $b = 50$, $c = 100$, $d = 50$

$\therefore E(T) = \frac{1}{5,000}\left(\frac{50^2+100^2}{50+100} + \frac{50^2+50^2}{50+50}\right) = .0267$ hrs.

Also, $L = \frac{\lambda}{\mu-\lambda} = \frac{24}{30-24} = 4$

$n = 2$: $\lambda = \frac{\lambda_p}{n} = 12$

Relabelling symmetric areas,



So $a = 50$, $b = 50$, $c = 25$, $d = 50$

$E(T) = \frac{1}{5000}\left(\frac{50^2+25^2}{50+25} + \frac{50^2+50^2}{50+50}\right) = .0183$ hrs.

Also, $L = \frac{\lambda}{\mu-\lambda} = \frac{12}{30-12} = \frac{2}{3}$

Hence, $E(TC) = n\left[(C_f + C_s) + C_t L + \frac{24}{n} C_t E(T)\right]$

n	λ	$E[T]$	L	C_f+C_s	$C_t L$	$\lambda C_t E(T)$	$E[TC]$
1	24	.0267	4	35	100	16	151
2	12	.0183	2/3	35	50/3	5.5	114.33

So there should be two Facilities.

18.4-25 From Table 17.4 and adjacent text

	s = 1	s = 2
w_1 - 1/μ	.024	.00037
w_2 - 1/μ	.154	.00793
w_3 - 1/μ	1.033	.06542

Note that $\lambda_1 = .2$, $\lambda_2 = .6$ and $\lambda_3 = 1.2$

s	E[waiting costs]				E[SC]	E[TC]
	critical	serious	stable	total		
1	480.00	92.40	12.40	584.80	40.00	624.80
2	7.40	4.76	0.79	12.95	80.00	92.95

So there should be two doctors.

18.4-26. The system is a non-preemptive queueing system with $\mu = 4$, $\lambda_1 = 6$, $\lambda_2 = 4$ and $\lambda_3 = 2$, where Class 1 is government jobs, Class 2 is commercial jobs and Class 3 is standard jobs. The problem is to determine whether $s = 4$ or $s = 5$.

For $s = 4$, $A = 31.41$, $B_0 = 1$, $B_1 = 5/8$, $B_2 = 3/8$, $B_3 = 1/4$

$W_1 = .30094$

$W_2 = .38585$

$W_3 = .58962$

For $s = 5$, $A = 52.35$, $B_0 = 1$, $B_1 = 7/10$, $B_2 = 1/2$, $B_3 = 2/5$

$W_1 = .27729$

$W_2 = .30458$

$W_3 = .34552$

s	E[WC]				E[SC]	E[TC]
	Government	Commercial	Standard	Total		
4	6771.15	3472.65	884.43	11128.23	5000	16128.23
5	6239.02	2741.22	518.28	9498.52	6250	15748.52

So two additional lathes should be obtained.

CASES

18-1 The operations of the records and benefits call center can be modeled as an M/M/s queueing system. We, therefore, use the template for the M/M/s queueing model throughout this case. The mean arrival rate equals 70 per hour, and the mean service rate of every representative equals 6 per hour. Mark needs at least s = 12 representatives answering phone calls to ensure that the queue does not grow indefinitely.

a) In order to solve this problem we have to determine the number of servers by "trial and error" until we find a number s such that the probability of waiting more than 4 minutes in the queue is above 35%.

For 13 servers we obtain the following results:

Template for M/M/s Queueing Model

Data			Results	
l =	70	(mean arrival rate)	L =	17.07963527
m =	6	(mean service rate)	L_q =	5.4129686
s =	13	(# servers)		
			W =	0.24399479
Pr(w>t) =	0.825608		W_q =	0.077328123
when t =	0.066667			
			r =	0.897435897
Prob(w_q>t) =	0.362914			
when t =	0.066667		P_0 =	5.32592E-06
			P_1 =	6.21358E-05
			P_2 =	0.000362459
			P_3 =	0.001409561
			P_4 =	0.004111221
			P_5 =	0.009592849
			P_6 =	0.018652761
			P_7 =	0.031087935
			P_8 =	0.045336573
			P_9 =	0.058769631
			P_{10} =	0.06856457
			P_{11} =	0.072719998
			P_{12} =	0.070699998
			P_{13} =	0.063448716
			P_{14} =	0.056941156
			P_{15} =	0.051101037
			P_{16} =	0.045859905
			P_{17} =	0.041156325
			P_{18} =	0.036935163
			P_{19} =	0.033146942
			P_{20} =	0.029747255
			P_{21} =	0.026696255
			P_{22} =	0.023958177
			P_{23} =	0.021500928
			P_{24} =	0.019295705
			P_{25} =	0.017316658

For 13 servers, the probability that a customer has to wait more than 4 minutes equals 36.3%.

If there are 12 servers, this probability would be 78%:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	12	(# servers)

Pr(w>t) =	0.944173
when t =	0.066667

Prob(w_q>t) =	0.779968
when t =	0.066667

If there are 14 servers, this probability would be less than 16.4%:

Template for M/M/s Queueing Model

Data

l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	14	(# servers)

Pr(w>t) =	0.75683
when t =	0.066667

Prob(w_q>t) =	0.163704
when t =	0.066667

It appears that Mark currently employs 13 servers.

b) Using the same procedure as in part (a) we find that for s = 18 servers the probability of waiting more than 1 minute drops below 5%:

Template for M/M/s Queueing Model

Data			Results	
l =	70	(mean arrival rate)	L =	11.77798802
m =	6	(mean service rate)	L_q =	0.111321353
s =	18	(# servers)		
			W =	0.168256972
Pr(w>t) =	0.909075		W_q =	0.001590305
when t =	0.016667			
			r =	0.648148148
Prob(w_q>t) =	0.032078			
when t =	0.016667		P_0 =	8.49029E-06
			P_1 =	9.90534E-05
			P_2 =	0.000577812
			P_3 =	0.002247045
			P_4 =	0.006553882
			P_5 =	0.015292391
			P_6 =	0.029735204
			P_7 =	0.049558673
			P_8 =	0.072273065
			P_9 =	0.093687307
			P_{10} =	0.109301858
			P_{11} =	0.115926213
			P_{12} =	0.11270604
			P_{13} =	0.101146446
			P_{14} =	0.084288705
			P_{15} =	0.065557882
			P_{16} =	0.047802622
			P_{17} =	0.032805721
			P_{18} =	0.021262967
			P_{19} =	0.013781553
			P_{20} =	0.008932488
			P_{21} =	0.005789576
			P_{22} =	0.003752503
			P_{23} =	0.002432178
			P_{24} =	0.001576411
			P_{25} =	0.001021748

c) Using the same "trial and error" method as before, we find the minimal number of servers necessary to ensure that 80% of customers wait one minute or less to be s = 15.

Template for M/M/s Queueing Model

Data		
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	15	(# servers)

Pr(w>t) =	0.926712
when t =	0.016667

Prob(w_q>t) =	0.194213
when t =	0.016667

The minimal number of servers to ensure that 95% of customers wait 90 seconds or less is s = 17.

Template for M/M/s Queueing Model

Data		
l =	70	(mean arrival rate)
m =	6	(mean service rate)
s =	17	(# servers)

Pr(w>t) =	0.870524
when t =	0.025

Prob(w_q>t) =	0.046459
when t =	0.025

When an employee of Cutting Edge calls the benefits center from work and has to wait on the phone, the company loses valuable work time for this customer. Mark should try to estimate the amount of work time employees lose when they have to wait on the phone. Then he could determine the cost of this waiting time and try to choose the number of representatives in such a fashion that he reaches a reasonable trade-off between the cost of employees waiting on the phone and the cost of adding new representatives.

Clearly, Mark's criteria would be different if he were dealing with external customers. While the internal customers might become disgruntled when they have to wait on the phone, they cannot call somewhere else. Effectively, the benefits center holds monopolistic power. On the contrary, if Mark were running a call center dealing with external customers, these customers could decide to do business with a competitor if they become angry from waiting on the phone.

d) If the representatives can only handle 6 calls per hour, then Mark needs to employ 18 representatives (see part b). If a representative can handle 8 calls per hour, then the minimal number of representatives equals 14:

Template for M/M/s Queueing Model

	Data	
l =	70	(mean arrival rate)
m =	8	(mean service rate)
s =	14	(# servers)

Pr(w>t) =	0.881748
when t =	0.016667

Prob(w_q>t) =	0.036649
when t =	0.016667

The cost of training 14 employees equals 14*$2500 = $35000 and saves Mark 4*$30000 = $120000 in annual salary. In the first year alone Mark would save $85000 if he chose to train all his employees so that they can handle 8 instead of 6 phone calls per hour.

e) Mark needs to carefully check the number of calls arriving at the call center per hour. In this case we have made the simplifying assumption that the arrival rate is constant. That assumption is unrealistic; clearly we would expect more calls during certain times of the day, during certain days of the week, and during certain weeks of the year. We might want to collect data on the number of calls received depending on the time. This data could then be used to forecast the number of calls the center will receive in the near future, which in turn would help to forecast the number of representatives needed.

Also, Mark should carefully check the number of phone calls a representative can answer per hour. Clearly, the length of a call will depend on the issue the caller wants to discuss. We might want to consider training representatives for special issues. These representatives could then always answer those particular calls. Using specialized representatives might increase the number of phone calls the entire center can handle.

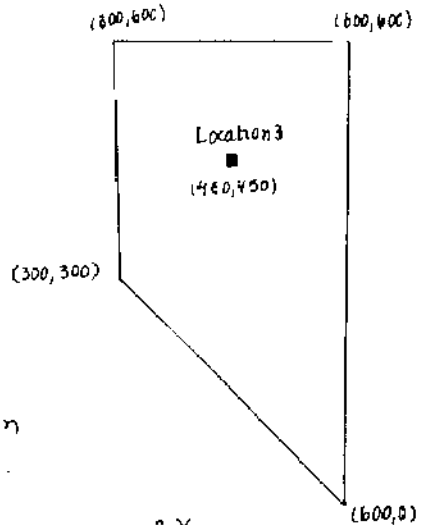
Finally, using an M/M/s model is clearly a great simplification. We need to evaluate whether the assumptions for an M/M/s model are at least approximately satisfied. If this is not the case, we should consider more general models such as M/G/s or G/G/s.

Supplement to Chapter 18

(Appears on the book's Web site, www.mhhe.com/hillier)

18S-1:

- Area assigned to tool crib in Location 3, in Alternative 3



- The first step is to relabel Location 3 as the origin (0,0) for an (x,y) coordinate system (subtract 450 from all coordinates shown in figure on the right).

- The probability density function of X is obtained by using the height of the area assigned to the tool crib at Location 3 for each possible value of $X = x$ and then dividing by the size of the area, as given in figure 1.(a) below. This then yields the uniform distribution of |X| shown in figure 1.(b).

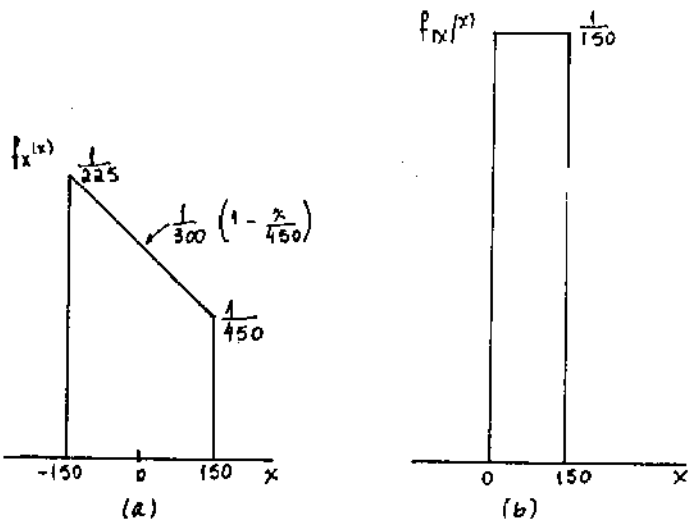


Fig. 1 Probability density functions of (a) X, and (b) |X|.

Thus, $E\{|X|\} = \frac{1}{150}\int_0^{150} x\,dx = 75$

185-1 continued

- The probability density function of Y is obtained by using the width of the area assigned to tool crib at Location 3 for each possible value of Y = y and then dividing by the size of the area, as given in figure 2.(a). This then leads to the probability density function of |Y| shown in figure 2.(b).

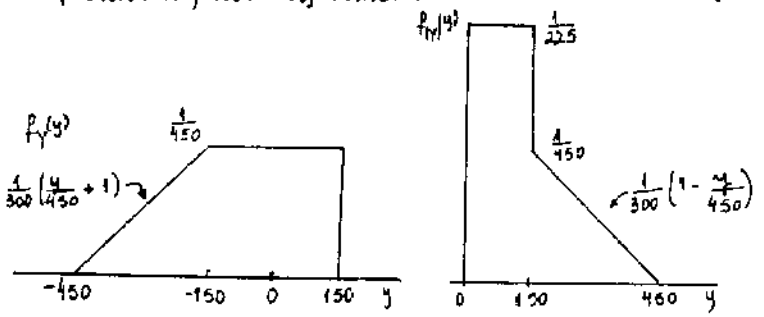


Fig. 2 Probability density functions of (a) Y, and (b) |Y|.

Thus,

$$E\{|Y|\} = \frac{1}{225}\int_0^{150} y\,dy + \frac{1}{300}\int_{150}^{450}\left(1-\frac{y}{450}\right) y\,dy =$$

$$= 133\ 1/3$$

Consequently, E(T) for the tool crib in Location 3 is

$$E(T) = \frac{2}{v}\left(E\{|X|\} + E\{|Y|\}\right) =$$

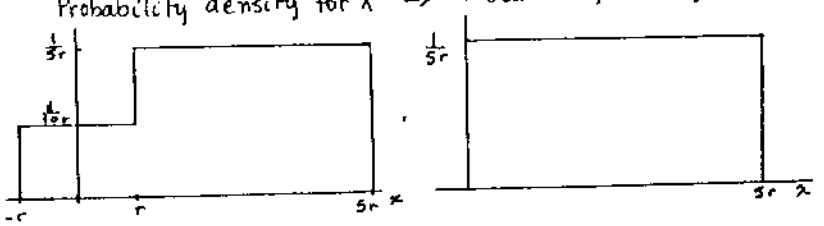
$$= \frac{2}{15{,}000}\left(75 + 133\ 1/3\right)$$

$$= 0.0278 \text{ hr.}$$

185-2.

(a) Total area $= (2r)^2 + (4r)^2 = 20r^2$

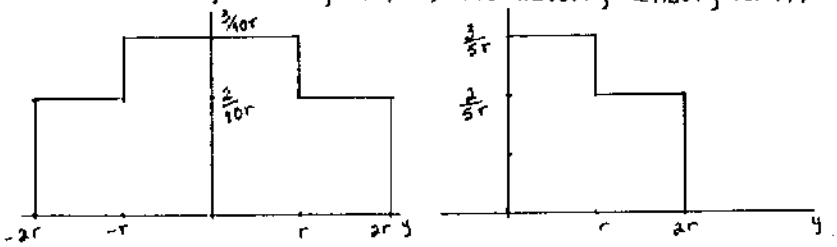
Probability density for X $\Rightarrow$ Probability density for |X|



So, $E\{|X|\} = \int_0^{5r}\left(\frac{1}{5r}\right)x\,dx = 2.5r$

18S-2(a) continued

Probability density for Y ⇒ Probability density for |Y|



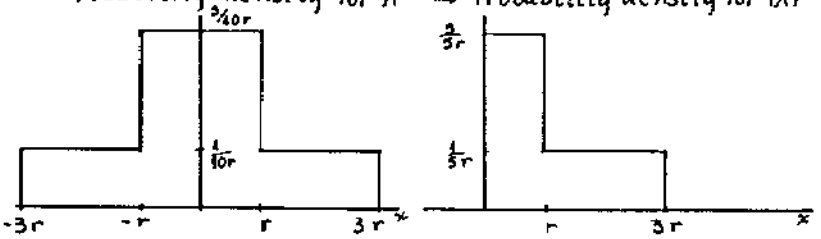
So, $E\{|Y|\} = \int_0^r \left(\frac{3}{5r}\right) y\,dy + \int_r^{2r} \left(\frac{2}{5r}\right) y\,dy = .9\,r$

Hence, $E[T] = \frac{2}{v}(2.5 + .9)r = \frac{6.8r}{v}$

(b) Area is symmetric about (0,0) so $E\{|X|\} = E\{|Y|\}$

Total area $= 5(2r)^2 = 20r^2$

Probability density for X ⇒ Probability density for |X|

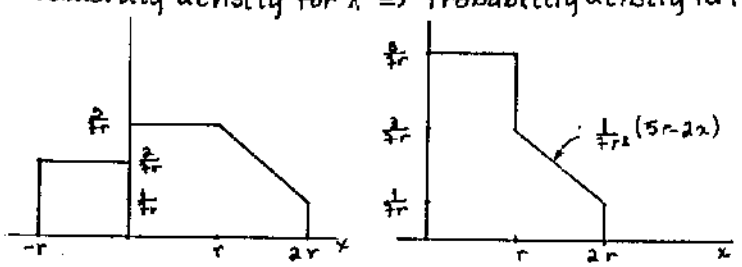


So, $E\{|X|\} = \int_0^r \left(\frac{3}{5r}\right) x\,dx + \int_r^{3r} \left(\frac{1}{5r}\right) x\,dx = 1.1\,r$

Hence, $E[T] = \frac{2r}{v}(1.1 + 1.1) = \frac{4.4r}{v}$

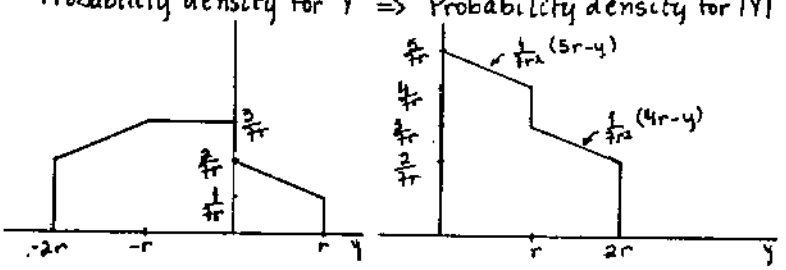
(c) Total area $= 2(2r^2 + r^2 + \frac{1}{2}r^2) = 7r^2$

Probability density for X ⇒ Probability density for |X|



So, $E\{|X|\} = \int_0^r \left(\frac{5}{7r}\right) x\,dx + \int_r^{2r} \left(\frac{1}{7r^2}\right)(5r - 2x)\,x\,dx = \frac{16}{21}\,r$

Probability density for Y ⇒ Probability density for |Y|



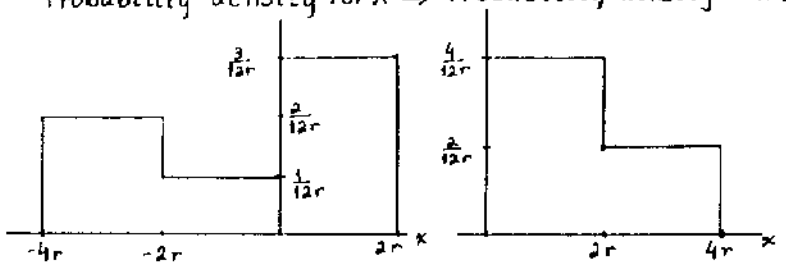
185-2. (c) (continued)

So, $E\{|Y|\} = \frac{1}{4r^2}\left\{\int_0^r (5r-y)y\,dy + \int_r^{2r}(4r-y)y\,dy\right\} = \frac{5r}{6}$

Hence, $E[T] = \frac{2r}{v}\left[\frac{(32+35)}{42}\right] = \frac{3.19\,r}{v}$

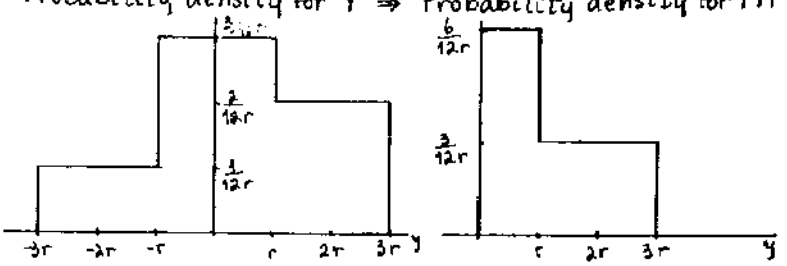
(d) Total area $= 6(4r^2) = 24r^2$

Probability density for X ⇒ Probability density for |X|



So, $E\{|X|\} = \int_0^{2r}\left(\frac{4}{12r}\right)x\,dx + \int_{2r}^{4r}\left(\frac{2}{12r}\right)x\,dx = \frac{5r}{3}$

Probability density for Y ⇒ Probability density for |Y|



So, $E\{|Y|\} = \int_0^r\left(\frac{6}{12r}\right)y\,dy + \int_r^{3r}\left(\frac{3}{12r}\right)y\,dy = \frac{5r}{4}$

So, $E(T) = \frac{2r}{v}\left(\frac{5}{3}+\frac{5}{4}\right) = \left(\frac{35}{6}\right)\frac{r}{v} = \frac{5.83\,r}{v}$

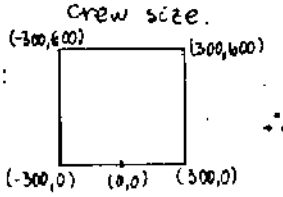
185-3.

Given: $C_f = 10$, $C_m = 15$, $C_l = 40$, $\lambda_p = 90$, $v = 20000$ feet/hour

E(loading time) = 1/20 hours

For unloading, $\mu_m = 30m$ where m is the crew size.

$m = 1$:

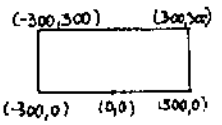


So $a = 300$, $b = 0$, $c = 300$ and $d = 600$

$\therefore E(T) = \frac{1}{20000}\left[\frac{300^2+300^2}{300+300} + \frac{600^2}{600}\right] = .045$ hrs.

Also, $L = \frac{\lambda}{\mu_m - \lambda} = \frac{90}{30m-90} = \frac{3}{m-3}$

185-3. (continued)



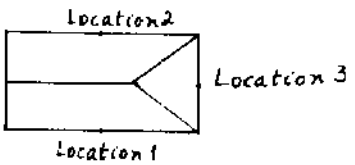
$n=2$: Labelling the two symmetric areas:

So $a=300$, $b=0$, $c=300$, $d=300$

So $E(T) = \frac{1}{20\,000}\left[\frac{300^2+300^2}{300+300} + \frac{300^2}{300}\right] = .030$ hrs.

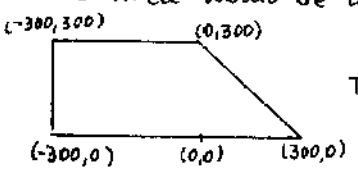
Also, $\lambda = \frac{\lambda_p}{n} = 45 \Rightarrow L = \frac{45}{30m-45} = \frac{3}{2m-3}$

$n=3$: The facilities would be located as follows



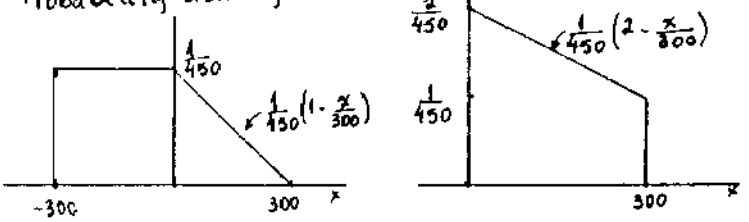
Consider Locations 1 and 2 that are symmetric

The area would be labelled as



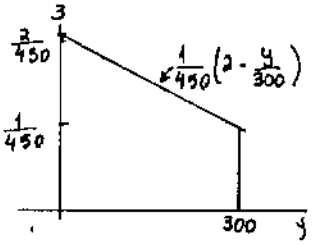
Total area = 135,000

Probability density of X $\Rightarrow$ Probability density of |X|



So $E\{|X|\} = \int_0^{300} \left(\frac{1}{450}\right)\left(2-\frac{x}{300}\right) x\,dx = \frac{400}{3}$

Probability density of $Y=|Y|$:



So $E\{|Y|\} = E\{|X|\} = 400/3$

$\therefore E(T) = \frac{2}{20\,000}\left[\frac{400}{3} + \frac{400}{3}\right] = \frac{4}{150} = .0267$

Also, $\lambda = 33\,3/4 \Rightarrow L = \frac{135/4}{30m-135/4} = \frac{135}{120m-135} = \frac{9}{8m-9}$

18S-3. (continued)

$n=4$: The areas served by the four facilities would be identical to that of Location 3 for $n=3$.

So $E(T) = 4/200 = .020$ hrs. and $L = \frac{3}{4m-3}$

Summarizing:

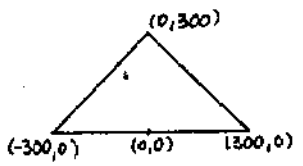
n	$E(T)$	L
1	.045 hrs.	$3/(m-3)$
2	.030 hrs.	$3/(2m-3)$
3	Locations 1,2; Location 3 .0267 hrs. .020 hrs.	Locations 1,2; Location 3 $9/(8m-9)$ $3/(4m-3)$
4	.020 hrs.	$3/(4m-3)$

$n=1$: $E(TC) = (C_f + mC_m) + C_t L + \lambda C_t E(T) + \lambda C_v/20$ where $\lambda = 90$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_v/20$	$E(TC)$
4	3	.045	70	120	162	180	532.00
5	1.5	.045	85	60	162	180	487.00
6	1	.045	100	40	162	180	482.00
7	.75	.045	115	30	162	180	487.00

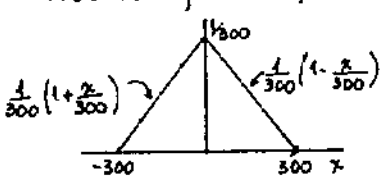
So for $n=1$, the minimum cost per hour is \$482.00 with $m=6$.

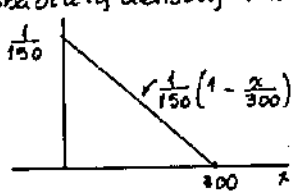
Now consider Location 3 : The area would be labelled



Total area = 90000

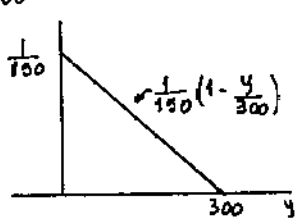
Probability density of X $\Rightarrow$ Probability density of $|X|$





So $E\{|X|\} = \int_0^{300} \frac{1}{150}\left(1-\frac{x}{300}\right) x\,dx = 100$

Probability density of $Y=|Y|$:



So $E\{|Y|\} = E\{|X|\} = 100$

$\therefore E(T) = \frac{2}{20000}(100+100) = \frac{4}{200} = .020$ hrs.

Also, $\lambda = 22\ 1/2 \Rightarrow L = \frac{45/2}{30m-45/2} = \frac{45}{60m-45} = \frac{3}{4m-3}$

18S-3. (continued)

$n = 2$: $E(TC) = 2[(C_f + mC_m) + C_t L + \lambda C_t E(T) + \lambda C_t/20]$, where $\lambda = 45$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
2	3	.030	40	120	54	90	608.00
3	1	.030	55	40	54	90	478.00
4	.6	.030	70	24	54	90	476.00
5	3/7	.030	85	17.14	54	90	492.29

So for $n=2$, the minimum cost per hour is \$476.00 with $m = 4$.

$n = 3$: At Locations 1 and 2 where $\lambda = 33\ 3/4$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
2	9/7	.0267	40	51.43	36	67.5	194.93
3	3/5	.0267	55	24	36	67.5	182.50
4	9/23	.0267	70	15.65	36	67.5	189.15

At Location 3, where $\lambda = 22\ 1/2$

m	L	$E(T)$	$C_f + mC_m$	$C_t L$	$\lambda C_t E(T)$	$\lambda C_t/20$	$E(TC)$
1	3	.020	25	120	18	45	208.00
2	3/5	.020	40	24	18	45	127.00
3	1/3	.020	55	13.33	18	45	131.33

So, for $n = 3$, the minimum cost per hour is $2(182.50) + 127.00 = 492.00$ with $m = 3$ at Locations 1 and 2, and $m = 2$ at Location 3.

$n = 4$: Since all areas served are symmetric to that of Location 3 for $n = 3$, the minimum cost per hour is $4(127.00) = 508.00$ with $m = 2$.

Summarizing,

n	m	$E(TC)$
1	6	482.00
2	4 at both Locations	476.00
3	3 at Locations 1 and 2 2 at Location 3	492.00
4	2 at all Locations	508.00

Therefore, there should be two facilities with crew size of 4.

CHAPTER 19

19-3.1

(a) $K=15,\ h=30,\ a=30 \Rightarrow Q^* = \sqrt{\frac{(2)(30)(15)}{0.30}} = 54.77$

$t^* = Q^*/a = 1.83$ months

(b) $p=3 \Rightarrow Q^* = \sqrt{\frac{(2)(30)(15)}{0.30}} \cdot \sqrt{\frac{3+0.30}{3}} = 57.45$

$S^* = \sqrt{\frac{(2)(30)(15)}{0.30}} \cdot \sqrt{\frac{3}{3+0.30}} = 52.22$

$t^* = Q^*/a = 1.91$ months

19-3-2

(a) $K=25,\ h=0.05,\ a=600 \Rightarrow Q^* = \sqrt{\frac{(2)(600)(25)}{0.05}} = 774.60$

$t^* = Q^*/a = 1.29$ weeks

(b) $p=2 \Rightarrow Q^* = \sqrt{\frac{(2)(600)(25)}{0.05}} \sqrt{\frac{2+0.05}{2}} = 784.22$

$S^* = \sqrt{\frac{(2)(600)(25)}{0.05}} \sqrt{\frac{2}{2+0.05}} = 765.09$

$t^* = Q^*/a = 1.31$ weeks

19.3-3

a)

Data

D =	676	(demand/year)
K =	$75	(setup cost)
h =	$600.00	(unit holding cost)
L =	3.5	(lead time in days)
WD =	365	(working days/year)

Results

Reorder Point =	6.5
Annual Setup Cost =	$10,140
Annual Holding Cost =	$1,500
Total Variable Cost =	$11,640

Decision

Q =	5	(order quantity)

b)

Q	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
	$10,140	$1,500	$11,640
5	$10,140	$1,500	$11,640
7	$7,243	$2,100	$9,343
9	$5,633	$2,700	$8,333
11	$4,609	$3,300	$7,909
13	$3,900	$3,900	$7,800
15	$3,380	$4,500	$7,880
17	$2,982	$5,100	$8,082
19	$2,668	$5,700	$8,368
21	$2,414	$6,300	$8,714
23	$2,204	$6,900	$9,104
25	$2,028	$7,500	$9,528

c)

Data		
D =	676	(demand/year)
K =	$75	(setup cost)
h =	$600.00	(unit holding cost)
L =	3.5	(lead time in days)
WD =	365	(working days/year)

Results	
Reorder Point =	6.48
Annual Setup Cost =	$3,900
Annual Holding Cost =	$3,900
Total Variable Cost =	$7,800

Decision		
Q =	13	(order quantity)

d)

Data		
D =	676	(demand/year)
K =	$75	(setup cost)
h =	$600	(unit holding cost)
L =	3.5	(lead time in days)
WD =	365	(working days/year)

Results	
Reorder Point =	6.48
Annual Setup Cost =	$3,900
Annual Holding Cost =	$3,900
Total Variable Cost =	$7,800

Decision		
Q =	13	(order quantity)

The results are the same as those obtained in part c).

e) $Q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2(75)(676)}{0.2(3000)}} = 13$ computers purchased with each order

f)

$$\text{Number of orders per year} = \frac{D}{Q} = \frac{676}{13} = 52$$

$ROP = D(LT) = (13)\left(\frac{1}{2}\right) = 6.5$ - inventory level when each order is placed

g) The optimal policy reduces the total variable inventory cost by $3,840 per year, which is a 33% reduction.

19.3-4

a)

Data		
D =	102000	(demand/year)
K =	$1,000	(setup cost)
h =	$0.12	(unit holding cost)
L =	0	(lead time in days)
WD =	365	(working days/year)

Results	
Reorder Point =	0
Annual Setup Cost =	$12,000.00
Annual Holding Cost =	$510.00
Total Variable Cost =	$12,510.00

Decision		
Q =	8500	(order quantity)

b)

Months	Q	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
		$12,000	$510	$12,510
1	8500	$12,000	$510	$12,510
2	17000	$6,000	$1,020	$7,020
3	25500	$4,000	$1,530	$5,530
4	34000	$3,000	$2,040	$5,040
5	42500	$2,400	$2,550	$4,950
6	51000	$2,000	$3,060	$5,060
7	59500	$1,714	$3,570	$5,284
8	68000	$1,500	$4,080	$5,580
9	76500	$1,333	$4,590	$5,923
10	85000	$1,200	$5,100	$6,300

c)

Data

D =	102000	(demand/year)
K =	$1,000	(setup cost)
h =	$0.12	(unit holding cost)
L =	0	(lead time in days)
WD =	365	(working days/year)

Results

Reorder Point =	0
Annual Setup Cost =	$2,473.86
Annual Holding Cost =	$2,473.86
Total Variable Cost =	$4,947.73

Decision

Q =	41231	(order quantity)

d)

Data

D =	102000	(demand/year)
K =	$1,000	(setup cost)
h =	$0.12	(unit holding cost)
L =	0	(lead time in days)
WD =	365	(working days/year)

Results

Reorder Point =	0
Annual Setup Cost =	$2,473.86
Annual Holding Cost =	$2,473.86
Total Variable Cost =	$4,947.73

Decision

Q =	41231	(order quantity)

The results are the same as those obtained in part c).

e)

$$Q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2(1000)(8500)}{0.01}} = 41{,}231 \text{ gallons purchased with each order}$$

19.3-5

a)

Data

D =	10400	(demand/year)
K =	$20	(setup cost)
h =	$0.20	(unit holding cost)
L =	3.5	(lead time in days)
WD =	365	(working days/year)

Results

Reorder Point =	99.73
Annual Setup Cost =	$144.22
Annual Holding Cost =	$144.22
Total Variable Cost =	$288.44

Decision

Q =	1442	(order quantity)

b)

Data		
D =	10400	(demand/year)
K =	$20	(setup cost)
h =	$0.15	(unit holding cost)
L =	3.5	(lead time in days)
WD =	365	(working days/year)

Results	
Reorder Point =	99.73
Annual Setup Cost =	$124.90
Annual Holding Cost =	$124.90
Total Variable Cost =	$249.80

Decision		
Q =	1665	(orcer quantity)

The order quantity is increased by 223 LCDs. If the order quantity obtained in part a) is still used the TVC increases to $252.39.

c)

Data		
D =	10400	(demand/year)
K =	$20	(setup cost)
h =	$0.25	(unit holding cost)
L =	3.5	(lead time in days)
WD =	365	(working days/year)

Results	
Reorder Point =	99.73
Annual Setup Cost =	$161.25
Annual Holding Cost =	$161.25
Total Variable Cost =	$322.49

Decision		
Q =	1290	(order quantity)

The order quantity is decreased by 152 LCDs. If the order quantity obtained in part a) is still used the TVC increases to $324.49.

d)

Cost of Capital	Unit Holding Cost	Q
		1290
10%	0.15	1665
12%	0.17	1564
14%	0.19	1480
16%	0.21	1407
18%	0.23	1345
20%	0.25	1290

e)

Setup Hours	Setup Cost	Q
		1665
0.5	$10	1178
0.75	$15	1442
1	$20	1665
1.25	$25	1862
1.5	$30	2040

f)

Unit Holding Cost	Optimal Order Quantity (Q)				
	Setup Cost				
1665.33	$10	$15	$20	$25	$30
0.15	1178	1442	1665	1862	2040
0.17	1106	1355	1564	1749	1916
0.19	1046	1281	1480	1654	1812
0.21	995	1219	1407	1574	1724
0.23	951	1165	1345	1504	1647
0.25	912	1117	1290	1442	1580

19.6 a) Q* will decrease by half.

b) Q* will double.

c) Q* remains the same.

d) Q* will double.

e) Q* remains the same.

19-7 a)

$$Q^* = \sqrt{\frac{2KD}{h}} \Rightarrow 50 = \sqrt{\frac{2(75)(50)}{h}} \Rightarrow \sqrt{h} = \frac{\sqrt{2(75)(50)}}{50} \Rightarrow h = \$3 \text{ per month}$$

which is 15% of the acquisition cost.

b) Optimal Order Quantity h = 20%

Data		
D =	50	(demand/month)
K =	$75	(setup cost)
h =	$4.00	(unit holding cost)
L =	5	(lead time in days)
WD =	25	(working days/month)

Results	
Reorder Point =	10
Annual Setup Cost =	$86.60
Annual Holding Cost =	$86.60
Total Variable Cost =	$173.21

Decision		
Q =	43	(order quantity)

Current Inventory Policy

Data		
D =	50	(demand/month)
K =	$75	(setup cost)
h =	$4.00	(unit holding cost)
L =	5	(lead time in days)
WD =	25	(working days/month)

Results	
Reorder Point =	10
Annual Setup Cost =	$75.00
Annual Holding Cost =	$100.00
Total Variable Cost =	$175.00

Decision		
Q =	50	(order quantity)

c) Reorder point is 10 as shown in the above spreadsheet.

d)

$$ROP = -S^* + D(LT) = -5 + (50)\left(\frac{1}{5}\right) = 5 \text{ hammers which adds \$20 to his TVC}$$

(5 hammers x $4 holding cost).

19.8 $Q^* = \sqrt{\frac{2KD}{h}} \Rightarrow 10 = \sqrt{\frac{2(200)(D)}{400}} \Rightarrow D = 10^2\left(\frac{400}{2(200)}\right) \Rightarrow D = 100$ per year

19.3-9

$K = 12{,}000,\ h = 0.30,\ a = 8{,}000,\ p = 5$

$$Q^* = \sqrt{\frac{(2)(8000)(12000)}{0.30}}\sqrt{\frac{5+0.30}{5}} = 26{,}046$$

$$S^* = \sqrt{\frac{(2)(8000)(12000)}{0.30}}\sqrt{\frac{5}{5+0.30}} = 24{,}572$$

$t^* = Q^*/a = 3.26$ months

19.3-10 a)

Data		
D =	3000	(demand/year)
K =	$200	(setup cost)
h =	$20	(unit holding cost)
L =	0	(lead time in days)
WD =	365	(working days/year)

Results	
Reorder Point =	0
Annual Setup Cost =	$2,449.49
Annual Holding Cost =	$2,449.49
Total Variable Cost =	$4,898.98

Decision		
Q =	245	(order quantity)

b)

Data		
D =	3000	(demand/year)
K =	$200	(setup cost)
h =	$20.00	(unit holding cost)
p =	$30.00	(unit shortage cost)

Results	
Max Inventory Level =	189.74
Annual Setup Cost =	$1,897.37
Annual Holding Cost =	$1,138.42
Annual Shortage Cost =	$758.95
Total Variable Cost =	$3,794.73

Decision		
Q =	316	(order quantity)
S =	126	(maximum shortage)

19.3-11 a)

Data		
D =	676	(demand/year)
K =	$75	(setup cost)
h =	$600	(unit holding cost)
p =	$200	(unit shortage cost)

Results	
Max Inventory Level =	6.00
Annual Setup Cost =	$1,950.00
Annual Holding Cost =	$415.38
Annual Shortage Cost =	$1,538.46
Total Variable Cost =	$3,903.85

Decision		
Q =	26	(order quantity)
S =	20	(maximum shortage)

This TVC is a reduction of $3900 or 50% from the TVC when shortages were not allowed.

b)

Q	Annual Setup Cost	Annual Holding Cost	Annual Shortage Cost	Total Variable Cost
	$1,950	$415	$1,538	$3,904
15	$3,380	$500	$2,667	$6,547
17	$2,982	$159	$2,353	$5,494
19	$2,668	$16	$2,105	$4,789
21	$2,414	$14	$1,905	$4,333
23	$2,204	$117	$1,739	$4,061
25	$2,028	$300	$1,600	$3,928
27	$1,878	$544	$1,481	$3,904
29	$1,748	$838	$1,379	$3,966
31	$1,535	$1,171	$1,290	$4,097
33	$1,536	$1,536	$1,212	$4,285
35	$1,449	$1,929	$1,143	$4,520

c)

S	Annual Setup Cost	Annual Holding Cost	Annual Shortage Cost	Total Variable Cost
	$1,950	$415	$1,538	$3,904
10	$1,950	$2,954	$385	$5,288
12	$1,950	$2,262	$554	$4,765
14	$1,950	$1,662	$754	$4,365
16	$1,950	$1,154	$985	$4,088
18	$1,950	$738	$1,246	$3,935
20	$1,950	$415	$1,538	$3,904
22	$1,950	$185	$1,862	$3,996
24	$1,950	$46	$2,215	$4,212
26	$1,950	$0	$2,600	$4,550
28	$1,950	$46	$3,015	$5,012
30	$1,950	$185	$3,462	$5,596

19.3-12

ratio of p to h	$Q^* = \sqrt{\frac{h+p}{p}}\sqrt{\frac{2KD}{h}}$	Maximum Inventory Level	Maximum Shortage
1/3	2,000	500	1,500
1	1,414	707	707
2	1,225	816	408
3	1,155	866	289
5	1,095	913	183
10	1,049	953	95

19.3-13 $S = .8Q \Rightarrow T = \frac{ak}{Q} + ac + 0.32hQ + 0.02pQ$

$= \frac{ak}{Q} + ac + (0.32h + 0.02p)Q$

$\frac{\partial T}{\partial Q} = -\frac{ak}{Q^2} + (0.32h + 0.02p) = 0$

$\Rightarrow Q^* = \sqrt{\frac{ak}{0.32h + 0.02p}}$

19.3-14

(a) Maximum inventory $= \frac{(b-a)Q}{b}$

Length of interval $I = \frac{Q}{b}$

Average inventory in interval $I = \frac{(b-a)Q}{2b}$

Length of interval $II = Q/a - Q/b$

Average inventory in interval $II = \frac{(b-a)Q}{2b}$

Thus, the average inventory per cycle is: $\frac{(b-a)Q}{2b}$

Thus, the holding cost per cycle is: $\frac{(b-a)Q}{2ab}$

$\Rightarrow T = \frac{-ak}{Q} + \frac{(b-a)hQ}{2b} + ac$

(b) $\frac{\partial T}{\partial Q} = \frac{-ak}{Q^2} + \frac{(b-a)h}{2b} = 0 \Rightarrow Q^* = \sqrt{\frac{2abk}{(b-a)h}}$

19.3-15 a)

Data

D =	5200	(demand/year)
K =	$50	(setup cost)
I =	0.2	(inventory holding cost rate)
N =	3	(# of discount categories)

Category	Price	Range of order quantities Lower Limit	Upper Limit	EOQ	Q*	Annual Purchase Cost	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
1	$100	0	99	161.25	99.00	$520,000.00	$2,626.26	$990.00	$523,616.26
2	$95	100	499	165.43	165.43	$494,000.00	$1,571.62	$1,571.62	$497,143.25
3	$90	500	10000000	169.97	500.00	$468,000.00	$520.00	$4,500.00	$473,020.00

Results

Optimal Q =	500
Total Variable Cost =	$473,020

b) Orders placed per year $= \frac{D}{Q} = \frac{5200}{500} = 10.4$

Time interval between orders $= \frac{Q}{D} = \frac{500}{5200} = (0.096 \text{ years})(52) = 5 \text{ weeks}$

19.3-16 a)

Data

D =	365	(demand/year)
K =	$5	(setup cost)
I =	0.2	(inventory holding cost rate)
N =	3	(# of discount categories)

(CONT'D)

(CONT'D)

Category	Price	Range of order quantities Lower Limit	Upper Limit	EOQ	Q*	Annual Purchase Cost	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
1	$4.00	0	49	67.55	49.00	$1,460.00	$37.24	$19.60	$1,516.84
2	$3.90	50	99	68.41	68.41	$1,423.50	$26.68	$26.68	$1,476.86
3	$3.80	100	10000000	69.30	100.00	$1,387.00	$18.25	$38.00	$1,443.25

Results	
Optimal Q =	100
Tota Variable Cost =	$1,443.25

b) Orders placed per year $= \frac{D}{Q} = \frac{365}{100} = 3.65$

Time interval between orders $= \frac{Q}{D} = \frac{100}{365} = (0.274 \text{ years})(52) = 14.25 \text{ weeks}$

19.3-17 a)

Discount Category	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	$TVC = (8.50)(400) + (80)\left(\frac{400}{Q}\right) + (0.2)(8.50)\left(\frac{Q}{2}\right)$
2	$TVC = (8)(400) + (80)\left(\frac{400}{Q}\right) + (0.2)(8)\left(\frac{Q}{2}\right)$
3	$TVC = (7.50)(400) + (80)\left(\frac{400}{Q}\right) + (0.2)(7.50)\left(\frac{Q}{2}\right)$

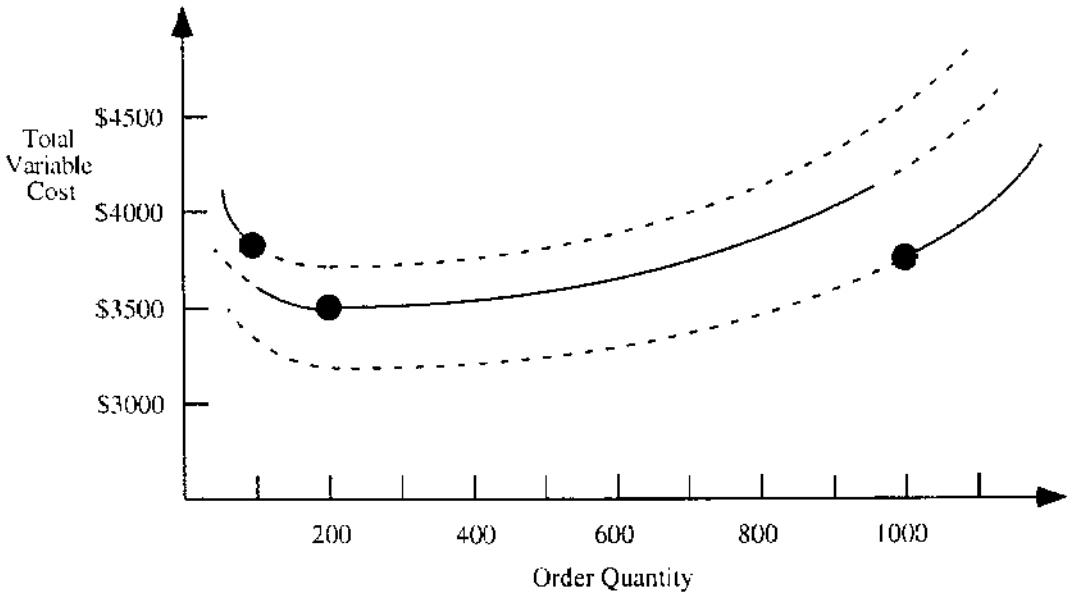
b)

Discount Category	$Q^* = \sqrt{\frac{2KD}{h}}$
1	$Q^* = \sqrt{\frac{2(80)(400)}{0.2(8.50)}} = 194$
2	$Q^* = \sqrt{\frac{2(80)(400)}{0.2(8)}} = 200$
3	$Q^* = \sqrt{\frac{2(80)(400)}{0.2(7.50)}} = 207$

c)

Discount Category	Feasible Q	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	99	$3,807.38
2	200	$3,520.00
3	1,000	$3,782.00

d)



e) $Q^* = 200$ with a TVC of \$3,520.

f)

Data		
D =	400	(demand/year)
K =	$80	(setup cost)
I =	0.2	(inventory holding cost rate)
N =	3	(# of discount categories)

Category	Price	Range of order quantities Lower Limit	Upper Limit	EOQ	Q*	Annual Purchase Cost	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
1	$8.50	0	99	194.03	99.00	$3,400.00	$323.23	$84.15	$3,807.38
2	$8.00	100	999	200.00	200.00	$3,200.00	$160.00	$160.00	$3,520.00
3	$7.50	1000	10000000	206.56	1000.00	$3,000.00	$32.00	$750.00	$3,782.00

Results	
Optimal Q =	200
Total Variable Cost =	$3,520

g) Since the value of Q that minimizes TVC for discount category 2 is feasible that means that this order quantity minimizes the annual setup and holding costs. Category 1 could therefore not possibly have lower annual setup and holding costs. Furthermore, since the purchase price per case is higher for category 1, it could not possibly have lower purchasing costs. Thus, we can eliminate category 1 as a candidate for providing the optimal order quantity.

h) Orders placed per year $= \frac{D}{Q} = \frac{400}{200} = 2$

Time interval between orders $= \frac{Q}{D} = \frac{200}{400} = (0.5 \text{ years})(12) = 6 \text{ months}.$

19.3-18
a)

Discount Category	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	$TVC = (1.00)(2400) + (4)\left(\frac{2400}{Q}\right) + (0.17)(1.00)\left(\frac{Q}{2}\right)$
2	$TVC = (0.95)(2400) + (4)\left(\frac{2400}{Q}\right) + (0.17)(0.95)\left(\frac{Q}{2}\right)$
3	$TVC = (0.90)(2400) + (4)\left(\frac{2400}{Q}\right) + (0.17)(0.90)\left(\frac{Q}{2}\right)$

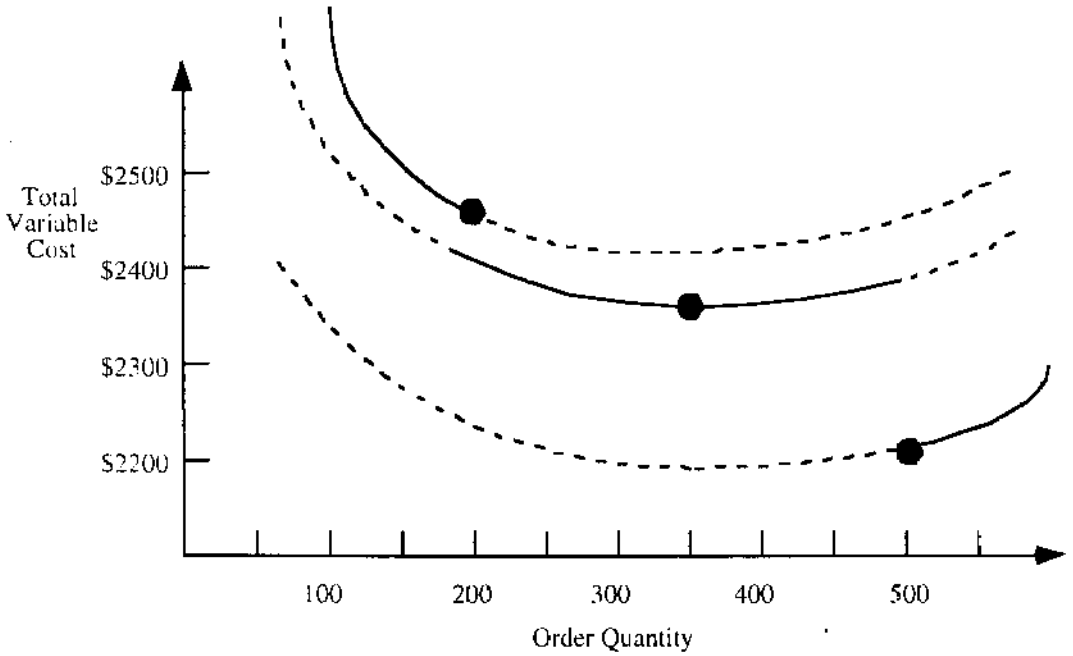
b)

Discount Category	$Q^* = \sqrt{\frac{2KD}{h}}$
1	$Q^* = \sqrt{\frac{2(4)(2400)}{0.17(1.00)}} = 336$
2	$Q^* = \sqrt{\frac{2(4)(2400)}{0.17(0.95)}} = 345$
3	$Q^* = \sqrt{\frac{2(4)(2400)}{0.17(0.90)}} = 354$

c)

Discount Category	Feasible Q	$TVC = cD + K\frac{D}{Q} + h\frac{Q}{2}$
1	199	\$2,465.16
2	345	\$2,335.68
3	500	\$2,217.45

d)



e) $Q^* = 500$ with a TVC of \$2,217.45.

f)

Data		
D =	2400	(demand/year)
K =	$4	(setup cost)
I =	0.17	(inventory holding cost rate)
N =	3	(# of discount categories)

Category	Price	Range of order quantities: Lower Limit	Upper Limit	EOQ	Q*	Annual Purchase Cost	Annual Setup Cost	Annual Holding Cost	Total Variable Cost
1	$1.00	0	199	336.07	199.00	$2,400.00	$48.24	$16.92	$2,465.16
2	$0.95	200	499	344.80	344.80	$2,280.00	$27.84	$27.84	$2,335.68
3	$0.90	500	10000000	354.25	500.00	$2,160.00	$19.20	$38.25	$2,217.45

Results	
Optimal Q =	500
Total Variable Cost =	$2,217.45

g) Since the value of Q that minimizes TVC for discount category 2 is feasible that means that this order quantity minimizes the annual setup and holding costs. Category 1 could therefore not possibly have lower annual setup and holding costs. Furthermore, since the purchase price per bag is higher for category 1, it could not possibly have lower purchasing costs. Thus, we can eliminate category 1 as a candidate for providing the optimal order quantity.

h) Orders placed per year $= \frac{D}{Q} = \frac{2400}{500} = 4.8$

Time interval between orders $= \frac{Q}{D} = \frac{500}{2400} = (0.21 \text{ years})(12) = 2.5 \text{ months}$

19.4-1

$c_5 = 4 + 3 = 7$

$c_4^{(4)} = 7 + 4 + 2 = 13$; $c_4^{(5)} = 4 + 5 + .3\,(3) = 9.9$; $\mathbf{c_4 = c_4^{(5)} = 9.9}$

$c_3^{(3)} = 9.9 + 4 + 2 = 15.9$; $c_3^{(4)} = 7 + 4 + 4 + .3\,(2) = 15.6$; $c_3^{(5)} = 4 + 7 + .3\,(2 + 6) = 13.4$

$\mathbf{c_3 = c_3^{(5)} = 13.5}$

$c_2^{(2)} = 13.5 + 4 + 4 = 21.5$; $c_2^{(3)} = 9.9 + 4 + 6 + .3\,(2) = 20.5$; $c_2^{(4)} = 7 + 4 + 8 + .3\,(2 + 4) = 20.8$

$c_2^{(5)} = 4 + 11 + .3\,(2 + 4 + 9) = 19.5$; $\mathbf{c_2 = c_2^{(5)} = 19.5}$

$c_1^{(1)} = 19.5 + 4 + 2 = 25.5$; $c_1^{(2)} = 13.5 + 4 + 6 + .3\,(4) = 24.7$; $c_1^{(3)} = 9.9 + 4 + 8 + .3\,(4 + 4) = 24.3$

$c_1^{(4)} = 7 + 4 + 10 + .3\,(4 + 4 + 6) = 25.2$; $c_1^{(5)} = 4 + 13 + .3\,(4 + 4 + 6 + 12) = 24.8$; $\mathbf{c_1 = c_1^{(3)} = 24.3}$

The optimal production schedule is to producee 8 in period 1 and 5 in period 4 at a cost of $24.30.

19.4-2

$k = 2, c = 1, h = 0.20, \quad (r_1, r_2, r_3, r_4) = (4, 3, 4, 3)$

$\mathbf{c_4 = 2 + 3 = 5}$

$c_3^{(3)} = 5 + 2 + 4 = 11$; $c_3^{(4)} = 2 + 7 + .2\,(3) = 9.6$; $\mathbf{c_3 = c_3^{(4)} = 9.6}$

$c_2^{(2)} = 9.6 + 2 + 3 = 14.6$; $c_2^{(3)} = 5 + 2 + 7 + .3\,(4) = 15.2$; $c_2^{(4)} = 2 + 10 + .3\,(4 + 6) = 15$;

$\mathbf{c_2 = c_2^{(2)} = 14.6}$

$c_1^{(1)} = 14.6 + 2 + 4 = 20.6$; $c_1^{(2)} = 9.6 + 2 + 7 + .2\,(3) = 19.2$;

$c_1^{(3)} = 5 + 2 + 11 + .2\,(3 + 8) = 20.2$; $\quad c_1^{(4)} = 2 + 14 + .2\,(3 + 8 + 9) = 20.0$; $\mathbf{c_1 = c_1^{(2)} = 19.2}$.

The optimal production schedule is to produce 7 in period 1 and 7 in period 3 at a cost of 19.2.

19.4-3

$k = 2, \quad (c_1, c_2, c_3, c_4) = (1.4, 1, 1.4, 1)$, $h = .2, \quad (r_1, r_2, r_3, r_4) = (3, 2, 3, 2)$

x_4	z_4	$c^*(x_4)$	z_4^*
0	2	4	2
1	1	3	1
2	0	0	0

x_3 \ z_3	0	1	2	3	4	5	$c^*(x_3)$	z_3^*
0	—	—	—	10.2	10.8	9.4	9.4	5
1	—	—	8.8	9.4	8.0	—	8.0	4
2	—	7.4	8.0	6.6	—	—	6.6	3
3	4	6.6	5.2	—	—	—	4	0
4	3.2	3.8	—	—	—	—	3.2	0
5	0.4	—	—	—	—	—	0.4	0

(CONT'D)

(CONT'D)

x_2 \ z_2	0	1	2	3	4	5	6	7	$c^*(x_2)$	z_2^*
0	—	—	13.4	13.2	14.0	11.6	13.0	10.4	10.4	7
1	—	12.4	12.2	13.0	10.6	12.0	9.4	—	9.4	6
2	9.4	11.2	12.0	9.6	11.0	8.4	—	—	8.4	5
3	8.2	11.0	8.6	10.0	7.4	—	—	—	7.4	4
4	7.0	7.6	9.0	6.4	—	—	—	—	6.4	3
5	4.6	8.0	5.4	—	—	—	—	—	4.6	0
6	4.0	4.4	—	—	—	—	—	—	4.0	0
7	1.4	—	—	—	—	—	—	—	1.4	0

x_1 \ z_1	3	4	5	6	7	8	9	10	$c^*(x_1)$	z_1^*
0	16.8	17.2	17.8	18.4	19.0	18.8	19.8	18.8	16.8	3

The optimal production schedule is to produce 3 units in the first period and 7 units in the second period at a cost of 16.8.

19.4-4 $h = 2$.

$$B(x_n, z_n) = \begin{cases} k_n + c_n z_n + 2 \cdot \max[0, z_n - 3] + h(x_n + z_n - r_n) & \text{for } 0 < z_n \le 4 \\ h \cdot (x_n - r_n) & \text{for } z_n = 0 \end{cases}$$

x_3	z_3	$C^*(x_3)$	z_3^*
0	4	47	4
1	3	36	3
2	2	27	2
3	1	18	1
4	0	4	0

x_2 \ z_2	0	1	2	3	4	$C^*(x_2)$	z_2^*
0	—	—	—	87	90	87	3
1	—	—	77	78	83	77	2
2	—	67	68	71	76	67	1
3	47	58	61	64	64	47	0
4	38	51	54	52	—	38	0

x_1 \ z_1	0	1	2	3	4	$C^*(x_1)$	z_1^*
1	87	92	92	82	85	82	3

The optimal production schedule is to produce 3 units in period 1 and 4 units in period 3, at a cost of 82.

19.4-5

$c_2 = 5 + 36 = \mathbf{41}$

$c_1^{(1)} = 41 + 5 + 27 = 73$; $c_1^{(2)} = 5 + 63 + .3(3) = 68.9$; $c_1 = c_1^{(2)} = \mathbf{68.9}$.

The optimal schedule is to produce all 7 in the first period at a cost of 68.9 .

19.5-1 a) $Q = \sqrt{\frac{h+p}{p}}\sqrt{\frac{2KD}{h}} = \sqrt{\frac{3000+1000}{1000}}\sqrt{\frac{2(1500)(900)}{3000}} = 60$

b) $R = \mu + K_L\sigma = 50 + 0.675(15) = 60$

c)

Data		
D =	900	(average demand/unit time)
K =	1500	(setup cost)
h =	3000	(unit holding cost)
p =	1000	(unit shortage cost)
L =	0.75	(service level)

Results	
Q=	60
R =	60.1173555

Demand During Lead Time	
Distribution =	Normal
mean =	50
stand. dev. =	15

d) Safety Stock = R – mean = 60 – 50 = 10

e) If demand during the delivery time exceeds 60 (the order quantity), then the reorder point will be hit again before the order arrives, triggering another order.

19.5-2

a) $$Q = \sqrt{\frac{h+p}{p}}\sqrt{\frac{2KD}{h}} = \sqrt{\frac{8+1}{1}}\sqrt{\frac{2(40)(40)}{8}} = 60$$

$$R = a + L(b-a) = 5 + 0.8(15-5) = 13$$

b)

Data		
D =	40	(average demand/unit time)
K =	40	(setup cost)
h =	8	(unit holding cost)
p =	1	(unit shortage cost)
L =	0.8	(service level)

Results	
Q=	60
R =	13

Demand During Lead Time		
Distribution =	Uniform	
a =	5	(lower endpoint)
b =	15	(upper endpoint)

c) Average number of orders per year = (40)(12) / 60 = 8.
Probability of a stockout before order received??= 0.2.
Average number of stockouts per year = (8)(0.2) = 1.6.

19.5-3

a)

L	Case 1 h=$1, σ=1	Case 2 h=$100, σ=1	Case 3 h=$1, σ=100	Case 4 h=$100, σ=100
0.5	0	0	0	0
0.75	0.675	67.5	67.5	6750
0.9	1.282	128.2	128.2	12,820
0.95	1.645	164.5	164.5	16,450
0.99	2.327	232.7	232.7	23,270
0.999	3.098	309.8	309.8	30,980

b)

ΔL	ΔC ($) Case 1 h=$1, σ=1	Case 2 h=$100, σ=1	Case 3 h=$1, σ=100	Case 4 h=$100, σ=100
0.25	0.675	67.5	67.5	6750
0.15	0.607	60.7	60.7	6070
0.05	0.363	36.3	36.3	3630
0.04	0.682	68.2	68.2	6820
0.009	0.771	77.1	77.1	7710

c) As the service level gets higher, increasing the service level further costs more for smaller increases. Thus, there will be diminishing returns when raising the service level further and further. You should balance the cost of the safety stock with the cost of stockouts to determine the best service level.

19.5-4

a) $C = hK_L\sigma = (\$100)(1.282)(100) = \$12{,}820.$

b) $\sigma = \sqrt{d}\sigma_1 \Rightarrow 100 = \sqrt{4}\sigma_1 \Rightarrow \sigma_1 = 50.$

If the lead time were 1 day, then $C = hK_L\sigma_1 = (\$100)(1.282)(50) = \$6410.$

This is a 50% reduction in the cost of the safety stock.

c) $\sigma = \sqrt{d}\sigma_1 = \sqrt{8}(50) = 141.4.$
$C = hK_L\sigma = (\$100)(1.282)(141.4) = \$18{,}127.$ This is a 41% increase.

d) The lead time would need to quadruple to 16 days.

19.5-5

a) The safety stock would drop to zero.

b) The safety stock would decrease.

c) The safety stock is unchanged for a given service level. However, with higher shortage costs, there will be an incentive to increase the service level (with a correspondingly higher level of safety stock).

d) The safety stock increases.

e) The safety stock doubles.

f) The safety stock doubles.

19.5-6 a) Ground Chuck:

Data		
D =	26000	(average demand/unit time)
K =	25	(setup cost)
h =	0.3	(unit holding cost)
p =	3	(unit shortage cost)
L =	0.95	(service level)

Results	
Q =	2183
R =	145

Demand During Lead Time		
Distribution =	Uniform	
a =	50	(lower endpoint)
b =	150	(upper endpoint)

(CONT'D)

(CONT'D)

Chuck Wagon:

Data		
D =	26000	(average demand/unit time)
K =	200	(setup cost)
h =	0.3	(unit holding cost)
p =	3	(unit shortage cost)
L =	0.95	(service level)

Results	
Q =	6175
R =	829

Demand During Lead Time	
Distribution =	Normal
mean =	500
stand. dev. =	200

b) Ground Chuck: $R = a + L(b-a) = 50 + 0.95(150-50) = 145.$
Chuck Wagon: $R = \mu + K_L\sigma = 500 + 1.645(200) = 829.$

c) Ground Chuck: Safety Stock = R – mean = 145 – 100 = 45.
Chuck Wagon: Safety Stock = R – mean = 829 – 500 = 329.

d) Ground Chuck:

$$\text{Average annual holding cost} = (\$0.30)\left(\frac{45+(2183+45)}{2}\right) = \$340.95$$

Chuck Wagon:

$$\text{Average annual holding cost} = (\$0.30)\left(\frac{329+(6175+329)}{2}\right) = \$3{,}416.50$$

e) Ground Chuck:

$$\text{Annual shipping cost} = K\left(\frac{D}{Q}\right) = (\$25)\left(\frac{26{,}000}{2183}\right) = \$297.76.$$

$$\text{Annual purchase cost} = (26{,}000)(\$1.49) = \$38{,}740$$

$$\text{Average annual acquisition cost} = \$297.76 + \$38{,}740 = \$39{,}037.76.$$

Chuck Wagon:

$$\text{Annual shipping cost} = K\left(\frac{D}{Q}\right) + (\$0.10)D$$

$$= (\$200)\left(\frac{26{,}000}{6175}\right) + (\$0.10)(26{,}000) = \$3442.11.$$

$$\text{Annual purchase cost} = (26{,}000)(\$1.35) = \$35{,}100.$$

$$\text{Average annual acquisition cost} = \$3442.11 + \$35{,}100 = \$38{,}542.11.$$

f) Ground Chuck: \$340.95 + \$39,037.76 = \$39,378.71.
Chuck Wagon: \$3416.50 + \$38,542.11 = \$41,958.61.

Jed should choose Ground Chuck as their supplier.

g) If Jed would like to use the beef within a month of receiving it, then Ground Chuck is the better choice. The order quantity with Ground Chuck is roughly one month's supply, whereas with Chuck Wagon the optimal order quantity is roughly three month's supply.

19.5-7. a) Option 1:

Data		
D =	26000	(average demand/unit time)
K =	130	(setup cost)
h =	6	(unit holding cost)
p =	100	(unit shortage cost)
L =	0.99	(service level)

Results	
Q =	1093
R =	366

Demand During Lead Time	
Distribution =	Normal
mean =	250
stand. dev. =	50

Option 2:

Data		
D =	26000	(average demand/unit time)
K =	50	(setup cost)
h =	6	(unit holding cost)
p =	100	(unit shortage cost)
L =	0.99	(service level)

Results	
Q =	678
R =	995

Demand During Lead Time		
Distribution =	Uniform	
a =	500	(lower endpoint)
b =	1000	(upper endpoint)

b) Option 1: $R = \mu + K_L\sigma = 250 + 2.327(50) = 366.$
Option 2: $R = a + L(b-a) = 500 + 0.99(1000 - 500) = 995.$

c) Option 1: Safety Stock $= R -$ mean $= 366 - 250 = 116.$
Option 2: Safety Stock $= R -$ mean $= 995 - 750 = 245.$

d) Option 1: Average annual holding cost $= (\$6)\left(\frac{116 + (1093 + 116)}{2}\right) = \$3975.$

Option 2: Average annual holding cost $= (\$6)\left(\frac{245 + (678 + 245)}{2}\right) = \$3504.$

e) Option 1:

$$\text{Annual shipping cost} = K\left(\frac{D}{Q}\right) + (\$3)D$$

$$= (\$130)\left(\frac{26{,}000}{1093}\right) + (\$3)(26{,}000) = \$81{,}092.$$

Annual purchase cost $= (26{,}000)(\$30) = \$780{,}000.$

Average annual acquisition cost $= \$81{,}092 + \$780{,}000 = \$861{,}092.$

(CONT'D)

(CONT'D)

Option 2:

$$\text{Annual shipping cost} = K\left(\frac{D}{Q}\right) + (\$3)D$$

$$= (\$50)\left(\frac{26{,}000}{678}\right) + (\$2)(26{,}000) = \$53{,}917.$$

Annual purchase cost $= (26{,}000)(\$30) = \$780{,}000.$

Average annual acquisition cost = \$53,917 + \$780,000 = \$833,9

f) Option 1: \$3975 + \$861,092 = \$865,067.
Option 2: \$3504 + \$833,917 = \$837,421.

Option 2 should be selected.

19.6-1

$$\Phi(y^{\square}) = \frac{y^{\square} - 200}{100} = \frac{p - c}{p + h} = \frac{25 - 18}{25 + 0.1}. \quad \text{So } y^{\square} = 100\left(2 + \frac{7}{25.1}\right) \approx 228$$

19.6-2 a)

Alternative	State of Nature (Purchase Requests) 15	16	17	18	Expected Payoff	
Order 15 copies	15	15	15	15	\$15.00	
Order 16 copies	14	16	16	16	\$15.20	Maximum
Order 17 copies	13	15	17	17	\$15.00	
Order 18 copies	12	14	16	18	\$14.20	
Prior Probability	0.4	0.2	0.3	0.1		

Freddie's most profitable alternative is to order 16 copies.

b)

Alternative	State of Nature (Purchase Requests) 15	16	17	18	Expected Cost	
Order 15 copies	0	1	2	3	1.10	
Order 16 copies	1	0	1	2	0.90	Minimum
Order 17 copies	2	1	0	1	1.10	
Order 18 copies	3	2	1	0	1.90	
Prior Probability	0.4	0.2	0.3	0.1		

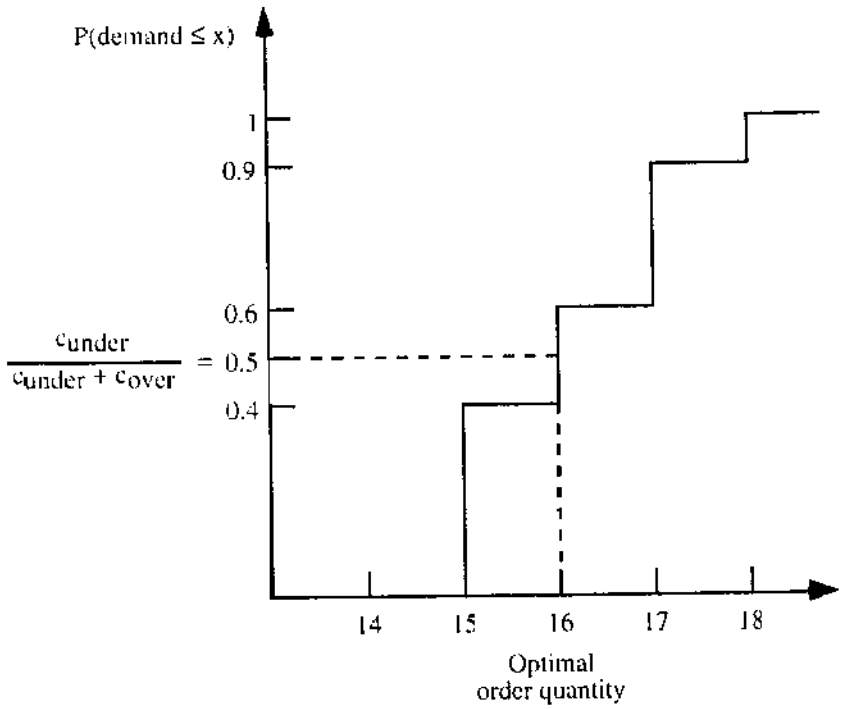
Freddie's most profitable alternative is to order 16 copies.

c) Service level if order 15 copies = 0.4
Service level if order 16 copies = 0.6
Service level if order 17 copies = 0.9
Service level if order 18 copies = 1

$$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{1}{1+1} = 0.5$$

Freddie should order 16 copies.

d)



19.6-3

a) C_{under}=($3)-($1) = $2 C_{over} = ($1)-($.50) = $.50

b)

Alternative	State of Nature (Purchase Requests) 0	1	2	3	4	5	Expected Cost	
Make 0	0.0	2.0	4.0	6.0	8.0	10.0	5.10	
Make 1	0.5	0.0	2.0	4.0	6.0	8.0	3.35	
Make 2	1.0	0.5	0.0	2.0	4.0	6.0	1.98	
Make 3	1.5	1.0	0.5	0.0	2.0	4.0	1.10	
Make 4	2.0	1.5	1.0	0.5	0.0	2.0	0.98	Minimum
Make 5	2.5	2.0	1.5	1.0	0.5	0.0	1.23	
Prior Probability	0.1	0.15	0.2	0.3	0.15	0.1		

Prepare 4 doughnuts each day to minimize costs.

c) Service level if 0 made = 0.1
Service level if 1 made = 0.25
Service level if 2 made = 0.45
Service level if 3 made = 0.75
Service level if 4 made = 0.9
Service level if 5 made = 1

$$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{2}{2 + 0.5} = 0.8$$

Prepare 4 doughnuts each day.

(CONT'D)

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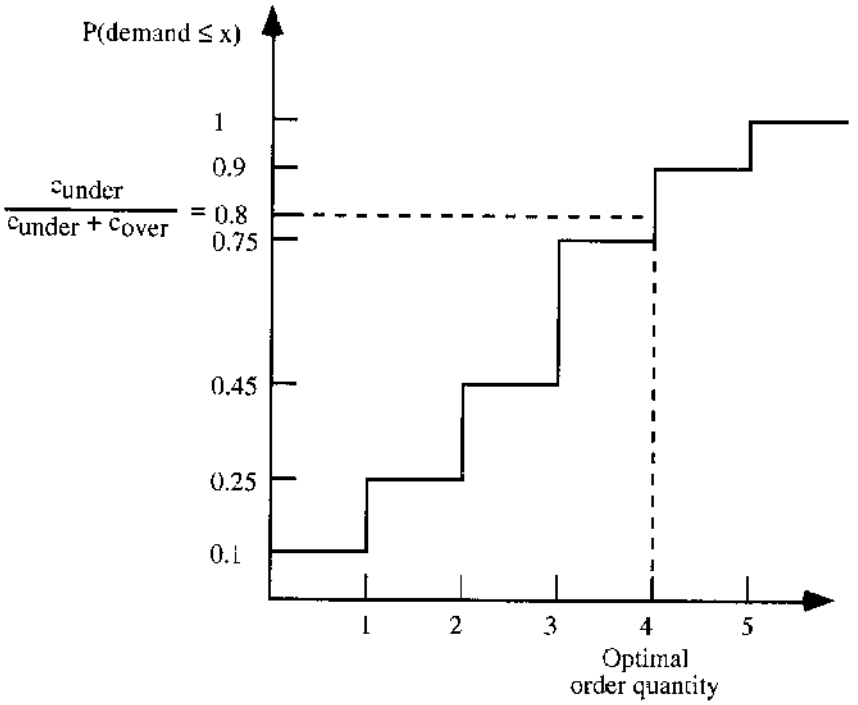


d) The probability of running short will be 1-0.9 = 0.1 = 10%

e) Before 5 doughnuts are prepared, the optimal service level needs to exceed 0.9.

Let g be the cost of lost customer goodwill. Then C_{under} = \$2 +g.

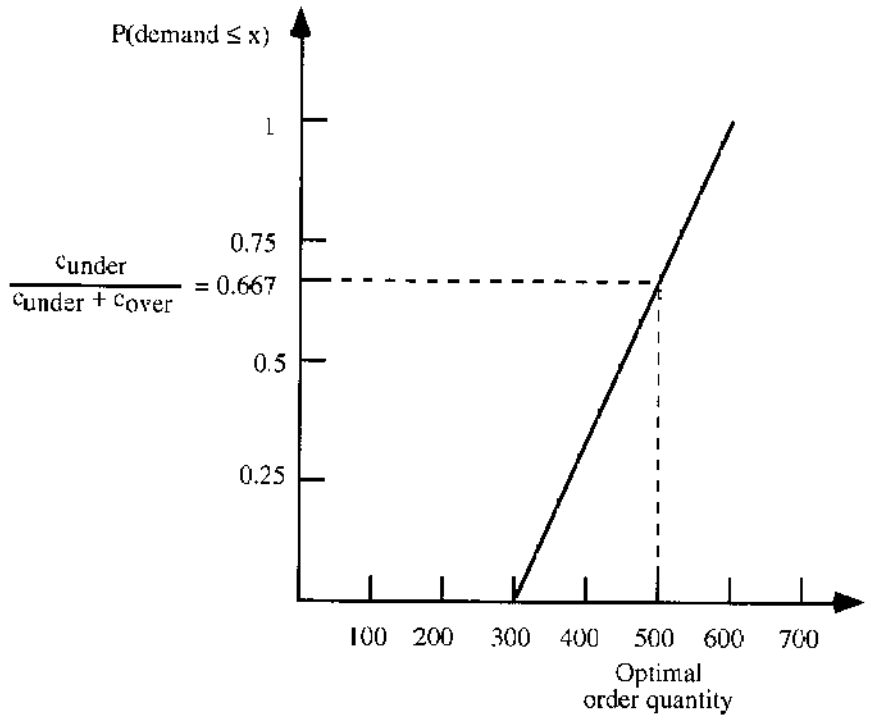
$$\text{Thus } \frac{C_{under}}{C_{under} + C_{over}} > 0.9 \quad \Leftrightarrow \quad \frac{\$2 + g}{\$2.5 + g} > 0.9 \quad \Leftrightarrow \quad g > \$2.50$$

The goodwill cost would need to be at least \$2.50 before 5 doughnuts should be prepared.

19.6-4 a)

$$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{1}{1 + 0.5} = 0.667$$

b)



c) Q* = 300+0.667(600-300) = 500

d) The probability of running short is 1-0.667 = 0.333 = 33.3%

e)

$$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{1+1.5}{1-1.5+0.5} = 0.833$$

Q* = 300+0.833(600-300) = 550
The probability of running short is 1-0.833 = 0.167 = 16.7%

19.6-5

a) Revenue (w/ shortages) = 500 ($3.00) = $1,500

b) Average number of loaves sold (w/out shortages) = $300 + \frac{500-300}{2} = 400$

Average daily revenue (w/out shortages) = 400($3.00) = $1,200

c) $1,500 x 0.333 = $500 (w/ shortages)
$1,200 x 0.667 = $800 (w/out shortages)
Average daily revenue over all days = $500 + $800 = $1,300

d) Average number of loaves not sold $= \frac{200-0}{2} = 100$

Average number of day-old loaves obtained over all days = 100 x 0.667 = 66.7
Average daily revenue from day-old bread = 66.7 x $1.50 = $100

e) Average total daily revenue = $1,300 + $100 = $1,400
Average daily profit = $1,400 – $2(500) = $400

f) Average daily profit (600 loaves)= $3(450)-$2(600)+$1.50(150) = $375

g) Average daily profit (550 loaves) = $375 + \frac{$400-$375}{2} = $387.50

h) Average size of shortage (550 loaves) = $\frac{600-550}{2} = 25$ loaves

Average daily shortage over all days = 25 x 0.167 = 4.167
Average daily cost of loss of goodwill = 4.167 x \$1.50 = \$6.25
Average daily profit (550 loaves & loss of goodwill) = \$387.50 - \$6.25 = \$381.25

i) Average size of shortage (500 loaves) = $\frac{100-0}{2} = 50$ loaves

Average daily shortage over all days = 50 x 0.333 = 16.67
Average daily cost of loss of goodwill = 16.67 x \$1.50 = \$25
Average daily profit (500 loaves & loss of goodwill) = \$400 - \$25 = \$375

19.6-6 a) $Q^* = a +$(service level) (b-a) = a + (0.667)(75) = a + 50

b) Probability of incurring shortage = 1 - 0.667 = 33.3% which is the same as in problem 13.3.

c) Maximum shortage = b – (a + 50) = 25
Maximum number of loaves that won't be sold = 50

The corresponding numbers for problem 13.3 are 100 and 200 respectively which are four times the amounts here.

d) The average daily costs of underordering and overordering for the new plan are 25% of the original costs. Thus, it is quite valuable to obtain as much information as possible about demand before placing the final order for a perishable product.

e) $Q^* = a +$(service level) (b-a) = a + (0.833)(75) = a + 62.5
Probability of incurring shortage = 1 - 0.833 = 16.67%
Maximum shortage = b – (a + 62.5) = 12.5
Maximum number of loaves that won't be sold = 62.5

19.6-7

(a) $y^{\alpha} = -\lambda \ln\left(\frac{c+h}{p+h}\right) = -50 \ln\left(\frac{1000+300}{10000+300}\right) \approx 103$

(b) $C(y) = c(y - x_0) + L(y)$ and so when we take the derivative with respect to y the term involving the initial inventory disappears, and so the optimal policy is to order up to the same amount as in part (a). So order 103 - 23 = 80 parts.

(c) We want the y such that $P\{D \le y\} = 0.90$

$$\Phi(y) = 1 - e^{-\frac{y}{50}} = 0.9$$

$$\Rightarrow e^{-\frac{y}{50}} = 0.1 \quad \Rightarrow -\frac{y}{50} = \ln(0.1) \quad \Rightarrow y = -50 \ln(0.1) \approx 115$$

(d) $\left(\frac{p-c}{p+h}\right) = 0.9 \Rightarrow \frac{p-1000}{p+300} = 0.9 \quad \Rightarrow 0.1p = 1,270 \quad \Rightarrow p = \$12,700$

19.6-8 a) This problem can be interpreted as an inventory problem with uncertain demand for a perishable product with euro-traveler's checks as the product. Once Stan gets back from his trip the checks are not good anymore so they are a perishable. He can redeposit the amount into his savings account but will incur a fee of lost interest. Stan must decide how many checks to buy without knowing how many he will need.

C_{under} = value of 1 day – cost of 1 day – cost of 1 check = \$49

C_{over} = cost of check + lost interest = \$3

b)

Alternative	State of Nature (Amount Needed) 1000	1100	1200	1300	1400	1500	1600	1700	Expected Cost	
Buy 0	0	0	0	50	100	150	200	250	80.00	
Buy 1	3	3	3	0	50	100	150	200	45.90	
Buy 2	6	6	6	3	0	50	100	150	25.05	
Buy 3	9	9	9	6	3	0	50	100	14.80	
Buy 4	12	12	12	9	6	3	0	50	9.85	Minimum
Buy 5	15	15	15	12	9	6	3	0	10.20	
Prior Probability	0.05	0.1	0.15	0.25	0.2	0.1	0.1	0.05		

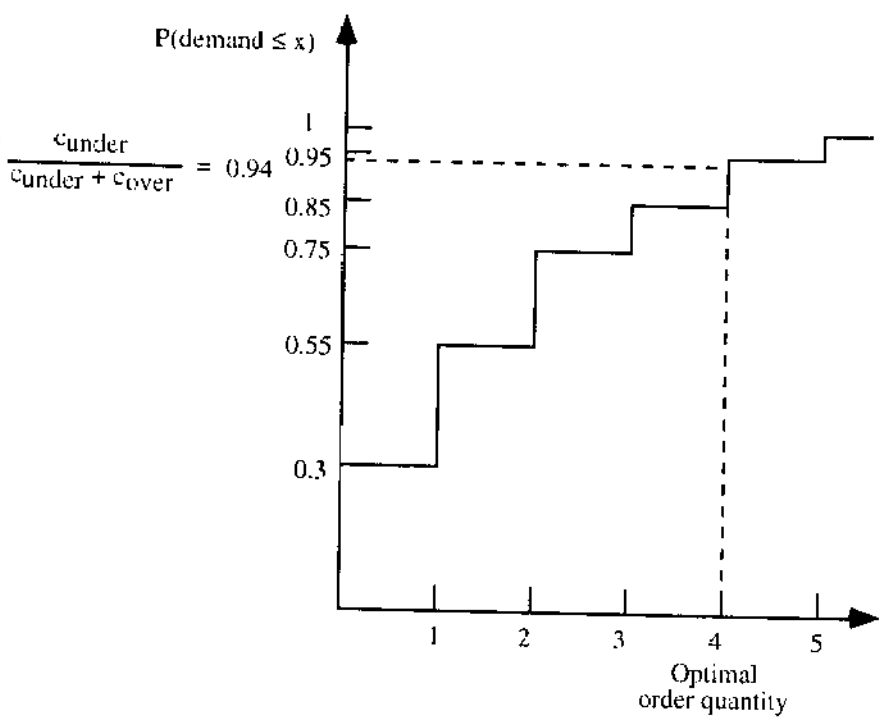
Purchase 4 additional checks.

c) Service level for buying 0 = 0.3
Service level for buying 1 = 0.55
Service level for buying 2 = 0.75
Service level for buying 3 = 0.85
Service level for buying 4 = 0.95
Service level for buying 5 = 1

$$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{49}{49+3} = 0.94$$

Buy 4 additional checks.

d)



1969 a) $$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{3000}{3000+1000} = 0.75$$

b) $Q = \mu + K_L\sigma = 50 + (0.675)(15) = 60$

19.6-10 a) When interpreting this problem as an inventory problem, overbooked reservations are the perishable products that are being placed into inventory.

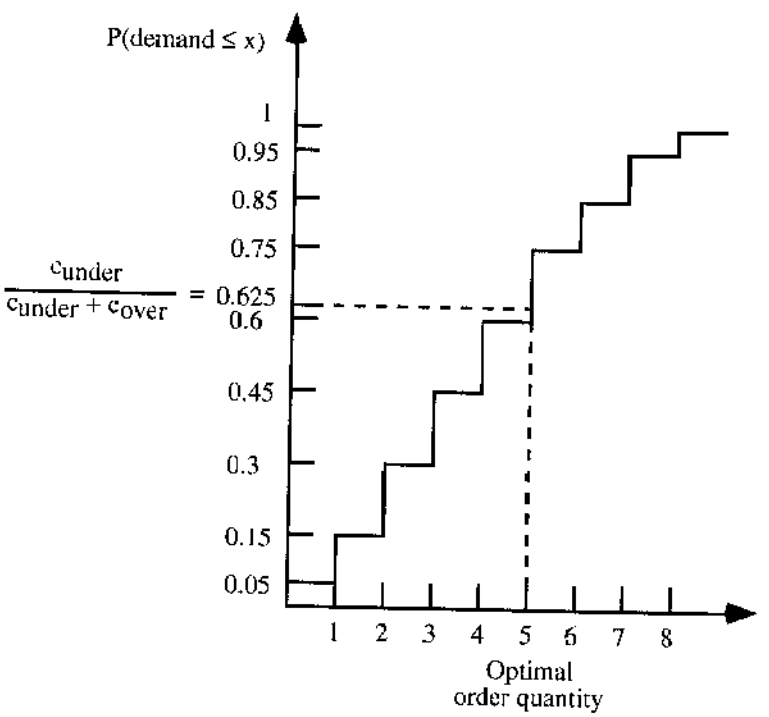
b) C_{under} = lost fare = \$250
C_{over} = cost of certificate = \$150

c) Service level for accepting 0 = 0.05
Service level for accepting 1 = 0.15
Service level for accepting 2 = 0.3
Service level for accepting 3 = 0.45
Service level for accepting 4 = 0.6
Service level for accepting 5 = 0.75
Service level for accepting 6 = 0.85
Service level for accepting 7 = 0.95
Service level for accepting 8 = 1

$$\text{Optimal service level} = \frac{C_{under}}{C_{under} + C_{over}} = \frac{250}{250 + 150} = 0.625$$

Accept 5 overbooked reservation.

d)



19.6-11

Following the development in Sec. 18.4

$$C(y) = cy + h\int_0^y (y-\xi)\vartheta_D(\xi)\,d\xi + p\int_y^\infty (\xi - y)\vartheta_D(\xi)\,d\xi + kP\{D \geq y\}$$

but if D is uniform on [a, b] then $P\{D \geq y\} = \frac{b-y}{b-a}$, and so

$$C(y) = cy + k\frac{b-y}{b-a} + L(y) \Rightarrow \frac{\delta(C_y)}{\delta y} = c - \frac{k}{b-a} + h\Phi(y) - p[1-\Phi(y)]$$

and so the minimum occurs when

$$\Phi(y) = \frac{p + \frac{k}{b-a} - c}{p+h}$$. In this case, p = c + 2, k = 14, h = -(c - 1), [a, b] = [40, 40]

and so: $\Phi(y) = \frac{y-40}{20} = \frac{2.7}{3} = 0.9 \Rightarrow y = 58$.

19.6-12

(a) $C(x,y) = c(y-x) + p\cdot P\{D>y\} = c(y-x) + p\,e^{-y} =$
$= (cy + p\,e^{-y}) - cx$

$\frac{dC(x,y)}{dy} = c - pe^{-y} = 0 \Rightarrow e^{-y} = c/p \Rightarrow y^* = -\ln(c/p)$

So order up to y^* if $x < y^*$, and do not order otherwise.

(b) $C(x,y) = \begin{cases} K + c(y-x) + pe^{-y} & \text{if } y > x \\ pe^{-x} & \text{if } y = x \end{cases}$

An (s,S) policy will be optimal with $S = -\ln(c/p)$
and s the smallest value such that $cs + pe^{-s} \leq K - c\ln(c/p) + c$

19.6-13

$L(y) = 1\int_0^y (y-\xi)\,d\xi/20 + 3\int_y^{20} (\xi - y)\,d\xi/20 = y^2/40 - 3y + 30$

So $cy + L(y) = y^2/40 - y + 30$

$\Phi(S) = S/20 = (p-c)/(p+h) = (3-2)/(3+1) = 1/4 \Rightarrow S = 5$

$cs + L(s) = K + cS + L(S) \Rightarrow s^2/40 - s + 30 = 1.5 + 27.5$
$\Rightarrow s^2/40 - s + 1 = 0$
$\Rightarrow s = 5 - \sqrt{15} = 1.13$

So the $(s,S) = (1.13, 5)$ policy is optimal.

19.6-14

Single Period Model With A Setup Cost:

Demand density is exponential, with lambda = 25.

Unit production or purchasing cost, c = 1

Unit inventory holding cost, h = 0.4

Unit shortage cost, p = 1.5

Setup Cost, K = 10

Results:

s = -11.e3

S = 7.63454

19.7-1 (a) Single Period Model With No Setup Cost:

Demand density is exponential, with lambda = 25.
Unit production or purchasing cost, c = 10
Unit inventory holding cost, h = 6
Unit shortage cost, p = 15

Results:

y(0) = 6.79834

(b) Two Period Model With No Setup Cost:

Demand density is exponential, with lambda = 25.
Unit production or purchasing cost, c = 10
Unit inventory holding cost, h = 6
Unit shortage cost, p = 15

Results:

y1(0) = 23.2932
y2(0) = 6.79834

19.7-2 (a) Single Period Model With No Setup Cost:

Demand density is uniform between 0 and 50.
Unit production or purchasing cost, c = 10
Unit inventory holding cost, h = 8
Unit shortage cost, p = 15

Results:

y(0) = 10.8696

(b) Two Period Model With No Setup Cost:

Demand density is uniform between 0 and 50.
Unit production or purchasing cost, c = 10
Unit inventory holding cost, h = 8
Unit shortage cost, p = 15

Results:

y1(0) = 9.26156
y2(0) = 10.8696

19.7-3 Infinite Period Model with no Setup Cost:
Demand density is exponential, with lambda = 25.
Unit production or purchasing cost, c = 1
Unit inventory holding cost, h = 0.25
Unit shortage cost, p = 2
Discount Factor = 0.9
Results: y(0) = 46.5188

19.7-4 Two Period Model With No Setup Cost:

Demand density is exponential, with lambda = 25.

Unit production or purchasing cost, c = 1

Unit inventory holding cost, h = 0.25

Unit shortage cost, p = 2

Results:

y1(0) = 36.521

y2(0) = 14.6947

19.7-5 Infinite Period Model With No Setup Cost:

Demand density is exponential, with lambda = 25.

Unit production or purchasing cost, c = 1

Unit inventory holding cost, h = 0.25

Unit shortage cost, p = 2

Discount Factor = 0.9

Results:

y(0) = 46.5188

19.7-6 Infinite Period Model With No Setup Cost:

Demand density is exponential, with lambda = 1.

Unit production or purchasing cost, c = 2

Unit inventory holding cost, h = 1

Unit shortage cost, p = 5

Discount Factor = 0.95

Results:

y(0) = 1.69645

19.7-7 Same as 19.7-6 above.

19.7-8. Infinite Period Model With No Setup Cost:
Demand density is uniform between 2,000 and 3,000.

Unit production or purchasing cost, c = 150

Unit inventory holding cost, h = 2

Unit shortage cost, p = 30

Discount Factor = 0.9

Results :

$y(0) = 2{,}468.15$

19.7-9 Infinite Period Model With No Setup Cost:
Demand density is exponential, with lambda = 1000.

Unit production or purchasing cost, c = 80

Unit inventory holding cost, h = 0.70

Unit shortage cost, p = 2

Discount Factor = 0.998 Results: $y(0) = 497$

19.7-10

$h = .3$, $p = 2.5$

$$G_1(y) = .3\int_0^y \frac{(y-\xi)}{25} e^{-\xi/25} d\xi + 2.5\int_y^\infty \frac{(\xi-y)}{25} e^{-\xi/25} d\xi =$$

$$= .3y + 70\, e^{-y/25} - 7.5$$

$G_1'(y) = .3 - 2.8\, e^{-y/25} = 0 \Rightarrow G(y)$ attains its minimum at $y = 55.84$, since $G_1''(y) = 2.8\, e^{-y/25} > 0$ for any y.

Now K must be found such that $G_1(K) = G_1(K+100)$, or

$.3K + 70\, e^{-K/25} - 7.5 = .3(K+100) + 70\, e^{-(K+100)/25} - 7.5$ or

$70\, e^{-K/25}(1 - e^{-4}) = 30$, so $K = 20.72 \approx 21$

Note that $K = 21 < y^0 = 55.84 < K + 100 = 121$, and $G_1(21) \approx G(121)$

Hence, the optimal policy is a $(K, Q) = (21, 100)$ policy.

19.7-11 Since $c \geq 0$, the answer is identical to that for problem 19.7-10

19.7-12

$$L(y) = \int_y^\infty p(\xi - y) f_D(\xi) d\xi + \int_0^y h(y - \xi) f_D(\xi) d\xi$$

$$\frac{dL(y)}{dy} = \int_y^\infty -p\, f_D(\xi) d\xi + \int_0^y h\, f_D(\xi) d\xi = -p[1 - \Phi(y)] + h\,\Phi(y)$$

So $\frac{dL(y)}{dy} + c(1-\alpha) = 0 \Rightarrow -p + p\,\Phi(y) + h\,\Phi(y) + c(1-\alpha) = 0$

$\Rightarrow \Phi(y) = \frac{p - c(1-\alpha)}{p+h}$, as desired.

Cases

19.1 a) Robert's problem can be solved using the basic EOQ model. The data for the case is as follows:

D = 12*250 = 3000
K = 18.75/3 = 6.25
h = 0.12*1.25 = 0.15
L = 0
WD = 12*30 = 360

Basic EOQ Model (Analytical)

Data

D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
L =	0	(lead time in days)
WD =	360	(working days/year)

Results

Reorder Point =	0
Annual Setup Cost =	$37.50
Annual Holding Cost =	$37.50
Total Variable Cost =	$75.00

Decision

Q =	500.00	(order quantity)

Robert should order 500 toothbrushes 6 times per year.

b) Now the lead time equals L = 6 days. We can use the basic EOQ model again:

Basic EOQ Model (Analytical)

Data

D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
L =	6	(lead time in days)
WD =	360	(working days/year)

Results

Reorder Point =	50
Annual Setup Cost =	$37.50
Annual Holding Cost =	$37.50
Total Variable Cost =	$75.00

Decision

Q =	500.00	(order quantity)

Whenever the inventory of toothbrushes drops down to 50, Robert should place an order for 500 toothbrushes. He needs to place an order 6 times a year.

c) Now we need to consider the EOQ model with planned shortages. The shortage costs are p = 1.50.

EOQ Model with Planned Shortages (Analytical)

Data		
D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
p =	$1.50	(unit shortage cost)

Decision		
Q =	524.40	(order quantity)
S =	47.67	(maximum shortage)

Results	
Max Inventory Level =	476.73
Annual Setup Cost =	$35.75
Annual Holding Cost =	$32.50
Annual Shortage Cost =	$3.25
Total Variable Cost =	$71.51

Robert should place orders of about 524 toothbrushes. Because of the lead time of L = 6 days the reorder point is –47.67 + (3000/360)*6 = 2.33. The maximum shortage that occurs would be roughly 48 toothbrushes.

d) We compute the inventory policies for the two extreme cases p = 0.85 and p = 25.

EOQ Model with Planned Shortages (Analytical)

Data		
D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
p =	$0.85	(unit shortage cost)

Decision		
Q =	542.33	(order quantity)
S =	81.35	(maximum shortage)

Results	
Max Inventory Level =	460.98
Annual Setup Cost =	$34.57
Annual Holding Cost =	$29.39
Annual Shortage Cost =	$5.19
Total Variable Cost =	$69.15

The reorder point when p = $0.85 equals –81.35 + (3000/360)*6 = -31.35

EOQ Model with Planned Shortages (Analytical)

Data		
D =	3000	(demand/year)
K =	$6.25	(setup cost)
h =	$0.15	(unit holding cost)
p =	$25.00	(unit shortage cost)

Decision		
Q =	501.50	(order quantity)
S =	2.99	(maximum shortage)

Results	
Max Inventory Level =	498.51
Annual Setup Cost =	$37.39
Annual Holding Cost =	$37.17
Annual Shortage Cost =	$0.22
Total Variable Cost =	$74.78

The reorder point when p = $25.00 equals –2.99 + (3000/360)*6 = 47.01. We can conclude that as the shortage cost increases, the reorder point increases.

e) Now we need to consider the EOQ model with quantity discounts.

There are 3 discount categories with prices of $1.25, $1.15, and $1.00 respectively. The holding cost rate equals I = 0.12.

EOQ Model with Quantity Discounts (Analytical)

Data		
D =	3000	(demand/year)
K =	$6.25	(setup cost)
I =	0.12	(inventory holding cost rate)
N =	3	(number of discount categories)

		Range of order quantities				Annual Purchase	Annual Setup	Annual Holding	Total Variable
Ctgry	Price	Lower Limit	Upper Limit	EOQ	Q*	Cost	Cost	Cost	Cost
1	$1.25	0	500	500.00	500.00	$3,750.00	$37.50	$37.50	$3,825.00
2	$1.15	501	999	521.29	521.29	$3,450.00	$35.97	$35.97	$3,521.94
3	$1.00	1000	10000000	559.02	1000.00	$3,000.00	$18.75	$60.00	$3,078.75

Results	
Optimal Q =	1000
TVC=	$3,078.75

The optimal order quantity equals Q =1000. Robert should order 3 times a year.

e) Option 1:

$$\text{Annual shipping cost} = K\left(\frac{D}{Q}\right) + (\$3)D$$

$$= (\$130)\left(\frac{26{,}000}{1093}\right) + (\$3)(26{,}000) = \$81{,}092.$$

Annual purchase cost = (26,000)(S30) = $780,000.

Average annual acquisition cost = S81,092 + $780,000 = S861,092.

Option 2:

$$\text{Annual shipping cost} = K\left(\frac{D}{Q}\right) + (\$3)D$$

$$= (\$50)\left(\frac{26{,}000}{678}\right) + (\$2)(26{,}000) = \$53{,}917.$$

Annual purchase cost = (26,000)($30) = $780,000.

Average annual acquisition cost = $53,917 + $780,000 = $833,917.

f) Option 1: S3975 + $861,092 = $865,067.
Option 2: S3504 + $833,917 = $837,421.

Option 2 should be selected.

19-2 For the analysis of this case we use the template for perishable products.

a) First we need to determine the optimal service level for Howie. The unit sale price equals $5, the unit purchase cost equal $3, and the unit salvage value equals 0.5*$3 - 0.5 = S1.

	Data
Unit sale price =	5
Unit purchase cost =	3
Unit salvage value =	1

Results	
Cost of overordering =	2
Cost of underordering =	2
Optimal Service Level =	0.5

Since Talia assumes that demand is uniformly distributed between 120 and 420 sets, Howie should order 120 + 0.5*300 = 270 sets.

b) If Leisure Limited refunds 75% of the purchase cost, then the unit salvage value for a returned set equals 0.75*$3-$0.5 = $1.75. We determine the new optimal service level:

Data	
Unit sale price =	5
Unit purchase cost =	3
Unit salvage value =	1.75

Results	
Cost of overordering =	1.25
Cost of underordering =	2
Optimal Service Level =	0.615385

The order quantity is now 120 + 0.615385*300 = 304.62. Note that Howie can now order more sets at one time than he could under the scenario of part (a) because he is not punished as severely as before when he fails to sell all sets.

For the case of a 25% refund the unit salvage value equals $0.25.

Data	
Unit sale price =	5
Unit purchase cost =	3
Unit salvage value =	0.25

Results	
Cost of overordering =	2.75
Cost of underordering =	2
Optimal Service Level =	0.421053

Now the order quantity equals 120 + 0.421053*300 = 246.32. Howie must now purchase fewer sets at one time (compared to the previous scenarios) because he is punished more severely if he fails to sell sets.

c) For a unit sale price of $6 and a 50% refund on returned firecracker sets, the optimal service level can be determined as follows:

Data	
Unit sale price =	6
Unit purchase cost =	3
Unit salvage value =	1

Results	
Cost of overordering =	2
Cost of underordering =	3
Optimal Service Level =	0.6

However, if Howie raises the price of a firecracker set, one would expect that the demand for his sets will decrease. Therefore, Talia should not use the same uniform demand distribution that she used for her previous calculations of the optimal order quantity.

d) Talia's strategy for estimating the demand is overly simplistic. She makes the very simplifying assumption that the demand is uniformly distributed between 120 and 420 sets. However, she does not take into account that the demand depends on the price of a firecracker set. She should expect that stands charging less than the average price of $5 per set typically sell more sets than stands charging $5 or even more than $5 per set. Talia should call Buddy again to try to obtain more detailed information, such as the range of sales and the average sale of stands charging $5 or $6 per set.

Talia should also reevaluate her assumption that the demand is uniformly distributed. She should check how her forecasts change if she uses other demand distributions, such as, for example, a normal distribution.

19.3 a) We can use the statistical functions in Excel to compute the sample mean and sample variance:

25	
31	
18	
22	
40	
19	
38	
21	
25	
36	
34	
28	
27	
28	**Average**
7.29154762	**stand.dev**

The sample mean equals 28, and the sample variance equals 53.1667.

814

b) Based on the findings of Scarlett Windermere, American Aerospace can use a (R,Q) policy for the inventory of part 10003487. The assumptions of this model are satisfied:

1. The part is a stable product
2. Its inventory level is under continuous review.
3. While the production of this part itself has no lead time, it is typically delayed by the lead time of one and a half months of the little steel part. Therefore, we can use this lead time for our analysis.
4. The demand for this part is the same as for the jet engine MX332, since it is only used for this particular engine. We therefore assume that the demand for the part is approximately normally distributed with a mean of 28 and a variance of 53.1667.
5. Excess demand is backlogged.
6. There is a fixed setup cost K = 5800, a holding cost h = 750 and a shortage cost of p = 3250.

Note that the average demand per year equals D = 12*28 = 336. The average demand during the one and a half months of lead time equals 1.5*28 = 42, and its standard deviation equals 1.5* 7.29154762 = 10.9373214.

Data			Results	
D =	336	(average demand/unit time)	Q =	79.9753808
K =	5800	(setup cost)	R =	53.3357995
h =	750	(unit holding cost)		
p =	3250	(unit shortage cost)		
L =	0.85	(service level)		

Demand During Lead Time		
Distribution =	N	(U=uniform; N=Normal)
mean =	42	
stand. dev. =	10.9373214	

American Aerospace should implement an (R,Q) policy with R = 53.34 and Q = 79.98.

c) For the computation of the approximate holding and setup cost we use the rounded values Q = 80 and R = 53.

The average inventory just before an order arrives equals 53 – 42 = 11, and just after the order has arrived it equals 11+80 = 91. So, the average inventory level equals 51 resulting in average yearly holding cost of (approximately) 51*750 = 38250.

The average number of setups per year equals 336/80 = 4.2 resulting in average yearly setup cost of 4.2*5800 = 24360.

d) The new service level equals L = 0.95.

Data			Results	
D =	336	(average demand/unit time)	Q =	79.9753808
K =	5800	(setup cost)	R =	59.990286
h =	750	(unit holding cost)		
p =	3250	(unit shortage cost)		
L =	0.95	(service level)		

Demand During Lead Time

Distribution =	N	(U=uniform; N=Normal)
mean =	42	
stand. dev. =	10.9373214	

Again, we round and use Q = 80 and R = 60.

The average inventory just before an order arrives equals 80 – 60 = 20, and just after the order has arrived it equals 20+80 = 100. So, the average inventory level equals 60 resulting in average yearly holding cost of (approximately) 60*750 = 45000. Notice that the average yearly holding cost has increased substantially. This should come as no surprise due to the increased safety stock (20 instead of 11).

The average number of setups per year is still 336/80 = 4.2, resulting again in an average yearly setup cost of 4.2*5800 = 24360.

e) Scarlett's independent analysis of the stationary part 10003487 can only be justified since there is only a single jet engine that needs this part, and this part appears to be the bottleneck in the production process. But more often, the same stationary part will be used for several jet engines. As a result the demand for stationary parts will depend on the demand for several jet engines, and a stockout in one stationary part affects the demand for other parts. These interdependencies cannot be captured by an independent analysis of each part. Therefore, Scarlett's approach will most likely result in rather inaccurate inventory policies for many other stationary parts.

f) Scarlett could try to forecast the demand for jet engines based on sales data from previous years.

CHAPTER 20

20.1-1

20.1-2

20.4-1

a) Forecast = last value = 39

b) Forecast = average of all data to date = $\frac{5+17+29+41+39}{5} = \frac{131}{5} = 26$

c) Forecast = average of last 3 values = $\frac{29+41+39}{3} = \frac{109}{3} = 36$

d) It appears as if demand is rising so the average forecasting method seems inappropriate because it uses older, out of date data.

20.4-2

a) Forecast = last value = 13

b) Forecast = average of all data to date = $\frac{15+18+12+17+13}{5} = \frac{75}{5} = 15$

c) Forecast = average of last 3 values = $\frac{12+17+13}{3} = \frac{42}{3} = 14$

d) The averaging method seems best since all five months of data are relevant in determining the forecast of sales for next month.

20.4-3 Forecast = $\frac{1977-1945}{4} = 8 + 2083 = 2091$

20.4-4 Forecast = $\frac{793-805}{3} = -4 + 782 = 778$

20.4-5 Forecast = $\frac{1532-1632}{10} = -10 + 1551 = 1541$

20.4-6 Forecast (0.1) = α(last value) + (1 - α)(last forecast) = (0.1)(792) + (1-0.1)(782) = 783

Forecast (0.3) = α(last value) + (1 - α)(last forecast) = (0.3)(792) + (1-0.3)(782) = 785

Forecast (0.5) = α(last value) + (1 - α)(last forecast) = (0.5)(792) + (1-0.5)(782) = 787

20.4-7 Forecast (0.1) = α(last value) + (1 - α)(last forecast) = (0.1)(1973) + (1-0.1)(2083) = 2072

Forecast (0.3) = α(last value) + (1 - α)(last forecast) = (0.3)(1973) + (1-0.3)(2083) = 2050

Forecast (0.5) = α(last value) + (1 - α)(last forecast) = (0.5)(1973) + (1-0.5)(2083) = 2028

20.4-8

$\alpha = 0 \Rightarrow F_{t+1} = F_t = \cdots = F_1$

in other words, the forecast remains equal to the best initial guess for the variable and never changes.

$\alpha = 1 \Rightarrow F_{t+1} = x_t$

hence, the forecast always equals the current value of the variable.

20.4-9

(a) $F_{t+1} = \alpha X_t + (1-\alpha) F_t \Rightarrow X_t = \frac{1}{\alpha}(F_{t+1} - (1-\alpha)F_t) = 2F_{t+1} - F_t$ in this case

$\Rightarrow$ Actual demand in April $= 2(390) - 380 = 400$

Actual demand in May $= 2(380) - 390 = 370$

(b) $F_{Feb} = .5 X_{Jan} + .5 F_{Jan}$

$F_{mar} = .5 X_{Feb} + .5 F_{Feb}$

$= .5 X_{Feb} + .5[.5 X_{Jan} + .5 F_{Jan}] = .5 X_{Feb} + .25 X_{Jan} + .25 F_{Jan}$

We know that $X'_{Jan} = X_{Jan} + 32$

$X'_{Feb} = X_{Feb}$

and $F'_{Jan} = F_{Jan}$

$\Rightarrow F'_{March} = F_{March} + .25(32) = 408$

and so we can complete the table

	Jan	Feb	March	April	May	June
Forecast			408	384	392	381
Actual	400		360	400	370	

20.5-1

a)

Quarter	Call Volume	Seasonal Factor
1	6809	$\frac{6809}{7027} = 0.97$
2	6465	$\frac{6465}{7027} = 0.92$
3	6569	$\frac{6569}{7027} = 0.93$
4	8266	$\frac{8266}{7027} = 1.18$

b)

Quarter	Seasonal Factor	Actual Call Volume	Seasonally Adjusted Call Volume
1	0.97	7257	$\frac{7257}{0.97} = 7481$
2	0.92	7064	$\frac{7064}{0.92} = 7678$
3	0.93	7784	$\frac{7784}{0.93} = 8370$
4	1.18	8724	$\frac{8724}{1.18} = 7393$

c)

Quarter	Two-Year Average	Seasonal Factor
1	7033	$\frac{7033}{7367} = 0.95$
2	6765	$\frac{6765}{7367} = 0.92$
3	7177	$\frac{7177}{7367} = 0.97$
4	8495	$\frac{8495}{7367} = 1.15$

d)

Quarter	Seasonal Factor	Actual Call Volume	Seasonally Adjusted Call Volume
1	0.95	6992	$\frac{6992}{0.95} = 7360$
2	0.92	6822	$\frac{6822}{0.92} = 7415$
3	0.97	7949	$\frac{7949}{0.97} = 8195$
4	1.15	9650	$\frac{9650}{1.15} = 8391$

20.S-2

a)

Quarter	Unemployment Rate	Seasonal Factor
1	0.062	$\frac{0.062}{0.063} = 0.98$
2	0.06	$\frac{0.06}{0.063} = 0.95$
3	0.075	$\frac{0.075}{0.063} = 1.19$
4	0.055	$\frac{0.055}{0.063} = 0.87$

b)

Quarter	Seasonal Factor	Actual Unemployment Rate	Seasonally Adjusted Call Volume
1	0.98	0.078	$\frac{0.078}{0.98} = 0.08$
2	0.95	0.074	$\frac{0.074}{0.95} = 0.078$
3	1.19	0.087	$\frac{0.087}{1.19} = 0.073$
4	0.87	0.061	$\frac{0.061}{0.87} = 0.07$

This progression indicates that the state's economy is improving with the unemployment rate decreasing from 8% to 7% (seasonally adjusted) over the four quarters.

20.5-3

a)

Quarter	Three-Year Average	Seasonal Factor
1	21	$\frac{21}{25} = 0.84$
2	23	$\frac{23}{25} = 0.92$
3	30	$\frac{30}{25} = 1.2$
4	26	$\frac{26}{25} = 1.04$

b) Seasonally adjusted value = $\frac{28}{1.04} = 27$, so the actual forecast = (27)(0.84) = 23.

c) Q1 = 23 as shown in part b)

Q2 = $\frac{23}{0.84} = 27$, so the actual forecast = (27)(0.92) = 25

Q3 = $\frac{25}{0.92} = 27$, so the actual forecast = (27)(1.2) = 33

Q4 = $\frac{33}{1.2} = 27$, so the actual forecast = (27)(1.04) = 28

d)

Quarter	Seasonal Factor	Average House Sales	Seasonally Adjusted Forecast
1	0.84	25	(25)(0.84)=21
2	0.92	25	(25)(0.92)=23
3	1.2	25	(25)(1.2)=30
4	1.04	25	(25)(1.04)=26

20.5-4 a) - b) - c) - d)

Year	Quarter	Sales	I	F	S
1996	1	6900	0.965		
	2	6700	0.937		
	3	7900	1.105		
	4	7100	0.993		7150
1997	1	8200	0.997	6900	7285
	2	7000	0.941	6826	7303
	3	7300	1.086	8069	7234
	4	7500	1.001	7183	7266
1998	1	9400	1.049	7245	7482
	2	9200	0.992	7043	7711
	3	9800	1.119	8372	7842
	4	9900	1.047	7849	8047
1999	1	11400	1.113	8442	8329
	2	10000	1.029	8260	8505
	3	9400	1.116	9513	8495
	4	8400	1.036	8892	8448
2000	1	8800		9402	8394
	2	7600		8633	8293
	3	7500		9256	8136
	4			8431	

$\alpha = 0.1$
$\beta = 0.2$

e) There is a seasonal effect: 1 → 2 ; and it is incorporated by the parameter I.

f) There is substantial error in our estimates; the constant level assumption is not good enough with $\alpha = 0.1$ $\beta = 0.2$

20.6-1 Forecast =

α(last value) + (1 - α)(last forecast)

+β[(α)(last value - next to last value) + (1 - α)(last forecast - next to last forecast)]

+(1 - β)(last est. of trend)

Forecast (year 1) = initial average + initial trend = 3900 + 700 = 4600

Forecast (year 2) = (0.25)(4600)+(1-0.25)(4600)+(0.25)[(0.25)(4600-3900)
+(1-0.25)(4600-3900)]+(1-0.25)(700) = 5300

Forecast (year 3) = (0.25)(5300)+(1-0.25)(5300)+(0.25)[(0.25)(5300-4600)
+(1-0.25)(5300-4600)]+(1-0.25)(700) = 6000

20.6-2 Forecast =

α(last value) + (1 - α)(last forecast)

+β[(α)(last value - next to last value) + (1 - α)(last forecast - next to last forecast)]

+(1 - β)(last est. of trend)

= (0.2)(550) + (1-0.2)(540) + (0.3)[(0.2)(550 – 535) + (1-0.2)(540 – 530)] + (1-0.3)(10)
= 552

20.6-3 Forecast =

α(last value) + (1 - α)(last forecast)

+β[(α)(last value - next to last value) + (1 - α)(last forecast - next to last forecast)]

+(1 - β)(last est. of trend)

= (0.1)(4935) + (1-0.1)(4975) + (0.2)[(0.1)(4935 – 4655) + (1-0.1) (4975 – 4720)]
+ (1-0.2)(240)
= 5215

20.6-4

Time Period	Data	Latest Trend	Estimated Trend	Exponential Smoothing Forecast	Forecasting Error
1	15		5	15	0
2	21	5	5	20	1
3	24	5	5	25	1
4	32	5	5	30	2
5	37	5	5	35	2
6	41	5	5	41	0
7	40	5	5	46	6
8	47	4	5	50	3
9	51	4	5	54	3
10	53	4	5	58	5
11		4	4	62	

Smoothing Constants	
α =	0.2
β =	0.2

Initial Estimates	
Average =	10
Trend =	5

Mean Absolute Deviation	
MAD =	2.272

Forecast for next production yield = 62%.

20.7-1

Quarter	Forecast	True Value	Error
1	327	345	18
2	332	317	15
3	328	336	8
4	330	311	19

$$\text{MAD} = \frac{\text{Sum of forecasting errors}}{\text{Number of forecasts}} = \frac{18+15+8+19}{4} = \frac{60}{4} = 15$$

$$\text{MSE} = \frac{\text{Sum of squares of Forecasting errors}}{\text{Number of forecasts}} = 243.5$$

20.7-2a) Method 1 $\text{MAD} = \frac{258+499+560+809+609}{5} = \frac{2735}{5} = 547$

Method 2 $\text{MAD} = \frac{374+471+293+906+396}{5} = \frac{2440}{5} = 488$

b) $\text{MSE}_1 = 330,905$ $\text{MSE}_2 = 285,044$

c) She can use the older data to calculate more forecasting errors and compare MSE and MAD for a longer time span. This may make her feel more comfortable with her decision.

20.7-3 a) Forecast (year 1) = initial estimate = 5000
Forecast (year 2) = α(last value) + (1 - α)(last forecast)
= (0.25)(4600) + (1-0.25)(5000) = 4900
Forecast (year 3) = (0.25)(5300) + (1-0.25)(4900) = 5000

b) $MAD = \frac{400 + 400 + 1000}{3} = 600$

c) $MSE = \frac{(400)^2 + (400)^2 + (1000)^2}{3} = 440{,}000$

d) Forecast (next year) = (0.25)(6000) + (1-0.25)(5000) = 5250

20.7-4 a) Since sales are relatively stable, the averaging method would be appropriate for forecasting future sales. This method uses a larger sample size than the last-value method, which should make it more accurate and since the older data is still relevant, it should not be excluded, as would be the case in the moving-average method.

b)

Time Period	Data	Last-Value Forecast	Forecasting Error
1	23		
2	24	23	1
3	22	24	2
4	28	22	6
5	22	28	6
6	27	22	5
7	20	27	7
8	26	20	6
9	21	26	5
10	29	21	8
11	23	29	6
12	28	23	5

Mean Absolute Deviation	
MAD =	5.182

c)

Time Period	Data	Averaging Forecast	Forecasting Error
1	23		
2	24	23	1
3	22	24	2
4	28	23	5
5	22	24	2
6	27	24	3
7	20	24	4
8	26	24	2
9	21	24	3
10	29	24	5
11	23	24	1
12	28	24	4

Mean Absolute Deviation	
MAD =	3

d)

Time Period	Data	Moving-Average Forecast	Forecasting Error
1	23		
2	24		
3	22		
4	26	23	5
5	22	25	3
6	27	24	3
7	20	26	6
8	26	23	3
9	21	24	3
10	29	22	7
11	23	25	2
12	26	24	4

Number of previous periods to consider: n = 3

Mean Absolute Deviation: MAD = 3.926

e) Considering the MAD values, the averaging method is the best one to use.

f) Unless there is reason to believe that sales will not continue to be relatively stable, the averaging method should be the most accurate in the future as well.

20.7-5 a) and b)

Smoothing Constant	MAD	MSE
0.1	2.70	9.44
0.2	2.82	10.24
0.3	2.97	11.20
0.4	3.13	12.35
0.5	3.32	13.75

Ben Swanson should choose 0.1 for the smoothing constant.

20.7-6 a) Since the sales level is shifting significantly from month to month, and there is no consistent trend, the last-value method seems like it will perform well. The averaging method will not do as well because it places too much weight on old data. The moving-average method will be better than the averaging method but will lag any short-term trends. The exponential smoothing method will also lag trends by placing too much weight on old data. Exponential smoothing with trend will likely not do well because the trend is not consistent.

b) <u>last-value method</u>

Time Period	Data	Last-Value Forecast	Forecasting Error
1	126		
2	137	126	11
3	142	137	5
4	150	142	8
5	153	150	3
6	154	153	1
7	148	154	6
8	145	148	3
9	147	145	2
10	151	147	4
11	159	151	8
12	166	159	7

Mean Absolute Deviation: MAD = 5.273

(CONT'D)

(CONT'D)

averaging method

Time Period	Data	Averaging Forecast	Forecasting Error
1	126		
2	137	126	11
3	142	132	10
4	150	135	15
5	153	139	14
6	154	142	12
7	148	144	4
8	145	144	1
9	147	144	3
10	151	145	6
11	159	145	14
12	166	147	19

Mean Absolute Deviation
MAD = 10

moving-average method

Time Period	Data	Moving-Average Forecast	Forecasting Error
1	126		
2	137		
3	142		
4	150	135	15
5	153	143	10
6	154	148	6
7	148	152	4
8	145	152	7
9	147	149	2
10	151	147	4
11	159	148	11
12	166	152	14

Number of previous periods to consider
n = 3

Mean Absolute Deviation
MAD = 8.111

Comparing MAD values, the last-value method is the best to use of these three options.

c)

Smoothing Constant	MAD
0.1	18.5
0.2	13.0
0.3	10.1
0.4	8.66
0.5	7.96

It appears that a high value for the smoothing constant is appropriate.

d)

α	β	MAD
0.1	0.1	25.4
0.1	0.3	21.2
0.1	0.5	17.7
0.3	0.1	13.5
0.3	0.3	9.75
0.3	0.5	8.80
0.5	0.1	8.42
0.5	0.3	6.70
0.5	0.5	6.49

It appears that a high value for both smoothing constants is appropriate.

e) Management should use the last-value method to forecast sales. Using this method the forecast for January of the new year will be 166.

20.7-7

a)

Smoothing Constant	MAD
0.1	1.51
0.2	1.62
0.3	1.73
0.4	1.84
0.5	1.95

Choose $\alpha = 0.1$

b)

Smoothing Constant	MAD
0.1	1.84
0.2	1.88
0.3	1.92
0.4	2
0.5	2.1

Choose $\alpha = 0.1$

c)

Smoothing Constant	MAD
0.1	2.82
0.2	2.54
0.3	2.26
0.4	2.06
0.5	1.9

Choose $\alpha = 0.5$

20.7-8

a)

Smoothing Constant	MAD
0.1	0.740
0.2	0.749
0.3	0.759
0.4	0.77
0.5	0.782

Choose $\beta = 0.1$

b)

Smoothing Constant	MAD
0.1	2.61
0.2	2.76
0.3	2.87
0.4	2.99
0.5	3.05

Choose $\beta = 0.1$

c)

Smoothing Constant	MAD
0.1	5.66
0.2	6.02
0.3	6.23
0.4	6.36
0.5	6.54

20.7-9 Choose $\beta = 0.1$

a) The time series is not stable enough for the moving-average method.

b)

Time Period	Data	Moving-Average Forecast	Forecasting Error
1	382		
2	405		
3	398		
4	421	395	26
5	426	408	18
6	415	415	0
7	443	421	22
8	451	428	23
9	446	436	10
10	464	447	17
11		454	

Number of previous periods to consider

n = 3

Mean Absolute Deviation

MAD = 16.619

c)

Time Period	Data	Exponential Smoothing Forecast	Forecasting Error
1	382	380	2
2	405	381	24
3	398	393	6
4	421	389	32
5	426	401	26
6	415	403	12
7	443	398	46
8	451	412	40
9	446	416	31
10	464	413	51
11		422	

Smoothing Constant	
α =	0.5

Initial Estimate	
Average =	380

Mean Absolute Deviation	
MAD =	26.75

d)

Time Period	Data	Latest Trend	Estimated Trend	Exponential Smoothing Forecast	Forecasting Error
1	382		10	380	2
2	405	11	10	391	14
3	398	14	11	405	7
4	421	9	11	414	7
5	426	12	11	427	1
6	415	11	11	438	23
7	443	5	10	441	2
8	451	10	10	451	0
9	446	10	10	461	15
10	464	6	9	466	2
11		8	9	474	

Smoothing Constants	
α =	0.25
β =	0.25

Initial Estimates	
Average =	370
Trend =	10

Mean Absolute Deviation	
MAD =	7.285

e) Exponential smoothing with trend should be used in the future.

20.7-10 a)

Year	Quarter	True Value
1	1	25
1	2	47
1	3	68
1	4	42
2	1	27
2	2	46
2	3	72
2	4	39
3	1	24
3	2	49
3	3	70
3	4	44

Type of Seasonality
Quarterly

Month	Estimate for Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

b)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	25	45			
1	2	47	46	45	47	0
1	3	68	45	46	70	2
1	4	42	46	45	40	2
2	1	27	49	46	26	1
2	2	46	45	49	50	4
2	3	72	47	45	68	4
2	4	39	43	47	43	4
3	1	24	44	43	24	0
3	2	49	48	44	45	4
3	3	70	46	48	72	2
3	4	44	49	46	42	2
4	1			49	27	
4	2					
4	3					
4	4					
5	1					
5	2					

Type of Seasonality: Quarterly

Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation

MAD = 2.392

MSE = 31/11

Forecast = 27 acre-feet

c) Winter = (49)(0.55) = 27
Spring = (49)(1.03) = 50
Summer = (49)(1.52) = 74
Fall = (49)(0.9) = 44

d)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	25	45			
1	2	47	46	45	47	0
1	3	68	45	46	69	1
1	4	42	46	45	41	1
2	1	27	49	46	25	2
2	2	46	45	46	48	2
2	3	72	47	46	70	2
2	4	39	43	46	42	3
3	1	24	44	46	25	1
3	2	49	48	46	47	2
3	3	70	46	46	70	0
3	4	44	49	46	41	3
4	1			46	25	
4	2					
4	3					
4	4					
5	1					
5	2					

Type of Seasonality: Quarterly

Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation

MAD = 1.570

MSE = 37/11

Forecast = 25 acre-feet

e)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	25	45			
1	2	47	46			
1	3	68	45			
1	4	42	46			
2	1	27	49	46	25	2
2	2	46	45	47	48	2
2	3	72	47	46	70	2
2	4	39	43	47	42	3
3	1	24	44	46	25	1
3	2	49	48	45	48	3
3	3	70	46	45	69	1
3	4	44	49	45	41	3
4	1			47	26	
4	2					
4	3					
4	4					
5	1					
5	2					
5	3					
5	4					
6						

Number of previous periods to consider
n = 4

Type of Seasonality
Quarterly

Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation
MAD = 2.169

MSE = 41/8

Forecast = 26 acre-feet

f)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	25	45	46	25	0
1	2	47	46	46	47	0
1	3	68	45	46	70	2
1	4	42	46	46	41	1
2	1	27	49	46	25	2
2	2	46	45	46	47	1
2	3	72	47	46	70	2
2	4	39	43	46	42	3
3	1	24	44	46	25	1
3	2	49	48	46	47	2
3	3	70	46	46	70	0
3	4	44	49	46	41	3
4	1			46	25	
4	2					
4	3					
4	4					
5	1					
5	2					
5	3					
5	4					
6	1					
6	2					
6	3					
6	4					

Smoothing Constant
α = 0.1

Initial Estimate
Average = 46

Type of Seasonality
Quarterly

Quarter	Seasonal Factor
1	0.550
2	1.027
3	1.519
4	0.904

Mean Absolute Deviation
MAD = 1.420

MSE = $\frac{37}{12}$ = 3.08

Forecast = 25 acre-feet

g) The exponential smoothing method results in the lowest MAD value (1.420).

h) Exponential smoothing gives the lowest MSE value (3.08)

20.7-11 a)

Year	Quarter	Value
1	1	23
1	2	22
1	3	31
1	4	26
2	1	19
2	2	21
2	3	27
2	4	24
3	1	21
3	2	26
3	3	32
3	4	28

Type of Seasonality: Quarterly

Quarter	Estimate for Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

b)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27			
1	2	22	24	27	25	3
1	3	31	26	24	29	2
1	4	26	25	26	27	1
2	1	19	23	25	21	2
2	2	21	23	23	21	0
2	3	27	23	23	27	0
2	4	24	23	23	23	1
3	1	21	25	23	19	2
3	2	26	28	25	23	3
3	3	32	27	28	34	2
3	4	28	27	27	28	0
4	1			27	23	
4	2					
4	3					
4	4					
5	1					
5	2					

Type of Seasonality: Quarterly

Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation: MAD = 1.485

c)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27			
1	2	22	24	27	25	3
1	3	31	26	26	31	0
1	4	26	25	26	27	1
2	1	19	23	26	21	2
2	2	21	23	25	23	2
2	3	27	23	25	30	3
2	4	24	23	24	25	1
3	1	21	25	24	20	1
3	2	26	28	24	22	4
3	3	32	27	25	30	2
3	4	28	27	25	26	2
4	1			25	21	
4	2					
4	3					
4	4					
5	1					
5	2					

Type of Seasonality: Quarterly

Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation: MAD = 1.942

d)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27			
1	2	22	24			
1	3	31	26			
1	4	26	25			
2	1	19	23	26	21	2
2	2	21	23	24	22	1
2	3	27	23	24	29	2
2	4	24	23	23	24	0
3	1	21	25	23	19	2
3	2	26	28	23	21	5
3	3	32	27	25	30	2
3	4	28	27	26	27	1
4	1			27	22	
4	2					
4	3					
4	4					
5	1					
5	2					
5	3					
5	4					
6	1					

Number of previous periods to consider

n = 4

Type of Seasonality

Quarterly

Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation

MAD = 1.983

e)

Year	Quarter	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27	25	21	2
1	2	22	24	26	24	2
1	3	31	26	25	30	1
1	4	26	25	25	26	0
2	1	19	23	25	21	2
2	2	21	23	25	23	2
2	3	27	23	24	29	2
2	4	24	23	24	25	1
3	1	21	25	24	20	1
3	2	26	28	24	22	4
3	3	32	27	25	30	2
3	4	28	27	25	26	2
4	1			26	22	
4	2					
4	3					
4	4					
5	1					
5	2					
5	3					
5	4					
6	1					
6	2					
6	3					
6	4					

Smoothing Constant

α = 0.25

Initial Estimate

Average = 25

Type of Seasonality

Quarterly

Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation

MAD = 1.561

f)

Year	Qtr	True Value	Seasonally Adjusted Value	Latest Trend	Est. Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	1	23	27		0	25	21	2
1	2	22	24	1	0	26	24	2
1	3	31	26	0	0	25	30	1
1	4	26	25	0	0	26	27	1
2	1	19	23	0	0	25	21	2
2	2	21	23	-1	0	25	23	2
2	3	27	23	-1	0	24	29	2
2	4	24	23	-1	0	23	24	0
3	1	21	25	0	0	23	19	2
3	2	26	28	0	0	23	21	5
3	3	32	27	1	0	25	29	3
3	4	28	27	1	0	25	26	2
4	1			1	0	26	22	

Smoothing Constants

α =	0.25
β =	0.25

Initial Estimates

Average =	25
Trend =	0

Type of Seasonality

Quarterly

Quarter	Seasonal Factor
1	0.840
2	0.920
3	1.200
4	1.040

Mean Absolute Deviation

MAD =	1.778

g) Using the last-value method with seasonality (MAD=1.485), the forecast for Q1 is 23 houses.

h) Q2 = (27)(0.92) = 25
Q3 = (27)(1.2) = 32
Q4 = (27)(1.04) = 28

20.1-12 a)

Method	MAD
Last-value	3.07
Averaging	3.12
Moving-average	2.18
Exponential smoothing	2.34

b)

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76			
1	Feb	71	81			
1	Mar	66	73			
1	Apr	72	77	76	71	1
1	May	77	80	77	74	3
1	June	85	78	77	84	1
1	July	94	80	79	92	2
1	Aug	96	83	80	91	5
1	Sep	80	82	81	78	2
1	Oct	73	80	82	75	2
1	Nov	84	80	82	86	2
1	Dec	89	82	81	87	2
2	Jan			81	73	
2	Feb					
2	Mar					
2	Apr					
2	May					
2	June					
2	July					
2	Aug					
2	Sep					

Number of previous periods to consider

n =	3

Type of Seasonality

Monthly

Month	Seasonal Factor
Jan	0.90
Feb	0.88
Mar	0.91
Apr	0.93
May	0.96
June	1.09
July	1.17
Aug	1.15
Sep	0.97
Oct	0.91
Nov	1.05
Dec	1.08

Mean Absolute Deviation

MAD =	2.183

According to the moving-average method, the forecast for Jan. is 73 passengers.

20.7-3a)

Method	MAD
Last-value	2.49
Averaging	7.06
Moving-average	2.79
Exponential smoothing	4.28

b)

Year	Month	True Value	Seasonally Adjusted Value	Latest Trend	Est. Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	75	83		2	82	74	1
1	Feb	76	86	2	2	84	74	2
1	Mar	81	89	2	2	87	79	2
1	Apr	84	90	3	2	90	83	1
2	May	85	89	2	2	92	88	3
2	Jun	99	91	2	2	93	102	3
2	Jul	107	91	2	2	95	111	4
2	Aug	108	94	1	2	96	110	2
3	Sep	94	97	1	2	97	95	-
3	Oct	90	99	2	2	99	90	0
3	Nov	106	101	2	2	101	106	0
3	Dec	110	102	2	2	103	111	1
4	Jan			2	2	104	94	

Type of Seasonality

Monthly

Month	Seasonal Factor
Jan	0.90
Feb	0.88
Mar	0.91
Apr	0.93
May	0.96
Jun	1.09
Jul	1.17
Aug	1.15
Sep	0.97
Oct	0.91
Nov	1.05
Dec	1.06

Smoothing Constants

α =	0.2
β =	0.2

Initial Estimates

Average =	80
Trend =	2

Mean Absolute Deviation

MAD =	1.656

MAD here is lower that all those found in part a).

c)

Year	Month	True Value	Seasonally Adjusted Value	Latest Trend	Est. Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	68	76		0	80	72	4
1	Feb	71	81	-1	0	79	69	2
1	Mar	66	73	0	0	79	72	6
1	Apr	72	77	-1	0	77	72	0
2	May	77	80	0	0	77	74	3
2	Jun	85	78	0	0	77	84	1
2	Jul	94	80	0	0	77	90	4
2	Aug	96	83	0	0	78	89	7
3	Sep	80	82	1	0	79	77	3
3	Oct	73	80	1	0	80	73	0
3	Nov	84	80	0	0	80	84	0
3	Dec	89	82	0	0	80	87	2
4	Jan	75	83	1	0	81	73	2
4	Feb	76	86	1	0	83	73	3
4	Mar	81	89	2	1	86	79	2
4	Apr	84	90	3	1	89	83	1
5	May	85	89	2	1	90	87	2
5	Jun	99	91	1	1	89	97	2
5	Jul	107	91	-1	1	91	106	1
5	Aug	108	94	2	1	91	105	3
6	Sep	94	97	1	1	94	91	3
6	Oct	90	99	3	1	97	88	2
6	Nov	106	101	3	2	99	104	2
6	Dec	110	102	2	2	101	109	1
7	Jan			2	2	102	92	

Type of Seasonality

Monthly

Month	Seasonal Factor
Jan	0.90
Feb	0.88
Mar	0.91
Apr	0.93
May	0.96
Jun	1.09
Jul	1.17
Aug	1.15
Sep	0.97
Oct	0.91
Nov	1.05
Dec	1.08

Smoothing Constants

α =	0.2
β =	0.2

Initial Estimates

Average =	80
Trend =	0

Mean Absolute Deviation

MAD =	2.289

d) Exponential smoothing with trend should be used.

20.7-14 a)

Month	Estimate for Seasonal Factor
Jan	0.808
Feb	0.807
Mar	0.876
Apr	0.921
May	1.016
June	1.105
July	1.018
Aug	1.189
Sep	0.806
Oct	0.761
Nov	1.437
Dec	1.256

b)

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan					
1	Feb					
1	Mar					
1	Apr					
1	May					
1	June					
1	July					
1	Aug					
1	Sep					
1	Oct	335	440			
1	Nov	594	413	440	633	39
1	Dec	527	420	427	536	9
2	Jan	364	450	424	343	21
2	Feb	343	425	428	345	2
2	Mar	391	446	432	378	13
2	Apr	437	474	441	406	31
2	May	458	451	449	456	2
2	June	494	447	457	505	11
2	July	468	460	457	466	2
2	Aug	555	467	453	538	17
2	Sep	387	480	458	369	18
2	Oct	364	478	469	357	7
2	Nov	662	461	475	683	21
2	Dec	581	463	473	594	13
3	Jan			467	377	

Number of previous periods to consider

n =	3

Type of Seasonality

Monthly

Month	Seasonal Factor
Jan	0.808
Feb	0.807
Mar	0.876
Apr	0.921
May	1.016
June	1.105
July	1.018
Aug	1.189
Sep	0.806
Oct	0.761
Nov	1.437
Dec	1.256

Mean Absolute Deviation

MAD =	14.761

c)

Year	Month	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	364	450	420	339	25
1	Feb	343	425	426	344	
1	Mar	391	446	426	373	18
1	Apr	437	474	430	396	41
1	May	458	451	439	446	12
1	June	494	447	441	488	6
1	July	468	460	442	450	18
1	Aug	555	467	446	530	25
1	Sep	387	480	450	363	24
1	Oct	364	478	456	347	17
1	Nov	662	461	461	662	0
1	Dec	581	463	461	578	3
2	Jan			461	372	
2	Feb					
2	Mar					
2	Apr					
2	May					
2	June					
2	July					
2	Aug					
2	Sep					
2	Oct					
2	Nov					
2	Dec					

Smoothing Constant

α =	0.2

Initial Estimate

Average =	420

Type of Seasonality

Monthly

Month	Seasonal Factor
Jan	0.808
Feb	0.807
Mar	0.876
Apr	0.921
May	1.016
June	1.105
July	1.018
Aug	1.189
Sep	0.806
Oct	0.761
Nov	1.437
Dec	1.256

Mean Absolute Deviation

MAD =	15.777

d)

Year	Month	True Value	Seasonally Adjusted Value	Latest Trend	Est. Trend	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
1	Jan	364	450		0	420	339	25
1	Feb	343	425	6	1	427	345	2
1	Mar	391	446	1	1	428	375	16
1	Apr	437	474	5	2	434	399	38
2	May	458	451	10	4	445	452	6
2	Jun	494	447	5	4	450	497	3
2	Jul	468	460	3	4	453	461	7
2	Aug	555	467	5	4	458	545	10
3	Sep	387	480	6	4	464	374	13
3	Oct	364	478	7	5	472	359	5
3	Nov	662	461	6	5	479	688	26
3	Dec	581	463	2	4	479	602	21
4	Jan			1	4	480	388	

Type of Seasonality

Monthly

Month	Seasonal Factor
Jan	0.808
Feb	0.807
Mar	0.876
Apr	0.921
May	1.016
Jun	1.105
Jul	1.018
Aug	1.139
Sep	0.806
Oct	0.761
Nov	1.437
Dec	1.256

Smoothing Constants

α =	0.2
β =	0.2

Initial Estimates

Average =	420
Trend =	0

Mean Absolute Deviation

MAD =	14.200

e) Exponential smoothing with trend results in the best MAD value (14.2).

f)

Month	Avg. Forecast	Forecasting Error
January	340	24
February	345	2
March	375	16
April	400	37
May	451	7
June	497	7
July	459	9
August	538	17
September	369	18
October	354	10
November	678	16
December	591	12

MAD = 14.58

g) Exponential smoothing with trend performed better than the average of all three so it should be used next year.

20.7-15

Quarter	Sales	Forecast a)	Squared Error a)	Forecast b)	Squared Error b)	Forecast c)	Squared Error c)	Forecast d)	Squared Error d)
1	6900								
2	6700								
3	7900			6880	1040400	6840	1123600	6500	1960000
4	7100			6982	13924	7158	3364	6846	64516
5	8200	7150	1102500	6994	1454918	7141	1122328	6871	1766082
6	7000	7475	225625	7114	13092	7458	210149	7338	114384
7	7300	7550	62500	7103	38818	7321	437	7275	637
8	7500	7400	10000	7123	142370	7315	34364	7323	31458
9	9400	7500	3610000	7160	5015754	7370	4119934	7432	3872612
10	9200	7800	1960000	7384	3296509	7979	1490434	8256	891432
11	9800	8350	2102500	7566	4991051	8345	2115813	8857	888430
12	9900	8975	855625	7789	4454884	8782	1250390	9543	127179
13	11400	9575	3330625	8000	11557236	9117	5210929	10086	1727552
14	10000	10075	5625	8340	2754386	9802	39173	11034	1068141
15	9400	10275	765625	8506	798647	9861	212940	11184	3182671
16	8400	10175	3150625	8596	38297	9723	1750377	10949	6496332
17	8800	9800	1000000	8576	50119	9326	276795	10255	2116296
18	7600	9150	2402500	8599	997030	9168	2459499	9758	4656936
19	7500	8550	1102500	8499	997327	8698	1434713	8856	1838858
	MSE		1445750		2214986		1344426		1811971

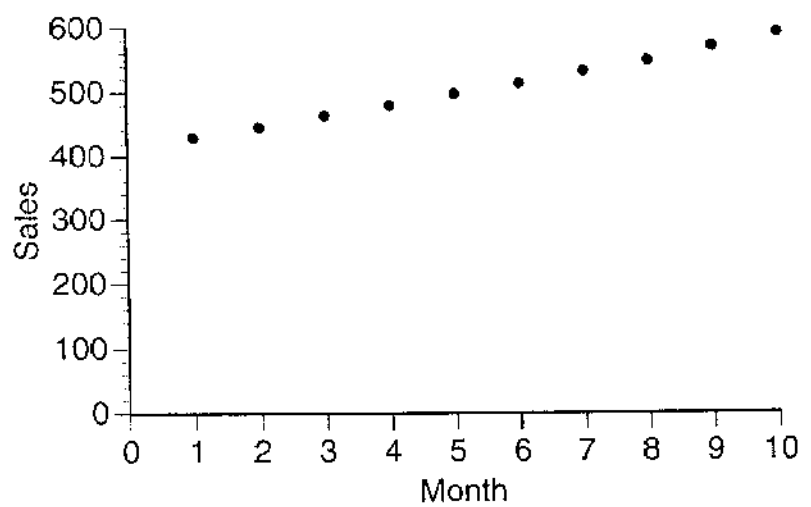
smallest.

20.7-16

Quarter	Sales	Forecast a)	Squared Error a)	Forecast b)	Squared Error b)	Forecast c)	Squared Error c)	Forecast d)	Squared Error d)
1	546								
2	528								
3	530			544	202	541	112	510	400
4	508			543	1210	537	866	500	67
5	647	528	14161	539	11599	523	15277	487	25665
6	594	553	1661	550	1930	560	1124	534	3622
7	665	570	9073	554	12218	631	1149	556	11828
8	630	604	702	566	4158	641	127	603	727
9	736	634	10404	572	26907	655	6642	628	11727
10	724	656	4590	588	18396	679	2030	687	1404
11	813	689	15438	602	44549	732	6496	727	7314
	MSE		8004		13463		3758		6973

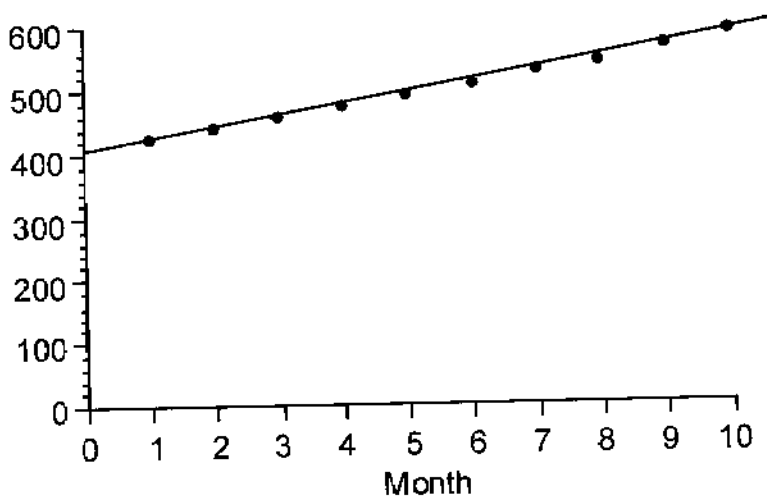
smallest.

20.9-1 a)



b) $y = 410 + 17.6x$

c)

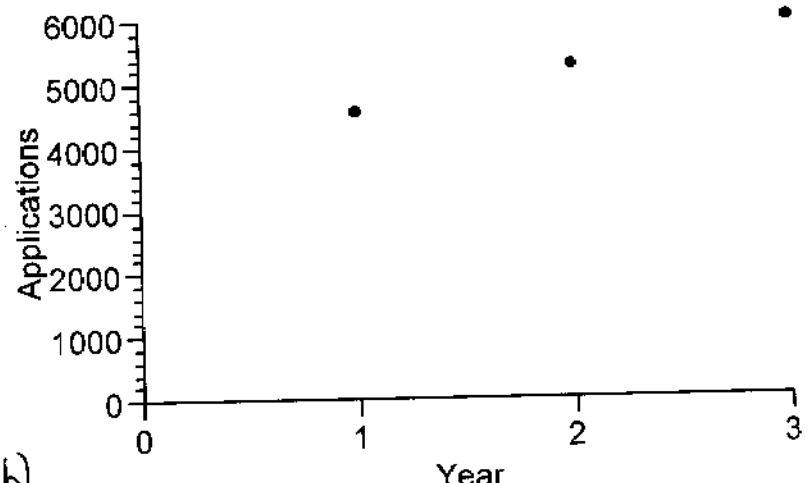


d) $y = 410 + (17.6)(11) = 604$

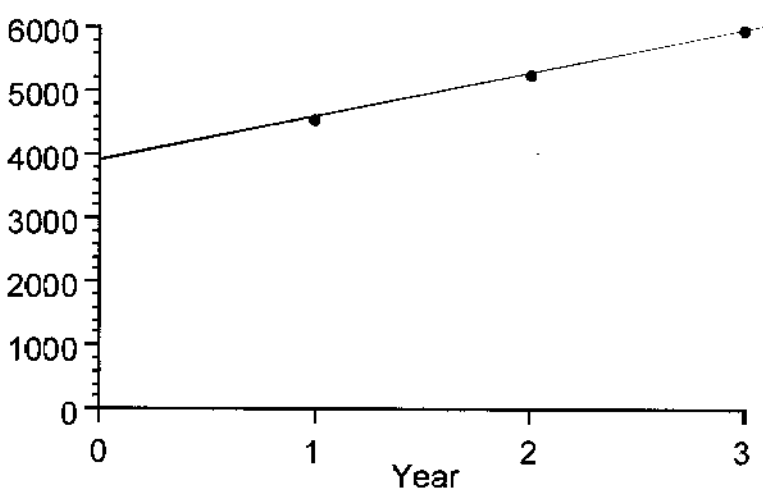
e) $y = 410 + (17.6)(20) = 762$

f) The average growth in sales per month is 17.6.

20.9-2 a)



b)



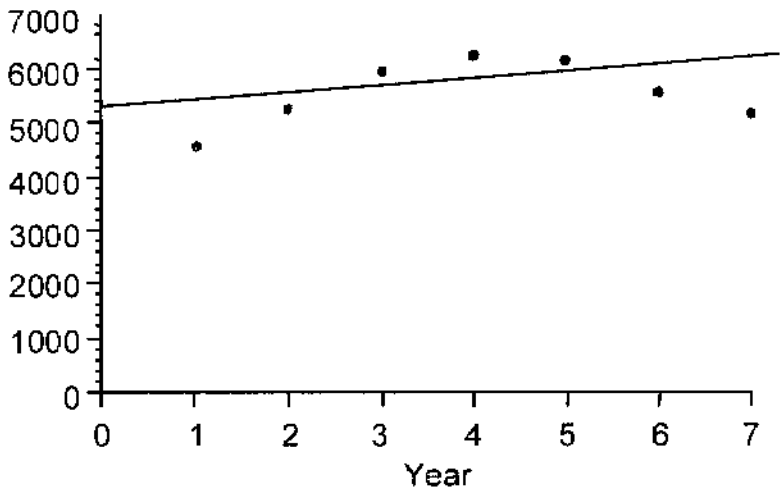
20-25

c) $y = 3900 + 700x$

d) y (year 4) $= 3900 + (700)(4) = 6700$
y (year 5) $= 3900 + (700)(5) = 7400$
y (year 6) $= 3900 + (700)(6) = 8100$
y (year 7) $= 3900 + (700)(7) = 8800$
y (year 8) $= 3900 + (700)(8) = 9500$

e) It does not make sense to use the forecast obtained earlier of 9500. The relationship between the variable has changed and, thus, the linear regression that was used is no longer appropriate.

f)



$y = 5228 + 92.9x$
$y = 5228 + (92.9)(8) = 5971$

The linear regression line does not provide a close fit to the data. Consequently, the forecast that it provides for year 8 is not likely to be accurate. It does not make sense to continue to use a linear regression line when changing conditions cause a large shift in the underlying trend in the data.

g)

Time Period	Data	Latest Trend	Estimated Trend	Exponential Smoothing Forecast	Forecasting Error
1	4600		700	4600	0
2	5300	700	700	5300	0
3	6000	700	700	6000	0
4	6300	700	700	6700	400
5	6200	500	600	7100	900
6	5600	150	375	7025	1425
7	5200	-338	19	6331	1131
8		-547	-264	5502	
9					

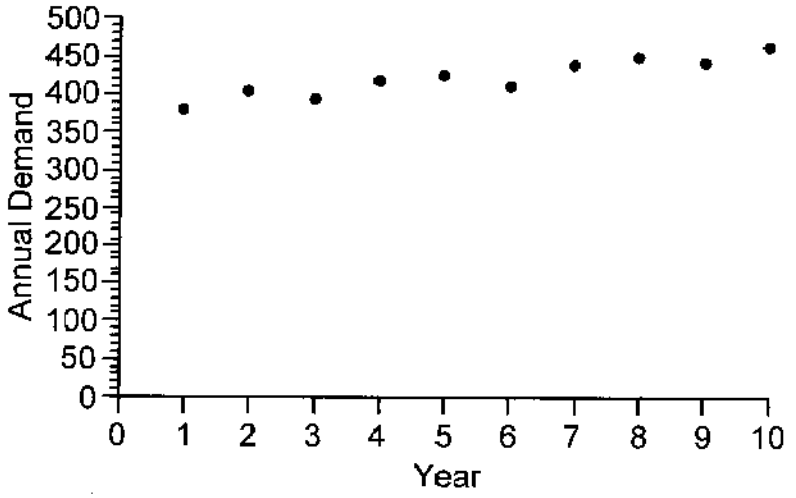
Smoothing Constants	
α =	0.5
β =	0.5

Initial Estimates	
Average =	3900
Trend =	700

Mean Absolute Deviation	
MAD =	550.893

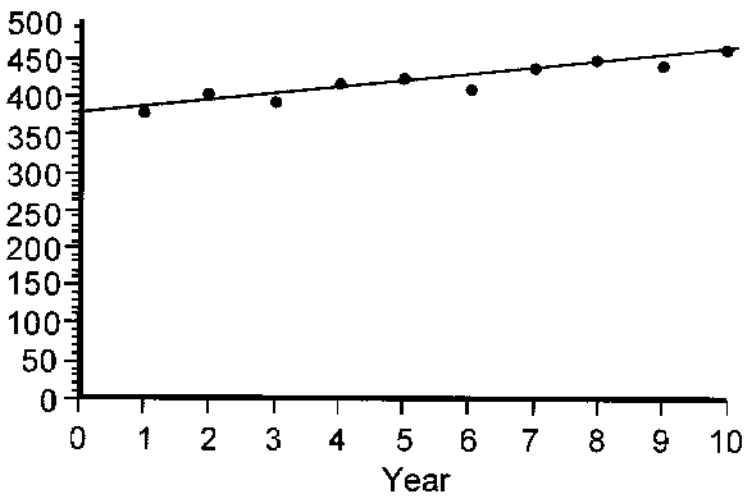
Causal forecasting takes all the data into account, even the data from before changing conditions cause a shift. Exponential smoothing with trend adjusts to shifts in the underlying trend.

20.9-3 a)



b) $y = 380 + 8.15x$

c)



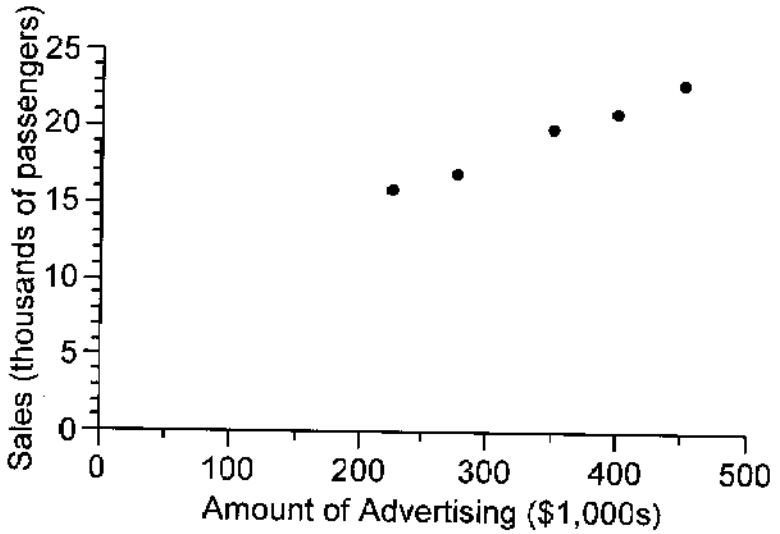
d) $y = 380 + (8.15)(11) = 470$

e) $y = 380 = (8.15)(15) = 503$

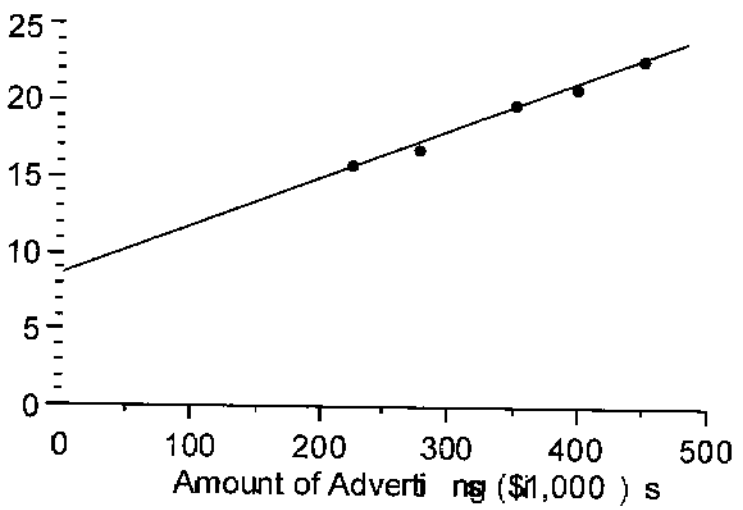
f) The average growth per year is 8.15 tons.

20.9-4 a) The amount of advertising is the independent variable and sales is the dependent variable.

b)



c) $y = 8.71 + 0.031x$



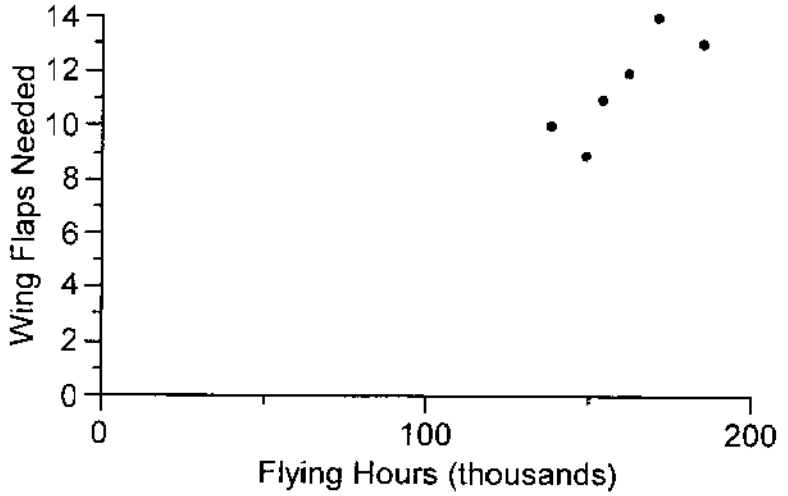
d) $y = 8.71 + (0.031)(300) = 18{,}000$ passengers

e) $22 = 8.71 + (0.031)(x)$ $x = \$429{,}000$

f) An increase of 31 passengers can be attained.

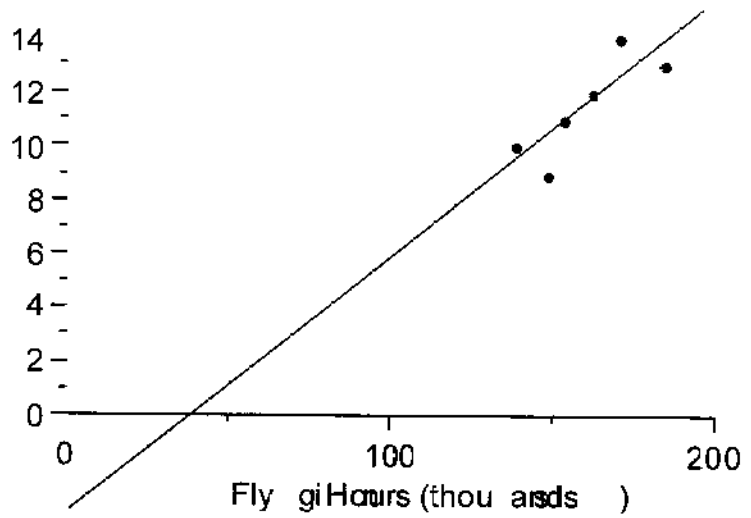
20.9-5 a) The number of flying hours is the independent variable and the number of wing flaps needed is the dependent variable.

b)



c) $y = -3.38 - 0.093x$

d)



e) $y = -3.38 + (0.093)(150) = 11$

f) $y = -3.38 + (0.093)(200) = 15$

20.9-6

Time Period	Independent Variable	Dependent Variable	Estimate	Estimation Error	Square of Error
1	323	24	22	2.48	6
2	359	23	25	2.02	4
3	396	28	29	0.63	0
4	421	32	31	0.93	1
5	457	34	35	0.57	0
6	472	37	36	0.97	1
7	446	33	34	0.50	0
8	407	30	30	0.30	0
9	374	27	26	0.51	0
10	343	22	23	1.47	2

Linear Regression Line
y = a + bx

a =	-9.954
b =	0.097

Estimator

If x =	200
then y=	9.533

Joe should use the linear regression line $y = -9.95 + 0.097x$ to develop a forecast for jobs in the future.

20.9-7. (a) $\hat{Y}(x) = 121.04 - 1.0346x$

$\hat{Y}(55) = 64.137$

(b) $t_{.025:5} = 2.571$, $S_{Y|X} = 6.34$

$\sqrt{1 + \frac{1}{7} + \frac{(x_i - \bar{x})^2}{\Sigma(x_i - \bar{x})^2}} = 1.0735$

$\Rightarrow$ the 95% prediction interval is [46.64, 81.64]

(c) by interpolation

$t_{.0125:5} = 3.365 - \frac{.0025}{.015}(3.365 - 2.571)$

$= 3.233$

$\Rightarrow$ the simultaneous 95% prediction interval is [42.13, 86.14]

(d) interpolating: $c^{**} = 10.722 + \frac{1}{2}(11.150 - 10.722)$

$= 10.936$

$\Rightarrow$ the simultaneous tolerance interval is [37.1, 91.2]

20.9-8

(a) $\Sigma x = 20$, $\Sigma y = 40$, $\Sigma xy = 242$ and $\Sigma x^2 = 120$

$\Rightarrow \hat{Y}(x) = -.2 + 2.05x$

$\hat{Y}(10) = 20.3$

(b) $S^2_{Y|X} = .6333$, $t_{.025:3} = 3.182$, and $\sqrt{\frac{1}{5} + \frac{(x_i - \bar{x})^2}{\Sigma(x_i - \bar{x})^2}} = \sqrt{1.1}$

$\Rightarrow$ [17.64, 22.9] is the 95% confidence interval

(c) $\sqrt{1 + \frac{1}{5} + \frac{(x_i - \bar{x})^2}{\Sigma(x_i - \bar{x})^2}} = \sqrt{2.1}$

$\Rightarrow$ [16.630, 23.970] is the 95% prediction interval

(d) interpolating: $c^{**} = 11.150 + \frac{1}{2}(14.953 - 11.150)$

$= 13.0515$

$\Rightarrow$ [9.406, 31.194] is the simultaneous tolerance interval

20.9-9 (a) $K = 19.96/10 = 1.996$

$\log q = .016 - (1.996)(0) = .016$

$\Rightarrow \log r = .016 + 1.996 \log t$

and at $\log t = 3$ $\log r = 6.004$

(b)

$\log t$	$\log r$	$\hat{E}(\log r)$
-2.0	-3.95	—
-1.0	-2.12	—
0.0	0.08	-3.767
1.0	2.20	-3.382
2.0	3.87	-2.824
3.0	—	-2.155

(c)

$\log t$	$\log r$	$\hat{E}(\log r)$	$\hat{E}(K)$	$\log r_{adj}$
-2.0	-3.95	-3.95	1.996	—
-1.0	-2.12	-1.971	1.994	—
0.0	0.08	0.029	1.995	0.024
1.0	2.20	2.042	1.997	2.024
2.0	3.87	4.022	1.995	4.039
3.0	—	—	—	6.017

20.9-10. $Q = \sum(y_i - bx_i)^2$

$\frac{dQ}{db} = \sum -2x_i(y_i - bx_i) = 0 \Rightarrow \beta = \frac{\sum x_i y_i}{\sum x_i^2}$

Cases

20-1 a) We need to forecast the call volume for each day separately.

1) To obtain the seasonally adjusted call volume for the past 13 weeks, we first have to determine the seasonal factors. Because call volumes follow seasonal patterns within the week, we have to calculate a seasonal factor for Monday, Tuesday, Wednesday, Thursday, and Friday. We calculate the seasonal factors using the following formula:

$$\frac{\text{Average for the Day}}{\text{Overall Average}}$$

The spreadsheet used to calculate the seasonal factors follows.

	Monday	Tuesday	Wednesday	Thursday	Friday	Total
Week 44	1130	851	859	828	726	4394
Week 45	1085	1042	892	840	799	4658
Week 46	1303	1121	1003	1113	1005	5545
Week 47	2652	2825	1841	0	0	7318
Week 48	1949	1507	989	990	1084	6519
Week 49	1260	1134	941	847	714	4896
Week 50	1002	847	922	842	784	4397
Week 51	823	0	0	401	429	1653
Week 52/1	1209	830	0	1082	841	3962
Week 2	1362	1174	967	930	853	5286
Week 3	924	954	1346	904	758	4886
Week 4	886	878	802	945	610	4121
Week 5	910	754	705	729	772	3870
Average	1268.846	1070.538	866.692	803.923	721.154	946.231
Seasonal Factors	**1.341**	**1.131**	**0.916**	**0.850**	**0.762**	

Now that we have found the seasonal factors, we can obtain the seasonally adjusted series. For each day within each of the 13 weeks, we need to calculate the seasonally adjusted daily call volumes using the following formula:

$$\frac{\text{Actual Daily Call Volume}}{\text{Seasonal Factor for the Corresponding Day of the Week}}$$

The spreadsheet for the seasonally adjusted call volumes follows.

	Monday	Tuesday	Wednesday	Thursday	Friday
Week 44	842.69	752.18	937.83	974.57	952.59
Week 45	809.13	921.01	973.86	988.69	1048.37
Week 46	971.70	990.83	1095.05	1310.02	1318.67
Week 47	1977.71	2496.97	2009.95	0.00	0.00
Week 48	1453.45	1332.01	1079.76	1165.25	1422.32
Week 49	939.63	1002.32	1027.36	996.93	936.84
Week 50	747.23	748.65	1006.61	991.05	1028.69
Week 51	613.74	0.00	0.00	471.98	562.89
Week 52/1	901.60	733.62	0.00	1273.53	1103.48
Week 2	1015.70	1037.68	1055.74	1094.63	1119.23
Week 3	689.06	843.22	1469.53	1064.02	994.58
Week 4	660.73	776.05	875.60	1112.28	800.39
Week 5	678.62	666.45	769.70	858.05	1012.95

2) To forecast the call volume for the next week using the last-value forecasting method, we need to use the Last Value with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753	843	952	101
44	Wed	859	938	753	690	169
44	Thurs	828	974	938	797	31
44	Fri	726	955	974	740	14
45	Mon	1085	809	955	1281	196
45	Tues	1042	922	809	914	128
45	Wed	892	974	922	845	47
45	Thurs	840	988	974	828	12
45	Fri	799	1051	988	751	48
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	998			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	814			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1056			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1489			
3	Thurs	904	1054			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016			
6	Mon	1362	1016	1016	1362	0
6	Tues	1148	1016	1016	1148	0
6	Wed	931	1016	1016	931	0
6	Thurs	864	1016	1016	864	0
6	Fri	773	1017	1016	773	0

Type of Seasonalit: Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

Weekly Demand for the next we 5077

The forecasted call volume for the next week is 5077 calls: 1362 calls are received on Monday, 1148 calls are received on Tuesday, 931 calls are received on Wednesday, 864 calls are received on Thursday, and 773 calls are received on Friday.

3) To forecast the call volume for the next week using the averaging forecasting method, we need to use the Averaging with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753			
44	Wed	856	938			
44	Thurs	828	974			
44	Fri	726	955			
45	Mon	1085	809			
45	Tues	1042	922			
45	Wed	892	974			
45	Thurs	840	986			
45	Fri	798	1051			
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1058			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	686	861			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016			
6	Mon	1270	947	947	1270	0
6	Tues	1070	947	947	1070	0
6	Wed	867	947	947	867	0
6	Thurs	805	947	947	805	0
6	Fri	720	947	947	720	0

Type of Seasonality: Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

Weekly Demand for the Next Week 4732

The forecasted call volume for the next week is 4732 calls: 1270 calls are received on Monday, 1070 calls are received on Tuesday, 867 calls are received on Wednesday, 805 calls are received on Thursday, and 720 calls are received on Friday.

4) To forecast the call volume for the next week using the moving-average forecasting method, we need to use the Moving Averaging with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753			
44	Wed	859	936			
44	Thurs	828	974			
44	Fri	726	955			
45	Mon	1085	809			
45	Tues	1042	922			
45	Wed	892	974			
45	Thurs	840	988			
45	Fri	799	1051			
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1095			
46	Thurs	1213	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1084	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	967	1056			
2	Thurs	930	1094			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1016			
6	Mon	1070	798	798	1070	0
6	Tues	928	822	822	928	0
6	Wed	781	853	853	781	0
6	Thurs	739	859	859	739	0
6	Fri	662	871	871	662	0

Number of previous periods to consider: n = 5

Type of Seasonality: Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

Weekly Demand for the Next Week 4189

The forecasted call volume for the next week is 4189 calls: 1070 calls are received on Monday, 928 calls are received on Tuesday, 781 calls are received on Wednesday, 739 calls are received on Thursday, and 662 calls are received on Friday.

5) To forecast the call volume for the next week using the exponential smoothing forecasting method, we need to use the Exponential with Seasonality template. However, we need to modify this template because it does not provide the option for daily seasonality. The modified spreadsheet follows.

Week	Day	True Value	Seasonally Adjusted Value	Seasonally Adjusted Forecast	Actual Forecast	Forecasting Error
44	Mon	1130	843			
44	Tues	851	753			
44	Wed	859	938			
44	Thurs	828	974			
44	Fri	726	955			
45	Mon	1085	809			
45	Tues	1042	922			
45	Wed	892	974			
45	Thurs	840	988			
45	Fri	789	1051			
46	Mon	1303	972			
46	Tues	1121	992			
46	Wed	1003	1086			
46	Thurs	1113	1309			
46	Fri	1005	1322			
47	Mon	2652	1978			
47	Tues	2825	2500			
47	Wed	1841	2010			
47	Thurs	0	0			
47	Fri	0	0			
48	Mon	1949	1453			
48	Tues	1507	1334			
48	Wed	989	1080			
48	Thurs	990	1165			
48	Fri	1064	1426			
49	Mon	1260	940			
49	Tues	1134	1004			
49	Wed	941	1027			
49	Thurs	847	996			
49	Fri	714	939			
50	Mon	1002	747			
50	Tues	847	750			
50	Wed	922	1007			
50	Thurs	842	991			
50	Fri	784	1032			
51	Mon	823	614			
51	Tues	0	0			
51	Wed	0	0			
51	Thurs	401	472			
51	Fri	429	564			
52/1	Mon	1209	902			
52/1	Tues	830	735			
52/1	Wed	0	0			
52/1	Thurs	1082	1273			
52/1	Fri	841	1107			
2	Mon	1362	1016			
2	Tues	1174	1039			
2	Wed	987	1056			
2	Thurs	930	1084			
2	Fri	853	1122			
3	Mon	924	689			
3	Tues	954	844			
3	Wed	1346	1469			
3	Thurs	904	1064			
3	Fri	758	997			
4	Mon	886	661			
4	Tues	878	777			
4	Wed	802	876			
4	Thurs	945	1112			
4	Fri	610	803			
5	Mon	910	679			
5	Tues	754	667			
5	Wed	705	770			
5	Thurs	729	858			
5	Fri	772	1010	946		
6	Mon	1278	953	953	1278	0
6	Tues	1077	953	953	1077	0
6	Wed	873	953	953	873	0
6	Thurs	810	953	953	810	0
6	Fri	724	953	953	724	0

Weekly Demand for the Next Week 4763

Smoothing Constant
α = 0.1

Initial Estimate
Average = 946.2307692

Type of Seasonality
Daily

Day	Seasonal Factor
Mon	1.34
Tues	1.13
Wed	0.92
Thurs	0.85
Fri	0.76

The forecasted call volume for the next week is 4763 calls: 1278 calls are received on Monday, 1077 calls are received on Tuesday, 873 calls are received on Wednesday, 810 calls are received on Thursday, and 724 calls are received on Friday.

b) To obtain the mean absolute deviation for each forecasting method, we simply need to subtract the true call volume from the forecasted call volume for each day in the sixth week. We then need to take the absolute value of the five differences. Finally, we need to take the average of these five absolute values to obtain the mean absolute deviation.

The spreadsheet for the calculation of the mean absolute deviation for the last-value forecasting method follows.

LAST VALUE

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1362	639
6	Tues	677	1148	471
6	Wed	521	931	410
6	Thurs	571	864	293
6	Fri	498	773	275

Mean Absolute Deviation

MAD =	417

This method is the least effective of the four methods because this method depends heavily upon the average seasonality factors. If the average seasonality factors are not the true seasonality factors for week 6, a large error will appear because the average seasonality factors are used to transform the Friday call volume in week 5 to forecasts for all call volumes in week 6. We calculated in part (a) that the call volume for Friday is 0.76 times lower than the overall average call volume. In week 6, however, the call volume for Friday is only 0.83 times lower than the average call volume over the week. Also, we calculated that the call volume for Monday is 1.34 times higher than the overall average call volume. In Week 6, however, the call volume for Monday is only 1.21 times higher than the average call volume over the week. These differences introduce a large error.

The spreadsheet for the calculation of the mean absolute deviation for the averaging forecasting method appears below.

Averaging

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1270	547
6	Tues	677	1070	393
6	Wed	521	867	346
6	Thurs	571	805	234
6	Fri	498	720	222

Mean Absolute Deviation

MAD =	348

This method is the second-most effective of the four methods. Again, the reason lies in the average seasonality factors. Applying the average seasonality factors to an average call volume yields a much more accurate result than applying average seasonality factors to only one call volume. This method is not the most effective method, however, because the centralized call center experiences not only daily seasonality, but also weekly seasonality. For example, the call volumes in weeks 45 and 46 are much greater than the call volumes in week 6. Therefore, these larger call volumes inflate the average call volume, which in turn inflates the forecasts for Week 6.

The spreadsheet for the calculation of the mean absolute deviation for the moving-average forecasting method appears below.

MOVING AVERAGE

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1070	347
6	Tues	677	928	251
6	Wed	521	781	260
6	Thurs	571	739	168
6	Fri	498	662	164

Mean Absolute Deviation	
MAD =	238

This method is the most effective of the four methods because this method only uses the average week 5 call volume to forecast the call volumes for week 6. Again, applying the average seasonality factors to an average call volume yields a much more accurate result than applying average seasonality factors to only one call volume. Also, the average call volume used in this method is not overly inflated since it is an average of the week 5 call volumes, which are closer to the week 6 call volumes than any other of the 13 weeks.

The spreadsheet for the calculation of the mean absolute deviation for exponential forecasting method follows.

EXPONENTIAL SMOOTHING

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	1278	555
6	Tues	677	1077	400
6	Wed	521	873	352
6	Thurs	571	810	239
6	Fri	498	724	226

Mean Absolute Deviation	
MAD =	355

This method is almost as effective as the averaging forecasting method because the smoothing constant used is 0.1. Therefore, the call volumes from earlier weeks are still weighted in calculating the call volume average. This method is a little less effective than the averaging forecasting method because the smoothing constant causes less weight to be placed on the call volumes in weeks 44 and 45. These call volumes are lower than volumes in weeks 46, 48, and 49, however, and they help lower the already inflated average call volume.

c) This problem is simply a linear regression problem.

1) To find a mathematical relationship, we use the Linear Regression template. The decentralized case volumes are the independent variables, and the centralized case volumes are the dependent variables. Substituting the case volume data, we obtain the following spreadsheet:

Week	Independent Variable	Dependent Variable	Estimate	Estimation Error	Square of Error
44	512	2052	2038	13.84	192
45	721	2170	2121	49.45	2445
46	693	2779	2099	679.61	461872
47	540	2334	1984	350.27	122690
48	1386	2514	2623	109.26	11938
49	577	1713	2012	298.70	89221
50	405	1927	1882	45.32	2054
51	441	1167	1909	741.89	550409
52/1	655	1549	2071	521.66	272132
2	572	2126	2008	118.08	13943
3	475	2337	1935	402.41	161932
4	530	1916	1976	60.17	3620
5	595	2098	2025	72.69	5284

Linear Regression Line
$y = a + bx$

a =	1575.516778
b =	0.755947559

Estimator

If x =	613
then y=	2038.912632

2) To forecast the week 6 call volume for the centralized call center, we simply input the week 6 decentralized case volume for the value of x in the Estimator section of the Linear Regression Spreadsheet. The value of y then represents the week 6 centralized case volume. We multiply this value of y by 1.5 to obtain the week 6 centralized call volume. The calculations in Excel appear below.

Estimator

If x =	613
then y=	2038.912632

Calls = 1.5 * y
3058.36894

We then break this weekly call volume into daily call volume. We do this conversion by dividing the weekly call volume by the sum of the seasonal factors calculated in part (a) and then multiplying this weekly call volume by the appropriate seasonal factor to find the call volume for each of the five days of the week. The spreadsheet showing these calculations follows:

Seasonal Factors	1.341	1.131	0.916	0.850	0.762

Week 6 Call Volume	3058
Sum of Seasonal Factors	5.000
Converted Week 6 Call Volume	611.6
Call Volume for:	
Monday	820.1237
Tuesday	691.9468
Wednesday	560.19
Thursday	519.6189
Friday	466.1206
Week 6 Call Volume	3058

The forecasted call volume for week 6 is 3058 calls: 820 calls are received on Monday, 692 calls are received on Tuesday, 560 calls are received on Wednesday, 520 calls are received on Thursday, and 466 calls are received on Friday.

3) To calculate the mean absolute deviation, we need to subtract the true call volume from the forecasted call volume for each day in the sixth week. We then need to take the absolute value of the five differences. Finally, we need to take the average of these five absolute values to obtain the mean absolute deviation.

The spreadsheet for the calculation of the mean absolute deviation follows.

CAUSAL FORECASTING

Week	Day	True Value	Actual Forecast	Forecasting Error
6	Mon	723	820	97
6	Tues	677	692	15
6	Wed	521	560	39
6	Thurs	571	520	51
6	Fri	498	466	32

Mean Absolute Deviation	
MAD =	47

This forecasting method is by far the most effective method. The centralized center performs the same services and serves the same population as the decentralized center. Therefore, the call volume trends are the same. Once we have a factor to scale the decentralized call volumes to the centralized call volumes, we have a very effective forecasting method.

d) We would definitely recommend using the causal forecasting method implemented in part (c) because it yields the lowest error. The causal method shows us that the call volume trends remain relatively the same year after year. We had to convert between case volumes and call volumes in part (c), however, and such a conversion introduces error. For example, what if a case generates a higher or lower number of calls? We therefore recommend that call volume data be meticulously recorded as the centralized center continues its operation. Once one year's worth of call volumes have been collected, the causal forecasting model should be updated. The model should be updated to use the historical centralized call volume data instead of the historical decentralized case volume data.

CHAPTER 21

21.2-1 a) State i = the number of customers at the facility, $i = 0, 1, 2$

decision 1 = use the slow service rate

decision 2 = use the fast service rate

expected net immediate cost $= C_{ij}$, $i = 1, 2$; $j = 1, 2$

$C_{11} = 3 - \frac{3}{5} \times 50 = -27$, $C_{12} = 9 - \frac{4}{5} \times 50 = -31$

$C_{21} = -27$, $C_{22} = -31$

b) The four possible stationary policies R_1, R_2, R_3, R_4 are

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
1	1	1	2	2
2	1	2	1	2

Transition matrix:

Under R_1: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$

Under R_2: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{5} \\ 0 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$

Under R_3: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \\ 0 & \frac{3}{5} & \frac{2}{5} \end{bmatrix}$

Under R_4: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \\ 0 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$

Expected long-term Average cost:

R_1: $-27(\pi_{11} + \pi_{21}) = C_1$

R_2: $-27\pi_{12} - 31\pi_{22} = C_2$

R_3: $-31\pi_{13} - 27\pi_{23} = C_3$

R_4: $-31(\pi_{14} + \pi_{24}) = C_4$

c) For R_1, $\pi_1 = 0.517$, $\pi_2 = 0.172$, $C_1 = -18.6$

For R_2, $\pi_1 = 0.541$, $\pi_2 = 0.135$, $C_2 = -18.79$

For R_3, $\pi_1 = 0.508$, $\pi_2 = 0.085$, $C_3 = -18.043$

For R_4, $\pi_1 = 0.519$, $\pi_2 = 0.065$, $C_4 = -18.104$.

Obviously, C_2 is the minimum. So, R_2 is the best policy.

21.2-2. a) States = $\begin{cases} 0 & \text{not dented} \\ 1 & \text{dented} \end{cases}$

Decisions at state 0 : street , two spaces , lot
Decisions at state 1 : repair , not repaired
So, there are 5 decisions .

$$C(ik): \begin{bmatrix} 0 & 45 & 5 & - & - \\ - & - & - & 50 & 9 \end{bmatrix}$$

b) There are 5 stationary deterministic policies

c)

state	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$	$d_i(R_5)$
0	1	1	2	2	3
1	4	5	4	5	-

R_1: $P = \begin{bmatrix} 0.9 & 0.1 \\ 1 & 0 \end{bmatrix}$ $\quad \pi = (0.909 , 0.091)$
$C_1 = \pi_0 \cdot 0 + \pi_1 \cdot 50 = 4.55$

R_2: $P = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 1 \end{bmatrix}$ $\quad \pi = (0, 1)$
$C_2 = 9$

R_3: $P = \begin{bmatrix} 0.98 & 0.02 \\ 1 & 0 \end{bmatrix}$ $\quad \pi = (0.98 , 0.02)$
$C_3 = 0.98 \times 4.5 + 0.02 \times 50 = 5.41$

R_4: $P = \begin{bmatrix} 0.98 & 0.02 \\ 0 & 1 \end{bmatrix}$ $\quad \pi = (0, 1)$
$C_4 = 9$

R_5: $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C_5 = 5$ if initial state of the car is undented

So, R_1, namely , parking in the street and have it repaired when dented , is the optimal policy .

21.2-3. a) Let state 0 = Low sales
1 = High sales

decision 1 = Do not advertise
2 = Advertise

cost matrix , C(ik):

$$\begin{bmatrix} -2 & -1 \\ -4 & -3 \end{bmatrix}$$

21.2-3 (continued)

b) The four possible stationary deterministic policies are:

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

R_1: $P = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$ $\quad C_1 = -2\pi_0 - 4\pi_1$

R_2: $P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$ $\quad C_2 = -2\pi_0 - 3\pi_1$

R_3: $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ $\quad C_3 = -\pi_0 - 4\pi_1$

R_4: $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$ $\quad C_4 = -\pi_0 - 3\pi_1$

c) R_1: $\pi = (0.667, 0.333)$, $C_1 = -2.667$

R_2: $\pi = (0.5, 0.5)$, $C_2 = -2.5$

R_3: $\pi = (0.5, 0.5)$, $C_3 = -2.5$

R_4: $\pi = (0.333, 0.667)$, $C_4 = -2.333$

So, policy R_1 is optimal.

21.2-4 a) Let state 0 = good mood, 1 = bad mood; decision 1 = take her out for dinner, 2 = not take her out

cost matrix, $C(i,k)$:

$$\begin{bmatrix} 14 & 0 \\ 14 & 75 \end{bmatrix}$$

b) The four possible stationary deterministic policies are

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

21.2-4 (continued)

R_1: $\begin{bmatrix} 0.875 & 0.125 \\ 0.875 & 0.125 \end{bmatrix}$ $\quad C_1 = 14\pi_0 + 14\pi_1$

R_2: $\begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}$ $\quad C_2 = 14\pi_0 + 75\pi_1$

R_3: $\begin{bmatrix} 0.125 & 0.875 \\ 0.875 & 0.125 \end{bmatrix}$ $\quad C_3 = 14\pi_1$

R_4: $\begin{bmatrix} 0.125 & 0.875 \\ 0.125 & 0.875 \end{bmatrix}$ $\quad C_4 = 75\pi_1$

c) R_1: $\pi = (0.875, 0.125)$, $C_1 = 14$

R_2: $\pi = (0.5, 0.5)$, $C_2 = 44.5$

R_3: $\pi = (0.5, 0.5)$, $C_3 = 7$

R_4: $\pi = (0.125, 0.875)$, $C_4 = 65.625$

So, optimal policy is R_3.

21.2-5. a) Let state 0 denote that the point is over and that there are two serves for the next point, and let state 1 denote that only one serve for the current point is remaining. Let decision 1 be to attempt an ace and decision 2 be to attempt a lob. The transition

Cost matrix, $C(i,k)$:

$C_{01} = \frac{3}{8}[\frac{2}{3}(-1) + \frac{1}{3}(1)] = -\frac{1}{8}$

$C_{02} = \frac{7}{8}[\frac{1}{3}(-1) + \frac{2}{3}(1)] = \frac{7}{24}$

$C_{11} = \frac{3}{8}[\frac{2}{3}(-1) + \frac{1}{3}(1)] + \frac{5}{8}(1) = \frac{1}{2}$

$C_{12} = \frac{7}{8}[\frac{1}{3}(-1) + \frac{2}{3}(1)] + \frac{1}{8}(1) = \frac{5}{12}$

b) There are four possible stationary deterministic policies:

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

R_1: $P = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ 1 & 0 \end{bmatrix}$ $\quad C_1 = -\frac{1}{8}\pi_0 + \frac{1}{2}\pi_1$

R_2: $P = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ 1 & 0 \end{bmatrix}$ $\quad C_2 = -\frac{1}{8}\pi_0 + \frac{5}{12}\pi_1$

R_3: $P = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} \\ 1 & 0 \end{bmatrix}$ $\quad C_3 = \frac{7}{24}\pi_0 + \frac{1}{2}\pi_1$

21.2-5 (b) (continued)

R_4: $P = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} \\ 1 & 0 \end{bmatrix}$, $C_4 = \frac{7}{24}\pi_0 + \frac{5}{12}\pi_1$

c) R_1: $\pi = (0.615, 0.385)$, $C_1 = 0.27$

R_2: $\pi = (0.615, 0.385)$, $C_2 = 0.237$

R_3: $\pi = (0.889, 0.111)$, $C_3 = 0.315$

R_4: $\pi = (0.889, 0.111)$, $C_4 = 0.306$

So, R_4 is the optimal policy. (0: attempt ace; 1: attempt lob)

21.2-6 (a) Let states 0, 1, 2 be that the market is currently at 11000, 12000, and 13000 respectively. Let decisions 1, 2 be to invest in the go-go fund and go-slow fund, respectively. NOTE: All \$ numbers are expressed in \$ thousands. (NOT the probabilities).

$C_{01} = 5/10(-20) + 2/10(-50) = -20$

$C_{02} = 5/10(-10) + 2/10(-20) = -9$

$C_{11} = 1/10(20) + 4/10(-20) = -6$

$C_{12} = 1/10(10) + 4/10(-10) = -3$

$C_{21} = 2/10(50) + 4/10(20) = 18$

$C_{22} = 2/10(20) + 4/10(10) = 8$

(b) There are 8 possible policies:

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$	$d_i(R_5)$	$d_i(R_6)$	$d_i(R_7)$	$d_i(R_8)$
0	1	1	1	1	2	2	2	2
1	1	1	2	2	2	1	1	2
2	1	2	2	1	1	2	1	2

R_i's share the same transition matrix

$$P = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.1 \end{bmatrix}$$

R_1: $C_1 = -20\pi_0 - 6\pi_1 + 18\pi_2$

R_2: $C_2 = -20\pi_0 - 6\pi_1 + 8\pi_2$

R_3: $C_3 = -20\pi_0 - 3\pi_1 + 8\pi_2$

R_4: $C_4 = -20\pi_0 - 3\pi_1 + 18\pi_2$

R_5: $C_5 = -9\pi_0 - 3\pi_1 + 18\pi_2$

R_6: $C_6 = -9\pi_0 - 6\pi_1 + 8\pi_2$

R_7: $C_7 = -9\pi_0 - 6\pi_1 + 18\pi_2$

R_8: $C_8 = -9\pi_0 - 3\pi_1 + 8\pi_2$

21.2-6. (continued)

c) From OR courseware, $\pi = (0.171, 0.463, 0.366)$

Such that $R_1 = 0.39$, $R_2 = -3.27$, $R_3 = -1.881$, $R_4 = 1.779$

$R_5 = 3.66$ $R_6 = -1.389$ $R_7 = 2.271$ $R_8 = 0$

R_2 has the lowest value. so, $\begin{pmatrix} 0 : \text{Go-go} \\ 1 : \text{Go-go} \\ 2 : \text{Go-slow} \end{pmatrix}$ is optimal

21.2-7. a) Let state 0 = machine broken down

1 = machine running

decision 1 = Buck works on machine

2 = Bill works on machine

Cost matrix, $C(i,k)$:

$$\begin{bmatrix} 0 & 0 \\ -1200 & -1200 \end{bmatrix}$$

b) There are 4 possible (stationary deterministic) policies

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

R_1: $\begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$ $C_1 = -1200\pi_1$

R_2: $\begin{bmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{bmatrix}$ $C_2 = -1200\pi_1$

R_3: $\begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}$ $C_3 = -1200\pi_1$

R_4: $\begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$ $C_4 = -1200\pi_1$

c) R_1: $\pi = (0.5, 0.5)$, $C_1 = -600$

R_2: $\pi = (0.4, 0.6)$, $C_2 = -720$

R_3: $\pi = (0.545, 0.455)$, $C_3 = -546$

R_4: $\pi = (0.444, 0.556)$, $C_4 = -667.2$

21.2-7. c) (continued)

The smallest value is -720.

So, policy R_2 is optimal.

21.2-8. a) Let state 0 = she is in $(20, \infty)$

state 1 = she is in $(10, 19)$

2 = she is in $(-10, 9)$

3 = she is in $(-20, -11)$

Let decision 1 = do not speed

2 = speed

$C_{01} = 0 = C_{02}$, $C_{11} = 4$, $C_{12} = \frac{1}{8} \times (20+4) = 3$,

$C_{21} = 4 = C_{22}$, $C_{31} = 8$, $C_{32} = \frac{1}{8} \times (20+8) + \frac{7}{8} \times 4 = 7$

b) There are four (stationary deterministic) policies.

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	1	1
1	1	1	2	2
2	1	1	1	1
3	1	2	1	2

We do not speed at state 0 and 2 since speeding or not, it doesn't help to avoid penalty of lateness.

R_1: $P = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}$ $C_1 = 4\pi_1 + 4\pi_2 + 8\pi_3$

R_2: $P = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{7}{8} & \frac{1}{4} & \frac{1}{8} & 0 \end{bmatrix}$ $C_2 = 4\pi_1 + 4\pi_2 + 7\pi_3$

R_3: $P = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \end{bmatrix}$ $C_3 = 3\pi_1 + 4\pi_2 + 8\pi_3$

21.2-8 (b) (continued)

$$R_4: \quad P = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{5}{8} & \frac{1}{4} & \frac{1}{8} & 0 \end{bmatrix}, \quad C_4 = 3\pi_1 + 4\pi_2 + 7\pi_3$$

(c) R_1: $\pi = (0.484, 0.25, 0.174, 0.092)$ $\quad C_1 = 2.432$

R_2: $\pi = (0.475, 0.25, 0.184, 0.091)$ $\quad C_2 = 2.373$

R_3: $\pi = (0.459, 0.25, 0.203, 0.089)$ $\quad C_3 = 2.274$

R_4: $\pi = (0.45, 0.25, 0.213, 0.087)$ $\quad C_4 = 2.225$

So, R_4 attains the minimum and thus is the optimal policy.

21.2-9

(a) Let the state be the number of items in inventory at the begining of the period.
Let decision 1 = order 0 items
2 = order 1 item
3 = order 2 items
(The strange labeling of decisions here is to conform to the software package standard).
The cost matrix is:

C_{ik}	1	2	3
0	40/3	56/3	24
1	4	19	—
2	4	—	—

The policy, R_5, is:

i	$d_i(R_5)$
0	3
1	1
2	1

$$P(R_6) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \Rightarrow \pi = (4/9, 3/9, 2/9)$$

$\Rightarrow$ the expected cost $= \frac{4}{9}C_{03} + \frac{3}{9}C_{11} + \frac{2}{9}C_{21}$

$= \frac{4}{9}(24) + \frac{3}{9}(4) + \frac{2}{9}(4) = \frac{116}{9} = \$12.87$/period

(b) The six feasible stationary deterministic policies (that is, the policies where you never exceed the storage capacity) are

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$	$d_i(R_5)$	$d_i(R_6)$
0	1	2	3	1	2	3
1	1	1	1	2	2	2
2	1	1	1	1	1	1

The remaining 21 stationary deterministic policies are infeasible:

21.2-9 (b) (Continued)

i \ $d_i(R_j)$	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
0	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
1	3	3	3	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
2	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3

21.3-1. (a) Minimize $3y_{01} + 9y_{02} + 3y_{11} + 9y_{12} + 28y_{21} + 34y_{22}$

Subject to
$$y_{01} + y_{02} + y_{11} + y_{12} + y_{21} + y_{22} = 1$$
$$y_{01} + y_{02} - (\tfrac{1}{3}y_{01} + \tfrac{3}{10}y_{11} + \tfrac{1}{3}y_{02} + \tfrac{2}{5}y_{12}) = 0$$
$$y_{11} + y_{12} - (\tfrac{1}{2}y_{01} + \tfrac{1}{2}y_{11} + \tfrac{3}{5}y_{21} + \tfrac{1}{2}y_{02} + \tfrac{1}{2}y_{12} + \tfrac{4}{5}y_{22}) = 0$$
$$y_{21} + y_{22} - (\tfrac{2}{10}y_{11} + \tfrac{2}{5}y_{21} + \tfrac{4}{10}y_{12} + \tfrac{1}{5}y_{22}) = 0$$
$$y_{ik} \geq 0 \text{ for } i = 0,1,2 \text{ and } k = 1,2$$

(b) Use simplex-method, we find $y_{01} = 0.32432$, $y_{11} = 0.54054$, $y_{22} = 0.13514$ and other y_{ik}'s are zero. This tells us that we should adopt decision 1 at state 0, decision 2 at state 2, decision at state 1, which means, policy 2 is optimal.

21.3-2. (a) Minimize $4.5y_{02} + 5y_{03} + 50y_{14} + 9y_{15}$

subject to
$$y_{01} + y_{02} + y_{03} + y_{14} + y_{15} = 1$$
$$y_{01} + y_{02} + y_{03} - (\tfrac{9}{10}y_{01} + \tfrac{49}{50}y_{02} + y_{03} + y_{14}) = 0$$
$$y_{14} + y_{15} - (\tfrac{1}{10}y_{01} + \tfrac{1}{50}y_{02} + y_{15}) = 0$$
$$y_{ik} \geq 0 \quad i = 0, k = 1,2,3 \; ; \; i = 1, k = 4,5.$$

(b) Use simplex method, we find y_{ik}'s are zero except that $y_{01} = 0.9090$, $y_{14} = 0.0909$. So, $R_1 = \begin{bmatrix} 0: 1 \\ 1: 4 \end{bmatrix}$ is optimal.

21.3-3. (a) Minimize $-4y_{01} - 2y_{02} - 3y_{11} - 4y_{12}$

Subject to:
$$y_{01} + y_{02} + y_{11} + y_{12} = 1$$
$$y_{01} + y_{11} - (\tfrac{3}{4}y_{01} + \tfrac{1}{4}y_{02} + \tfrac{1}{2}y_{11} + \tfrac{1}{2}y_{12}) = 0$$
$$y_{02} + y_{12} - (\tfrac{1}{2}y_{01} + \tfrac{1}{2}y_{02} + \tfrac{1}{4}y_{11} + \tfrac{3}{4}y_{12}) = 0$$
$$y_{ik} \geq 0 \qquad i = 0, 1 \quad k = 1, 2$$

(b) Use simplex method, we find y_{ik}'s are zero except that $y_{01} = 0.6667$, $y_{11} = 0.3333$. So $R_1 = \begin{bmatrix} 0: \text{decision 1} \\ 1: \text{decision 1} \end{bmatrix}$ is optimal.

21.3-4.(a) Minimize $75y_{11} + 14y_{02} + 14y_{12}$

subject to
$$y_{01} + y_{11} + y_{02} + y_{12} = 1$$
$$y_{01} + y_{11} - (\tfrac{1}{8}y_{01} + \tfrac{1}{8}y_{11} + \tfrac{7}{8}y_{02} + \tfrac{7}{8}y_{12}) = 0$$
$$y_{02} + y_{12} - (\tfrac{7}{8}y_{01} + \tfrac{7}{8}y_{11} + \tfrac{1}{8}y_{02} + \tfrac{1}{8}y_{12}) = 0$$
$$y_{ik} \geq 0 \quad i = 0,1 \;, \; k = 1,2$$

(b) Solve this by simplex-method, we get $y_{02} = y_{11} = 0.5$, $y_{01} = y_{12} = 0$

So, optimal policy is $R_3 = \begin{bmatrix} 0, \text{ decision 2} \\ 1, \text{ decision 1} \end{bmatrix}$

21.3-5.(a) Minimize $-\tfrac{1}{8}y_{01} + \tfrac{7}{24}y_{02} + \tfrac{1}{2}y_{11} + \tfrac{5}{12}y_{12}$

subject to
$$y_{01} + y_{02} + y_{11} + y_{12} = 1$$
$$y_{01} + y_{02} - (\tfrac{3}{8}y_{01} + y_{11} + \tfrac{7}{8}y_{02} + y_{12}) = 0$$
$$y_{11} + y_{12} - (\tfrac{5}{8}y_{01} + \tfrac{1}{8}y_{02}) = 0$$
$$y_{ik} \geq 0 \quad i = 0,1 \;, \; k = 1,2$$

(b) The Simplex-method yields $y_{02} = 0.8889$, $y_{11} = 0.1111$, $y_{01} = y_{12} = 0$

So, optimal policy is $R_3 = \begin{bmatrix} 0 : \text{attempt ace} \\ 1 : \text{attempt lob} \end{bmatrix}$.

21.3-6.(a) Note: All $ terms are in $ thousands.

Minimize $-20y_{01} - 9y_{02} - 6y_{11} - 3y_{12} + 18y_{21} + 8y_{22}$

subject to
$$y_{01} + y_{02} + y_{11} + y_{12} + y_{21} + y_{22} = 1$$
$$y_{01} + y_{02} - [\tfrac{3}{10}(y_{01} + y_{02}) + \tfrac{1}{10}(y_{11} + y_{12}) + \tfrac{2}{10}(y_{21} + y_{22})] = 0$$
$$y_{11} + y_{12} - [\tfrac{3}{10}(y_{01} + y_{02}) + \tfrac{5}{10}(y_{11} + y_{12}) + \tfrac{4}{10}(y_{21} + y_{22})] = 0$$
$$y_{21} + y_{22} - [\tfrac{3}{10}(y_{01} + y_{02}) + \tfrac{4}{10}(y_{11} + y_{12}) + \tfrac{4}{10}(y_{21} + y_{22})] = 0$$
$$y_{ik} \geq 0 \quad i = 0,1,2 \quad k = 1,2$$

(b) The Simplex-method yields $y_{01} = 0.171$, $y_{11} = 0.463$, $y_{22} = 0.366$ and $y_{02} = y_{12} = y_{21} = 0$.

So, optimal policy is $R_2 = \begin{bmatrix} 0 : \text{go-go} \\ 1 : \text{go-go} \\ 2 : \text{go-slow} \end{bmatrix}$

21.3-7(a) Minimize $-1200y_{01} - 1200y_{12}$

Subject to:
$$y_{01} + y_{02} + y_{11} + y_{12} = 1$$
$$y_{01} + y_{02} - (.4y_{01} + .6y_{11} + .5y_{02} + .4y_{12}) = 0$$
$$y_{11} + y_{12} - (.6y_{01} + .4y_{11} + .5y_{02} + .6y_{12}) = 0$$

$y_{ik} \geq 0$ for $i = 0,1$ and $k = 1,2$.

(b) The simplex-method yields $y_{01} = 0.4$, $y_{12} = 0.6$, $y_{02} = y_{11} = 0$

So, optimal policy is $R_2 = \begin{bmatrix} 0 : \text{Buck} \\ 1 : \text{Bill} \end{bmatrix}$.

21.3-8(a) Minimize $4y_{01} + 4y_{21} + 8y_{31} + 3y_{12} + 7\ y_{32}$

Subject to

$$y_{01} + y_{11} + y_{12} + y_{21} + y_{31} + y_{32} = 1$$
$$y_{01} - (\tfrac{3}{8}y_{01} + \tfrac{1}{2}y_{11} + \tfrac{5}{8}y_{21} + \tfrac{3}{4}y_{31} + \tfrac{3}{8}y_{12} + \tfrac{5}{8}y_{32}) = 0$$
$$y_{11} + y_{12} - \tfrac{1}{4}(y_{01} + y_{11} + y_{21} + y_{31} + y_{12} + y_{32}) = 0$$
$$y_{21} - (\tfrac{1}{4}y_{01} + \tfrac{1}{8}y_{11} + \tfrac{1}{8}y_{21} + \tfrac{1}{4}y_{12} + \tfrac{1}{8}y_{32}) = 0$$
$$y_{31} + y_{32} - (\tfrac{1}{8}y_{01} + \tfrac{1}{8}y_{11} + \tfrac{1}{8}y_{12}) = 0$$
$$y_{ik} \geq 0 \quad i = 0, 1, 2, 3, \quad k = 1, 2$$

(b) Simplex-method yields $y_{01} = 0.45$, $y_{12} = 0.25$, $y_{21} = 0.213$, $y_{32} = 0.087$ and other y_{ik}'s are zero. So, the optimal policy is R_4.

21.3-9(a)

Minimize $\frac{40}{3}y_{01} + \frac{56}{3}y_{02} + 24y_{03} + 4y_{11} + 19y_{12} + 4y_{21}$

subject to:

$$y_{01} + y_{02} + y_{03} + y_{11} + y_{12} + y_{21} = 1$$
$$y_{01} + y_{02} + y_{03} - (y_{01} + \tfrac{2}{3}y_{11} + \tfrac{1}{3}y_{21} + \tfrac{2}{3}y_{02} + \tfrac{1}{3}y_{12} + \tfrac{1}{3}y_{03}) = 0$$
$$y_{11} + y_{12} - (\tfrac{1}{3}y_{11} + \tfrac{1}{3}y_{21} + \tfrac{1}{3}y_{02} + \tfrac{1}{3}y_{12} + \tfrac{1}{3}y_{03}) = 0$$
$$y_{21} - (\tfrac{1}{3}y_{21} + \tfrac{1}{3}y_{12} + \tfrac{1}{3}y_{03}) = 0$$
$$y_{ik} \geq 0 \quad \text{for} \quad i = 0, 1, 2 \quad \text{and} \quad k = 1, 2, 3$$

(b) Simplex-method yields $y_{03} = 0.4444$, $y_{11} = 0.3333$, $y_{21} = 0.2222$, and $y_{01} = y_{02} = y_{12} = 0$. So, optimal policy is R_6.

21.4-1.

Number of states: 3

Number of decisions: 2

Cost Matrix, C_{ik}:
$$\begin{bmatrix} 0 & 0 \\ -27 & -31 \\ -27 & -31 \end{bmatrix}$$

$$P_{ij}(1) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.6 & 0.4 \end{bmatrix} \qquad P_{ij}(2) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

Initial Policy:
$d_0(R_1) = 1$
$d_1(R_1) = 1$
$d_2(R_1) = 2$

Discount Factor = 1

21.4-1. (continued)

Iteration # 1

Value Determination:

$$g(R_1) = 0 + 0.5v_0(R_1) + 0.5v_1(R_1) + 0v_2(R_1) - v_0(R_1)$$
$$g(R_1) = -27 + 0.3v_0(R_1) + 0.5v_1(R_1) + 0.2v_2(R_1) - v_1(R_1)$$
$$g(R_1) = -31 + 0v_0(R_1) + 0.8v_1(R_1) + 0.2v_2(R_1) - v_2(R_1)$$

Solution of Value Determination Equations:

$$g(R_1) = -18.8$$
$$v_0(R_1) = 52.84$$
$$v_1(R_1) = 15.27$$
$$v_2(R_1) = 0$$

Policy Improvement:

State 0:

$$0 + 0.5(52.84) + 0.5(15.27) + (0) - (52.84) = -18.8$$
$$0 + 0.5(52.84) + 0.5(15.27) + (0) - (52.84) = -18.8$$

State 1:

$$-27 + 0.3(52.84) + 0.5(15.27) + (0) - (15.27) = -18.8$$
$$-31 + 0.4(52.84) + 0.5(15.27) + (0) - (15.27) = -17.5$$

State 2:

$$-27 + 0(52.84) + 0.6(15.27) + (0) - (0) = -17.8$$
$$-31 + 0(52.84) + 0.8(15.27) + (0) - (0) = -18.8$$

Optimal Policy:

$$d_0(R_2) = 1$$
$$d_1(R_2) = 1$$
$$d_2(R_2) = 2$$

$$g(R_2) = -18.8$$
$$v_0(R_2) = 52.84$$
$$v_1(R_2) = 15.27$$
$$v_2(R_2) = 0$$

21.4-2. Number of states = 2

Number of decisions = 5

Cost Matrix, C(ik):

$$\begin{bmatrix} 0 & 4.5 & 5 & \frac{}{50} & \frac{}{9} \\ \frac{}{} & \frac{}{} & \frac{}{} & & \end{bmatrix}$$

Transition Matrix, p(ij)[1]:

$$\begin{bmatrix} 0.9 & 0.1 \\ 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[2]:

$$\begin{bmatrix} 0.98 & 0.02 \\ 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[3]:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[4]:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[5]:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

21.4-2. (continued)

Initial Policy:

d0(R1) = 1
d1(R1) = 4

Discount Factor = 1

ITERATION # 1

Value Determination:

g(R1) = 0 + 0.9v0(R1) + 0.1v1(R1) - v0(R1)
g(R1) = 50 + 1v0(R1) + 0v1(R1) - v1(R1)

Solution of Value Determination Equations:
g(R1) = 4.545
v0(R1) = -45.5
v1(R1) = 0

Policy Improvement:

State 0:

0	+ 0.9	(-45.5)	+ 0.1	(0)	- (-45.5)	= 4.545
4.5	+ 0.98	(-45.5)	+ 0.02	(0)	- (-45.5)	= 5.409
5	+ 1	(-45.5)	+ 0	(0)	- (-45.5)	= 5
---	+ 0	(-45.5)	+ 0	(0)	- (-45.5)	= ---
---	+ 0	(-45.5)	+ 0	(0)	- (-45.5)	= ---

State 1:

---	+ 0	(-45.5)	+ 0	(0)	- (0)	= ---
---	+ 0	(-45.5)	+ 0	(0)	- (0)	= ---
---	+ 0	(-45.5)	+ 0	(0)	- (0)	= ---
50	+ 1	(-45.5)	+ 0	(0)	- (0)	= 4.545
9	+ 0	(-45.5)	+ 1	(0)	- (0)	= 9

Optimal Policy:
d0(R2) = 1
d1(R2) = 4

g(R1) = 4.545
v0(R1) = -45.5
v1(R1) = 0

21.4-3.

Number of states: 2

Number of decisions: 2

Cost Matrix, C_{ik}: $\begin{bmatrix} -2 & -1 \\ -4 & -3 \end{bmatrix}$

$p_{ij}(2) = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$ $\quad p_{ij}(1) = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$

Initial Policy:
$d_0(R_1) = 1$
$d_1(R_1) = 1$

Iteration # 1

Value Determination:

$g(R_1) = -2 + 0.75v_0(R_1) + 0.25v_1(R_1) - v_0(R_1)$
$g(R_1) = -4 + 0.5v_0(R_1) + 0.5v_1(R_1) - v_1(R_1)$

21.4-3. (continued)

Solution of Value Determination Equations:

$g(R_1) = -2.67$
$v_0(R_1) = 2.667$
$v_1(R_1) = 0$

Policy Improvement:

State 0:
$-2 + 0.75(2.667) + (0) - (2.667) = -2.67$
$-1 + 0.5(2.667) + (0) - (2.667) = -2.33$

State 1:
$-4 + 0.5(2.667) + (0) - (0) = -2.67$
$-3 + 0.25(2.667) + (0) - (0) = -2.33$

Optimal Policy:
$d_0(R_2) = 1$
$d_1(R_2) = 1$

$g(R_2) = -2.67$
$v_0(R_2) = 2.667$
$v_1(R_2) = 0$

21.4-4.

Number of states = 2

Number of decisions = 2

Cost Matrix, C(ik):

$$\begin{bmatrix} 0 & 14 \\ 75 & 14 \end{bmatrix}$$

Transition Matrix, p(ij)[1]:

$$\begin{bmatrix} 0.125 & 0.875 \\ 0.125 & 0.875 \end{bmatrix}$$

Transition Matrix, p(ij)[2]:

$$\begin{bmatrix} 0.875 & 0.125 \\ 0.875 & 0.125 \end{bmatrix}$$

Initial Policy:

d0(R1) = 1
d1(R1) = 2

Discount Factor = 1

ITERATION # 1

Value Determination:

g(R1) = 0 +0.125v0(R1) + 0.875v1(R1) - v0(R1)
g(R1) = 14 +0.875v0(R1) + 0.125v1(R1) - v1(R1)

Solution of Value Determination Equations:
g(R1) = 7
v0(R1) = -8
v1(R1) = 0

Policy Improvement:

State 0:
0 + 0.125(-8) + 0.875(0) - (-8) = 7
14 + 0.875(-8) + 0.125(0) - (-8) = 15

21.4-4. (continued)

State 1:
75 + 0.125(-8) + 0.875(0) - (0) = 74
14 + 0.875(-8) + 0.125(0) - (0) = 7

Optimal Policy:
d0(R2) = 1
d1(R2) = 2

g(R1) = 7
v0(R1) = -8
v1(R1) = 0

21.4-5

Number of states: 2

Number of decisions: 2

Cost Matrix, C_{ik}: $\begin{bmatrix} -0.12 & 0.292 \\ 0.5 & 0.417 \end{bmatrix}$

$P_{ij}(1) = \begin{bmatrix} 0.375 & 0.625 \\ 1 & 0 \end{bmatrix}$ $P_{ij}(2) = \begin{bmatrix} 0.875 & 0.125 \\ 1 & 0 \end{bmatrix}$

Initial Policy:
$d_0(R_1) = 1$
$d_1(R_1) = 1$

Iteration # 1

Value Determination:

$g(R_1) = -0.12+0.375v_0(R_1) + 0.625v_1(R_1) - v_0(R_1)$
$g(R_1) = 0.5 + 1v_0(R_1) + 0v_1(R_1) - v_1(R_1)$

Solution of Value Determination Equations:

$g(R_1) = 0.115$
$v_0(R_1) = -0.38$
$v_1(R_1) = 0$

Policy Improvement:

State 0:
-0.12+ 0.375(-0.38) +(0) - (-0.38) = 0.115
0.292+ 0.875(-0.38) +(0) - (-0.38) = 0.34

State 1:
0.5 + 1(-0.38) +(0) - (0) = 0.115
0.417+ 1(-0.38) +(0) - (0) = 0.032

New Policy:
$d_0(R_2) = 1$
$d_1(R_2) = 2$

Iteration # 2

Value Determination:

$g(R_2) = -0.12+0.375v_0(R_2) + 0.625v_1(R_2) - v_0(R_2)$
$g(R_2) = 0.417+ 1v_0(R_2) + 0v_1(R_2) - v_1(R_2)$

Solution of Value Determination Equations:

21.4-5 (continued)

$g(R_2) = 0.083$
$v_0(R_2) = -0.33$
$v_1(R_2) = 0$

Policy Improvement:

State 0:
$-0.12 + 0.375(-0.33) + (0) - (-0.33) = 0.083$
$0.292 + 0.875(-0.33) + (0) - (-0.33) = 0.333$

State 1:
$0.5 + 1(-0.33) + (0) - (0) = 0.167$
$0.417 + 1(-0.33) + (0) - (0) = 0.083$

New Policy:
$d_0(R_3) = 2$
$d_1(R_3) = 1$

Iteration # 3

Value Determination:

$g(R_3) = -0.12 + 0.375v_0(R_3) + 0.625v_1(R_3) - v_0(R_3)$
$g(R_3) = 0.5 + 1v_0(R_3) + 1v_1(R_3) - v_1(R_3)$

Solution of Value Determination Equations:

$g(R_3) = 0.115$
$v_0(R_3) = -0.38$
$v_1(R_3) = 0$

Policy Improvement:

State 0:
$-0.12 + 0.375(-0.38) + (0) - (-0.38) = 0.115$
$0.292 + 0.875(-0.38) + (0) - (-0.38) = 0.34$

State 1:
$0.5 + 1(-0.38) + (0) - (0) = 0.115$
$0.417 + 1(-0.38) + (0) - (0) = 0.032$

Optimal Policy:
$d_0(R_4) = 2$
$d_1(R_4) = 1$

$g(R_4) = 0.115$
$v_0(R_4) = -0.38$
$v_1(R_4) = 0$

21.4-6. Note: All $ terms are in $ thousands.

Number of states: 3

Number of decisions: 2

Cost Matrix, C_{ik}: $\begin{bmatrix} -20 & -9 \\ -6 & -3 \\ 18 & 8 \end{bmatrix}$

21.4-6. (continued)

$$p_{ij}(1) = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \quad p_{ij}(2) = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

Initial Policy:

$d_0(R_1) = 1$
$d_1(R_1) = 1$
$d_2(R_1) = 2$

Iteration # 1

Value Determination:

$$g(R_1) = -20 + 0.3v_0(R_1) + 0.5v_1(R_1) + 0.2v_2(R_1) - v_0(R_1)$$
$$g(R_1) = -6 + 0.1v_0(R_1) + 0.5v_1(R_1) + 0.4v_2(R_1) - v_1(R_1)$$
$$g(R_1) = 8 + 0.2v_0(R_1) + 0.4v_1(R_1) + 0.4v_2(R_1) - v_2(R_1)$$

Solution of Value Determination Equations:

$g(R_1) = -3.27$
$v_0(R_1) = -32.4$
$v_1(R_1) = -12$
$v_2(R_1) = 0$

Policy Improvement:

State 0:

$-20 + 0.3(-32.4) + 0.5(-12) + (0) - (-32.4) = -3.27$
$-9 + 0.3(-32.4) + 0.5(-12) + (0) - (-32.4) = 7.732$

State 1:

$-6 + 0.1(-32.4) + 0.5(-12) + (0) - (-12) = -3.27$
$-3 + 0.1(-32.4) + 0.5(-12) + (0) - (-12) = -0.27$

State 2:

$18 + 0.2(-32.4) + 0.4(-12) + (0) - (0) = 6.732$
$8 + 0.2(-32.4) + 0.4(-12) + (0) - (0) = -3.27$

Optimal Policy:

$d_0(R_2) = 1$
$d_1(R_2) = 1$
$d_2(R_2) = 2$

$g(R_2) = -3.27$
$v_0(R_2) = -32.4$
$v_1(R_2) = -12$
$v_2(R_2) = 0$

21.4-7. Number of states = 2

Number of decisions = 2

Cost Matrix, C(ik):

0	0
-1200	-1200

Transition Matrix, p(ij)[1]:

0.4	0.6
0.6	0.4

Transition Matrix, p(ij)[2]:

0.5	0.5
0.4	0.6

21.4-7 (continued)

```
Initial Policy:

d0(R1) = 1
d1(R1) = 1                    Discount Factor = 1

ITERATION # 1

Value Determination:

g(R1) = 0     +  0.4v0(R1) +    0.6v1(R1) - v0(R1)
g(R1) = -1200+  0.6v0(R1) +    0.4v1(R1) - v1(R1)

Solution of Value Determination Equations:
g(R1)  = -600
v0(R1) = 1000
v1(R1) = 0

Policy Improvement:

State 0:
0     + 0.4  ( 1000) + 0.6  (0) - (1000 ) = -600
0     + 0.5  ( 1000) + 0.5  (0) - (1000 ) = -500

State 1:
-1200+ 0.6  ( 1000) + 0.4  (0) - (0) = -600
-1200+ 0.4  ( 1000) + 0.6  (0) - (0) = -800

New Policy:
  d0(R2) = 1
  d1(R2) = 2


ITERATION # 2

Value Determination:

g(R2) = 0     +  0.4v0(R2) +    0.6v1(R2) - v0(R2)
g(R2) = -1200+  0.4v0(R2) +    0.6v1(R2) - v1(R2)

Solution of Value Determination Equations:
g(R2)  = -720
v0(R2) = 1200
v1(R2) = 0

Policy Improvement:

State 0:
0     + 0.4  ( 1200) + 0.6  (0) - (1200 ) = -720
0     + 0.5  ( 1200) + 0.5  (0) - (1200 ) = -600

-1200+ 0.6  ( 1200) + 0.4  (0) - (0) = -480
-1200+ 0.4  ( 1200) + 0.6  (0) - (0) = -720

Optimal Policy:
  d0(R3) = 1
  d1(R3) = 2


g(R2)  = -720
v0(R2) = 1200
v1(R2) = 0
```

21.4-8

Markov Decision Processes Model:

Number of states: 4

Number of decisions: 2

Cost Matrix, C_{ik}:

$$\begin{bmatrix} 0 & --- \\ 4 & 3 \\ 4 & --- \\ 8 & 7 \end{bmatrix}$$

21-8 (continued)

$$P_{ij}(1) = \begin{bmatrix} 0.375 & 0.25 & 0.25 & 0.125 \\ 0.5 & 0.25 & 0.125 & 0.125 \\ 0.625 & 0.25 & 0.125 & 0 \\ 0.75 & 0.25 & 0 & 0 \end{bmatrix} \qquad P_{ij}(2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.375 & 0.25 & 0.25 & 0.125 \\ 0 & 0 & 0 & 0 \\ 0.625 & 0.25 & 0.125 & 0 \end{bmatrix}$$

Initial Policy:

$d_0(R_1) = 1$
$d_1(R_1) = 2$
$d_2(R_1) = 1$
$d_3(R_1) = 2$

Iteration # 1

Value Determination:

$$g(R_1) = 0 + 0.375v_0(R_1) + 0.25v_1(R_1) + 0.25v_2(R_1) + 0.125v_3(R_1) - v_0(R_1)$$
$$g(R_1) = 3 + 0.375v_0(R_1) + 0.25v_1(R_1) + 0.25v_2(R_1) + 0.125v_3(R_1) - v_1(R_1)$$
$$g(R_1) = 4 + 0.625v_0(R_1) + 0.25v_1(R_1) + 0.125v_2(R_1) + 0v_3(R_1) - v_2(R_1)$$
$$g(R_1) = 7 + 0.625v_0(R_1) + 0.25v_1(R_1) + 0.125v_2(R_1) + 0v_3(R_1) - v_3(R_1)$$

Solution of Value Determination Equations:

$g(R_1) = 2.213$
$v_0(R_1) = -5.9$
$v_1(R_1) = -2.9$
$v_2(R_1) = -3$
$v_3(R_1) = 0$

Policy Improvement:

State 0:

$$0 + 0.375(-5.9) + 0.25(-2.9) + 0.25(-3) + (0) - (-5.9) = 2.213$$
$$--- + 0(-5.9) + 0(-2.9) + 0(-3) + (0) - (-5.9) = ---$$

State 1:

$$4 + 0.5(-5.9) + 0.25(-2.9) + 0.125(-3) + (0) - (-2.9) = 2.85$$
$$3 + 0.375(-5.9) + 0.25(-2.9) + 0.25(-3) + (0) - (-2.9) = 2.213$$

State 2:

$$4 + 0.625(-5.9) + 0.25(-2.9) + 0.125(-3) + (0) - (-3) = 2.213$$
$$--- + 0(-5.9) + 0(-2.9) + 0(-3) + (0) - (-3) = ---$$

State 3:

$$8 + 0.75(-5.9) + 0.25(-2.9) + 0(-3) + (0) - (0) = 2.85$$
$$7 + 0.625(-5.9) + 0.25(-2.9) + 0.125(-3) + (0) - (0) = 2.213$$

Optimal Policy:

$d_0(R_2) = 1$
$d_1(R_2) = 2$
$d_2(R_2) = 1$
$d_3(R_2) = 2$

$g(R_2) = 2.213$
$v_0(R_2) = -5.9$
$v_1(R_2) = -2.9$
$v_2(R_2) = -3$
$v_3(R_2) = 0$

21-9

Markovian Decision Processes Model:

Number of states = 3

Number of decisions = 3

Cost Matrix, C(ik):

$$\begin{bmatrix} 13.33 & 18.67 & 24 \\ 4 & 19 & --- \\ 4 & --- & --- \end{bmatrix}$$

Transition Matrix, p(ij)[1]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.667 & 0.333 & 0 \\ 0.333 & 0.333 & 0.333 \end{bmatrix}$$

Transition Matrix, p(ij)[2]:

$$\begin{bmatrix} 0.667 & 0.333 & 0 \\ 0.333 & 0.333 & 0.333 \\ 0 & 0 & 0 \end{bmatrix}$$

Transition Matrix, p(ij)[3]:

$$\begin{bmatrix} 0.333 & 0.333 & 0.333 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Initial Policy:

d0(R1) = 3
d1(R1) = 1
d2(R1) = 1

Discount Factor = 1

Average Cost Policy Improvement Algorithm:

ITERATION # 1

Value Determination:

g(R1) = 24 +0.333v0(R1) + 0.333v1(R1) + 0.333v2(R1) - v0(R1)
g(R1) = 4 +0.667v0(R1) + 0.333v1(R1) + 0v2(R1) - v1(R1)
g(R1) = 4 +0.333v0(R1) + 0.333v1(R1) + 0.333v2(R1) - v2(R1)

Solution of Value Determination Equations:
g(R1) = 12.89
v0(R1) = 20
v1(R1) = 6.667
v2(R1) = 0

Policy Improvement:

State 0:
13.33+ 1 (20) + 0 (6.667) + 0 (0) - (20) = 13.33
18.67+ 0.667(20) + 0.333(6.667) + 0 (0) - (20) = 14.22
24 + 0.333(20) + 0.333(6.667) + 0.333(0) - (20) = 12.89

State 1:
4 + 0.667(20) + 0.333(6.667) + 0 (0) - (6.667) = 12.89
19 + 0.333(20) + 0.333(6.667) + 0.333(0) - (6.667) = 21.22
--- + 0 (20) + 0 (6.667) + 0 (0) - (6.667) = ---

State 2:
4 + 0.333(20) + 0.333(6.667) + 0.333(0) - (0) = 12.89
--- + 0 (20) + 0 (6.667) + 0 (0) - (0) = ---
--- + 0 (20) + 0 (6.667) + 0 (0) - (0) = ---

New Policy:
d0(R2) = 3
d1(R2) = 1
d2(R2) = 1

21-5-1. Let states 0, 1 and 2 denote \$600, \$800 and \$1000 offers, respectively, and let state 3 be that the car has already been sold (state ∞ of hint). Let decisions 1 and 2 be to reject and accept the offer, respectively.

$c_{01} = c_{11} = c_{21} = 60$, $c_{02} = -600$, $c_{12} = -800$ and $c_{22} = -1000$

21.5-1. (continued)

The transition probabilities are:

$$P(1)=\begin{pmatrix} 5/8 & 1/4 & 1/8 & 0\\ 5/8 & 1/4 & 1/8 & 0\\ 5/8 & 1/4 & 1/8 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad P(2)=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Let the initial guess be: $\begin{pmatrix} \$600 \text{ offer} : \text{reject}\\ \$800 \text{ offer} : \text{accept}\\ \$1000 \text{ offer} : \text{accept}\end{pmatrix}$

The relevant system of equation is: $v_0 = 60 + .95(5/8\, v_0 + 1/4\, v_1 + 1/8\, v_2)$

$v_1 = -800 + .95 v_3$

$v_2 = -1000 + .95 v_3$

$v_3 = .95 v_3$

The solution is $(v_0, v_1, v_2, v_3) = (-7960/13, -800, -1000, 0)$

Policy improvement:

state 0 with decision 2: $-600 + .95 v_3 = -600 > v_0$

state 1 with decision 1: $60 + .95(5/8\, v_0 + 1/4\, v_1 + 1/8\, v_2) = -7960/13 > v_1$

state 2 with decision 1: $60 + .95(5/8\, v_0 + 1/4\, v_1 + 1/8\, v_2) = -7960/13 > v_2$

Hence, $\begin{pmatrix} \$600 \text{ offer: reject}\\ \$800 \text{ offer: accept}\\ \$1000 \text{ offer: accept}\end{pmatrix}$ is optimal.

21.5-2(a) Minimize $60(y_{01}+y_{11}+y_{21}) - 600\, y_{02} - 800\, y_{12} - 1000\, y_{22}$

subject to $y_{01} + y_{02} - (.95)(5/8)(y_{01}+y_{11}+y_{21}) = 1/3$

$y_{11} + y_{12} - (.95)(1/4)(y_{01}+y_{11}+y_{21}) = 1/3$

$y_{21} + y_{22} - (.95)(1/8)(y_{01}+y_{11}+y_{21}) = 1/3$

$y_{ik} \geq 0 \quad i = 0,1,2 \quad k = 1,2.$

(b) The simplex-method yields $y_{01} = 0.81979$, $y_{12} = 0.5277$, $y_{22} = 0.43056$ and $y_{02} = y_{11} = y_{21} = 0$.

So, policy $\begin{pmatrix} 0 : 1\\ 1 : 2\\ 2 : 2\end{pmatrix} = \begin{pmatrix} \$600 : \text{reject}\\ \$800 : \text{accept}\\ \$1000 : \text{accept}\end{pmatrix}$ is optimal

21.5-3. $V_i^n = \min\left\{ \begin{array}{ll} 60 + .95(5/8\, V_0^{n-1} + 1/4\, V_1^{n-1} + 1/8\, V_2^{n-1}) & \text{for } K=1\\ -(\text{offer}) & \text{for } K=2 \end{array} \right\} \quad i = 0,1,2$

and $V_i^0 = 0$ for $i = 0,1,2$.

Iteration 1: $V_i^1 = \min\{60, -(\text{offer})\} = -(\text{offer})$ for $i = 0,1,2$

Iteration 2: $V_0^2 = \min\{-605, -600\} = -605$ with K = reject

$V_1^2 = \min\{-605, -800\} = -800$ with K = accept

$V_2^2 = \min\{-605, -1000\} = -1000$ with K = accept

Iteration 3: $V_0^3 = \min\{-607.97, -600\} = -607.97$ with K = reject

$V_1^3 = \min\{-607.97, -800\} = -800$ with K = accept

$V_2^3 = \min\{-607.97, -1000\} = -1000$ with K = accept

21.5-3. (Continued)

The approximate optimal solution is $\begin{pmatrix} \$600 \text{ offer: reject} \\ \$800 \text{ offer: accept} \\ \$1000 \text{ offer: accept} \end{pmatrix}$

(which, by Problem 22, we know is in fact optimal).

21.5-4. Let states 0, 1 and 2 denote a selling price of \$10, \$20 and \$30, respectively, and let state 3 denote that the stock has already been sold. Let decisions 1 and 2 be to hold and sell the stock, respectively.

$c_{01} = c_{11} = c_{21} = 0$, $c_{02} = -10$, $c_{12} = -20$ and $c_{22} = -30$.

The transition probabilities are:

$$P(1) = \begin{pmatrix} 4/5 & 1/5 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P(2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let the initial guess be $\begin{pmatrix} \$10: \text{hold} \\ \$20: \text{hold} \\ \$30: \text{sell} \end{pmatrix}$

The relevant system of equations is:

$$\begin{aligned} v_0 &= 0 + .9(4/5 v_0 + 1/5 v_1) \\ v_1 &= 0 + .9(1/4 v_0 + 1/4 v_1 + 1/2 v_2) \\ v_2 &= -30 + .9 v_3 \\ v_3 &= 0 + .9 v_3 \end{aligned}$$

The solution is $(v_0, v_1, v_2, v_3) = (-4860/353, -7560/353, -30, 0)$

Policy improvement:

State 0 with decision 2: $-10 + .9 v_3 = -10 > v_0$

State 1 with decision 2: $-20 + .9 v_3 = -20 > v_0$

State 2 with decision 1: $0 + .9[3/4 v_1 + 1/4 v_2] = -21.21 > v_2$

Hence, $\begin{pmatrix} \$10: \text{hold} \\ \$20: \text{hold} \\ \$30: \text{sell} \end{pmatrix}$ is optimal.

21.5-5. (a) Minimize $-10 y_{02} - 20 y_{12} - 30 y_{22}$

subject to

$$\begin{aligned} y_{01} + y_{02} - (.9)(4/5 y_{01} + 1/4 y_{11}) &= 1/3 \\ y_{11} + y_{12} - (.9)(1/5 y_{01} + 1/4 y_{11} + 3/4 y_{21}) &= 1/3 \\ y_{21} + y_{22} - (.9)(1/2 y_{11} + 1/4 y_{21}) &= 1/3 \\ y_{ik} \geq 0 \quad i = 0, 1, 2, \; k = 1, 2 & \end{aligned}$$

(b) The simplex method yields: $y_{01} = 1.96059$, $y_{11} = 0.95851$, $y_{22} = 0.76463$

So, the optimal policy is $\begin{pmatrix} 0: 1 \\ 1: 1 \\ 2: 2 \end{pmatrix} = \begin{pmatrix} \$10; \text{hold} \\ \$20: \text{hold} \\ \$30: \text{sell} \end{pmatrix}$

21.5-6. $V_0^n = \min \begin{cases} \cdot 9\,(4/5\ V_0^{n-1} + 1/5\ V_1^{n-1}) & \text{for } K=1 \\ -10 & \text{for } K=2 \end{cases}$

$V_1^n = \min \begin{cases} \cdot 9\,(1/4\ V_0^{n-1} + 1/4\ V_1^{n-1} + 1/2\ V_2^{n-1}) & \text{for } K=1 \\ -20 & \text{for } K=2 \end{cases}$

$V_2^n = \min \begin{cases} \cdot 9\,(3/4\ V_1^{n-1} + 1/4\ V_2^{n-1}) & \text{for } K=1 \\ -30 & \text{for } K=2 \end{cases}$

and $V_i^0 = 0$ for $i = 0, 1, 2$.

Iteration 1: $V_0^1 = \min\{0, -10\} = -10$ with $K=2$

$V_1^1 = \min\{0, -20\} = -20$ with $K=2$

$V_2^1 = \min\{0, -30\} = -30$ with $K=2$

Iteration 2: $V_0^2 = \min\{-10.8, -10\} = -10.8$ with $K=1$

$V_1^2 = \min\{-20.25, -20\} = -20.25$ with $K=1$

$V_2 = \min\{-20.25, -30\} = -30$ with $K=2$

Iteration 3: $V_0^3 = \min\{-11.42, -10\} = -11.42$ with $K=1$

$V_1^3 = \min\{-20.49, -20\} = -20.49$ with $K=1$

$V_2^3 = \min\{-20.42, -30\} = -30$ with $K=2$

So the indicated approximation to the optimal solution is
($10 : hold
$20 : hold
$30 : sell) which happens also to be optimal.

21.5-7 (a) Let state 0 = chemical 0 produced this month
1 = chemical 1 produced this month

decision 1 = use process A next month
2 = use process B next month

The four stationary deterministic policies are:

i	$d_i(R_1)$	$d_i(R_2)$	$d_i(R_3)$	$d_i(R_4)$
0	1	1	2	2
1	1	2	1	2

(b) Markovian Decision Processes Model:

Number of states = 2

Number of decisions = 2

Cost Matrix, C(ik):

28	26
37	24

Transition Matrix, p(ij)[2]:

0.2	0.8
0.3	0.7

Transition Matrix, p(ij)[1]:

0.2	0.8
0.3	0.7

Initial Policy:

d3(R1) = 2
d1(R1) = 2

Discounted Cost Policy Improvement Algorithm:

Discount Factor = 0.5

ITERATION # 1

Value Determination:

g(R1) = 26 + (0.5) [0.2V0(R1) + 0.8V1(R1)]
g(R1) = 24 + (0.5) [0.3V0(R1) + 0.7V1(R1)]

Solution of Value Determination Equations:
V1(R1) = 50.48
V2(R1) = 48.57

21.5-7. (continued)

```
Policy Improvement:

State 0:
28   + (0.5) [0.2  (50.48) + 0.8  (48.57) ] = 52.48
26   + (0.5) [0.2  (50.48) + 0.8  (48.57) ] = 50.48

State 1:
37   + (0.5) [0.3  (50.48) + 0.7  (48.57) ] = 61.57
24   + (0.5) [0.3  (50.48) + 0.7  (48.57) ] = 48.57

Optimal Policy:
  d0(R2) = 2          V0(R1) = 50.48
  d1(R2) = 2          V1(R1) = 48.57
```

21.5-8 (a) Minimize $28y_{01} + 26y_{02} + 46y_{11} + 22y_{12}$

Subject to: $y_{01} + y_{02} - \frac{1}{2}(\frac{1}{5}y_{01} + \frac{3}{10}y_{11} + \frac{1}{5}y_{02} + \frac{3}{10}y_{12}) = \frac{1}{2}$

$y_{11} + y_{12} - \frac{1}{2}(\frac{4}{5}y_{01} + \frac{7}{10}y_{11} + \frac{4}{5}y_{02} + \frac{7}{10}y_{12}) = \frac{1}{2}$

$y_{01}, y_{02}, y_{11}, y_{12} \geq 0$

(b) Simplex method yields: $y_{02} = 0.7619$, $y_{12} = 1.2381$, $y_{01} = y_{11} = 0$.

So, policy $\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$ = (chemical 0: process B; chemical 1: process B) is optimal.

21.5-9.

```
Markovian Decision Processes Model:   Number of states = 2
                                      Number of decisions = 2

Cost Matrix, C(ik):
|_ 28    26 _|
|_ 37    24 _|

Transition Matrix, p(ij)[1]:
|_ 0.2   0.8 _|
|_ 0.3   0.7 _|

Transition Matrix, p(ij)[2]:
|_ 0.2   0.8 _|
|_ 0.3   0.7 _|

Initial Policy:

d0(R1) = 1
d1(R1) = 2          Discount Factor = 0.5

Method of Successive Approximations:


Initial V(j):
  V(0) = 0
  V(1) = 0

ITERATION # 1

State 0:
28   + (0.5) [0.2  (    0) + 0.8  (    0) ] = 28
26   + (0.5) [0.2  (    0) + 0.8  (    0) ] = 26

State 1:
37   + (0.5) [0.3  (    0) + 0.7  (    0) ] = 37
24   + (0.5) [0.3  (    0) + 0.7  (    0) ] = 24

New Policy and New V(j):
  d0(R2) = 2,         V(0) = 26
  d1(R2) = 2,         V(1) = 24
```

21.5-9. (continued)

ITERATION # 2

State 0:
28 + (0.5) [0.2 (26) + 0.8 (24)] = 40.2
26 + (0.5) [0.2 (26) + 0.8 (24)] = 38.2

State 1:
37 + (0.5) [0.3 (26) + 0.7 (24)] = 49.3
24 + (0.5) [0.3 (26) + 0.7 (24)] = 36.3

New Policy and New V(j):
d0(R3) = 2, V(0) = 38.2
d1(R3) = 2, V(1) = 36.3

21.5-10.

Markovian Decision Processes Model: Number of states = 2

Number of decisions = 2

Cost Matrix, C(ik):

$$\begin{bmatrix} 28 & 26 \\ 37 & 24 \end{bmatrix}$$

Transition Matrix, p(ij)[1]:

$$\begin{bmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{bmatrix}$$

Transition Matrix, p(ij)[2]:

$$\begin{bmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{bmatrix}$$

Initial Policy:

d0(R1) = 1
d1(R1) = 2

Discount Factor = 0.5

Method of Successive Approximations:

Initial V(j):
V(0) = 0
V(1) = 0

ITERATION # 1

State 0:
28 + (0.5) [0.2 (0) + 0.8 (0)] = 28
26 + (0.5) [0.2 (0) + 0.8 (0)] = 26

State 1:
37 + (0.5) [0.3 (0) + 0.7 (0)] = 37
24 + (0.5) [0.3 (0) + 0.7 (0)] = 24

New Policy and New V(j):
d0(R2) = 2, V(0) = 26
d1(R2) = 2, V(1) = 24

ITERATION # 2

State 0:
28 + (0.5) [0.2 (26) + 0.8 (24)] = 40.2
26 + (0.5) [0.2 (26) + 0.8 (24)] = 38.2

State 1:
37 + (0.5) [0.3 (26) + 0.7 (24)] = 49.3
24 + (0.5) [0.3 (26) + 0.7 (24)] = 36.3

New Policy and New V(j):
d0(R3) = 2, V(0) = 38.2
d1(R3) = 2, V(1) = 36.3

ITERATION # 3

21.5-10. (continued)

```
State 0:
28   + (0.5) [0.2  ( 38.2) + 0.8  ( 36.3) ] = 46.34
26   + (0.5) [0.2  ( 38.2) + 0.8  ( 36.3) ] = 44.34

State 1:
37   + (0.5) [0.3  ( 38.2) + 0.7  ( 36.3) ] = 55.43
24   + (0.5) [0.3  ( 38.2) + 0.7  ( 36.3) ] = 42.43

New Policy and New V(j):
  d0(R4) = 2,          V(0) = 44.34
  d1(R4) = 2,          V(1) = 42.43
```

The optimal solution is, therefore, to use process B in all periods, regardless of the chemical produced.

21.5-11.

$$V_0^n = \min\begin{cases} 0 + .90(\tfrac{7}{8} V_1^{n-1} + \tfrac{1}{16} V_2^{n-1} + \tfrac{1}{16} V_3^{n-1}) & \text{for } k=1 \\ 4000 + .90\, V_1^{n-1} & \text{for } k=2 \\ 6000 + .90\, V_0^{n-1} & \text{for } k=3 \end{cases}$$

$$V_1^n = \min\begin{cases} 1000 + .90(\tfrac{3}{4} V_1^{n-1} + \tfrac{1}{8} V_2^{n-1} + \tfrac{1}{8} V_3^{n-1}) & \text{for } k=1 \\ 4000 + .90\, V_1^{n-1} & \text{for } k=2 \\ 6000 + .90\, V_0^{n-1} & \text{for } k=3 \end{cases}$$

$$V_2^n = \min\begin{cases} 3000 + .90(\tfrac{1}{2} V_2^{n-1} + \tfrac{1}{2} V_3^{n-1}) & \text{for } k=1 \\ 4000 + .90\, V_1^{n-1} & \text{for } k=2 \\ 6000 + .90\, V_0^{n-1} & \text{for } k=3 \end{cases}$$

$$V_3^n = 6000 + .90\, V_0^{n-1} \quad \text{for } k=3$$

and $V_0^0 = V_1^0 = V_2^0 = V_3^0 = 0$

Iteration 1: $V_0^1 = \min\{0, 4000, 6000\} = 0$ when $k=1$
$V_1^1 = \min\{1000, 4000, 6000\} = 1000$ when $k=1$
$V_2^1 = \min\{3000, 4000, 6000\} = 3000$ when $k=1$
$V_3^1 = 6000$ when $k=3$

Iteration 2: $V_0^2 = \min\{1293.75, 4900, 6000\} = 1293.75$ when $k=1$
$V_1^2 = \min\{2687.5, 4900, 6000\} = 2687.5$ when $k=1$
$V_2^2 = \min\{7050, 4900, 6000\} = 4900$ when $k=2$
$V_3^2 = 6000$ when $k=3$

Iteration 3: $V_0^3 = \min\{2729.53, 6418.75, 7164.38\} = 2729.53$ when $k=1$
$V_1^3 = \min\{4040.31, 6418.75, 7164.38\} = 4040.31$ when $k=1$
$V_2^3 = \min\{7905, 6418.75, 7164.38\} = 6418.75$ when $k=2$
$V_3^3 = 7164.38$ when $k=3$

Iteration 4: $V_0^4 = \min\{3945.80, 7636.28, 8456.58\} = 3945.80$ when $k=1$
$V_1^4 = \min\{5255.31, 7636.28, 8456.58\} = 5255.31$ when $k=1$
$V_2^4 = \min\{9112.41, 7636.28, 8456.58\} = 7636.28$ when $k=2$
$V_3^4 = 8456.58$ when $k=3$

21.5-11 (continued)

So the optimal policy is:

in periods 1,2,3 : (0: leave alone, 1: leave alone, 2: overhaul, 3: replace) and in period 4: (0: leave alone, 1: leave alone, 2: leave alone, 3: replace)

CHAPTER 22

22.1-1 a) Let the numbers 0.0000 to 0.4999 correspond to heads and the numbers 0.5000 to 0.9999 correspond to tails. The random observations for throwing an unbiased coin are 0.7520 = tails, 0.4184 = heads, 0.4189 = heads, 0.5982 = tails, 0.9559 = tails, and 0.1403 = heads.

b) Let the numbers 0.0000 to 0.5999 correspond to strikes and the numbers 0.6000 to 0.9999 correspond to balls. The random observations for pitches are 0.7520 = ball, 0.4184 = strike, 0.4189 = strike, 0.5982 = strike, 0.9559 = ball, and 0.1403 = strike.

c) Let the numbers 0.0000 to 0.3999 correspond to green lights, the numbers 0.4000 to 0.4999 correspond to yellow lights, and the numbers 0.5000 to 0.9999 correspond to red lights. The random observations for lights are 0.7520 = red, 0.4184 = yellow, 0.4189 = yellow, 0.5982 = red, 0.9559 = red, and 0.1403 = green.

22.1-2 a) If it is raining then let the numbers 0.0000 to 0.5999 correspond to rain for the next day and 0.6000 to 0.9999 correspond to clear for the next day. If it is clear then let the numbers 0.0000 to 0.7999 correspond to clear for the next day and the numbers 0.8000 to 0.9999 correspond to rain for the next day.

Day	Random Number	Weather
1	0.9559	rain
2	0.1403	rain
3	0.9345	clear
4	0.0801	clear
5	0.6892	clear
6	0.5146	clear
7	0.6290	clear
8	0.1612	clear
9	0.0989	clear
10	0.1155	clear

b)

Day	Random Number	Weather
		Clear
1	0.8212	Rain
2	0.1449	Rain
3	0.1762	Rain
4	0.7318	Clear
5	0.9218	Rain
6	0.1237	Rain
7	0.2881	Rain
8	0.8235	Clear
9	0.5954	Clear
10	0.8405	Rain

22.1-3 a) $P(2)=\frac{4}{25},\ P(3)=\frac{7}{25},\ P(4)=\frac{8}{25},\ P(5)=\frac{5}{25},\ P(6)=\frac{1}{25}$

b) $\text{Mean} = (2)\frac{4}{25}+(3)\frac{7}{25}+(4)\frac{8}{25}+(5)\frac{5}{25}+(6)\frac{1}{25}=3.68$ stoves

c) Random numbers can be assigned to the various sales levels to simulate daily sales. Let the numbers 0.000 to 0.1599 correspond to 2 stoves being sold in a day, the numbers 0.1600 to 0.4399 correspond to 3 stoves, the numbers 0.4400 to 0.7599 correspond to 4 stoves, the numbers 0.7600 to 0.9599 correspond to 5 stoves, and the numbers 0.9600 to 0.9999 correspond to 6 stoves.

d) Sales are 4 stoves, 6 stoves, and 2 stoves on the three days. The average is 4, which is 0.32 higher than the mean obtained in part (b).

e) Answers will vary. The following 300-day simulation yielded an average demand of 3.72.

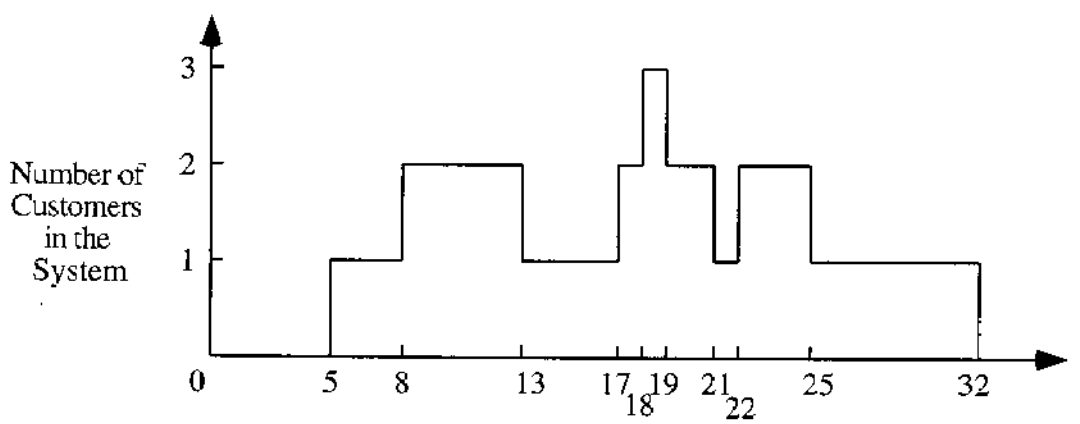
Day	Random Number	Demand
1	0.5475	4
2	0.3597	3
3	0.6539	4
4	0.6263	4
5	0.9576	5
6	0.8396	5
7	0.1005	2
297	0.5809	4
298	0.3673	3
299	0.4453	4
300	0.1361	2
	Average =	3.720

Distribution of Demand

Probability	Cumulative	Demand
0.16	0	2
0.28	0.16	3
0.32	0.44	4
0.20	0.76	5
0.04	0.96	6

22.1-4

a)



b)

$$P_0 = \frac{5}{32} = 0.156 \qquad P_1 = \frac{3+4+1+7}{32} = 0.469$$

$$P_2 = \frac{5+1+2+3}{32} = 0.344 \qquad P_3 = \frac{1}{32} = 0.031$$

$$Est\{L\} = \sum_{0}^{3} nP_n = (0)(0.156) + (1)(0.469) + (2)(0.344) + (3)(0.031) = 1.25 \text{ cust.}$$

$$Est\{L_q\} = \sum_{1}^{3} (n-1)P_n = (1-1)(0.469) + (2-1)(0.344) + (3-1)(0.031) = 0.406 \text{ cust.}$$

(CONT'D)

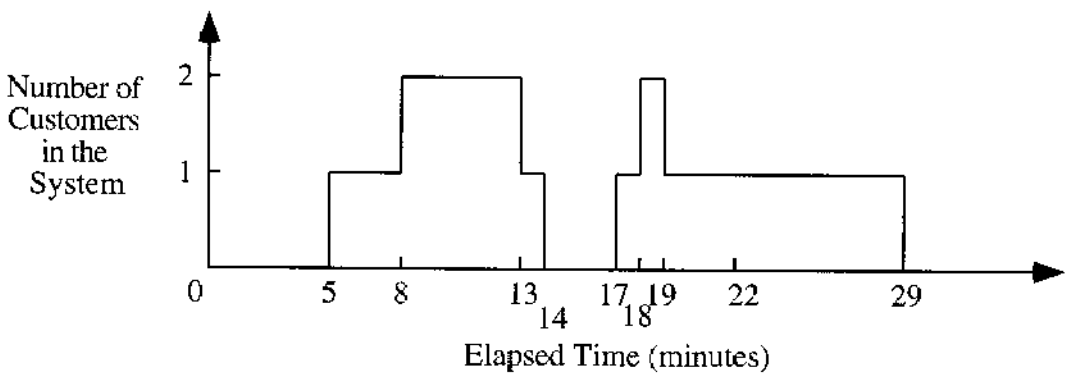
(CONT'D)

Customers	Arrival Time	Service Time	Departure Time	System Time	Wait Time
1	5	8	13	8	0
2	8	6	19	11	5
3	17	2	21	4	2
4	18	4	25	7	3
5	22	7	32	10	3

$$Est\{W_q\} = \frac{\text{sum of observed waiting times}}{\text{number of observed waiting times}} = \frac{13}{5} = 2.6 \text{ minutes}$$

$$Est\{W\} = \frac{\text{sum of observed system times}}{\text{number of observed system times}} = \frac{40}{5} = 8 \text{ minutes}$$

c)



d)

$$P_0 = \frac{5+3}{29} = 0.276 \qquad P_1 = \frac{3+1+1+3+7}{29} = 0.517 \qquad P_2 = \frac{5+1}{29} = 0.207$$

$$Est\{L\} = \sum_{0}^{2} nP_n = (0)(0.276) + (1)(0.517) + (2)(0.207) = 0.931 \text{ customers}$$

$$Est\{L_q\} = \sum_{1}^{2} (n-1)P_n = (1-1)(0.517) + (2-1)(0.207) = 0.207 \text{ customers}$$

Customers	Arrival Time	Service Time	Departure Time	System Time	Wait Time
1	5	8	13	8	0
2	8	6	14	6	0
3	17	2	19	2	0
4	18	4	22	4	0
5	22	7	29	7	0

$$Est\{W_q\} = \frac{\text{sum of observed waiting times}}{\text{number of observed waiting times}} = \frac{0}{5} = 0 \text{ minutes}$$

$$Est\{W\} = \frac{\text{sum of observed system times}}{\text{number of observed system times}} = \frac{27}{5} = 5.4 \text{ minutes}$$

22.1-5

(a) Arrival time ~ exp(½ per minute)
Departure time ~ exp(⅙ per minute)
Time until next arrival is given by $x = -12\ln(1-r)$
Time until next departure is given by $x = -6\ln(1-r)$

Time (in minutes)	Customers in system	r for arrival	r for departure	Minutes until next Arrival	Minutes until next Departure
0	0	.096	—	1.211	—
1.211	1	.596	.665	10.100	6.562
7.773	0	—	—	3.538	
11.311	1	.764	.842	17.327	11.071
22.382	0				

(b) $P\{\text{arrival in six-minute period}\} = 1 - e^{-\frac{6}{12}} = .393$
$P\{\text{departure in six-minute period}\} = 1 - e^{-1} = .632$

So, $0 \le r_a \le .392$ means there was an arrival
and $0 \le r_d \le .631$ means there was a departure

Time (in minutes)	Customers in system	r_a	r_d	Arrival?	Departure?
0	0	.096	—	yes	—
6	1	.569	.665	no	no
12	1	.764	.842	no	no
18	1	.492	.224	no	yes
24	0	.950	—	no	—
30	0	.610	—	no	—
36	0	.145	—	yes	—
42	1	.484	.552	no	yes
48	0	.350	—	yes	—

c) Distr. of interarrival times: Exponential Mean = 0.2
Distr. of service times: Exponential Mean = 0.1

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	0.4674	0.01815
0.01815	0	No	0.4674	---
0.4674	0	Yes	1.35969	0.76469
0.76469	0	No	1.35969	---
1.35969	0	Yes	1.37764	1.38405
1.37764	1	Yes	1.59748	1.38405
1.38405	0	Yes	1.59748	1.40728
1.40728	0	No	1.59748	---
1.59748	0	Yes	1.81374	1.9353
1.81374	1	Yes	2.07844	1.9353
1.9353	0	Yes	2.07844	2.09178
2.07844	1	Yes	2.09584	2.09178
2.09178	0	Yes	2.09584	2.09842
2.09584	1	Yes	2.3323	2.09842
2.09842	0	Yes	2.3323	2.26042
2.26042	0	No	2.3323	---

(CONT'D)

(CONT'D)

2.3323	0	Yes	2.33589	2.33749
2.33589	1	Yes	2.35022	2.33749
2.33749	0	Yes	2.35022	2.34943
2.34943	0	No	2.35022	---
2.35022	0	Yes	2.42298	2.40413
2.40413	0	No	2.42298	---
2.42298	0	Yes	2.42362	2.54099
2.42362	1	Yes	2.55037	2.54099
2.54099	0	Yes	2.55037	2.61055
2.55037	1	Yes	2.74492	2.61055
2.61055	0	Yes	2.74492	2.64271
2.64271	0	No	2.74492	---
2.74492	0	Yes	3.31753	2.80169
2.80169	0	No	3.31753	---
3.31753	0	Yes	3.32939	3.42686
3.32939	1	Yes	3.42362	3.42686
3.42362	2	Yes	3.89049	3.42686
3.42686	1	Yes	3.89049	3.65956
3.65956	0	Yes	3.89049	3.74341
3.74341	0	No	3.89049	---

The results from the simulation run are as follows:

Average number waiting to begin service: 0.237795

Average number waiting or in service: 0.753969

Average: 0.04015 0.15169

d)

Data

Number of Servers = 1

Interarrival Times

Distribution = Exponential

Mean = 0.2

Service Times

Distribution = Exponential

Mean = 0.1

Length of Simulation Run

Number of Arrivals = 10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.0328644	0.9648188	1.10051
L_q =	0.5279548	0.4717765	0.5841332
W =	0.2060475	0.194558	0.2175371
W_q =	0.1053428	0.0951514	0.1155342
P_0 =	0.4852904	0.4810073	0.5095735
P_1 =	0.2479705	0.2411367	0.2548043
P_2 =	0.1222497	0.1164324	0.1280671
P_3 =	0.0658451	0.0604186	0.0712716
P_4 =	0.032752	0.0284869	0.0370171
P_5 =	0.01899	0.0152969	0.0226831
P_6 =	0.0086706	0.0063485	0.0109928
P_7 =	0.0040461	0.0025727	0.0055196
P_8 =	0.0024146	0.0012389	0.0035904
P_9 =	0.0011045	0.0003739	0.001835
P_{10} =	0.000353	-4.053E-05	0.0007465

e)

$\lambda = 5$ $L = 1$

$\mu = 10$ $L_q = 0.5$

$s = 1$ $W = 0.2$

$W_q = 0.1$

every measure is inside the 95% confidence interval.

22.1-6

a) 1. A description of the components of the system. The system is a single-server queueing system with the crew as the server and machines as the customers. The service time has a uniform distribution over the interval from 0 to twice the mean, and interarrival time distribution has an exponential distribution with a mean of 5 hours.

2. A simulation clock will record the amount of simulated time that will elapse.

3. The state of the system will be defined as N(t) = number of machines needing repair at time t.

4. The breakdowns and repairs that occur over time are randomly generated by generating random observations from the distributions of interarrival and service times.

5. Adjust the state of the system when a breakdown or repair occurs by:

$$\text{Reset } N(t) = \begin{cases} N(t)+1, & \text{if a breakdown occurs at time t} \\ N(t)-1, & \text{if a repair occurs at time t} \end{cases}$$

6. Advance the time on the simulation clock by using the next-event time advance procedure.

b)

Time (in hours)	Machines Broken	r_a	r_d	Hours until next arrival	Hours until next departure
0	0	.136	—	9.97	—
9.97	1	.987	.057	.07	.45
10.04	2	3.26	—	5.61	.38
10.42	1	—	.193	5.23	1.54
11.96	0	—	—	3.69	—
15.65	1	.14	.792	9.81	6.34

c)

$P(\text{arrival in 1 hour period}) = 1 - e^{-0.2} = .181$

$P(\text{departure in 1 hour period}) = .125$

So:

$0 \le r_a \le .181$ means there was an arrival

$0 \le r_d \le .125$ means there was a departure.

(CONT'D)

(cont'd)

Time (in hours)	Machines broken	r_a	r_d	Arrival?	Departure?
0	0	.443	–	NO	–
1	0	.526	–	NO	–
2	0	.186	–	NO	–
3	0	.205	–	No	–
4	0	.811	–	NO	–
5	0	.072	–	yes	–
6	1	.043	.378	Yes	NO
7	2	.574	.555	No	NO
8	2	.823	.881	NO	NO
9	2	.321	.948	NO	NO
10	2	.710	.438	NO	NO
11	2	.837	.882	NO	NO
12	2	.220	.167	NO	NO
13	2	.44	.183	YES	NO
14	3	.422	.341	NO	NO
15	3	.349	.114	NO	YES
16	2	.315	.874	NO	NO
17	2	.595	.894	NO	NO
18	2	.918	.498	NO	NO
19	2	.464	.538	NO	NO
20	2	.319	.312	NO	NO

d) crew size = 4

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	0.60886	4.73613
0.60886	1	Yes	3.33478	4.73613
3.33478	2	Yes	13.3965	4.73613
4.73613	1	Yes	13.3965	8.21202
8.21202	0	Yes	13.3965	10.7508
10.7508	0	No	14.4768	15.6734
14.4768	0	Yes	15.9811	16.2081
15.9811	0	Yes	38.8955	21.5422
38.8955	0	Yes	44.2451	40.699
44.2451	0	Yes	50.6593	46.5535
50.6593	0	Yes	52.5093	53.9709
53.9709	0	Yes	52.5093	59.5861
52.5093	0	Yes	54.436	53.8193

CONT'D

9.11

d) (CONT'D)

crew size = 3

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	0.60886	6.31484
0.60886	1	Yes	3.33478	6.31484
3.33478	2	Yes	13.3965	6.31484
6.31484	1	Yes	13.3965	10.9494
10.9494	0	Yes	13.3965	14.3343
13.3965	1	Yes	17.1226	14.3343
14.3343	1	No	17.1226	16.6426
16.6426	1	No	17.1226	24.0574
17.1226	2	No	22.4722	24.0574
22.4722	2	Yes	28.8864	25.5501
28.8864	2	Yes	30.7364	33.3018
30.7364	2	Yes	38.6232	38.2233
38.6232	2	Yes	40.5499	40.3699
40.5499	2	Yes	41.1736	46.8118

crew size = 2

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	0.60886	3.15742
0.60886	1	Yes	3.33478	3.15742
3.15742	0	Yes	3.33478	6.52381
3.33478	1	Yes	3.64433	6.52381
3.64433	2	Yes	6.79651	6.52381
6.52381	1	Yes	6.79651	9.8056
6.79651	2	Yes	—	9.8056
6.79651	3	Yes	29.7109	9.8056
9.8056	2	Yes	29.7109	11.0079
11.0079	1	Yes	29.7109	12.5469
12.5469	0	Yes	29.7109	14.7546
14.7546	0	Yes	29.7109	18.4981
29.7109	1	Yes	31.6376	30.5842
30.5842	1	No	31.6376	33.7152
31.6376	1	Yes	31.7936	33.8986
31.7936	1	Yes	38.4414	34.3012
38.4414	1	Yes	43.6121	41.5592
43.6121	1	Yes	60.9818	44.9606
60.9818	1	Yes	66.9273	63.1735
66.9273	1	Yes	76.0143	68.1042

e)

2 Servers:

Data

Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	5

Service Times

Distribution =	Uniform
Minimum =	0
Maximum =	8

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.818	2.524	3.112
L_q =	2.016	1.736	2.296
W =	14.045	12.756	15.335
W_q =	10.049	8.778	11.319
P_0 =	0.198	0.181	0.216
P_1 =	0.209	0.194	0.224
P_2 =	0.164	0.153	0.175
P_3 =	0.123	0.114	0.132
P_4 =	0.0879	0.0803	0.0956
P_5 =	0.0637	0.0556	0.0719
P_6 =	0.0473	0.0388	0.0559
P_7 =	0.0342	0.027	0.0415
P_8 =	0.0240	0.0177	0.0304
P_9 =	0.0162	0.0102	0.0222
P_{10} =	0.0128	0.00689	0.0187

3 Servers:

Data

Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	5

Service Times

Distribution =	Uniform
Minimum =	0
Maximum =	6

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.172	1.105	1.24
L_q =	0.0577	0.519	0.634
W =	5.915	5.648	6.182
W_q =	2.909	2.652	3.166
P_0 =	0.404	0.391	0.417
P_1 =	0.291	0.283	0.299
P_2 =	0.16	0.153	0.166
P_3 =	0.0771	0.0712	0.083
P_4 =	0.038	0.0332	0.0427
P_5 =	0.0168	0.0132	0.0203
P_6 =	0.00719	0.00467	0.00972
P_7 =	0.00224	0.00106	0.00342
P_8 =	0.0018	0.000588	0.00309
P_9 =	0.00152	0.000221	0.00282
P_{10} =	0.000428	-0.00021	0.00107

(CONT'D)

(CONT'D)

4 Servers:

Data	
Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	5

Service Times

Distribution =	Uniform
Minimum =	0
Maximum =	4

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	0.57	0.549	0.592
L_q =	0.172	0.158	0.187
W =	2.882	2.809	2.956
W_q =	0.87	0.807	0.934
P_0 =	0.602	0.593	0.611
P_1 =	0.273	0.268	0.279
P_2 =	0.0906	0.0862	0.0949
P_3 =	0.246	0.0219	0.0272
P_4 =	0.00766	0.00602	0.0093
P_5 =	0.00178	0.001	0.00255
P_6 =	0.000162	0	0.000329
P_7 =	0.000115	-0.0001	0.000341
P_8 =	0.000058	-0.00006	0.00017
P_9 =	0.000059	-0.00006	0.000174
P_{10} =	0	0	0

For these simulation runs, 3 servers was enough to get the average waiting time before repair below 3 hours.

f)

2 Servers:

Data		
λ =	0.2	(mean arrival rate)
$1/\mu$ =	4	(expected service time)
σ =	2.3094011	(standard deviation)
s =	1	(# servers)

Results

L =	2.93333333
L_q =	2.13333333
W =	14.6666667
W_q =	10.6666667

3 Servers:

Data		
λ =	0.2	(mean arrival rate)
$1/\mu$ =	3	(expected service time)
σ =	1.7320508	(standard deviation)
s =	1	(# servers)

Results

L =	1.2
L_q =	0.6
W =	6
W_q =	3

4 Servers:

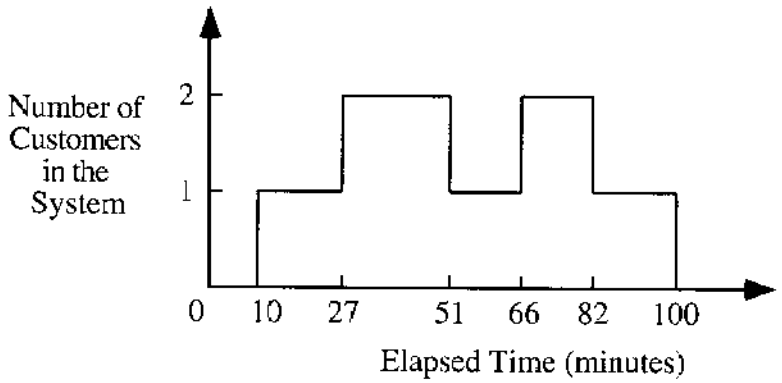
Data		
λ =	0.2	(mean arrival rate)
$1/\mu$ =	2	(expected service time)
σ =	1.1547005	(standard deviation)
s =	1	(# servers)

Results

L =	0.57777778
L_q =	0.17777778
W =	2.88888889
W_q =	0.88888889

3 servers is enough to get the average waiting time before repair down to 3 hours.

22.1-7 a)



b)

$$P_0 = \frac{10}{100} = 0.1; \qquad P_1 = \frac{17+15+18}{100} = 0.4;$$

$$P_2 = \frac{24+16}{100} = 0.4; \qquad P_3 = \frac{0}{100} = 0.$$

c) $Est(L) = \sum_{n=1}^{3} nP_n = 1(0.4) + 2(0.4) + 3(0) = 1.2.$

$$Est(L_q) = \sum_{n=2}^{3} (n-1)P_n = 1(0.4) + 2(0) = 0.4.$$

d) $$Est\{W\} = \frac{\text{sum of observed system times}}{\text{number of observed system times}} = \frac{41+55+34}{3} = 43.33 \text{ minutes.}$$

$$Est\{W_q\} = \frac{\text{sum of observed waiting times}}{\text{number of observed waiting times}} = \frac{0+24+16}{3} = 13.33 \text{ minutes.}$$

22.1-8

a) Distr. of interarrival times: Translated Exp. Min = 0.5 Mean = 1
Distr. of service times: Erlang Mean = 1.5 k = 4

Current Time	Number of Customers in Queue	Customer Being Served Server 1	Server 2	Next Arrival	Next Service Completion Server 1	Server 2
0	0	Yes	No	1.6685	3.00911	---
1.6685	0	Yes	Yes	2.2903	3.00911	4.06113
2.2903	1	Yes	Yes	3.45204	3.00911	4.06113
3.00911	0	Yes	Yes	3.45204	4.31305	4.06113
3.45204	1	Yes	Yes	3.99204	4.31305	4.06113
3.99204	2	Yes	Yes	5.08213	4.31305	4.06113
4.06113	1	Yes	Yes	5.08213	4.31305	4.82208
4.31305	0	Yes	Yes	5.08213	7.01408	4.82208
4.82208	0	Yes	No	5.08213	7.01408	---
5.08213	0	Yes	Yes	5.80875	7.01408	6.56763
5.80875	1	Yes	Yes	6.42612	7.01408	6.56763
6.42612	2	Yes	Yes	8.45996	7.01408	6.56763
6.56763	1	Yes	Yes	8.45996	7.01408	7.793
7.01408	0	Yes	Yes	8.45996	9.25094	7.793
7.793	0	Yes	No	8.45996	9.25094	---
8.45996	0	Yes	Yes	8.45996	9.25094	8.95073

(CONT'D)

(cont'd, 22.1-8 a)

8.45996	1	Yes	Yes	9.01185	9.25094	8.95073
8.95073	0	Yes	Yes	9.01185	9.25094	10.3732
9.01185	1	Yes	Yes	10.7538	9.25094	10.3732
9.25094	0	Yes	Yes	10.7538	11.0051	10.3732
10.3732	0	Yes	No	10.7538	11.0051	---
10.7538	0	Yes	Yes	11.8319	11.0051	11.9901
11.0051	0	No	Yes	11.8319	---	11.9901
11.8319	0	Yes	Yes	12.5131	13.3238	11.9901
11.9901	0	Yes	No	12.5131	13.3238	---
12.5131	0	Yes	Yes	15.8697	13.3238	14.986
13.3238	0	No	Yes	15.8697	---	14.986
14.986	0	No	No	15.8697	---	---
15.8697	0	Yes	No	18.1124	16.8485	---
16.8485	0	No	No	18.1124	---	---
18.1124	0	Yes	No	19.0569	18.8949	---
18.1124	0	Yes	No	19.0569	18.8949	---
18.8949	0	No	No	19.0569	---	---
19.0569	0	Yes	No	19.8234	21.8863	---
19.8234	0	Yes	Yes	20.7688	21.8863	21.3164
20.7688	0	Yes	Yes	---	21.8863	21.3164

The results from the simulation run are as follows:

Average number waiting to begin service: 0.186891

Average number waiting or in service: 1.522408

Average: 0.18669 1.87597

b)

Data

Number of Servers =	2

Interarrival Times

Distribution =	Translated Exp.
Minimum =	0.5
Mean =	1

Service Times

Distribution =	Erlang
Mean =	1.5
k =	4

Length of Simulation Run

Number of Arrivals =	5,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.796	1.73	1.862
L_q =	0.303	0.26	0.345
W =	1.803	1.748	1.858
W_q =	0.304	0.263	0.345
P_0 =	0.082	0.0742	0.0903
P_1 =	0.343	0.326	0.359
P_2 =	0.352	0.34	0.363
P_3 =	0.161	0.148	0.174
P_4 =	0.0483	0.0382	0.0584
P_5 =	0.012	0.00708	0.017
P_6 =	0.00204	0.000479	0.0036
P_7 =	0.000222	-0.00017	0.000612
P_8 =	0	0	0
P_9 =	0	0	0
P_{10} =	0	0	0

c)

Data

Number of Servers =	3

Interarrival Times

Distribution =	Translated Exp.
Minimum =	0.5
Mean =	1

Service Times

Distribution =	Erlang
Mean =	1.5
k =	4

Length of Simulation Run

Number of Arrivals =	5,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.498	1.469	1.527
L_q =	0.0101	0.00773	0.0124
W =	1.5075	1.487	1.528
W_q =	0.0101	0.00781	0.0125
P_0 =	0.108	0.1	0.116
P_1 =	0.411	0.4	0.422
P_2 =	0.365	0.355	0.375
P_3 =	0.106	0.0981	0.113
P_4 =	0.00971	0.007551	0.0119
P_5 =	0.00018	0.000003	0.000357
P_6 =	0	0	0
P_7 =	0	0	0
P_8 =	0	0	0
P_9 =	0	0	0
P_{10} =	0	0	0

d) Two Tellers:

Data

Number of Servers =	2

Interarrival Times

Distribution =	Translated Exp.
Minimum =	0.5
Mean =	0.9

Service Times

Distribution =	Erlang
Mean =	1.5
k =	4

Length of Simulation Run

Number of Arrivals =	5,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.355	2.188	2.522
L_q =	0.671	0.532	0.809
W =	2.103	1.968	2.238
W_q =	0.599	0.479	0.718
P_0 =	0.04	0.0334	0.0465
P_1 =	0.236	0.213	0.259
P_2 =	0.343	0.321	0.365
P_3 =	0.212	0.198	0.227
P_4 =	0.0946	0.0773	0.112
P_5 =	0.0427	0.028	0.0573
P_6 =	0.0197	0.011	0.0284
P_7 =	0.0088	0.00296	0.0146
P_8 =	0.00253	0.000469	0.00459
P_9 =	0.000453	-0.00018	0.00108
P_{10} =	0.000005	-0.000005	0.000016

(CONT'D)

(CONT'D) part d)

Three Tellers:

Data

Number of Servers =	3

Interarrival Times

Distribution =	Translated Exp.
Minimum =	0.5
Mean =	0.9

Service Times

Distribution =	Erlang
Mean =	1.5
k =	4

Length of Simulation Run

Number of Arrivals =	5,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.684	1.652	1.716
L_q =	0.0217	0.0169	0.0264
W =	1.516	1.494	1.538
W_q =	0.0195	0.0153	0.0237
P_0 =	0.0702	0.064	0.0764
P_1 =	0.361	0.349	0.373
P_2 =	0.404	0.395	0.414
P_3 =	0.143	0.134	0.152
P_4 =	0.0196	0.0158	0.0235
P_5 =	0.001	0.000342	0.00173
P_6 =	0	0	0
P_7 =	0	0	0
P_8 =	0	0	0
P_9 =	0	0	0
P_{10} =	0	0	0

e)

Two tellers provides reasonable wait times, with both the original arrival wait ($W_q = 0.3$ minutes) and the higher arrival rate in part c ($W_q = 0.6$ minutes).

22.1-9

```
Priority Class 1 (higher priority) customers
    Distr. of interarrival times:   Uniform          Min  = 1      Max  = 3
    Distr. of service times:        Erlang           Mean = 1.5    k   = 4

Priority Class 2 (lower priority) customers
    Distr. of interarrival times:   Translated Exp.  Min  = 0.5    Mean = 1
    Distr. of service times:        Erlang           Mean = 1.5    k   = 4
```

Current Time	# of Customers in Line Class 1	Class 2	Class of Customer Being Served Server 1	Server 2	Next Arrival Class 1	Class 2	Next Service Completion Server 1	Server 2
0	0	0	1	idle	1.19323	0.59076	4.05587	---
0.59076	0	0	1	2	1.19323	1.98438	4.05587	1.75598
1.19323	1	0	1	2	4.03947	1.98438	4.05587	1.75598
1.75598	0	0	1	1	4.03947	2.57211	4.05587	2.6547
2.57211	0	1	1	1	4.03947	4.35852	4.05587	2.6547
2.6547	0	0	1	2	4.03947	4.35852	4.05587	5.02524
4.03947	1	0	1	2	6.60605	4.35852	4.05587	5.02524
4.05587	0	0	1	2	6.60605	4.35852	6.31076	5.02524
4.35852	0	1	1	2	6.60605	5.60471	6.31076	5.02524
5.02524	0	0	1	2	6.60605	5.60471	6.31076	6.5263
5.60471	0	1	1	2	6.60605	6.32351	6.31076	6.5263
6.31076	0	0	2	2	6.60605	6.32351	7.80267	6.5263
6.32351	0	1	2	2	6.60605	7.67972	7.80267	6.5263

(CONT'D)

(CONT'D) 22.1-9

6.5263	0	0	2	2	6.60605	7.67972	7.80267	7.41307
6.60605	1	0	2	2	7.66733	7.67972	7.80267	7.41307
7.66733	2	0	2	2	9.48954	7.67972	7.80267	7.41307
7.41307	1	0	2	1	9.48954	7.67972	7.80267	8.0084
7.67972	1	1	2	1	9.48954	8.7606	7.80267	8.0084
7.80267	0	1	1	1	9.48954	8.7606	9.39632	8.0084
8.0084	0	0	1	2	9.48954	8.7606	9.39632	10.085
8.7606	0	1	1	2	9.48954	9.54627	9.39632	10.085
9.39632	0	0	2	2	9.48954	9.54627	11.8025	10.085
9.48954	1	0	2	2	11.0103	9.54627	11.8025	10.085
9.54627	1	1	2	2	11.0103	10.484	11.8025	10.085
10.085	0	1	2	1	11.0103	10.484	11.8025	10.6113
10.484	0	2	2	1	11.0103	11.2066	11.8025	10.6113
10.6113	0	1	2	2	11.0103	11.2066	11.8025	11.1199
11.0103	1	1	2	2	13.2226	11.2066	11.8025	11.1199
11.1199	0	1	2	1	13.2226	11.2066	11.8025	12.4916
11.2066	0	2	2	1	13.2226	11.804	11.8025	12.4916
11.804	0	3	2	1	13.2226	12.6438	11.8025	12.4916
11.8025	0	2	2	1	13.2226	12.6438	13.5255	12.4916
12.4916	0	1	2	2	13.2226	12.6438	13.5255	13.1055
12.6438	0	2	2	2	13.2226	13.1919	13.5255	13.1055
13.1055	0	1	2	2	13.2226	13.1919	13.5255	14.1874
13.1919	0	2	2	2	13.2226	14.1007	13.5255	14.1874
13.2226	1	2	2	2	15.2685	14.1007	13.5255	14.1874
13.5255	0	2	1	2	15.2685	14.1007	15.9147	14.1874

The results from the simulation run are as follows:

Class 1 customers:

Average number waiting to begin service: 0.209215

Average number waiting or in service: 1.007613

Average: 1.07381 2.59826

Class 2 customers:

Average number waiting to begin service: 1.406468

Average number waiting or in service: 2.575706

Average: 0.38188 1.91684

22.1-10

a) - A description of the components of the system. For parts b) through f) each type of car corresponds to an M/M/1 system, one independent of the other. For parts g), h) and i) the sysem is an M/M/2 system.
Both arrival and service rates have exponential distributions.
- A simulation clock will record the amount of simulated time that elapses.

(CONT'D)

(cont.) a)

- The state of the system is defined as:

  $N_j(t)$: # japanese cars needing repair at time t.

  $N_g(t)$: # german " " " " " "

- The breakdowns and repairs that occur over time are generated by random observations with exponential distributions

- The state of the system follows the dynamics:

$$N_j(t) = \begin{cases} N_j(t)+1 & \text{if a japanese car arrives to the shop} \\ N_j(t)-1 & \text{" " " " is repaired.} \end{cases}$$

  same thing for $N_g(t)$.

- Advance the time on the simulation clock by using the next-event time advance procedure.

b)

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	0.03044	0.04731
0.03044	1	Yes	0.16674	0.04731
0.04731	0	Yes	0.16674	0.0818
0.0818	0	No	0.16674	—
0.16674	0	Yes	0.32435	0.33876
0.32435	1	Yes	—	0.33876
0.32435	2	Yes	—	0.33876
0.32435	2	Yes	1.47007	0.33954
1.47007	2	Yes	1.73755	1.71047
1.73755	2	Yes	2.05826	1.92858
2.05826	2	Yes	2.15076	2.17713
2.15076	2	Yes	2.5451	2.16401
2.5451	2	Yes	2.64143	2.84944
2.64143	2	Yes	2.67262	2.69043

c) & d)

German cars:

Data	
Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	0.25

Service Times

Distribution =	Exponential
Mean =	0.2

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	3.963	3.302	4.625
L_q =	3.165	2.52	3.809
W =	0.994	0.84	1.149
W_q =	0.794	0.642	0.946
P_0 =	0.201	0.179	0.224
P_1 =	0.158	0.143	0.174
P_2 =	0.128	0.116	0.14
P_3 =	0.104	0.0937	0.113
P_4 =	0.0849	0.0762	0.0936
P_5 =	0.062	0.0542	0.0698
P_6 =	0.0518	0.0451	0.0586
P_7 =	0.0434	0.0362	0.0507
P_8 =	0.033	0.0269	0.0391
P_9 =	0.0257	0.0197	0.0316
P_{10} =	0.0232	0.0167	0.0297

Japanese cars:

Data	
Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	0.5

Service Times

Distribution =	Exponential
Mean =	0.2

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	0.639	0.605	0.674
L_q =	0.25	0.225	0.276
W =	0.325	0.31	0.34
W_q =	0.127	0.115	0.139
P_0 =	0.611	0.6	0.622
P_1 =	0.233	0.228	0.239
P_2 =	0.097	0.0919	0.102
P_3 =	0.0368	0.033	0.0405
P_4 =	0.0131	0.0108	0.0155
P_5 =	0.00501	0.00346	0.00657
P_6 =	0.00174	0.000963	0.00252
P_7 =	0.00104	0.000218	0.00187
P_8 =	0.000431	-0.00008	0.000938
P_9 =	0.000262	-0.00004	0.000568
P_{10} =	0.000042	-0.00004	0.000125

e)

Current Time	Number of Customers in Queue	Customer Being Served Server 1	Server 2	Next Arrival	Next Service Completion Server 1	Server
0	0	Yes	No	0.03044	0.04731	---
0.03044	0	Yes	Yes	0.76195	0.04731	0.06493
0.04731	0	No	Yes	0.76195	---	0.06493
0.06493	0	No	No	0.76195	---	---
0.76195	0	Yes	No	0.83716	1.01054	---
0.83716	1	Yes	No	0.85615	1.05115	1.07757
0.85615	1	Yes	No	1.17686	1.04719	0.93015
1.17686	1	Yes	No	1.32545	1.49234	1.19012
1.32545	1	Yes	No	1.42178	1.62979	1.3504
1.42178	1	Yes	No	1.48302	1.42802	1.53588
1.48302	1	Yes	No	1.81541	1.57642	1.68985
1.81541	1	Yes	No	1.8777	2.5102	2.03288

B)

Data

Number of Servers = 2

Interarrival Times

Distribution = Exponential
Mean = 0.25

Service Times

Distribution = Exponential
Mean = 0.2

Length of Simulation Run

Number of Arrivals = 10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	0.925	0.887	0.963
L_q =	0.14	0.1211	0.159
W =	0.234	0.227	0.242
W_q =	0.0355	0.0309	0.0401
P_0 =	0.439	0.427	0.451
P_1 =	0.337	0.329	0.345
P_2 =	0.138	0.133	0.144
P_3 =	0.0516	0.0473	0.0558
P_4 =	0.0213	0.0182	0.0245
P_5 =	0.00766	0.00592	0.0094
P_6 =	0.00343	0.00221	0.00474
P_7 =	0.00122	0.000569	0.00188
P_8 =	0.000407	-0.00014	0.000949
P_9 =	0.000081	-0.00005	0.000211
P_{10} =	0.000025	-0.00002	0.000074

g) This option significantly decreases the waiting time for German cars without the added cost of an additional mechanic.

Data

Number of Servers =	2

Interarrival Times

Distribution =	Exponential
Mean =	0.1667

Service Times

Distribution =	Exponential
Mean =	0.22

Length of Simulation Run

Number of Arrivals =	20,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.31	2.175	2.445
L_q =	0.996	0.88	1.111
W =	0.385	0.365	0.405
W_q =	0.166	0.148	0.184
P_0 =	0.208	0.198	0.218
P_1 =	0.27	0.26	0.279
P_2 =	0.178	0.172	0.184
P_3 =	0.119	0.114	0.124
P_4 =	0.0779	0.0731	0.0827
P_5 =	0.05	0.0459	0.0542
P_6 =	0.0345	0.0305	0.0384
P_7 =	0.0224	0.0191	0.0256
P_8 =	0.0141	0.0113	0.0168
P_9 =	0.00925	0.00679	0.0117
P_{10} =	0.0056	0.00384	0.00735

h)

Question Part	Simulated Estimate for W	Analytical Value for W
b	0.91 days	1.00 days
c	0.32 days	0.33 days
d (German)	0.99 days	1.00 days
d (Japanese)	0.33 days	0.33 days
e	0.23 days	0.24 days
f	0.39 days	0.39 days

The results of the simulation were quite accurate for all but part b, where only 100 arrivals were simulated. This suggests that more than 100 arrivals should be simulated.

i) Answers will vary. The option of training the two current mechanics significantly decreases the waiting time for German cars, without a significant impact on the wait for German cars, and does so without the added cost of a third mechanic. Adding a third mechanic lowers the average wait for German cars even more, but comes at an added cost for the third mechanic.

22.1-11

a) - A description of the parameters of the system. There are two independent G/M/1 systems: printers and monitors. For printers the arrival stream is deterministic. For monitors the arrival process is $\sim U[10,20]$. The inspection time is exponentially distributed with mean 10 minutes.

(CONT'D)

(CONT'D a)

- A simulation clock will record the amount of simulated time that ellapses.
- The state of the system is defined as:

  $N_M(t)$: # of monitors in the inspection station

  $N_P(t)$: # of printers " " " "

- The arrivals to the stations and the inspection times are generated by sampling distributions as specified in the first bullet point.
- The dynamics of the system are:

$$N_M(t) = \begin{cases} N_M(t)+1 & \text{if a monitor arrives to the inspection station} \\ N_M(t)-1 & \text{: if a monitor completes its inspection} \end{cases}$$

- Advance the time on the simulation clock by using the next-event time advance procedure.

b)

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	18.8535	2.3654
2.3654	0	No	18.8535	—
18.8535	0	Yes	30.1903	20.578
20.578	0	No	30.1903	—
30.1903	0	Yes	45.5138	38.7912
38.7912	0	No	45.5138	40.7702
45.5138	0	Yes	62.9157	57.9432
62.9157	0	Yes	73.018	63.6754
73.018	0	Yes	86.4483	85.0383
86.4483	0	Yes	99.2208	96.0002
99.2208	0	Yes	116.128	105.164
116.128	0	Yes	128.193	116.791
128.193	0	Yes	144.996	143.41
144.996	0	Yes	163.823	147.445

c)

Current Time	Number of Customers in Queue	Customer Being Served	Next Arrival	Next Service Completion
0	0	Yes	15	1.21772
1.21772	0	No	15	—
15	0	Yes	30	20.4518
20.4518	0	No	30	—
30	0	Yes	45	50.1234
45	0	Yes	60	46.7245
60	0	Yes	75	60.6191
75	0	Yes	90	80.4591
90	0	Yes	105	96.3044
105	0	Yes	120	113.601
120	0	Yes	135	127.452
135	0	Yes	150	136.979

d) Monitors:

Data

Number of Servers =	1

Interarrival Times

Distribution =	Uniform
Minimum =	10
Maximum =	20

Service Times

Distribution =	Exponential
Mean =	13

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.149	1.083	1.216
L_q =	0.488	0.433	0.543
W =	17.22	16.226	18.214
W_q =	7.31	6.483	8.138
P_0 =	0.339	0.325	0.352
P_1 =	0.375	0.365	0.386
P_2 =	0.16	0.152	0.168
P_3 =	0.0762	0.0683	0.084
P_4 =	0.0312	0.0253	0.0372
P_5 =	0.0131	0.00894	0.0172
P_6 =	0.00428	0.00227	0.0063
P_7 =	0.00109	0.000123	0.00206
P_8 =	0.000164	-0.00009	0.000412
P_9 =	0.000041	-0.00004	0.00012
P_{10} =	0	0	0

(CONT'D)

(CONT'D)

Printers:

Data

Number of Servers =	1

Interarrival Times

Distribution =	Constant
Value =	15

Service Times

Distribution =	Exponential
Mean =	10

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.178	1.111	1.246
L_q =	0.506	0.45	0.563
W =	17.675	16.661	18.688
W_q =	7.595	6.744	8.446
P_0 =	0.328	0.315	0.341
P_1 =	0.382	0.372	0.393
P_2 =	0.16	0.151	0.168
P_3 =	0.0744	0.0664	0.0823
P_4 =	0.0356	0.0291	0.0421
P_5 =	0.0134	0.00944	0.0174
P_6 =	0.0043	0.00237	0.00623
P_7 =	0.00129	0.000398	0.00217
P_8 =	0.00073	-0.00006	0.00152
P_9 =	0.000406	-0.00025	0.00106
P_{10} =	0	0	0

e) Monitors:

Data

Number of Servers =	1

Interarrival Times

Distribution =	Uniform
Minimum =	10
Maximum =	20

Service Times

Distribution =	Erlang
Mean =	10
k =	4

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	0.753	0.74	0.766
L_q =	0.087	0.0796	0.0944
W =	11.302	11.114	11.489
W_q =	1.305	1.195	1.415
P_0 =	0.334	0.327	0.341
P_1 =	0.582	0.577	0.587
P_2 =	0.0809	0.0749	0.0869
P_3 =	0.00291	0.00177	0.00405
P_4 =	0.000077	-0.00004	0.00019
P_5 =	0	0	0
P_6 =	0	0	0
P_7 =	0	0	0
P_8 =	0	0	0
P_9 =	0	0	0
P_{10} =	0	0	0

(CONT'D)

(CONT'D)

Printers:

Data

Number of Servers =	1

Interarrival Times

Distribution =	Constant
Value =	15

Service Times

Distribution =	Erlang
Mean =	10
k =	4

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	0.729	0.718	0.74
L_q =	0.0636	0.0579	0.0693
W =	10.934	10.767	11.102
W_q =	0.954	0.868	1.0393
P_0 =	0.335	0.328	0.341
P_1 =	0.603	0.598	0.608
P_2 =	0.0614	0.0563	0.0665
P_3 =	0.00109	0.000496	0.00168
P_4 =	0	0	0
P_5 =	0	0	0
P_6 =	0	0	0
P_7 =	0	0	0
P_8 =	0	0	0
P_9 =	0	0	0
P_{10} =	0	0	0

The new inspection equipment would drastically reduce the average waiting time for both monitors (7.3 minutes to 1.3 minutes) and printers (7.6 minutes to less than 1 minute).

22.2-1.

22.2-2.

22.3-1.

(a)

n	x_n	x_n+7	$\frac{x_n+7}{10}$	x_{n+1}
0	2	5	5/10	5
1	5	8	8/10	8
2	8	11	1 1/10	1
3	1	4	4/10	4
4	4	7	7/10	7
5	7	10	1	0
6	0	3	3/10	3
7	3	6	6/10	6
8	6	9	9/10	9
9	9	12	1 2/10	2

(b)

n	x_n	$5x_n+1$	$\frac{5x_n+1}{8}$	x_{n+1}
0	1	6	6/8	6
1	6	31	3 7/8	7
2	7	36	4 4/8	4
3	4	21	2 5/8	5
4	5	26	3 2/8	2
5	2	11	1 3/8	3
6	3	16	2	0
7	0	1	1/8	1

(c)

n	x_n	$61x_n+27$	$\frac{61x_n+27}{100}$	x_{n+1}
0	10	637	6 37/100	37
1	37	2284	22 84/100	84
2	84	5151	51 51/100	51
3	51	3138	31 38/100	38
4	38	2345	23 45/100	45

22.3-2

(a) $u_{n+1} = \dfrac{x_{n+1} + \frac{1}{2}}{10}$ $\quad n = 0, \cdots, 9$

(b) $u_{n+1} = \dfrac{x_{n+1} + \frac{1}{2}}{8}$ $\quad n = 0, \cdots, 7$

(c) $u_{n+1} = \dfrac{x_{n+1} + \frac{1}{2}}{100}$ $\quad n = 0, \cdots, 99$

22.3-3

n	x_n	$41x_n+33$	$\frac{41x_n+33}{100}$	x_{n+1}
0	48	2001	20 1/100	01
1	01	74	74/100	74
2	74	3067	30 67/100	67
3	67	2780	27 80/100	80
4	80	3313	33 13/100	13

22.3-4

n	x_n	$201x_n+503$	$\frac{201x_n+503}{1000}$	x_{n+1}
0	485	97988	97 988/1000	988
1	988	199091	199 91/1000	091
2	91	18794	18 794/1000	794

22.3-5

(a)

n	x_n	$13x_n+15$	$\frac{13x_n+15}{32}$	x_{n+1}
0	14	197	6 5/32	5
1	5	80	2 16/32	16
2	16	223	6 31/32	31
3	31	418	13 2/32	2
4	2	41	1 9/32	9

(b) Use $u_{n+1} = \dfrac{x_{n+1} + \frac{1}{2}}{32}$, we get

$(0.1719, 0.5156, 0.9844, 0.0781, 0.2696)$

22.3-6

(a) $X_1=7$, $X_2=10$, $X_3=5$, $X_4=9$, $X_5=11$, $X_6=12$, $X_7=6$, $X_8=3$
$X_9=8$, $X_{10}=4$, $X_{11}=2$, $X_{12}=1$

(b) Each integer appears only once in part (a)

(c) $X_{13}, X_{14}, \ldots$ will repeat X_1 to X_{12} with cycle length 12.

22.4-1

a) Answers will vary.

b) The formula in cell D10 is =VLOOKUP(C10, $J$8:$K$9, 2).

	B	C	D	E	F	G	H	I	J	K
3	**Summary of Game**							**Distribution of Coin Flips**		
4	Number of Flips =			2						
5	Winnings =			$6						
6										
7								Probability	Cumulative	Result (0=Tails, 1=Heads)
8			Result					0.5	0	0
9	Flip	Random Number	(0=Tails, 1=Heads)	Total Heads	Total Tails	Stop?		0.5	0.5	1
10	1	0.1921	0	0	1					
11	2	0.4894	0	0	2					
12	3	0.4593	0	0	3	Stop				
13	4	0.1960	0	0	4	NA				
14	5	0.4498	0	0	5	NA				
58	49	0.4746	0	21	28	NA				
59	50	0.7349	1	22	28	NA				

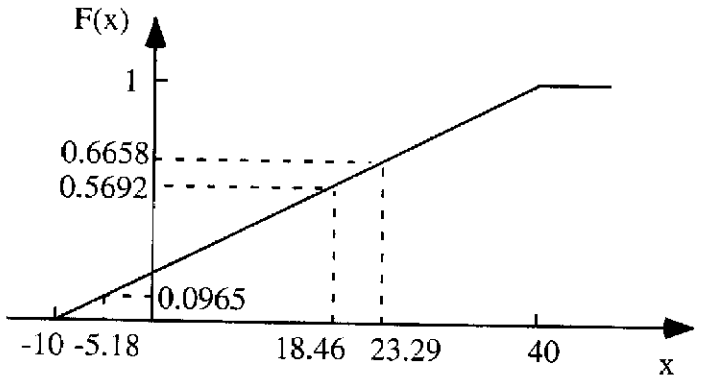
c) A simulation with 14 replications:

Play	Number of Flips	Winnings
	7	$1
1	11	-$3
2	5	$3
3	5	$3
4	9	-$1
5	7	$1
6	7	$1
7	5	$3
8	3	$5
9	17	-$9
10	5	$3
11	5	$3
12	3	$5
13	9	-$1
14	7	$1
Average	7	$1.00

d) A simulation with 1000 replications:

Play	Number of Flips	Winnings
	11	-$3
1	5	$3
2	13	-$5
3	15	-$7
4	5	$3
5	19	-$11
6	11	-$3
7	3	$5
8	3	$5
9	3	$5
10	7	$1
995	5	$3
996	3	$5
997	3	$5
998	3	$5
999	21	-$13
1000	5	$3
Average	9.15	-$1.15

22.4-2 a)



Claim

b)

$$F(x) = \frac{x+10}{50} = 0.0965 \text{ when } x = -5.18$$

$$F(x) = \frac{x+10}{50} = 0.5692 \text{ when } x = 18.46$$

$$F(x) = \frac{x+10}{50} = 0.6658 \text{ when } x = 23.29$$

22.4-3 (a) $r = Pr\{X \le x\} = \int_{25}^{x} \frac{dt}{50} = \frac{x-25}{50} \Rightarrow x = 50r + 25$

r	$x = 50r+25$
.096	29.80
.569	53.45
.665	58.25

(b) $r = Pr\{X \le x\} = \int_{-1}^{x} \frac{(t+1)^3}{4} dt = \frac{(x+1)^4}{16} \Rightarrow x = 2r^{1/4} - 1$

r	$x = 2r^{1/4} - 1$
.096	0.113
.569	0.737
.665	0.806

(c) $r = Pr\{X \le x\} = \int_{40}^{x} \frac{(t-40)}{200} dt = \frac{(x-40)^2}{400} \Rightarrow x = 20(2 + \sqrt{r})$

r	$x = 20(2+\sqrt{r})$
.096	46.197
.569	55.086
.665	56.310

22.4-4 (a) To determine whether $X = 0$ or X is distributed uniformly between -5 and 15, look at a 3 digit random number from table 23.1.
$000 \le r \le 499 \Rightarrow X = 0$
$500 \le r \le 999 \Rightarrow X$ is uniformly distributed
If $X = 0$, nothing else need be done.
If $X \sim U(-5, 15)$, then use the next 3-digit random number as a decimal to generate X as follows:

$r = Pr\{X \le x\} = \int_{-5}^{x} \frac{dt}{20} = \frac{x+5}{20} \Rightarrow x = 20r - 5$

r	(from table 23.1)
096	$\Rightarrow x_1 = 0$
569	$\Rightarrow x_2 \sim U(-5, 15)$
.665	$\Rightarrow x_2 = 20(.665) - 5 = 8.3$
764	$\Rightarrow x_3 \sim U(-5, 15)$
.842	$\Rightarrow x_3 = 20(.842) - 5 = 11.84$

Hence, the sequence is (0, 8.3, 11.84)

10

(b) $\Pr\{1 \le x \le 2\} = \int_1^2 (t-1)\,dt = \frac{1}{2}$

$\Pr\{2 \le x \le 3\} = \int_2^3 (3-t)\,dt = \frac{1}{2}$

So for $0 \le r \le 1/2$, $r = \int_1^x (t-1)\,dt = \frac{(x-1)^2}{2} \Rightarrow x = \sqrt{2r}+1$

and for $1/2 \le r \le 1$, $r = 1/2 + \int_2^x (3-t)\,dt = \frac{1}{2} - \frac{(3-x)^2}{2} \Rightarrow x = 3-\sqrt{2-2r}$

$x = \begin{cases} \sqrt{2r}+1 & \text{when } 0 \le r \le 1/2 \\ 3-\sqrt{2-2r} & \text{when } 1/2 \le r \le 1 \end{cases}$

r	x
.096	1.438
.569	2.072
.665	2.181

(c) Let Z be a Bernoulli random variable with $p = 1/3$; that is, $\Pr\{Z=1\} = 1/3$ and $\Pr\{Z=0\} = 2/3$. Then X is a random variable denoting the number of trials until the Bernoulli random variable takes on the value 1.

Let $000 \le r \le 332$ denote $Z=1$

and $333 \le r \le 999$ denote $Z=0$

r	z	x
096	1	1
569	0	
665	0	
764	0	
842	0	
492	0	
224	1	6
950	0	
610	0	
145	1	3

Hence, the sequence is 1, 6, 3

22.4-5

(a) $r = \Pr\{X \le x\} = \int_0^x 2t\,dt = x^2 \Rightarrow x = \sqrt{r}$

b)

Uniform Random Number	Random Observation
0.0956	0.309
0.5629	0.750
0.6695	0.818
0.7634	0.874
0.8426	0.918

c)

Random Observation	Uniform Random Number Used
0.09	0.0081
0.64	0.4096
0.49	0.2401

d) =SQRT(RAND()).

22.4-6

a) Answers will vary

b) Let the numbers 0-0.4999 correspond to heads and the numbers 0.5000 to 0.9999 correspond to tails. So we get:

Group 1: H-H-H Group 2: THH Group 3: HTT Group 4: THT
Group 5: THH Group 6: HHT Group 7: THT Group 8: TTH
Groups with 0 heads: 0, with 1 heads: 4. Groups with 2 heads: 3
Groups with 3 heads: 1

c)

Flip	Random Number	Result (0=Tails, 1=Heads)
1	0.6447	1
2	0.6897	1
3	0.1961	0

Total Number of Heads = 2

d) Answers will vary. The following 8 replications have 2 replications with 0 heads (1/4), 4 replications with 1 head (1/2), 1 replication with 2 heads (1/8), and 1 replication with 3 heads (1/8). This is not very close to the expected probability distribution.

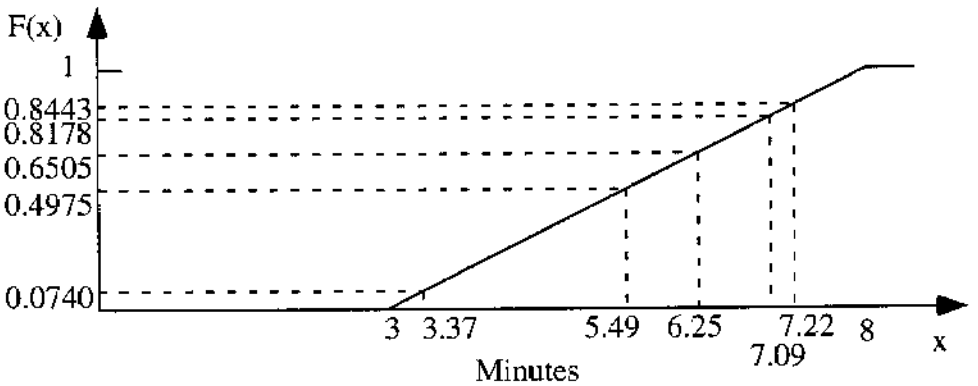
Replication	Number of Heads
	1
1	1
2	3
3	2
4	0
5	1
6	1
7	1
8	0

e) Answers will vary. With the following 800 replications, 93 have 0 heads (93/800), 303 have 1 head (303/800), 309 have 2 heads (309/800), and 95 have 3 heads (95/800). This is quite close to the expected probability distribution.

Replication	Number of Heads
	1
1	1
2	2
3	1
4	1
5	1
798	0
799	2
800	1

Number with 0 heads =	93
Number with 1 head =	303
Number with 2 heads =	309
Number with 3 heads =	95

22.4-7 a)



b)

$$F(x) = \frac{x-3}{5} = 0.6505 \text{ when } x = 6.25$$

$$F(x) = \frac{x-3}{5} = 0.0740 \text{ when } x = 3.37$$

$$F(x) = \frac{x-3}{5} = 0.8443 \text{ when } x = 7.22$$

$$F(x) = \frac{x-3}{5} = 0.4975 \text{ when } x = 5.49$$

$$F(x) = \frac{x-3}{5} = 0.8178 \text{ when } x = 7.09$$

c) Average = 5.88 which is higher that the mean of 5.5

d) Results will vary. The following 500-day simulation yielded an average of 5.479.

Day	Random Number	Service Time
1	0.1348	3.674
2	0.6798	6.399
3	0.7941	6.971
4	0.1825	3.913
5	0.6502	6.251
6	0.1088	3.544
497	0.9319	7.660
498	0.6662	6.331
499	0.8760	7.380
500	0.2933	4.467
	Average =	5.479

22.4-8

a)

Summary of Results:

Win? (1=Yes, 0=No)	0
Number of Tosses =	3

Simulated Tosses

Toss	Die 1	Die 2	Sum
1	4	2	6
2	3	2	5
3	6	1	7
4	5	2	7
5	4	4	8
6	1	4	5
7	2	6	8

Results

Win?	Lose?	Continue?
0	0	Yes
0	0	Yes
0	1	No
NA	NA	No
NA	NA	No
NA	NA	No
NA	NA	No

b) Answers will vary. Here is one simulation of 25 replications.

Game	Win?
	0
1	0
2	0
3	1
4	0
5	0
6	0
7	0
8	1
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	1
17	0
18	1
19	0
20	1
21	1
22	1
23	1
24	1
25	0

c) we have 9 wins and 16 loses. Hence $P(\text{win}) = \frac{9}{25}$ and $P(\text{lose}) = \frac{16}{25}$

d)

$\frac{\bar{x}-.493}{(\frac{.5}{\sqrt{n}})} \sim N(0,1) \Rightarrow Pr\left\{\frac{\bar{x}-.493}{(\frac{.5}{\sqrt{n}})} \le 1.64\right\} = .95$

So $Pr\{\bar{x} \le \frac{.82}{\sqrt{n}} + .493\} = .95$

Let $\frac{.82}{\sqrt{n}} + .493 = .5$. Solving for n gives $\sqrt{n} = 117$.
or $n \approx 13,689$.

22.4-9

$$F(x) = \begin{cases} \frac{1}{8}x & 0 \le x < 2 \\ \frac{3}{4} & x = 2 \\ \frac{3}{4} + \frac{1}{4}(x-2) & 2 < x \le 3 \end{cases}$$

$$\Rightarrow x_i = F^{-1}(u_i) = \begin{cases} 8u_i & 0 \le u_i < \frac{1}{4} \\ 2 & \frac{1}{4} \le u_i \le \frac{3}{4} \\ 4u_i - 1 & \frac{3}{4} < u_i \le 1 \end{cases}$$

So, when $u = \{\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{7}{8}\}$, $x = \{2, 2, 2, \frac{5}{4}\}$ with average $= \frac{29}{16}$

22.4-10. $r = Pr\{X \le x\} = Pr\left\{\frac{X-1}{2} \le \frac{x-1}{2}\right\} = 1 - \Phi(\frac{x-1}{2})$

$\Rightarrow x = 2\Phi^{-1}(1-r) + 1$

r	$\Phi^{-1}(1-r)$	x
.096	1.308	3.616
.569	-0.175	0.650
.665	-0.425	0.150
.764	-0.719	-0.438
.842	-1.001	-1.002
.492	0.02	1.04
.224	0.762	2.524
.950	-1.645	-2.290
.610	-0.280	0.440
.145	1.059	3.118
	Average =	0.7808

22.4-11

r_i^1	r_i^2	r_i^3
.096	.764	.224
.569	.842	.950
.665	.492	.610

$\sum_{i=1}^{3} r_i^n$	$x_n = 20(\sum_{i=1}^{3} r_i^n) - 20$
1.330	6.6
2.098	22.0
1.784	15.7

(b) We pick first three uniform random numbers in 2-13.

Since $X = 5\Phi^{-1}(1-r) + 10$, we get

$X_1 = 16.54,\ X_2 = 9.125,\ X_3 = 7.875$

22.4-12 (a)

r_i^1	r_i^2	r_i^3	r_i^4
.096	.764	.224	.145
.569	.842	.950	.484
.665	.492	.610	.552

$\sum_{i=1}^{3} r_i^n$	$x_K = 2\left(\sum_{i=1}^{3} r_i^n\right) - 3$
1.330	−.340
2.098	1.196
1.784	.568
1.181	−.638

Let Z_i (for $i = 1,2$) denote the chi-square observations.
Then $Z_1 = X_1^2 + X_2^2 = 1.546$
$Z_2 = X_3^2 + X_4^2 = 0.730$

(b)

r	$\Phi^{-1}(1-r) = x$
0.096	1.304
0.569	−0.175
0.665	−0.425
0.764	−0.719

(c) $Y = X_1^2 + X_2^2$

From (a), $Y_1 = 1.546,\quad Y_2 = 0.7297$

From (b), $Y_1 = 1.731,\quad Y_2 = 0.6946$

22.4-13 (a)

r	$x = -4\ln(1-r)$
.096	.404
.569	3.367

(b)

r_1	r_2	$x = -2\ln[(1-r_1)(1-r_2)]$
.096	.569	1.885
.665	.764	5.075

(c)

r_i^1	r_i^2	$\sum_{i=1}^{4} r_i^n$	$x_K = 4\left(\sum_{i=1}^{4} r_i^n\right) - 8$
.096	.224	3.428	5.71
.569	.950	2.965	3.860
.665	.610		
.764	.145		
.842	.484		
.492	.552		

22.4-14

a)

Uniform Random Number	Random Observation (minutes)
0.2655	9.22
0.3472	9.49
0.0248	7.25
0.9205	12.21
0.6130	10.38

b) If cell C4 contains the uniform random number, then the excel function would be
=IF(C4<0.2, 7+(2/0.2)*C4, IF(C4<0.8, 9+(2/0.6)*(C4-0.2), 11+(2/0.2)*(C4-0.8))).

22.4-15

r	$x_n = -1 \ln(1-r)$
.096	.101
.569	.842
.665	1.094
.764	1.444

Hence, the Erlang observation is $\sum_{n=1}^{4} x_n = 3.481$

22.4-16 (a) $X = \{1, 8, 11, 10, 5, 12, 15, 14, 9, 0, 3, 2, 13, 4, 7, 6, 1\}$

(b) $X = \{1, 8, 5, 9, 0, 3, 2, 4, 7, 6\}$

(c) No. Since the numbers above are not independent of each other. The change we could make is to use a vastly larger modulo in (a) and get last digits of those integers.

(d) $X = \{0, 5, 7, 6, 3, 7, 9, 9, 5, 0\}$

(e) No. Some numbers (like 4) never gains a probability to appear in the sequence. We can use a vastly larger modulo and then apply (d) to alleviate the problem.

22.4-17 (a) True. Both r_i and $1-r_i$ are uniformly distributed

(b) False. Numerically, $\Pi r_i \neq \Pi(1-r_i) \Rightarrow \Sigma x_i \neq \Sigma y_i$

(c) True. Erlang distribution is sum of exponential distributions.

22.4-18 (a) Not valid. Since $\Pr\{\frac{8}{8} \leq r_i < \frac{9}{8}\} = 0$ and r_i wouldn't reach 8.
Modify it as $\frac{n-1}{8} \leq r_i < \frac{n}{8}$

(b) Valid. When $\frac{n-1}{8} \leq r_i < \frac{n}{8}$, $n \leq 1 + 8r_i < n+1$.

(c) Not Valid. $X = \{4, 3, 6, 5, 0, 7, 2, 1\}$ It doesn't cover 8.
Adjustment. Let $X' = X + 1$.

22.4-19

r_1	X	r_2	$f(x)$	Accept?
.096	.192	.569	.192	no
.665	1.330	.764	.670	no
.842	1.684	.492	.316	no
.224	.448	.950	.448	no
.610	1.220	.145	.780	yes
.484	.968	.552	.968	yes
.350	.700	.590	.700	yes

So, the three samples from the triangular distribution are 1.220, .968 and .700.

22.4-20

Let $x = 10r_1 + 10$

r_1	x	r_2	$f(x)$	Accept?
.096	10.96	.569	.0192	no
.665	16.65	.764	.1350	no
.842	18.42	.492	.1684	no
.224	12.24	.950	.0448	no
.610	16.10	.145	.1220	no
.484	14.84	.552	.0968	no
.350	13.50	.590	.0700	no
.430	14.30	.041	.0860	yes
.802	18.02	.471	.1604	no
.255	12.55	.799	.0510	no
.608	16.08	.577	.1216	no
.347	13.47	.933	.0694	no
.581	15.81	.173	.1162	no
.603	16.03	.040	.1206	yes
.605	16.05	.842	.1210	no
.720	17.20	.449	.1440	no
.076	10.76	.407	.0152	no
.202	12.02	.963	.0404	no
.412	14.12	.369	.0824	no
.976	19.76	.171	.1952	yes

So the three samples from the given distribution are 14.30, 16.03 and 19.76.

22.4-21 We decide on size of risk by

$$\text{size} = \begin{cases} 0 & \text{if } 0 \le u < 0.7 \\ 1 & \text{if } 0.7 \le u < 0.9 \\ 2 & \text{if } 0.9 \le u < 1 \end{cases}$$

Size of loss by

$$x = \begin{cases} (20u)^2 & \text{if } 0 \le u \le \frac{1}{2} \\ 200u & \text{if } u \ge \frac{1}{2} \end{cases}$$

run 1

u	size
0.096	0
0.569	0
0.665	0
0.764	1

run 2

u	size
.492	0
.224	0
.950	2
.610	0

u	x
.842	164.4

	x
.145	8.41
.484	91.09

Total loss $= \sum_{i=1}^{4} \cdot I_{(\text{size}>0)} \cdot \sum_{j=1}^{\text{size}} x_{ij}$. Two simulation runs give us 164.4 and 99.5 Actually, 100 runs give us 145.

22.4-22 Since number (#) of employees incurring medical expenses has a binomial distribution, with $p=0.9$ and $n=3$,

$P\{\#=0\} = C_3^0 0.1^3 = 0.001$,

$P\{\#=1\} = C_3^1 0.9 \cdot 0.1^2 = 0.027$

$P\{\#=2\} = C_3^2 \cdot 0.9^2 \cdot 0.1 = 0.243$

$P\{\#=3\} = C_3^3 0.9^3 = 0.729$

Let $P_0=0$, $P_1=0.001$, $P_2=0.028$, $P_3=0.271$, $P_4=1$

In simulation, $\#=i$ if $P_i \le U < P_{i+1}$

So, $0.01 \Rightarrow \#=1$, $0.20 \Rightarrow \#=2$

Also, total amount = $\$100$ if $0 \le U < 0.9$; $\$10{,}000$ if $0.9 \le U < 1$

So, only 0.95 causes actual payment from the insurance company

So the total payment is $5,000.

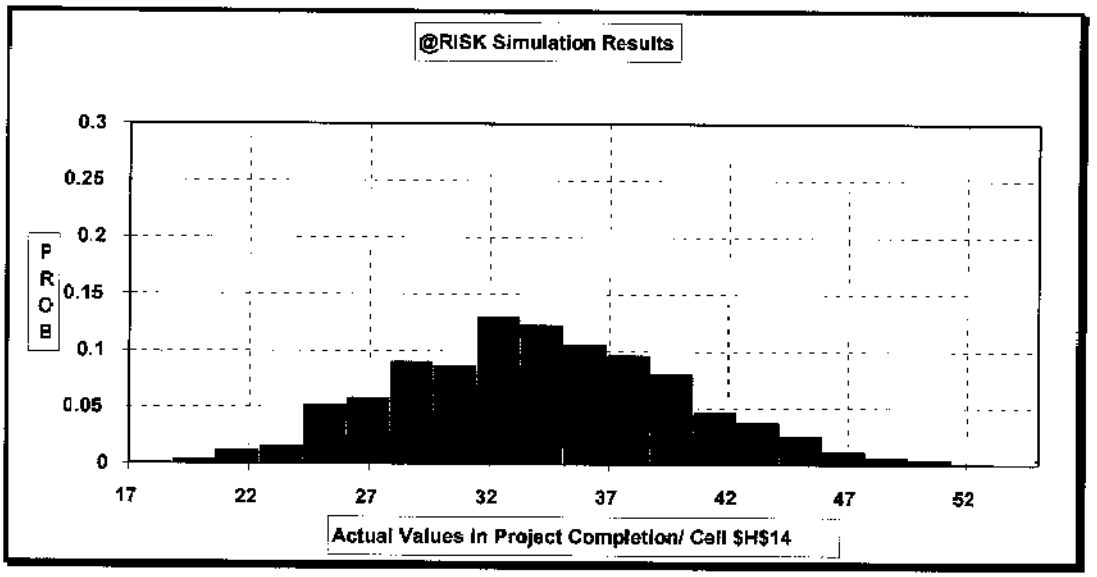
22.6-1 The simulation spreadsheet is shown below, and the @Risk results follow (1000 iterations). The probability of finishing in 22 months appears to be less than 2%.

	A	B	C	D	E	F	G	H
1		Immediate	Time Estimates			Start	Activity	Finish
2	Activity	Predecessor	o	m	p	Time	Time	Time
3	A	—	1.5	2	15	0	3.575679171	3.575679171
4	B	—	2	3.5	21	0	10.05252124	10.05252124
5	C	A	1	1.5	18	3.575679171	2.730574217	6.306253388
6	D	B	0.5	1	15	10.05252124	0.816185138	10.86870638
7	E	A	3	5	24	3.575679171	13.39355115	16.96923032
8	F	C	1	2	16	6.306253388	8.758827924	15.06508131
9	G	D	0.5	1	14	10.86870638	5.042382869	15.91108925
10	H	B	2.5	3.5	25	10.05252124	6.723277281	16.77579852
11	I	E, F	1	3	18	16.96923032	13.42055391	30.38978423
12	J	G, H	2	3	18	16.77579852	2.915871432	19.69166995
13								
14						Project Completion =		30.38978423
15						Project Deadline =		22
16						Deadline met (1=yes, 0=no)?		0

	F	G	H
1	Start	Activity	Finish
2	Time	Time	Time
3	0	=RiskTriang(C3,D3,E3)	=F3+G3
4	0	=RiskTriang(C4,D4,E4)	=F4+G4
5	=H3	=RiskTriang(C5,D5,E5)	=F5+G5
6	=H4	=RiskTriang(C6,D6,E6)	=F6+G6
7	=H3	=RiskTriang(C7,D7,E7)	=F7+G7
8	=H5	=RiskTriang(C8,D8,E8)	=F8+G8
9	=H6	=RiskTriang(C9,D9,E9)	=F9+G9
10	=H4	=RiskTriang(C10,D10,E10)	=F10+G10
11	=MAX(H7,H8)	=RiskTriang(C11,D11,E11)	=F11+G11
12	=MAX(H9,H10)	=RiskTriang(C12,D12,E12)	=F12+G12
13			
14		Project Completion =	=MAX(H11,H12)
15		Project Deadline =	22
16		Deadline met (1=yes, 0=no)?	=IF(H14<=H15,1,0)

(CONT'D)

(CONT'D) 22.6-1



Simulation Statistics *Date: 7/7/99 at 21:35* *Iterations: 1000* *Simulations: 1*		
Worksheet: [Prob22]		**[Prob22]**
Output Range: Project Completion		**Deadline Met?**
Cell: $H$14		**$H$16**
Minimum=	17.96710014	0
Maximum=	54.29407501	1
Mean=	33.42208051	0.019
Std Deviation=	5.911767542	0.136524723
Variance=	34.94899547	0.018639
Skewness=	0.242782767	7.046342806
Kurtosis=	2.90791179	50.65094694
Percentile Values		
5Perc=	24.10403433	0
10Perc=	25.70333328	0
15Perc=	27.18526602	0
20Perc=	28.23194313	0
25Perc=	29.16425562	0
30Perc=	30.25081844	0
35Perc=	31.08764973	0
40Perc=	31.71703453	0
45Perc=	32.37423325	0
50Perc=	33.16358757	0
55Perc=	33.97784405	0
60Perc=	34.6772171	0
65Perc=	35.58364754	0
70Perc=	36.4365715	0
75Perc=	37.35981274	0
80Perc=	38.30871277	0
85Perc=	39.35188961	0
90Perc=	41.20889206	0
95Perc=	43.43574772	0

22.6-2 The simulation spreadsheet is shown below, and the @Risk results follow (500 iterations). The mean claim appears to be approximately $2625.

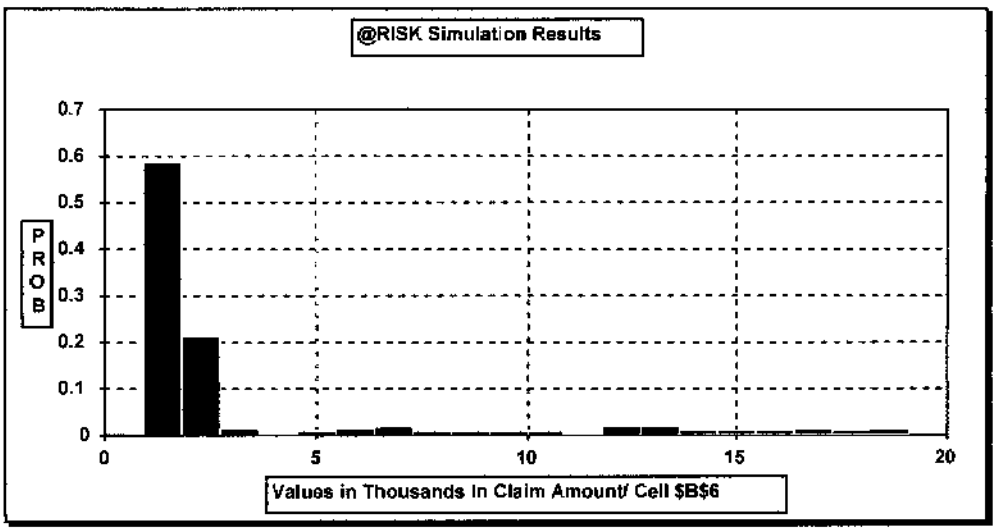
	A	B
1	Claim Distribution	
2	0	0.4
3	$ 1,000.00	0.4
4	$ 11,000.00	0.2
5		
6	Claim Amount =	$ 1,000.00

	A	B
1	Claim Distribution	
2	0	0.4
3	=RiskUniform(0,2000)	0.4
4	=RiskUniform(2000,20000)	0.2
5		
6	Claim Amount =	=RiskDiscrete(A2:A4,B2:B4)

Simulation Statistics	
Date: 7/7/99 at 21:59	
Iterations: 500	
Simulations: 1	
Worksheet: [Prob23]	
Output Range: Claim Amount	
Cell: $B$6	
Minimum=	0
Maximum=	19849.88281
Mean=	2624.683288
Std Deviation=	4848.025365
Variance=	23503349.94
Skewness=	2.180795137
Kurtosis=	6.595068631
Percentile Values	
5Perc=	0
10Perc=	0
15Perc=	0
20Perc=	0
25Perc=	0
30Perc=	0
35Perc=	0
40Perc=	0
45Perc=	262.7447815
50Perc=	556.0119629
55Perc=	803.9009399
60Perc=	1061.196899
65Perc=	1298.03125
70Perc=	1547.103271
75Perc=	1754.229674
80Perc=	1994.602295
85Perc=	6324.703125
90Perc=	11616.77246
95Perc=	15360.12598

(CONT'D)

22.6-2 (CONT'D)



22.6-3

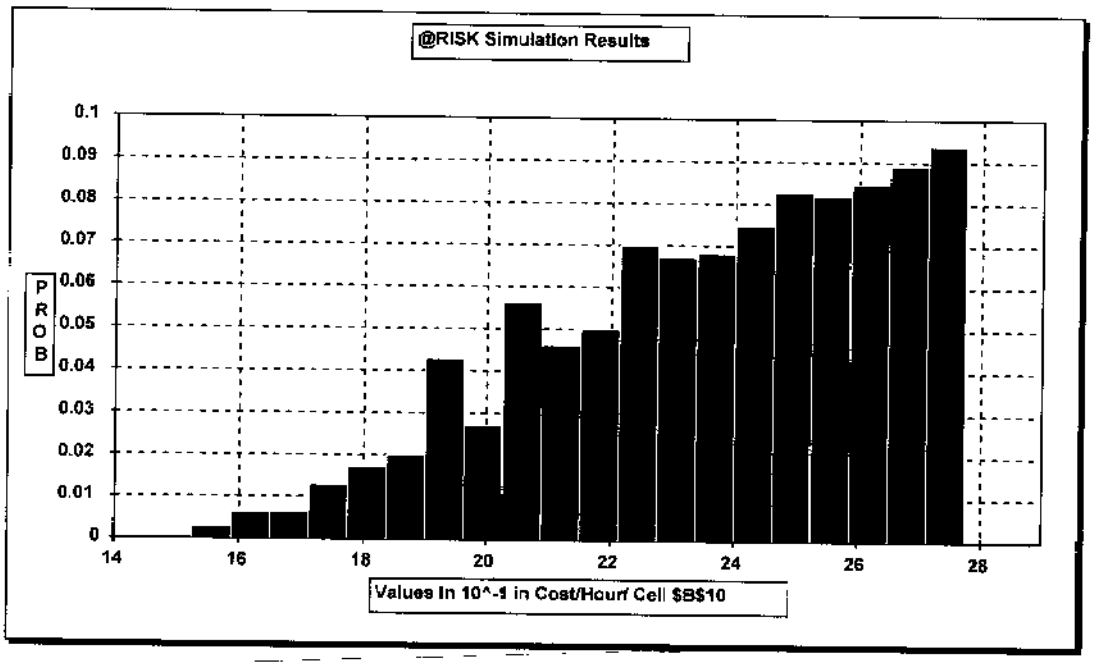
The simulation spreadsheet and @Risk results are shown below. The expected cost with the proposed system of replacing all relays with the first failure is approximately $2.37 / hour. This is cheaper than their current system of replacing each relay as it fails. Therefore, they should replace all four relays with the first failure.

	A	B
1		Time to Failure (hours)
2	Relay 1	1500
3	Relay 2	1500
4	Relay 3	1500
5	Relay 4	1500
6		
7	First Failure Time =	1500
8	Time to End of Shutdown =	1502
9	Cost =	$2,800
10	Cost / Hour =	$1.86

	B
2	=RiskUniform(1000,2000)
3	=RiskUniform(1000,2000)
4	=RiskUniform(1000,2000)
5	=RiskUniform(1000,2000)
6	
7	=MIN(B2:B5)
8	=B7+2
9	2800
10	=B9/B8

(CONT'D)

(CONT'D)
22.6-3

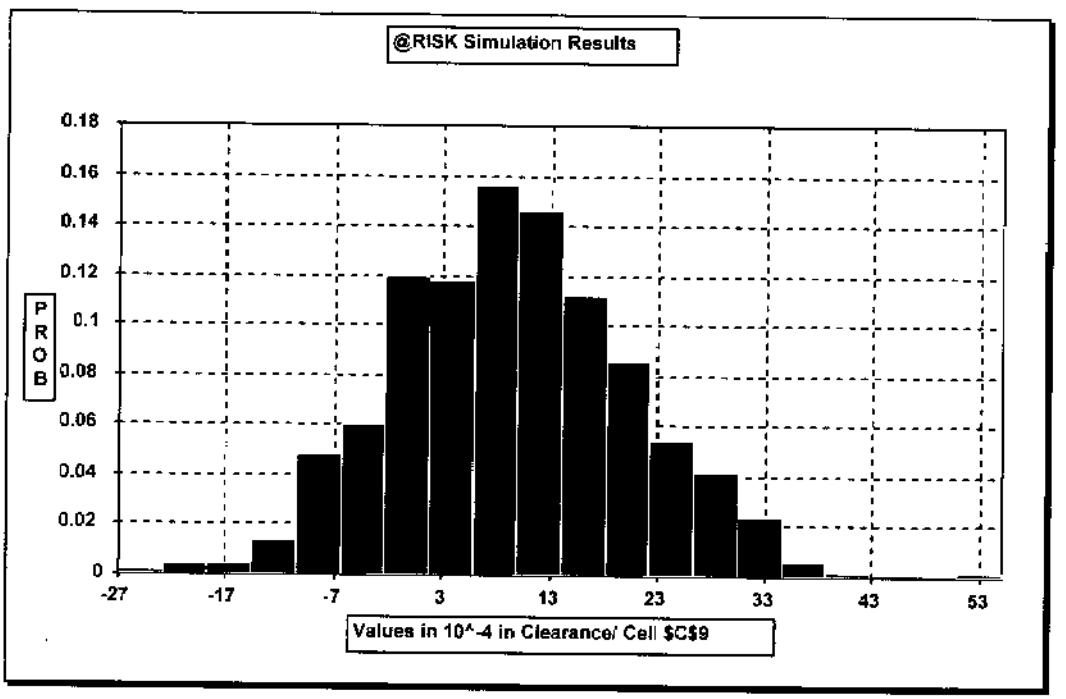


Simulation Statistics	
Date: 7/8/99 at 0:49	
Iterations: 1000	
Simulations: 1	
Worksheet: [Prob25]	
Output Range: Cost/Hour	
Cell: $B$10	
Minimum=	1.5042596
Maximum=	2.7938926
Mean=	2.3702155
Std Deviation=	0.2903318
Variance=	0.0842926
Skewness=	-0.5387744
Kurtosis=	2.4701799

Percentile Values	
5Perc=	1.8312686
10Perc=	1.9456343
15Perc=	2.0391303
20Perc=	2.0948352
25Perc=	2.1654532
30Perc=	2.227929
35Perc=	2.2687374
40Perc=	2.3186806
45Perc=	2.3694554
50Perc=	2.413556
55Perc=	2.4565992
60Perc=	2.4974302
65Perc=	2.5359856
70Perc=	2.5756141
75Perc=	2.6146076
80Perc=	2.6522719
85Perc=	2.6884112
90Perc=	2.7248517
95Perc=	2.7591258

22.64

	A	B	C	D	E
1		Shaft Distribution			Bushing Distribution
2		(Triangular)			(Normal)
3	Minimum =	1.000		Mean =	1.002
4	Most Likely =	1.001		Standard Dev. =	0.001
5	Maximum =	1.002			
6					
7	Shaft Radius =	1.001		Bushing Radius =	1.002
8					
9			Clearance =	0.001	
10		Interference? (1=yes, 0=no)		0	

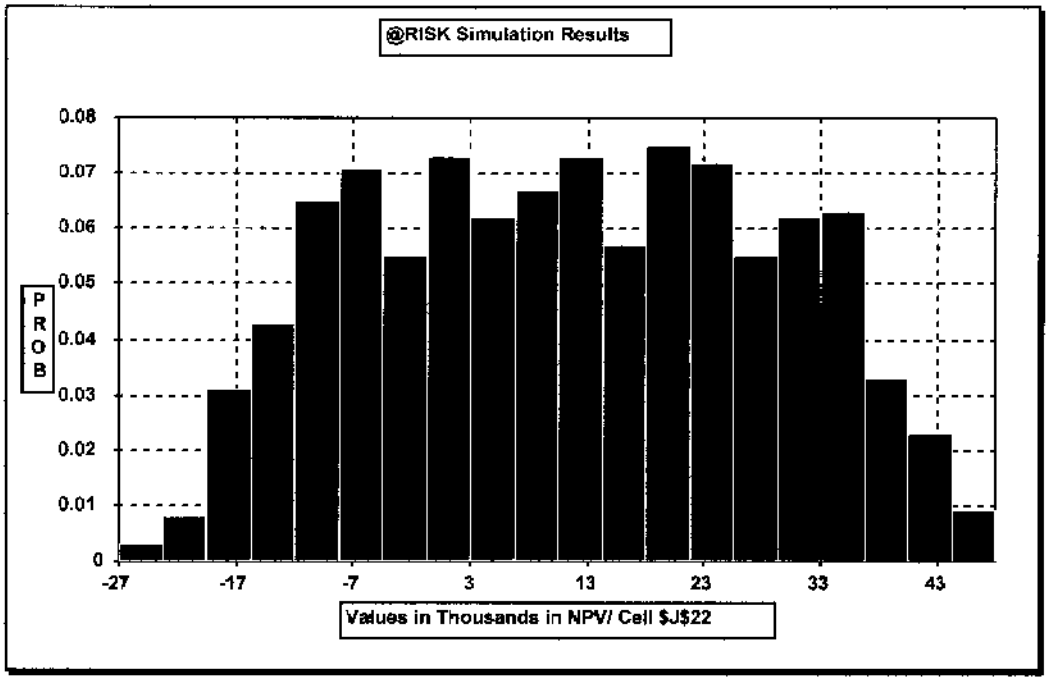


Simulation Statistics		
Date: 7/7/99 at 22:52		
Iterations: 500		
Simulations: 1		
Worksheet: [Prob26]		[Prob26]
Output Range: Clearance		Interference
Cell: $C$9		$C$10
Minimum=	-0.002580988	0
Maximum=	0.005446819	1
Mean=	0.001001341	0.174
Std Deviation=	0.001088255	0.379109483
Variance=	1.1843E-06	0.143724
Skewness=	0.129383011	1.719819813
Kurtosis=	3.224675187	3.95778019

Percentile Values		
5Perc=	-0.000767469	0
10Perc=	-0.00032581	0
15Perc=	-6.7149E-05	0
20Perc=	5.32986E-05	0
25Perc=	0.00022788	0
30Perc=	0.000408646	0
35Perc=	0.000573805	0
40Perc=	0.000749844	0
45Perc=	0.000888586	0
50Perc=	0.00099161	0
55Perc=	0.00109613	0
60Perc=	0.001228148	0
65Perc=	0.00137	0
70Perc=	0.001513866	0
75Perc=	0.001731504	0
80Perc=	0.001875703	0
85Perc=	0.002145935	1
90Perc=	0.002475443	1
95Perc=	0.002878371	1

22.6-5

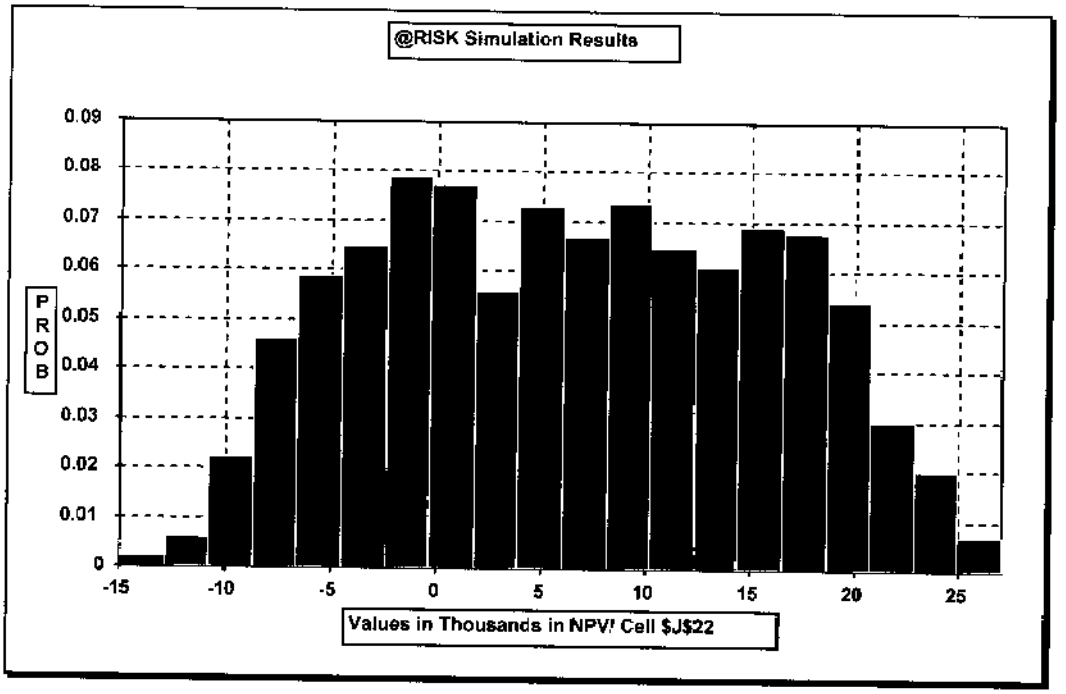
a) Option 2:



Simulation Statistics	
Date: 7/7/99 at 23:06	
Iterations: 1000	
Simulations: 1	
Worksheet: [Prob27]	
Output Range: NPV	
Cell: $J$22	
Minimum=	-25791.32617
Maximum=	46580.41406
Mean=	11547.39918
Std Deviation=	16386.60767
Variance=	268520911
Skewness=	-0.001998889
Kurtosis=	1.986458314

Percentile Values	
5Perc=	-13890.48525
10Perc=	-10431.59629
15Perc=	-7682.970483
20Perc=	-5201.45332
25Perc=	-2111.829041
30Perc=	501.982785
35Perc=	3289.138159
40Perc=	6137.576953
45Perc=	9038.039697
50Perc=	11793.09717
55Perc=	13929.08638
60Perc=	17264.95313
65Perc=	19709.53594
70Perc=	22204.24961
75Perc=	24534.8125
80Perc=	27785.38398
85Perc=	30690.3582
90Perc=	33634.53438
95Perc=	37353.37305

b) Option 3:



Simulation Statistics	
Date: 7/7/99 at 23:12	
Iterations: 1000	
Simulations: 1	
Worksheet: [Prob27]	
Output Range: NPV	
Cell: $J$22	
Minimum=	-14124.72363
Maximum=	26172.2168
Mean=	6580.300169
Std Deviation=	8834.92538
Variance=	78055906.46
Skewness=	0.035087097
Kurtosis=	1.984228551

Percentile Values	
5Perc=	-7179.637549
10Perc=	-5117.790088
15Perc=	-3601.592346
20Perc=	-2010.522158
25Perc=	-731.961853
30Perc=	626.2370483
35Perc=	1854.629413
40Perc=	3552.943408
45Perc=	4882.023315
50Perc=	6519.886475
55Perc=	8024.979858
60Perc=	9371.307617
65Perc=	10809.82715
70Perc=	12406.08984
75Perc=	14020.00659
80Perc=	15549.94551
85Perc=	16852.32041
90Perc=	18397.16484
95Perc=	20468.98721

c) Option 1 appears to be the best. It has the highest expected NPV ($18,130 vs. $11,547 vs. $6580) *and* there is less chance of losing money (almost 20% vs. greater than 25% for options 2 and 3).

22.6-6
a-c)

	A	B	C	D	E	F	G
2	Win (1=yes, 0=no)?	1					
3							
4	Toss	Die 1	Die 2	Sum	Win?	Lose?	Continue?
5	1	3	3	6	0	0	Yes
6	2	3	3	6	1	0	No
7	3	3	3	6	NA	NA	No
8	4	3	3	6	NA	NA	No
9	5	3	3	6	NA	NA	No
10	6	3	3	6	NA	NA	No
11	7	3	3	6	NA	NA	No
12	8	3	3	6	NA	NA	No
13	9	3	3	6	NA	NA	No
14	10	3	3	6	NA	NA	No
15	11	3	3	6	NA	NA	No
16	12	3	3	6	NA	NA	No
17	13	3	3	6	NA	NA	No
18	14	3	3	6	NA	NA	No
19	15	3	3	6	NA	NA	No
20	16	3	3	6	NA	NA	No
21	17	3	3	6	NA	NA	No
22	18	3	3	6	NA	NA	No
23	19	3	3	6	NA	NA	No
24	20	3	3	6	NA	NA	No
25	21	3	3	6	NA	NA	No
26	22	3	3	6	NA	NA	No
27	23	3	3	6	NA	NA	No
28	24	3	3	6	NA	NA	No
29	25	3	3	6	NA	NA	No
30	26	3	3	6	NA	NA	No
31	27	3	3	6	NA	NA	No
32	28	3	3	6	NA	NA	No
33	29	3	3	6	NA	NA	No
34	30	3	3	6	NA	NA	No

	B	C	D
5	=RiskDuniform({1,2,3,4,5,6})	=RiskDuniform({1,2,3,4,5,6})	=B5+C5
6	=RiskDuniform({1,2,3,4,5,6})	=RiskDuniform({1,2,3,4,5,6})	=B6+C6
7	=RiskDuniform({1,2,3,4,5,6})	=RiskDuniform({1,2,3,4,5,6})	=B7+C7
8	=RiskDuniform({1,2,3,4,5,6})	=RiskDuniform({1,2,3,4,5,6})	=B8+C8
9	=RiskDuniform({1,2,3,4,5,6})	=RiskDuniform({1,2,3,4,5,6})	=B9+C9

	E	F	G
5	=IF(OR(D5=7,D5=11),1,0)	=IF(OR(D5=2,D5=3,D5=12) 1,0)	=IF(E5+F5=1 "No","Yes")
6	=IF(G5="Yes",IF(D6=$D$5,1,0),"NA")	=IF(G5="Yes",IF(D6=7,1,0),"NA")	=IF(G5="Yes",IF(E6+F6=1,"No","Yes"),"No")
7	=IF(G6="Yes",IF(D7=$D$5,1,0),"NA")	=IF(G6="Yes",IF(D7=7,1,0),"NA")	=IF(G6="Yes",IF(E7+F7=1,"No","Yes"),"No")
8	=IF(G7="Yes",IF(D8=$D$5,1,0),"NA")	=IF(G7="Yes",IF(D8=7,1,0),"NA")	=IF(G7="Yes",IF(E8+F8=1,"No","Yes"),"No")
9	=IF(G8="Yes",IF(D9=$D$5,1,0),"NA")	=IF(G8="Yes",IF(D9=7,1,0),"NA")	=IF(G8="Yes",IF(E9+F9=1,"No","Yes"),"No")

d) Answers will vary. There is a fair amount of variability in the number of wins, so a large number of iterations (e.g., 10,000) are necessary to predict the true probability.

22.7-1. (a) $r = Pr\{X \le x\} = \int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x} \Rightarrow x = \frac{1}{1-r}$

r	$x = \frac{1}{1-r}$
.096	1.106
.569	2.320
.665	2.985
.764	4.237
.842	6.329
.492	1.969
.224	1.289
.950	20.000
.610	2.564
.145	1.170

sum = 43.969 $\Rightarrow \hat{\mu} = \frac{43.969}{10} = 4.3969$

(b) For stratum 1, $r' = 0.0 + 0.6r$
stratum 2, $r' = 0.6 + 0.3r$
stratum 3, $r' = 0.9 + 0.1r$

Stratum	r	r'	$x = \frac{1}{1-r'}$	Sampling Weight	$\frac{x}{w}$
1	.096	.058	1.062	1/2	2.124
	.569	.341	1.517		3.034
	.665	.399	1.664		3.328
2	.764	.829	5.848	1	5.848
	.842	.853	6.803		6.803
	.492	.748	3.968		3.968
3	.224	.922	12.821	4	3.205
	.950	.995	200.000		50.000
	.610	.961	25.641		6.410
	.145	.915	11.765		2.941

Sum: 87.661

$\hat{\mu} = 87.661/10 = 8.7661$

(c) $r' = 1 - r \Rightarrow x' = \frac{1}{1-r'} = \frac{1}{r}$

r	$x = \frac{1}{1-r}$	$x' = \frac{1}{r}$
.096	1.106	10.417
.569	2.320	1.757
.665	2.985	1.504
.764	4.237	1.309
.842	6.329	1.188
.492	1.969	2.033
.224	1.289	4.464
.950	20.000	1.053
.610	2.564	1.639
.145	1.170	6.897
Sum	43.969	32.261
$\hat{\mu}$	4.3969	3.2261

$\hat{\mu} = \frac{4.3969 + 3.2261}{2} = 3.8115$

22.7-2.

Stratum	x	x^2	w	$\frac{x}{w}$	$\frac{x^2}{w}$
1	8	64	18/10	80/18	640/18
	5	25		50/18	250/18
	1	1		10/18	10/18
	6	36		60/18	360/18
	3	9		30/18	90/18
	7	49		70/18	490/18
2	3	9	9/10	60/18	180/18
	5	25		100/18	500/18
	2	4		40/18	80/18
3	2	4	3/10	120/18	240/18
			Sum:	620/18	2840/18

Hence, $\hat{\mu} = \frac{620/18}{10} = 3\frac{2}{9}$ and $E[\hat{x^2}] = \frac{2840/18}{10} = 17\frac{3}{9}$

22.7-3

(a) Let $X = \begin{cases} 0 & \text{if } 0.1 \le r_i < 1 \\ 100 r_i + 5 & \text{if } 0 \le r_i < 0.1 \end{cases}$

r_i	0.096	0.665	0.842	0.224	0.610
X	14.5	0	0	0	0

Then, Average $= \bar{E}X = 2.9$

(b)

stratum	r	r^*	X	w	$\frac{X}{w}$	
1	0.096	0.0864	0	$\frac{2}{9}$	0	← $r^* = 0.9r$
	0.665	0.9665	11.65		1.46	
	0.842	0.9842	13.42		1.68	
	0.224	0.9224	7.24		0.905	← $r^* = 0.9 + 0.1r$, $X = (r^* - 0.9) \times 100 + 0.5$
2	0.610	0.9610	11.10	$\frac{8}{9}$	1.39	
				sum	5.435	

Thus, $\hat{E}X = \frac{5.435}{5} = 1.08$

22.7-4

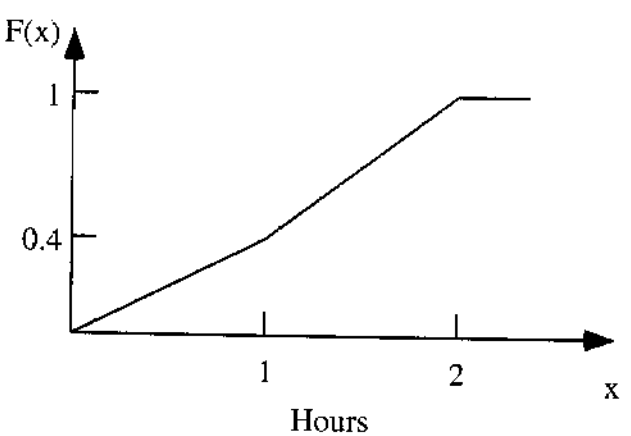
a) Let the numbers 0.0000 to 0.3999 correspond to a minor repair and 0.4000 to 0.9999 correspond to a major repair. The random numbers then indicate 0.7256 = major, 0.0817 = minor, 0.4392 = major.

Using the random numbers, the average length of the three repairs is then

1.2243 hours (first major repair)
0.9503 hours (minor repair)
1.6104 hours (second major repair)

The average repair time is then (1.224+ 0.950 + 1.610) / 3 = 1.26 hours.

b)



c) $F(x) = \begin{cases} 0.4x & \text{when } 0 \le x \le 1 \\ 0.4 + 0.6(x-1) & \text{when } x \ge 1 \end{cases}$

$F(x) = 0.2243 \quad \Rightarrow \quad x = 0.561$ hours

$F(x) = 0.9503 \quad \Rightarrow \quad x = 1.917$ hours

$F(x) = 0.6104 \quad \Rightarrow \quad x = 1.351$ hours

The average repair time is then (0.561 + 1.917 + 1.351) / 3 = 1.28 hours.

d)

$F(x) = 0.7757 \quad \Rightarrow \quad x = 1.626$ hours

$F(x) = 0.0497 \quad \Rightarrow \quad x = 0.124$ hours

$F(x) = 0.3896 \quad \Rightarrow \quad x = 0.974$ hours

The average repair time is then (1.626 + 0.124 + 0.974) / 3 = 0.91 hours.

e) The average repair time is
(0.561 + 1.917 + 1.351 + 1.626 + 0.124 + 0.974) / 6 = 1.09 hours.

f) The method of complementary random numbers in part (e) gave the closest estimate. It performs well because using complements helps counteract the more extreme random numbers (such as 0.9503).

g) Results will vary. The following 300-day simulation using the method of complementary random numbers yielded an overall average service time of 1.095 minutes. This is very close to the true mean of 1.1 minutes.

Day	Random Number	Service Time	Complimentary Random Number	Complimentary Service Time
1	0.1348	0.337	0.8652	1.775
2	0.6798	1.466	0.3202	0.800
3	0.7941	1.657	0.2059	0.515
4	0.1825	0.456	0.8175	1.696
5	0.6502	1.417	0.3498	0.874
6	0.1088	0.272	0.8912	1.819
7	0.1153	0.288	0.8847	1.808
297	0.5456	1.243	0.4544	1.091
298	0.3514	0.878	0.6486	1.414
299	0.8990	1.832	0.1010	0.253
300	0.1544	0.386	0.8456	1.743
	Average =	1.102		1.088

Overall Average = 1.095

h) We get 0.7256, 0.2744, 0.0817, 0.9183, 0.4387, 0.5608 for minor repair times and 1.2243, 1.7757, 1.9503, 1.0497, 1.6104, 1.3896 for major repair times. The weight for minor repair times is $\frac{6/12}{0.4} = 1.25$ and for major repair times it is $\frac{6/12}{0.6} = 1.083$. Dividing each sample by its corresponding weight we get that our estimate of the mean of the overall distr. of repair times is = 1.1

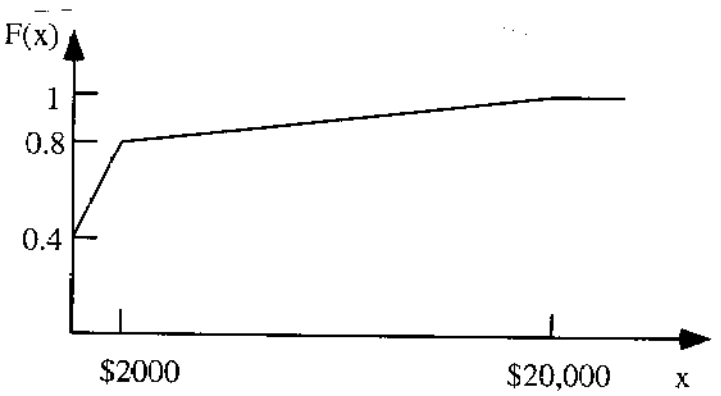
22.7-5 a) Let the numbers 0.0000 to 0.3999 correspond to no claims filed, 0.4000 to 0.7999 correspond to small claims filed, and 0.8000 to 0.9999 correspond to large claims filed. The random numbers then indicate 0.7256 = small, 0.0817 = no, 0.4392 = small.

Using the random numbers, the average size of each claim is then

(0.2243)(2000)=$448.60 (small claim)
$0 (no claim)
(0.6104)(2000)=$1,220.80 (small claim)

The average claim is then (448.60+1,220.80) /2 = $834.70.

b)



c)

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.4 + \dfrac{0.4x}{2,000} & \text{when } 0 \le x \le 2,000 \\ 0.8 + \dfrac{0.2(x - 2,000)}{18,000} & \text{when } 2,000 \le x \le 20,000 \end{cases}$$

$F(x) = 0.2243 \quad \Rightarrow \quad x = \$0$

$F(x) = 0.9503 \quad \Rightarrow \quad x = \$15,527$

$F(x) = 0.6104 \quad \Rightarrow \quad x = \$1,052$

The average claim is then ($0+$15,527+$1052) / 3 = $5526.33.

d)

$F(x) = 0.776 \quad \Rightarrow \quad x = \$1,880$

$F(x) = 0.050 \quad \Rightarrow \quad x = \$0$

$F(x) = 0.390 \quad \Rightarrow \quad x = \$0$

The average claim is then ($1,880 + $0 + $0) / 3 = $626.67 hours.

e) The average claim is
($0 + $15,527 + S1052 + $1880 + $0 + $0) / 6 = $3076.50.

f) The method of complementary random numbers in part (e) gave the closest estimate. It performs well because using complements helps counteract the more extreme random numbers (such as 0.9503).

g) Results will vary. The following 300-day simulation using the method of complementary random numbers yielded an overall average claim size of \$2547.15. This is quite close to the true mean of \$2600.

Day	Random Number	Size of Claim	Complimentary Random Number	Complimentary Size of Claim
1	0.2837	\$0.00	0.7163	\$1,581.27
2	0.4067	\$33.50	0.5933	\$966.50
3	0.4202	\$101.18	0.5798	\$898.82
4	0.3473	\$0.00	0.6527	\$1,263.54
5	0.9728	\$17,550.20	0.0272	\$0.00
6	0.8839	\$9,547.73	0.1161	\$0.00
7	0.6365	\$1,182.61	0.3635	\$0.00
297	0.1141	\$0.00	0.8859	\$9,734.86
298	0.3657	\$0.00	0.6343	\$1,171.49
299	0.7641	\$1,820.30	0.2359	\$0.00
300	0.0532	\$0.00	0.9468	\$15,215.71
	Average =	\$2,648.11		\$2,446.19

Overall Average = \$2,547.15

h. The six random small claims are: 1451.2 163.4 878.4 548.8 1836.6 1121.6
and the six random major claims are: 6037.4 19105.4 12987.2 15962.6 2894.6 9012.8
The weight for the random small claims is $\frac{6/12}{0.4}=1.25$ and for major claims it is $\frac{6/12}{0.2}=2.5$. Summing every sample divided by its corresponding weight, and dividing the sum by 12 we get as our estimate: \$2600.

22.7-6

$$F(x)=\int_{-\infty}^{x} f(y)\,dy=\begin{cases}0 & x<-1\\ \frac{1}{2}(x+1)^2 & -1\le x<0\\ 1-\frac{1}{2}(1-x)^2 & 0\le x<1\\ 1 & x\ge 1\end{cases}$$

$$\Rightarrow x=\begin{cases}\sqrt{2r}-1 & \text{if } 0\le r<\frac{1}{2}\\ 1-\sqrt{2(1-r)} & \text{if } \frac{1}{2}\le r<1\end{cases}$$

r	x	1-r	x
0.096	-0.5618	0.904	0.5618
0.569	0.0716	0.431	-0.0716

So, sample mean = 0

22.7-7.

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 0 & x < -1 \\ \frac{x^3+1}{2} & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$\Rightarrow x = \sqrt[3]{2u-1}$$

r	x	1-r	x
0.096	-0.9384	0.904	0.9314
0.569	0.5168	0.431	-0.5168

Then, Sample mean = 0

22.7-8 (a)

$$P\{X=k\} = \begin{cases} .125 & k=0 \\ .375 & k=1 \\ .375 & k=2 \\ .125 & k=3 \end{cases}$$

$$\Rightarrow X = \begin{cases} 0 & 0 \le r < .125 \\ 1 & .125 \le r < .5 \\ 2 & .5 \le r < .875 \\ 3 & .875 \le r < 1 \end{cases}$$

r	X	1-r	X
.096	0	.904	3
.569	2	.431	1
.665	2	.335	1

$\frac{0+2+2}{3} = 1.33$ = sample mean

(b) Sample mean $= \frac{4+5}{6} = 1.5$

(c) Let $X = \begin{cases} \text{head} & 0 \le r < \frac{1}{2} \\ \text{tail} & \frac{1}{2} \le r < 1 \end{cases}$

$r_1 = \{.096\ .569\ .665\} \Rightarrow x_1 = 1$, $r_2 = \{.764\ .802\ .492\} \Rightarrow x_2 = 1$

$r_3 = \{.224\ .950\ .610\} \Rightarrow x_3 = 1$ So, sample mean $= \frac{3}{3} = 1$

(d) We also have $r_1^* = \{.904\ .431\ .335\} \Rightarrow x_1^* = 2$,

$r_2^* = \{.236\ .158\ .508\} \Rightarrow x_2^* = 2$,

$r_3^* = \{.776\ .050\ .390\} \Rightarrow x_3^* = 2$,

Sample mean $= \frac{3+6}{3} = 3$

22.7-9 (a) For the shaft radius, $r_s = \int_1^s 400e^{-400(t-1)}\,dt = 1 - e^{-400(s-1)}$

so the shaft radius is given by $s = 1 + \frac{\ln(1-r_s)}{-400}$

For the bushing radius, $r_b = \int_1^b 100\,dt = 100(b-1)$

so the bushing radius is given by $b = 1 + \frac{r_b}{100}$

r_s	$s = 1 + \frac{\ln(1-r_s)}{-400}$	r_b	$b = 1 + \frac{r_b}{100}$	Interference? ($s > b$?)
.096	1.000252	.569	1.00569	no
.665	1.002734	.764	1.00764	no
.842	1.004613	.492	1.00492	no
.224	1.000634	.950	1.00950	no
.610	1.002354	.145	1.00145	yes
.484	1.001654	.552	1.00552	no
.350	1.001077	.590	1.00590	no
.430	1.001405	.041	1.00041	yes
.802	1.004048	.471	1.00471	no
.255	1.000736	.799	1.00799	no

Estimated probability of interference = 1/5

(b)

Stratum	Portion of Distribution	Stratum Random Number	Sample Size	Sample Weight
1	$0 \le F(b) \le 0.2$	$r_b' = .2\,r_b$	6	1/3
2	$0.2 \le F(b) \le 0.6$	$r_b' = .2 + .4\,r_b$	2	1/2
3	$0.6 \le F(b) \le 1.0$	$r_b' = .6 + .4\,r_b$	2	1/2

Stratum	r_s	$s = 1 + \frac{\ln(1-r_s)}{-400}$	r_b	r_b'	$b = 1 + \frac{r_b'}{100}$	Interference Weight
	.096	1.000252	.569	.114	1.00114	0
	.665	1.002734	.764	.153	1.00153	1/3
	.842	1.004613	.492	.098	1.00098	1/3
	.224	1.000634	.950	.190	1.00190	0
	.610	1.002354	.145	.029	1.00029	1/3
1	.484	1.001654	.552	.110	1.00110	1/3
	.350	1.001077	.590	.436	1.00436	0
2	.430	1.001405	.041	.216	1.00216	0
	.802	1.004048	.471	.788	1.00788	0
3	.255	1.000736	.799	.920	1.00920	0

Estimated Probability of interference = 4/30 = 2/15

c)

r_s	$s = 1 + \frac{\ln(1-r_s)}{-400}$	r_b	$b = 1 + \frac{r_b}{100}$	Interference? $s > b$?	$s' = 1 + \frac{\ln r_s}{-400}$	$b' = 1 + \frac{r_b}{100}$	Interference? $s' > b'$?
.096	1.000252	.569	1.00569	no	1.005859	1.00431	yes
.665	1.002734	.764	1.00764	no	1.001020	1.00236	no
.842	1.004613	.492	1.00492	no	1.000430	1.00508	no
.224	1.000634	.950	1.00950	no	1.003740	1.00050	yes
.610	1.002354	.145	1.00145	yes	1.001336	1.00855	no
.484	1.001654	.552	1.00552	no	1.001814	1.00448	no
.350	1.001077	.590	1.00590	no	1.002625	1.00410	no
.430	1.001405	.041	1.00041	yes	1.002110	1.00959	no
.802	1.004048	.471	1.00471	no	1.000552	1.00529	yes
.255	1.000736	.799	1.00799	no	1.003416	1.00201	yes

Estimated Probability of interference = $\frac{1}{2}\left(\frac{1}{5} + \frac{2}{5}\right) = 3/10$

Summarizing:

Method:	Monte Carlo	Stratified Sampling	Complementary
Pr{interference}:	1/5	2/15	3/10

22.8-1 (a) $y_1 = 0+5+4 = 9$; $z_1 = 3$
$y_2 = 0+2 = 2$; $z_2 = 2$
$y_3 = 0+3+1+6 = 10$; $z_3 = 4$
$\bar{y} = 21/3 = 7$; $\bar{z} = 9/3 = 3$
So $Est[\omega_s] = 7/3 = 2\ 1/3$
$s_{11}^2 = (81+4+100)/2 - (9+2+10)^2/6 = 19$
$s_{22}^2 = (9+4+16)/2 - (3+2+4)^2/6 = 1$
$s_{12}^2 = (27+4+40)/2 - (21)(9)/6 = 4$
$s^2 = 19 - 2(7/2)4 + (7/2)^2 = 5.778 \Rightarrow s = 2.404$
$1-2\alpha = .90 \Rightarrow \alpha = .05 \Rightarrow K_\alpha = 1.645$
So $Pr\{1.572 \le \omega_s \le 3.094\} = .90$

(b) $y_1 = 0+3+2 = 5$; $z_1 = 3$
$y_2 = 0+3+1+5 = 9$; $z_2 = 4$
$y_3 = 0$; $z_3 = 1$
$y_4 = 0+2+4 = 6$; $z_4 = 3$
$y_5 = 0+3+5+2 = 10$; $z_5 = 4$
$\bar{y} = 30/5 = 6$; $\bar{z} = 15/5 = 3$
So $Est[\omega_s] = 6/3 = 2$

$s_{11}^2 = (25+81+36+100)/4 - (10+6+0+9+5)^2/20 = 15\frac{1}{2}$
$s_{21}^2 = (9+16+1+9+16)/4 - (3+4+1+3+4)^2/20 = 1\frac{1}{2}$
$s_{12}^2 = (15+36+0+18+40)/4 - (30)(15)/20 = 4\frac{3}{4}$
$s^2 = 15\frac{1}{2} - (2)(2)(4\frac{3}{4}) + (2)^2(1\frac{1}{2}) = 2\frac{1}{2} \Rightarrow s = 1.581$
$1-2\alpha = .90 \Rightarrow \alpha = .05 \Rightarrow K_\alpha = 1.645$
So $Pr\{1.612 \le \omega_s \le 2.388\} = .90$

22.8-2. When a service completion occurs, ξ minutes have passed since the last arrival, where $0 \le \xi \le 25$. The time until the next arrival is uniformly distributed from $\bar{\xi}$ to $25-\xi$ where $\bar{\xi} = \max\{0, 5-\xi\}$. Thus, the probabilistic structure of when future arrivals will occur is dependent on the previous history, so this cannot be a regeneration point.

22.8-3 a) For any new tube, the time of the next failure is given by (current time + 1000 + 1000r) where r is a random number from Table 16.1.
At each shutdown, 1 hour is added to the time of the next failure for all tubes when simulating the status quo and 2 hours are added when simulating the proposal

Time	r_1	r_2	r_3	r_4	Time of failure of tube 1	tube 2	tube 3	tube 4
0	.096	.569	.665	.764	1096	1569	1665	1764
1096	.842	—	—	—	2939	1570	1666	1765
1570	—	.492	—	—	2940	3063	1667	1766
1667	—	—	.224	—	2941	3064	2892	1767
1767	—	—	—	.950	2942	3065	2893	3718
2893	—	—	.610	—	2943	3066	4504	3719
2943	.145	—	—	—	4089	3067	4505	3720
3067	—	.484	—	—	4090	4552	4506	3721
3721	—	—	—	.552	4091	4553	4507	5274
4091	.350	—	—	—	5442	4554	4508	5275
4508	—	—	.590	—	5443	4555	6099	5276
4555	—	.430	—	—	5444	5986	6100	5277
5000	—	—	—	—	5444	5986	6100	5277

Estimated cost of status quo = 11 × $1200 = $13,200

Simulation of proposal:

Time	r_1	r_2	r_3	r_4	First tube to fail	time of failure
0	.096	.569	.665	.764	Tube 1	1096
1096	.842	.492	.224	.950	Tube 3	2322
2322	.610	.145	.484	.552	Tube 2	3469
3469	.350	.590	.430	.041	Tube 4	4512
4512	.802	.471	.255	.799	Tube 3	5769

Estimated cost of proposal = 4 × $2800 = $11,200

(b) Based on the simulations in part (a), the proposal should be accepted.

(c) For the proposed policy, each shutdown is a regeneration point because all tubes are replaced, and the process begins anew. For the status quo, the process never repeats itself because tubes are replaced one at a time.

(d)

cycle	cycle cost	cycle length
1	$2800	1096
2	$2800	1226
3	$2800	1147
4	$2800	1043

$\bar{y} = \$2{,}800 \quad \bar{x} = 1128$

$\text{Est}\{\text{cost/hour}\} = 2800/1128 = \$2.482$

$S_{11}^2 = (4 \times 2800^2)/3 - (4 \times 2800)^2/12 = 0$

$S_{22}^2 = (1096^2 + 1226^2 + 1147^2 + 1043^2)/3 - (1096+1226+1147+1043)^2/12 = 6071\ 1/3$

$S_{12}^2 = 2800 \times (1096+1226+1147+1043)/3 - 4 \times 2800 \times (1096+1226+1147+1043)/12 = 0$

$S^2 = 0 - 2.482 \times 0 \times 2 + (2.482)^2 \times 6071\ 1/3 = 37410 \Rightarrow S = 193.4$

$1 - 2\alpha = 0.95 \Rightarrow \alpha = 0.025 \Rightarrow K_\alpha = 1.96$

$Pr\{2.314 \le \text{cost/hour} \le 2.650\} = 0.95$

22.8-4

a) i)

Data

Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	1.25

Service Times

Distribution =	Exponential
Mean =	1

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	4.0964599	3.4723718	4.720548
L_q =	3.297912	2.6914251	3.904399
W =	5.1344677	4.415052	5.8538834
W_q =	4.1335746	3.4239079	4.8432412
P_0 =	0.2014521	0.1791539	0.2237503
P_1 =	0.1562561	0.1405239	0.1719884
P_2 =	0.1238376	0.1119227	0.1357526
P_3 =	0.0991518	0.0901947	0.108109
P_4 =	0.0793526	0.0716784	0.0870269
P_5 =	0.062689	0.0552717	0.0701063
P_6 =	0.0488265	0.0421353	0.0555177
P_7 =	0.0433603	0.0361443	0.0505763
P_8 =	0.0367126	0.0299892	0.043436
P_9 =	0.0310377	0.0244672	0.0376082
P_{10} =	0.021833	0.0164072	0.0272588

(CONT'D)

(CONT'D) 22.B-4

ii)

Data

Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	1.25

Service Times

Distribution =	Erlang
Mean =	1
k =	4

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.578107	2.2926834	2.8635306
L_q =	1.7827864	1.5107674	2.0548055
W =	3.2298633	2.91602	3.5437066
W_q =	2.2334823	1.9225413	2.5444233
P_0 =	0.2046794	0.1873432	0.2220156
P_1 =	0.2197453	0.2043736	0.235117
P_2 =	0.1747516	0.1638331	0.18567
P_3 =	0.1248149	0.1161342	0.1334955
P_4 =	0.0888187	0.0805596	0.0970779
P_5 =	0.0640678	0.0561015	0.0720342
P_6 =	0.0435159	0.0356854	0.0513464
P_7 =	0.0275738	0.0209489	0.0341987
P_8 =	0.0172859	0.0117553	0.0228165
P_9 =	0.0122166	0.0072187	0.0172144
P_{10} =	0.00871	0.0028389	0.014581

iii)

Data

Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	1.25

Service Times

Distribution =	Constant
Value =	1

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.3267194	2.1264927	2.5269461
L_q =	1.5263968	1.3372653	1.7155282
W =	2.9072268	2.6952243	3.1192293
W_q =	1.9072268	1.6952243	2.1192293
P_0 =	0.1996774	0.1846231	0.2147317
P_1 =	0.2361344	0.222148	0.2501208
P_2 =	0.1947171	0.1852665	0.2041677
P_3 =	0.1375172	0.1290396	0.1459948
P_4 =	0.0924868	0.0839415	0.1010321
P_5 =	0.0562203	0.0492638	0.0631768
P_6 =	0.0324712	0.0262022	0.0387402
P_7 =	0.0180219	0.0132993	0.0227444
P_8 =	0.0104366	0.0068661	0.0140071
P_9 =	0.007966	0.0043986	0.0115334
P_{10} =	0.0056259	0.0023837	0.0088682

So $L_{q2}/L_{q1} = 1.78/3.30 = 0.54$

and $L_{q3}/L_{q1} = 1.63/3.30 = 0.46$

(b) $L_q = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$, $L = \rho + L_q$, $W_q = \frac{L_q}{\lambda}$, $W = W_q + \frac{1}{\mu}$

(i) $L_{q1} = \frac{0.64+0.64}{2\times 0.2} = 3.2$. $L_1 = 4$, $W_{q1} = 4$, $W_1 = 5$

$L_{q2} = \frac{0.64\times 0.25+0.64}{2\times 0.2} = 2$, $L_2 = 2.8$, $W_{q2} = 2.5$, $W_2 = 3.5$

$L_{q3} = \frac{0.64}{2\times 0.2} = 1.6$, $L_3 = 2.4$, $W_{q3} = 2$, $W_3 = 3$

$L_{q2}/L_{q1} = 0.675$, $L_{q3}/L_{q1} = \frac{1.6}{3.2} = 0.5$

They all fall into 95% confidence intervals in (a).

22.8-5

i)

Data

Number of Servers =	2

Interarrival Times

Distribution =	Exponential
Mean =	0.625

Service Times

Distribution =	Exponential
Mean =	1

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	4.6403996	3.7949461	5.4858511
L_q =	3.025417	2.2140863	3.8367477
W =	2.8919312	2.4089495	3.374913
W_q =	1.8854622	1.4084976	2.3624268
P_0 =	0.1073721	0.0932623	0.121482
P_1 =	0.1702732	0.1518527	0.1886936
P_2 =	0.1403936	0.125757	0.1550301
P_3 =	0.1148071	0.1037198	0.1258944
P_4 =	0.0910806	0.0824708	0.0996904
P_5 =	0.0722107	0.064574	0.0798474
P_6 =	0.0570938	0.0493916	0.064796
P_7 =	0.0468421	0.0398067	0.0538775
P_8 =	0.0350372	0.029091	0.0409835
P_9 =	0.0267432	0.0209395	0.032547
P_{10} =	0.0234001	0.0171027	0.0296976

ii)

Data

Number of Servers =	2

Interarrival Times

Distribution =	Erlang
Mean =	0.625
k =	4

Service Times

Distribution =	Exponential
Mean =	1

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	3.0299726	2.7669635	3.2929816
L_q =	1.4284437	1.1900918	1.6667957
W =	1.8719363	1.7176409	2.0262317
W_q =	0.8825016	0.7391642	1.025839
P_0 =	0.0806072	0.0720243	0.0891901
P_1 =	0.2372568	0.2199123	0.2546013
P_2 =	0.2137762	0.2000592	0.2274931
P_3 =	0.1488911	0.1392218	0.1585604
P_4 =	0.1029385	0.0946101	0.1112669
P_5 =	0.0695247	0.0612776	0.0777719
P_6 =	0.0473264	0.040051	0.0546018
P_7 =	0.0315265	0.0252751	0.0377779
P_8 =	0.0259208	0.0195809	0.0322607
P_9 =	0.0160099	0.0105627	0.0214571
P_{10} =	0.0109931	0.0063586	0.0156276

iii)

Data

Number of Servers =	2

Interarrival Times

Distribution =	Constant
Value =	0.625

Service Times

Distribution =	Exponential
Mean =	1

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.7358123	2.4512332	3.0203914
L_q =	1.14358	0.8830385	1.4041214
W =	1.7098827	1.5320207	1.8877446
W_q =	0.7147375	0.5518991	0.8775759
P_0 =	0.0731526	0.0657448	0.0805604
P_1 =	0.2614625	0.2423098	0.2806152
P_2 =	0.2563404	0.2414023	0.2712785
P_3 =	0.157098	0.1473874	0.1668086
P_4 =	0.0909031	0.0821201	0.0996861
P_5 =	0.0546979	0.0462018	0.063194
P_6 =	0.034087	0.0271133	0.0410606
P_7 =	0.0238301	0.0167202	0.0309399
P_8 =	0.0146206	0.0092269	0.0200144
P_9 =	0.0102025	0.0048136	0.0155914
P_{10} =	0.0083484	0.0031743	0.0135225

$L_{q2}/L_{q1} = \frac{1.53}{3.025} = 0.51$ $L_{q3}/L_{q1} = \frac{1.14}{3.025} = 0.38$

22.8-6

Data

Number of Servers =	1

Interarrival Times

Distribution =	Exponential
Mean =	1

Service Times

Distribution =	Exponential
Mean =	0.8

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	3.894759	3.1901989	4.5993191
L_q =	3.0976213	2.4081928	3.7870499
W =	3.9143169	3.230127	4.5985067
W_q =	3.1131763	2.4390993	3.7872534
P_0 =	0.2028623	0.1814813	0.2242434
P_1 =	0.1630081	0.147632	0.1783842
P_2 =	0.1290416	0.1177962	0.140287
P_3 =	0.1007961	0.0915579	0.1100344
P_4 =	0.0850072	0.0764865	0.093528
P_5 =	0.0677124	0.0601917	0.075233
P_6 =	0.05203	0.0452794	0.0587805
P_7 =	0.0408908	0.0338484	0.0479333
P_8 =	0.0345285	0.0272432	0.0418138
P_9 =	0.0259406	0.0199831	0.031898
P_{10} =	0.0187916	0.0134539	0.0241292

Data

Number of Servers =	1

Interarrival Times

Distribution =	Erlang
Mean =	1
k =	4

Service Times

Distribution =	Exponential
Mean =	0.8

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.8833767	2.5057286	3.2610249
L_q =	2.06611	1.7038045	2.4284155
W =	2.894756	2.5216494	3.2678626
W_q =	2.074264	1.7148919	2.433636
P_0 =	0.1827333	0.1642703	0.2011963
P_1 =	0.2335019	0.2129687	0.2540372
P_2 =	0.1586323	0.146593	0.1706716
P_3 =	0.1176148	0.1076587	0.127573
P_4 =	0.0848304	0.0767402	0.0929206
P_5 =	0.0603899	0.0520811	0.0686987
P_6 =	0.0452507	0.0362098	0.0542916
P_7 =	0.0325425	0.0240011	0.0410838
P_8 =	0.0236739	0.0169689	0.0303789
P_9 =	0.0206918	0.0132305	0.0281531
P_{10} =	0.0147396	0.0083984	0.0210809

Data

Number of Servers =	1

Interarrival Times

Distribution =	Constant
Value =	1

Service Times

Distribution =	Exponential
Mean =	0.8

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	2.0794874	1.8820166	2.2769583
L_q =	1.2823888	1.0966006	1.468177
W =	2.0794874	1.8820166	2.2769583
W_q =	1.2823888	1.0966006	1.468177
P_0 =	0.2029014	0.1880822	0.2177205
P_1 =	0.3004745	0.2833294	0.3176197
P_2 =	0.1876719	0.1782299	0.1971138
P_3 =	0.122003	0.1130305	0.1309755
P_4 =	0.0747585	0.0661909	0.0833261
P_5 =	0.0418572	0.0349302	0.0487841
P_6 =	0.025796	0.0193757	0.0322163
P_7 =	0.0176922	0.0118091	0.0235753
P_8 =	0.0108518	0.0066022	0.0151214
P_9 =	0.0066882	0.0034027	0.0099736
P_{10} =	0.0036447	0.0011988	0.0060907

$L_{q2}/L_{q1} = \frac{2.07}{3.10} = 0.668$ $L_{q3}/L_{q1} = \frac{1.28}{3.10} = 0.41$

Case Problems

22-1 Current Situation: A simulation run (shown below) indicates that the average number of jobs in the system is 1.9. Of these, half will be platen castings (0.95) and half will be housing castings (0.95). The waiting cost is therefore ($200)(0.95) + ($100)(0.95) = $285 / hour.

Current Situation

Data	
Number of Servers =	2
Interarrival Times	
Distribution =	Exponential
Mean =	15
Service Times	
Distribution =	Translated Exponential
Minimum Value =	10
Mean =	20
Length of Simulation Run	
Number of Arrivals =	10,000

Run Simulation

Results	Point Estimate	95% Confidence Interval Low	High
L =	1.90473734	1.81035793	1.99911674
L_q =	0.59841056	0.52743245	0.66938867
W =	29.0126292	27.9716675	30.0535909
W_q =	9.114886	8.1501552	10.0796168
P_0 =	0.20397713	0.19264639	0.21530787
P_1 =	0.28571897	0.27485167	0.29658627
P_2 =	0.2218313	0.2143161	0.22934649
P_3 =	0.13716649	0.13013886	0.14419411
P_4 =	0.07180874	0.06554342	0.07807405
P_5 =	0.03911113	0.03335364	0.04486862
P_6 =	0.01984847	0.01572564	0.0239713
P_7 =	0.01055801	0.00758385	0.01353217
P_8 =	0.00531947	0.00315077	0.00748817
P_9 =	0.00231608	0.00116563	0.00346654
P_{10} =	0.00146818	0.00041766	0.00251871

Proposal 1: A simulation run (shown below) indicates that the average number of jobs in the system with three planers is approximately 1.46. Of these, half will be platen castings (0.73) and half will be housing castings (0.73). The waiting cost is therefore ($200)(0.73) + ($100)(0.73) = $219 / hour. The savings ($66 / hour) is substantially more than the added cost of the third planer ($30 / hour), so this looks to be worthwhile. The net savings would be $36 / hour.

Proposal 1

Data	
Number of Servers =	3
Interarrival Times	
Distribution =	Exponential
Mean =	15
Service Times	
Distribution =	Translated Exponential
Minimum Value =	10
Mean =	20
Length of Simulation Run	
Number of Arrivals =	10,000

Run Simulation

Results	Point Estimate	95% Confidence Interval Low	High
L =	1.4567689	1.41377127	1.49976653
L_q =	0.10973775	0.09249252	0.12698298
W =	21.7295877	21.3625419	22.0966335
W_q =	1.63688018	1.39559645	1.87816392
P_0 =	0.24863392	0.23901401	0.25825383
P_1 =	0.33856279	0.33024864	0.34687694
P_2 =	0.22994152	0.22306044	0.23682261
P_3 =	0.1130194	0.1074762	0.11856259
P_4 =	0.04531033	0.04124076	0.0493799
P_5 =	0.01487792	0.01250839	0.01724745
P_6 =	0.00557272	0.00402145	0.00712399
P_7 =	0.00269386	0.00159608	0.00379164
P_8 =	0.00118696	0.00037629	0.00199763
P_9 =	0.0001609	-2.8773E-05	0.00035057
P_{10} =	3.9681E-05	-3.7977E-05	0.00011734

Proposal 2: A simulation run (shown below) indicates that the average number of jobs in the system with constant interarrival times is approximately 1.40. Of these, half will be platen castings (0.70) and half will be housing castings (0.70). The waiting cost is therefore ($200)(0.70) + ($100)(0.70) = $210 / hour. The savings ($75 / hour) is somewhat more than the added cost of changing the preceding production cost ($60 / hour). The net savings ($15) is less than for proposal 1, so this option is less worthwhile.

Proposal 2

Data

Number of Servers =	2

Interarrival Times

Distribution =	Constant
Value =	15

Service Times

Distribution =	Translated Exponential
Minimum Value =	10
Mean =	20

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.40021685	1.37657912	1.42385458
L_q =	0.06087211	0.04688633	0.07485789
W =	21.0032527	20.6486868	21.3578187
W_q =	0.91308164	0.70329497	1.12286832
P_0 =	0.0507294	0.0485373	0.05292149
P_1 =	0.55919647	0.5490678	0.56932514
P_2 =	0.33758206	0.32938548	0.34577865
P_3 =	0.04717064	0.04194527	0.05239601
P_4 =	0.00402596	0.00223602	0.0058159
P_5 =	0.00037505	-2.6602E-05	0.00077669
P_6 =	0.00026694	-0.00025512	0.000789
P_7 =	0.00046426	-0.0004437	0.00137222
P_8 =	0.00018922	-0.00018084	0.00055929
P_9 =	0	0	0
P_{10} =	0	0	0

Proposal 1 and 2: A simulation run (shown below) indicates that the average number of jobs in the system with both three planers and constant interarrival times is approximately 1.33. Of these, half will be platen castings (0.665) and half will be housing castings (0.665). The waiting cost is therefore ($200)(0.665) + ($100)(0.665) = $200 / hour. The savings ($85 / hour) is less than the combined cost of adding a third planer and changing the preceding production cost ($90 / hour), so this combined option does not appear to be worthwhile.

Proposal 1 and 2

Data

Number of Servers =	3

Interarrival Times

Distribution =	Constant
Value =	15

Service Times

Distribution =	Translated Exponential
Minimum Value =	10
Mean =	20

Length of Simulation Run

Number of Arrivals =	10,000

Run Simulation

Results

	Point Estimate	95% Confidence Interval Low	High
L =	1.3337841	1.32084457	1.34672363
L_q =	0.00087734	0.00026928	0.00148541
W =	20.0067615	19.8126686	20.2008545
W_q =	0.01316016	0.00403913	0.0222812
P_0 =	0.05617206	0.05408543	0.05825869
P_1 =	0.58758437	0.5796862	0.59548254
P_2 =	0.32340833	0.31585866	0.330958
P_3 =	0.03197872	0.02902062	0.03493681
P_4 =	0.00083571	0.00026183	0.00140959
P_5 =	2.0817E-05	-1.9966E-05	6.16E-05
P_6 =	0	0	0
P_7 =	0	0	0
P_8 =	0	0	0
P_9 =	0	0	0
P_{10} =	0	0	0

Overall recommendation: Proposal 1 appears to be the most worthwhile, with a net savings of about $36 / hour over the current situation. Other proposals that may be worth looking into should include giving priority to platen castings, because of the higher waiting cost for that type of job.

22-2 a) Before we begin the formal problem, we must first calculate the mean μ and standard deviation σ of the normally distributed random variable N. We are told that the annual interest rate will be used to estimate μ and the historical annual volatility will be used to estimate σ. Because the case is simulating weekly – not yearly – change, we must convert these yearly values to weekly values.

We first convert the annual interest rate r = 8 percent to a weekly interest rate w with the following formula:

$$\begin{aligned} w &= (1 + r)^{(1/52)} - 1 \\ &= (1 + 0.08)^{(1/52)} - 1 \\ &= (1.08)^{(1/52)} - 1 \\ &= 0.00148 \end{aligned}$$

We next convert the annual volatility $V_a = 0.30$ to a weekly volatility V_w with the following formula:

$$\begin{aligned} V_w &= V_a / \sqrt{52} \\ &= 0.30 / \sqrt{52} \\ &= 0.0416 \end{aligned}$$

Once we have the weekly interest rate and volatility, we can calculate μ and σ.

$$\begin{aligned} \mu &= w - 0.5(V_w)^2 \\ &= 0.00148 - 0.5(0.0416)^2 \\ &= 0.0006 \end{aligned}$$

$$\begin{aligned} \sigma &= V_w \\ &= 0.0416 \end{aligned}$$

1. One component appears in this system: the stock price. The stock price in the previous week is used to calculate the stock price in the next week. The relationship between the stock price in the previous week and the stock price in the next week is given by $s_n = e^N s_c$.

2. State of the system: P(t) = price of the stock at time t.

3. This simulation requires generating a series of random observations from the normal distribution. Each random observation is a normally distributed random variable that determines the increase or decrease of the stock price at the end of next week. The random variable is substituted for N in the following equation:

$$s_n = e^N s_c$$

To generate a series of random variables, we repeatedly use the function for generating a normally distributed random variable provided by the @RISK software add-in for Excel. This function is RiskNormal(μ, σ) where $\mu = 0.0006$ and $\sigma = 0.0416$.

4. The formula $s_n = e^N s_c$ gives us a procedure for changing the price (the state of the system) when an event occurs.

5. In this simulation, the time periods are fixed. We have a twelve-week period, and we need to calculate the change in the stock price each week. We have a formula $s_n = e^N s_c$ that relates the stock price at the end of the next week to the stock price at the end of the previous week. Thus, we do not have to worry about advancing the clock. We simply have to generate N for each of the twelve weeks.

6. We need to build a spreadsheet using the @RISK software add-in for Excel. We start with the current stock price of \$42.00. We then use the formula $s_n = e^N s_c$ to calculate the stock price at the end of each of the twelve weeks. We substitute the @RISK function RiskNormal(μ, σ) for N.

We then use the stock price at the end of the twelfth week to calculate the value of the option at the end of the twelfth week. If the stock price at the end of the twelfth week is greater than the exercise price of \$44.00, the value of the option is the difference between the value of the stock at the end of the twelfth week and the exercise price. If the stock price at the end of the twelfth week is less than or equal to the exercise price of \$44.00, the value of the option is \$0.

Finally, we need to discount the value of the option at the end of the twelfth week to the value of the option in today's dollars using the following formula:

$$\frac{\text{Value of the option at the end of the twelfth week}}{(1.00148)^{12}}$$

The spreadsheet formulas are shown below:

d) No, a random walk does not completely describe the price movement of the stock because the random walk assumes a consistent lognormal increase or decrease in the price of the stock. The price of the stock could change according to a different distribution, however, especially if an event occurs to trigger a dramatic increase or decrease in the stock. In this case, the European Space Agency may award Fellare the International Space Station contract. The award notice would most likely trigger a dramatic movement in the stock. The random walk does not take into account this dramatic event.

Current Stock Price	42	Annual IR	0.08
Price at End of Week:		Weekly IR	=((1+Annual_IR)^(1/52))-1
One	=EXP(RiskNormal(Mu, Sigma))*B2	Mu	=Weekly_IR-0.5*(Weekly_V^2)
Two	=EXP(RiskNormal(Mu,Sigma))*B4		
Three	=EXP(RiskNormal(Mu,Sigma))*B5	Annual V	0.3
Four	=EXP(RiskNormal(Mu,Sigma))*B6	Weekly V	=Annual_V/(52^(1/2))
Five	=EXP(RiskNormal(Mu,Sigma))*B7	Sigma	=Weekly_V
Six	=EXP(RiskNormal(Mu,Sigma))*B8		
Seven	=EXP(RiskNormal(Mu,Sigma))*B9		
Eight	=EXP(RiskNormal(Mu,Sigma))*B10		
Nine	=EXP(RiskNormal(Mu,Sigma))*B11		
Ten	=EXP(RiskNormal(Mu,Sigma))*B12		
Eleven	=EXP(RiskNormal(Mu,Sigma))*B13		
Twelve	=EXP(RiskNormal(Mu,Sigma))*B14		
Price of Option at End of Week Twelve	=IF(End_Price>44, End_Price-44, 0)		
Price of Option Today	=Option/(1+Weekly_IR)^12		

b) After running the appropriate number of simulation iterations, the mean of the cell labeled the "Price of Option Today" is the price of the option in today's dollars.

Number of Iterations	Price of Option
100	$1.70
500	$1.95
1000	$1.84

c) Using the Black-Scholes Formula, the price of the option is $1.88. The spreadsheet of formulas used to calculate the Black-Scholes Formula in Excel follows:

		Black-Scholes	
Annual IR	0.08	d1	=(LN(Current_Price/44/(1+Weekly_IR)^12))/(Weekly_V*12^(1/2)) + (Weekly_V*12^(1/2))/2
Weekly IR	=((1+Annual_IR)^(1/52))-1	d2	=_d1-Weekly_V*12^(1/2)
Mu	=Weekly_IR-0.5*(Weekly_V^2)		
		N[d1]	=NORMSDIST(_d1)
Annual V	0.3	N[d2]	=NORMSDIST(_d2)
Weekly V	=Annual_V/(52^(1/2))		
Sigma	=Weekly_V	V	=N_d1*Current_Price-N_d2*(44/(1+Weekly_IR)^12)

The spreadsheet showing the values used in the Black-Scholes formula follows:

		Black-Scholes	
Annual IR	**0.08**	d1	-0.127503153
Weekly IR	**0.001481116**	d2	-0.271618491
Mu	**0.000615731**		
		N[d1]	0.449271051
Annual V	**0.3**	N[d2]	0.39295775
Weekly V	**0.041602515**		
Sigma	**0.041602515**	**V**	**$1.88**

The price of the option obtained by simulation and the price of the option obtained by the Black-Scholes formula are amazingly close. The 1000-iteration simulation price is off by just four cents!

Problem Solutions for Chapter 23

(Appears on the book's Web site, www.mhhe.com/hillier)

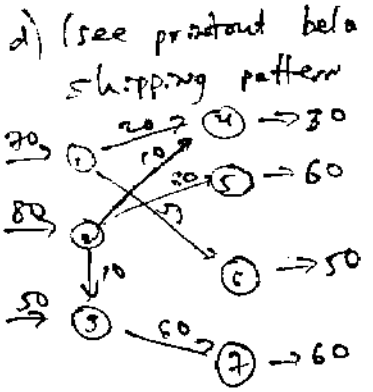
23.1-1

(a) supply centers: locations 1, 2 and 3.
receiving centers: locations 4, 5, 6 and 7.
Also, shipments can be sent via intermediate points.

(b)

	1	2	3	4	5	6	7	s_i
1	0	21	50	62	93	77	M	270
2	21	0	17	54	67	M	48	280
3	50	17	0	60	98	67	25	250
4	62	54	60	0	27	M	38	200
5	93	67	98	27	0	47	42	200
6	77	M	67	M	47	0	35	200
7	M	48	25	38	42	35	0	200
d_j	200	200	200	230	260	250	260	

(c) (see printout below)
shipping pattern



Optimal Solution: The main body of the table shows the optimal number of units (if not zero) to be sent from each source to each destination.

		Destination							
		1	2	3	4	5	6	7	Supply
Source	1	200			20		50		270
	2		200	10	10	60			280
	3			190				60	250
	4				200				200
	5					200			200
	6						200		200
	7							200	200
Demand		200	200	200	230	260	250	260	Cost is 11320

23.1-2 (a) Let the supply center be year 0 with a supply of 1.
Let the receiving center be year 3 with a demand of 1.
Let years 1 and 2 be transshipment points.
The following transshipment problem is obtained:

	Year 0	Year 1	Year 2	Year 3	Supply
Year 0	0	8	18	31	1
Year 1	M	0	10	21	0
Year 2	M	M	0	12	0
Year 3	M	M	M	0	0
Demand	0	0	0	1	

(b) The transportation problem is the same as above, with all supplies and demands increased by 1.

(c) Vogel's approximation

				u_i
0 (1)	8 (1)	18 0	31 2	8
M M+8	0 (0)	10 (0)	21 (1)	0
M M+18	M M+10	0 (1)	12 1	-10
M M+29	M M+21	M M+11	0 (1)	-21
v_j	-8	0	10	21

(d) Vogel's approximation prices out optimal.

23.1-3 (a) Let c_{ij}^k be the cost of buying a very old car ($k=1$) or a moderately old car ($k=2$) at the beginning of year i and trading it in at the end of year j.

c_{ij}^k = purchase price + (operating and maintenance cost for years $1, 2, \ldots, j-i+1$) − (trade in value after $j-i+1$ years)

c_{ij}^1

	1	2	3	4
1	2400	4800	7400	10300
2	M	2400	4800	7400
3	M	M	2400	4800
4	M	M	M	2400

c_{ij}^2

	1	2	3	4
1	3000	5000	7200	10700
2	M	3000	5000	7200
3	M	M	3000	5000
4	M	M	M	3000

Let $c_{ijn} = \min\{c_{ij}^1, c_{ij}^2\}$. Let the supply center be year 1 with supply of 1 and the demand center be year 5 with demand of 1. Years 2, 3 and 4 are transshipment points. $c_{ii}=0$ and $c_{i1}=M$ for $i>1$, $c_{5j}=M$ for $j<5$.

We have the following transshipment problem.

	Year j 1	2	3	4	5	Supply
1	0	2400	4800	7200	10300	1
2	M	0	2400	4800	7200	0
Year i 3	M	M	0	2400	4800	0
4	M	M	M	0	2400	0
5	M	M	M	M	0	0
Demand	0	0	0	0	1	

(b) The transportation problem's cost and requirements table is identical to the one above with all supplies and demands increased by 1.

(c) **Optimal Solution:** The main body of the table shows the optimal number of units (if not zero) to be sent from each source to each destination.

	Destination 1	2	3	4	5	Supply
1	1	1				2
2					1	1
Source 3			1			1
4				1		1
5					1	1
Demand	1	1	1	1	2	Cost is 9600

The optimal solution is to purchase a very old car for year 1 and a moderately old car for years 2, 3 and 4.

Cost = $9600

23.1-4 Suppose there are m supply centers, n receiving centers and p other transshipment centers.

Minimize $\sum_{i=1}^{m+n+p} \sum_{\substack{j=1 \\ j \neq i}}^{m+n+p} c_{ij} x_{ij}$

subject to $\sum_{\substack{j=1 \\ j \neq i}}^{m+n+p} (x_{ij} - x_{ji}) = \begin{cases} s_i & \text{for } i = 1, 2, \ldots, m \\ -d_i & \text{for } i = m+1, \ldots, m+n \\ 0 & \text{for } i = m+n+1, \ldots, m+n+p \end{cases}$

$x_{ij} \geq 0$ for all $i \neq j$

This model has the special structure that each decision variable appears in exactly 2 constraint equations, once with a coefficient of +1 and once with a coefficient of -1. The table of constraint coefficients is:

x_{12}	x_{13}	$\cdots$	$x_{1,m+n+p}$	x_{21}	x_{23}	$\cdots$	$x_{2,m+n+p}$	$\cdots$	$x_{m+n+p,1}$	$x_{m+n+p,2}$	$\cdots$	$x_{m+n+p,m+n+p-1}$
1	1	$\cdots$	1	-1	0	$\cdots$	0	$\cdots$	-1	0	$\cdots$	0
-1	0	$\cdots$	0	1	1	$\cdots$	1	$\cdots$	0	-1	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
0	0	$\cdots$	-1	0	0	$\cdots$	-1	$\cdots$	1	1	$\cdots$	1

23.2-1 (a) Maximize $2x_1 + 4x_2 + 3x_3 + 2x_4 - 5x_5 + 3x_6$

Master problem	$5x_1$	$-2x_2$	$+3x_3$	$+4x_4$	$+2x_5$	$+x_6$	≤ 20	
	$2x_1$	$+4x_2$		$+2x_4$		$+3x_6$	≤ 60	
Sub-problem 1	$3x_1$	$+2x_2$	$+3x_3$				≤ 30	
	$5x_1$		$-x_3$				≤ 30	
	$-x_1$	$+2x_2$	$+x_3$				≥ 20	
Sub-problem 2				x_4			≤ 15	
				x_4			≥ 3	
Sub-problem 3					$2x_5$	$-x_6$	≤ 20	
					$2x_5$	$+3x_6$	≤ 40	

$x_j \geq 0$ for $j = 1, 2, \ldots, 6$

(b) Converting $\geq$ inequalities to $\leq$ inequalities, the table of constraint coefficients becomes:

x_1	x_2	x_3	x_4	x_5	x_6
5	-2	3	4	2	1
2	4		2		3
3	2	3			
5		-1			
1	-2	-1			
			1		
			-1		
				2	-1
				2	3

23.2-2

Constraint	x_1	x_4	x_2	x_5	x_7	x_3	x_6	
master-problem	4	2	3	4	1	-2	0	
	6	0	5	1	4	3	-2	← upper bound
sub-problem 1	2	0	1					← GUB
	5	1	1					
	4	2	4					← GUB
sub-problem 2	1		1	1	1			
	8		2	1	3			
sub-problem 3	4					2	4	
	7					0	1	← upper bound

23.2-3 (a) Maximize $7x_1 + 3x_2 + 5x_3 + 4x_4 + 7x_5 + 5x_6$

Subject to $16x_1 + 7x_2 + 13x_3 + 8x_4 + 20x_5 + 10x_6 \leq 150$

$10x_1 + 3x_2 + 7x_3 \leq 50$

$4x_1 + 2x_2 + 5x_3 \leq 30$

$6x_4 + 13x_5 + 9x_6 \leq 45$

$3x_4 + 8x_5 + 2x_6 \leq 25$

$x_j \geq 0 \quad j = 1, 2, \ldots, 6$

(b)

x_1	x_2	x_3	x_4	x_5	x_6
16	7	13	8	20	10
10	3	7			
4	2	5			
			6	13	9
			3	8	2

23.3-1 Wyndor Glass Co.

$$A = \left[\begin{array}{cc|cc} 3 & 2 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 2 & 1 \end{array}\right] \qquad A_1 = (3) \quad A_2 = (2)$$

$$A_3 = (1) \quad A_4 = (2)$$

$c_1 = (3) \quad c_2 = (5) \quad \vec{x}_1 = (x_1) \quad \vec{x}_2 = (x_2) \quad b_0 = 18$

$b_1 = 4$

③ $x_B = \begin{pmatrix} y_{13} \\ y_{11} \\ y_{22} \end{pmatrix}$ $c_B = (125/2 \quad 0 \quad 40)$

$B = \begin{pmatrix} 105/2 & 0 & 20 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 2/105 & 0 & -40/105 \\ -2/105 & 1 & 40/105 \\ 0 & 0 & 1 \end{pmatrix}$

$W_1 = \frac{70}{2} x_1 + \frac{20}{2} x_2$, same, $W_1 = -14.28$, using x_2^*

$W_2 = -\frac{68}{2} x_3 + \frac{1}{7} x_4 - \frac{500}{2} + 40$, $x_{22}^* = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \Rightarrow W_2 = 0$

$c_B B^{-1} \geq 0$ $\Rightarrow$ y_{12} enters

$B^{-1} A_k' = \begin{pmatrix} 60/105 \\ 55/105 \\ 0 \end{pmatrix}$ $B^{-1} b' = \begin{pmatrix} 40/105 \\ 65/105 \\ 1 \end{pmatrix}$ min. ratio $= \frac{40}{60}$ $r = 1$

y_{13} leaves

④ $x_B = \begin{pmatrix} y_{12} \\ y_{11} \\ y_{22} \end{pmatrix}$ $c_B = (50 \quad 0 \quad 40)$ $B = \begin{pmatrix} 30 & 0 & 20 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 1/30 & 0 & -2/3 \\ -1/30 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}$

$W_1 = 10/3 \, x_2$ x_{11}^* or x_{12}^* both give $W_1 = 0$

$W_2 = -4/3 x_3 + 3 x_4 - 100/3 + 40$

$x_{22}^* = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \Rightarrow W_2 = 0$

$c_B B^{-1} = 5/3 > 0$

optimal

$x_B = \begin{pmatrix} y_{12} \\ y_{11} \\ y_{22} \end{pmatrix} = B^{-1} b' = \begin{pmatrix} 2/3 \\ 1/3 \\ 1 \end{pmatrix}$

$\Rightarrow \vec{x}_1 = \frac{1}{3} \begin{pmatrix} 0 \\ 6 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 0 \end{pmatrix}$

$\vec{x}_2 = 1 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

So $x_1 = 10/3$, $x_2 = 0$, $x_3 = 5$, $x_4 = 0 \rightarrow z = \frac{220}{3}$

23.3-3 Good Foods Corp.

3 subproblems, 2 linking constraints

Initial basis: ① $x_B = \begin{pmatrix} x_{s1} \\ x_{s2} \\ y_{11} \\ y_{21} \\ y_{31} \end{pmatrix}$ $B = B^{-1} = I$, $c_B = 0$

Solve subproblems:

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23.3-3 cont.

Sub 1 min. $-8x_1 - 5x_2 - 6x_3$
s.t. $2x_1 + 4x_2 + 3x_3 \le 10$
$7x_1 + 3x_2 + 6x_3 \le 15$
$5x_1 + 3x_3 \le 12$
$x_1, x_2, x_3 \ge 0$
$\rightarrow X_{1K}^* = \begin{pmatrix} 15/11 \\ 20/11 \\ 0 \end{pmatrix}$ $w_1 = -20$

Sub 2 min. $-9x_4 - 7x_5 - 9x_6$
s.t. $3x_4 + x_5 + 2x_6 \le 7$
$2x_4 + 4x_5 + 3x_6 \le 9$
$x_4, x_5, x_6 \ge 0$
$\rightarrow X_{2K}^* = \begin{pmatrix} 3/5 \\ 0 \\ 13/5 \end{pmatrix}$ $w_2 = -28.8$

Sub 3 min. $-6x_7 - 5x_8$
s.t. $8x_7 + 5x_8 \le 25$
$7x_7 + 9x_8 \le 30$
$6x_7 + 4x_8 \le 20$
$x_7, x_8 \ge 0$
$\rightarrow X_{3K}^* = \begin{pmatrix} 75/37 \\ 65/37 \end{pmatrix}$ $w_3 = -20.95$

w_2 is smallest, so let β_{2K} enter.

$$A'_K = \begin{pmatrix} A_2 X_{2K}^* \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left(\begin{pmatrix} 2 & 0 & 3 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 0 \\ 13/5 \end{pmatrix} \right) = \begin{pmatrix} 9 \\ 9/5 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$B^{-1}b = \begin{pmatrix} 30 \\ 20 \\ \vdots \\ 1 \end{pmatrix}$$ min. ratio $= \frac{1}{1}$ $r=4$, so β_{21} leaves

② $X_B = \begin{pmatrix} x_{51} \\ x_{52} \\ \vdots \\ \beta_{2K} \\ \beta_{31} \end{pmatrix}$ $C_B = \begin{pmatrix} 0 & 0 & 0 & \frac{144}{5} & 0 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & -9/5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$w_1 =$ (same) $w_1 = -20$
$w_2 = \begin{pmatrix} -9 & -7 & -9 \end{pmatrix} \vec{x}_1 + \frac{144}{5} \rightarrow w_2 = 0$
$w_3 =$ (same) $w_3 = -20.95$, smallest, so let β_{3K} enter

23.3-3 cont.

$$f'_k = \begin{pmatrix} f_{3\,} x_{3k}^{*} \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4 & 6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 25/7 \\ 15/7 \end{pmatrix} \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 18.65 \\ 2.03 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f''_1 = \begin{pmatrix} 2 \\ 9/5 \\ 1 \\ 1 \end{pmatrix} \quad \text{m.v. ratio} = \frac{1}{1} \quad r=5 \text{ , so } f_{31} \text{ leaves}$$

let $x_B = \begin{pmatrix} x_{51} \\ x_{52} \\ f_{11} \\ f_{12} \\ f_{32} \end{pmatrix}$ and continue...

It's clear that x_{51} will leave next, being replaced by f_{12}. After that, you're on your own! (Try it and use the computer to do most of the work.)

23.4-1

Constraint	x_3	x_6	x_7	x_1	x_2	x_4	x_5	x_8	x_9	x_{10}
1	0	0	0	3	1					
2	-1	0	0	1	2					
3	0	0	0			1	5			
4	1	-1	-1			2	-1			
5	0	0	0			0	1			
6	1	1	1					1	3	2
7	0	0	0					2	-1	1

23.4-2 (a) Let x_{ij} denote the number of units of product i to be produced in year j for $i=1,2$ and $j=1,2,3$

Let y_{ij} denote the number of units of product i to be sold in year j for $i=1,2$ and $j=1,2,3$

Let z_{ijk} denote the number of units of product i to be produced and stored in year j and sold in year k for $i=1,2$, and $j=1,2,3$ and $k=j+1, j+2, \ldots 3$

(cont.)

23.4-2 (a) cont.

Maximize $3y_{11} + 5y_{21} + 4y_{12} + 4y_{22} + 5y_{13} + 8y_{23}$
$-2z_{112} - 2z_{212} - 4z_{113} - 4z_{213} - 2z_{123} - 2z_{223}$

subject to $x_{11} \le 4$

$2x_{21} \le 12$

$3x_{11} + 2x_{21} \le 18$

$x_{11} - y_{11} - z_{112} - z_{113} = 0$

$x_{21} - y_{21} - z_{212} - z_{213} = 0$

$x_{12} \le 6$

$2x_{22} \le 12$

$3x_{12} + 2x_{22} \le 24$

$z_{112} + x_{12} - y_{12} - z_{123} = 0$

$z_{112} - y_{12} \le 0$

$z_{212} + x_{22} - y_{22} - z_{223} = 0$

$z_{212} - y_{22} \le 0$

$x_{13} \le 3$

$2x_{23} \le 10$

$3x_{13} + 2x_{23} \le 15$

$z_{113} + z_{123} + x_{13} - y_{13} = 0$

$z_{213} + z_{223} + x_{23} - y_{23} = 0$

$x_{ij} \ge 0$, $y_{ij} \ge 0$, $z_{ijk} \ge 0$ for all i, j, k

(b) Table of Constraint Coefficients:

z_{112} z_{113} z_{212} z_{213} z_{123} z_{223}	x_{11} y_{11} x_{21} y_{21}	x_{12} y_{12} x_{22} y_{22}	x_{13} y_{13} x_{23} y_{23}
0 0 0 0 0 0	1 0 0 0		
0 0 0 0 0 0	0 0 2 0		
0 0 0 0 0 0	3 0 2 0		
-1 -1 0 0 0 0	1 -1 0 0		
0 0 -1 -1 0 0	0 0 1 -1		
0 0 0 0 0 0		1 0 0 0	
0 0 0 0 0 0		0 0 2 0	
0 0 0 0 0 0		3 0 2 0	
1 0 0 0 -1 0		1 -1 0 0	
1 0 0 0 0 0		0 -1 0 0	
0 0 1 0 0 -1		0 0 1 -1	
0 0 1 0 0 0		0 0 0 -1	
0 0 0 0 0 0			1 0 0 0
0 0 0 0 0 0			0 0 2 0
0 0 0 0 0 0			3 0 2 0
0 1 0 0 1 0			1 -1 0 0
0 0 0 1 0 1			0 0 1 -1

23.5-1

Constraint	X_2	X_8	X_1	X_4	X_3	X_7	X_5	X_9	X_{10}	X_6
3	-1	0	5	-1	2	-3	-1	0	4	0
7	1	1	2	3	0	0	0	-1	0	2
1	0	1	2	3						
6	0	0	1	1						
2	1	2			1	2				
8	-1	-1			2	1				
5	-1	-2					2	5	3	
9	0	0					1	2	1	
10	-1	0					4	1	5	
4	0	-1								1

23.5-2(a) Let types 1 and 2 denote raw lumber and plywood respectively.

Let x_{ij} denote the thousand board feet of type i to be purchased in season j for $i=1,2$ and $j=1,2,3,4$.

Let y_{ij} denote the thousand board feet of type i to be sold in season j for $i=1,2$ and $j=1,2,3,4$

Let z_{ijk} denote the thousand board feet of type i to be purchased and stored in season j and sold in season k for $i=1,2$ and $j=1,2,3,4$ and $k=j+1, j+2, \ldots, 4$.

Maximize $-410x_{11}+425y_{11}-17z_{112}-27z_{113}-37z_{114}$
$-660x_{21}+705y_{21}-24z_{212}-42z_{213}-60z_{214}$
$-430x_{12}+440y_{12}-17z_{123}-27z_{124}$
$-715x_{22}+730y_{22}-24z_{223}-42z_{224}$
$-460x_{13}+465y_{13}-17z_{134}$
$-760x_{23}+770y_{23}-24z_{234}$
$-450x_{14}+455y_{14}$
$-740x_{24}+750y_{24}$

subject to
$x_{11}-y_{11}-z_{112}-z_{113}-z_{114}=0$
$x_{21}-y_{21}-z_{212}-z_{213}-z_{214}=0$
$x_{11}+x_{21}\le 2000$
$y_{11}\le 1000$
$y_{21}\le 600$

(model cont. next page)

23.5-2(a) cont.

$$z_{112} + x_{12} - y_{12} - z_{123} - z_{124} = 0$$
$$z_{112} - y_{12} \le 0$$
$$z_{212} + x_{22} - y_{22} - z_{223} - z_{224} = 0$$
$$z_{212} - y_{22} \le 0$$
$$z_{112} + z_{113} + z_{114} + z_{212} + z_{213} + z_{214} + x_{12} + x_{22} \le 2000$$
$$y_{12} \le 1400$$
$$y_{22} \le 1200$$

$$z_{113} + z_{123} + x_{13} - y_{13} - z_{134} = 0$$
$$z_{113} + z_{123} - y_{13} \le 0$$
$$z_{213} + z_{223} + x_{23} - y_{23} - z_{234} = 0$$
$$z_{213} + z_{223} - y_{23} \le 0$$
$$z_{113} + z_{114} + z_{123} + z_{124} + z_{213} + z_{214} + z_{223} + z_{224} + x_{13} + x_{23} \le 2000$$
$$y_{13} \le 2000$$
$$y_{23} \le 1500$$

$$z_{114} + z_{124} + z_{134} + x_{14} - y_{14} = 0$$
$$z_{214} + z_{224} + z_{234} + x_{24} - y_{24} = 0$$
$$z_{114} + z_{124} + z_{134} + z_{214} + z_{224} + z_{234} + x_{14} + x_{24} \le 2000$$
$$y_{14} \le 1600$$
$$y_{24} \le 100$$

$x_{ij} \ge 0$, $y_{ij} \ge 0$, and $z_{ijk} \ge 0$ for all i, j, and k

b)

z_{112}	z_{113}	z_{114}	z_{212}	z_{213}	z_{214}	z_{123}	z_{124}	z_{223}	z_{224}	z_{134}	z_{234}	x_{11}	y_{11}	x_{21}	y_{21}	x_{12}	y_{12}	x_{22}	y_{22}	x_{13}	y_{13}	x_{23}	y_{23}	x_{14}	y_{14}	x_{24}	y_{24}	RHS
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	≤ 2000
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	≤ 2000
0	1	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	≤ 2000
0	0	1	0	0	1	0	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	≤ 2000
-1	-1	-1	0	0	0	0	0	0	0	0	0	1	-1															= 0
0	0	0	0	0	0	0	0	0	0	0	0	0	1															≤ 1000
0	0	0	-1	-1	-1	0	0	0	0	0	0			1	-1													= 0
0	0	0	0	0	0	0	0	0	0	0	0			0	1													≤ 800
1	0	0	0	0	0	-1	-1	0	0	0	0					1	-1											= 0
1	0	0	0	0	0	0	0	0	0	0	0					0	-1											≤ 0
0	0	0	0	0	0	0	0	0	0	0	0					0	1											≤ 1400
0	0	0	1	0	0	0	0	-1	-1	0	0							1	-1									= 0
0	0	0	1	0	0	0	0	0	0	0	0							0	-1									≤ 0
0	0	0	0	0	0	0	0	0	0	0	0							0	1									≤ 1200
0	1	0	0	0	0	1	0	0	0	-1	0									1	-1							= 0
0	1	0	0	0	0	1	0	0	0	0	0									0	-1							≤ 0
0	0	0	0	0	0	0	0	0	0	0	0									0	1							≤ 2000
0	0	0	0	1	0	0	0	0	1	0	0	-1										1	-1					= 0
0	0	0	0	1	0	0	0	0	1	0	0	0										0	-1					≤ 0
0	0	0	0	0	0	0	0	0	0	0	0	0										0	1					≤ 1500
0	0	1	0	0	0	0	1	0	0	1	0													1	-1			= 0
0	0	0	0	0	0	0	0	0	0	0	0													0	1			≤ 1600
0	0	0	0	0	1	0	0	0	1	0	1															1	-1	= 0
0	0	0	0	0	0	0	0	0	0	0	0															0	1	≤ 100

23.6-1

(a) Max $20x_1 + 30x_2 + 25x_3$

Subject to
$$3x_1 + 2x_2 + x_3 \le 29$$
$$2x_1 + 4x_2 + 2x_3 \le 48$$
$$x_1 + 3x_2 + 5x_3 \le 57$$

and $x_i \ge 0$

(b) Let x_{21}, x_{22}, x_{23} be the values of x_2 when 29, 30, 31 are observed, respectively, for b_1, and $x_{31}, x_{32}, \ldots, x_{39}$ be the values of x_3 when the values for $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ respectively are

$$\begin{pmatrix} 29 \\ 48 \end{pmatrix}, \begin{pmatrix} 29 \\ 50 \end{pmatrix}, \begin{pmatrix} 29 \\ 52 \end{pmatrix}, \begin{pmatrix} 30 \\ 48 \end{pmatrix}, \begin{pmatrix} 30 \\ 50 \end{pmatrix}, \begin{pmatrix} 30 \\ 52 \end{pmatrix}, \begin{pmatrix} 31 \\ 48 \end{pmatrix}, \begin{pmatrix} 31 \\ 50 \end{pmatrix}, \begin{pmatrix} 31 \\ 52 \end{pmatrix}$$

$$\begin{aligned} \text{Max } 20x_1 &+ \left(\tfrac{1}{4}\right)\left[30x_{21} + 25\left[\left(\tfrac{1}{4}\right)x_{31} + \left(\tfrac{1}{2}\right)x_{32} + \left(\tfrac{1}{4}\right)x_{33}\right]\right] \\ &+ \left(\tfrac{1}{2}\right)\left[30x_{22} + 25\left[\left(\tfrac{1}{4}\right)x_{34} + \left(\tfrac{1}{2}\right)x_{35} + \left(\tfrac{1}{4}\right)x_{36}\right]\right] \\ &+ \left(\tfrac{1}{4}\right)\left[30x_{23} + 25\left[\left(\tfrac{1}{4}\right)x_{37} + \left(\tfrac{1}{2}\right)x_{38} + \left(\tfrac{1}{4}\right)x_{39}\right]\right] \end{aligned}$$

Subject to

$$3x_1 + 2x_{21} + x_{31} \le 29$$
$$3x_1 + 2x_{21} + x_{32} \le 29$$
$$3x_1 + 2x_{21} + x_{33} \le 29$$
$$3x_1 + 2x_{22} + x_{34} \le 30$$
$$3x_1 + 2x_{22} + x_{35} \le 30$$
$$3x_1 + 2x_{22} + x_{36} \le 30$$
$$3x_1 + 2x_{23} + x_{37} \le 31$$
$$3x_1 + 2x_{23} + x_{38} \le 31$$
$$3x_1 + 2x_{23} + x_{39} \le 31$$

$$2x_1 + 4x_{21} + 2x_{31} \le 48$$
$$2x_1 + 4x_{21} + 2x_{32} \le 50$$
$$2x_1 + 4x_{21} + 2x_{33} \le 52$$
$$2x_1 + 4x_{22} + 2x_{34} \le 48$$
$$2x_1 + 4x_{22} + 2x_{35} \le 50$$
$$2x_1 + 4x_{22} + 2x_{36} \le 52$$
$$2x_1 + 4x_{23} + 2x_{37} \le 48$$
$$2x_1 + 4x_{23} + 2x_{38} \le 50$$
$$2x_1 + 4x_{23} + 2x_{39} \le 52$$

$$x_1 + 3x_{21} + 5x_{31} \le 57$$
$$x_1 + 3x_{21} + 5x_{32} \le 57$$
$$x_1 + 3x_{21} + 5x_{33} \le 57$$
$$x_1 + 3x_{22} + 5x_{34} \le 57$$
$$x_1 + 3x_{22} + 5x_{35} \le 57$$
$$x_1 + 3x_{22} + 5x_{36} \le 57$$
$$x_1 + 3x_{23} + 5x_{37} \le 57$$
$$x_1 + 3x_{23} + 5x_{38} \le 57$$
$$x_1 + 3x_{23} + 5x_{39} \le 57$$

and

$$x_1 \ge 0,\ x_{2i} \ge 0,\ x_{3j} \ge 0 \quad \text{for } i = 1,2,3;\ j = 1, \ldots, 9.$$

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23.7-1 (a) $P\{7+(\frac{44}{3})+\frac{19}{3} \le b_1\} = P\{-2 \le Z\} = 0.9772$

$P\{\frac{14}{3}+\frac{88}{3}+\frac{32}{3} \le b_2\} = P\{-\frac{5}{3} \le Z\} = 0.9515$

$P\{\frac{2}{3}+22+\frac{95}{3} \le b_3\} = P\{-\frac{4}{3} \le Z\} = 0.9082$

$P\{$all constraints satisfied$\} = 0.8450$

The solution is feasible.

(b) Max $20x_1 + 30x_2 + 25x_3$

Subject to:

$3x_1 + 2x_2 + x_3 \le 28.04$

$2x_1 + 4x_2 + 2x_3 \le 46.71$

$x_1 + 3x_2 + 5x_3 \le 56.16$

$x_1, x_2, x_3 \ge 0$

(c) Max $20y_1 - 30y_2 - 25y_3$

subject to: $3y_1 - 2y_2 - y_3 \le -26.96$

$2y_1 - 4y_2 - 2y_3 \le -63.29$

$y_1 - 3y_2 - 5y_3 \le -90.15$

$y_1, y_2, y_3 \ge 0$

Since:

$\mu_1 = 30 - 2[(\frac{1}{4})30] - 1[(\frac{1}{2})(30) + (\frac{1}{2})(50)] = -25$

$\mu_2 = 50 - 4[(\frac{1}{4})30] - 2[(\frac{1}{2})(30) + (\frac{1}{2})(50)] = -60$

$\mu_3 = 60 - 3[(\frac{1}{4})30] - 5[(\frac{1}{2})(30) + (\frac{1}{2})(50)] = -\frac{330}{4}$

$\sigma_1^2 = [\frac{1}{2}]^2(4) + [1 - \frac{1}{2} - \frac{1}{2}]^2 \cdot 1 = 1$

$\sigma_2^2 = [4(\frac{1}{4}) + 2(\frac{1}{2})]^2(1) + [1 - 2(\frac{1}{2})]^2(4) = 4$

$\sigma_3^2 = [3(\frac{1}{4}) + 5(\frac{1}{2})]^2(1) + (5(\frac{1}{2}))^2(4) = \frac{569}{16}$

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23.7-2 (a) $b_i - \sum_{j=1}^{n} a_{ij} x_j$ is distributed as a normal random variable with mean $E(b_i) - \sum_{j=1}^{n} x_j E(a_{ij})$

and variance $\sigma^2(b_i) + \sum_{j=1}^{n} x_j^2 \sigma^2(a_{ij})$

Hence, $P\{0 \le b_i - \sum_{j=1}^{n} a_{ij} x_j\} = P\left\{\frac{-E(b_i) + \sum_{j=1}^{n} x_j E(a_{ij})}{(\sigma^2(b_i) + \sum_{j=1}^{n} x_j^2 \sigma^2(a_{ij}))^{1/2}} \le z\right\} \ge \alpha_i$

if and only if

$$-E(b_i) + \sum_{j=1}^{n} x_j E(a_{ij}) \le K_{\alpha_i} (\sigma^2(b_i) + \sum_{j=1}^{n} x_j^2 \sigma^2(a_{ij}))^{1/2}.$$

(b) $b_i - \sum_{j=1}^{n} a_{ij} x_j = b_i - \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} b_k d_{jk}$

is normal with mean $E(b_i) - \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} d_{jk} E(b_k)$

and variance $\sigma^2(b_i) + \sum_{j=1}^{n} a_{ij}^2 \sum_{k=1}^{m} d_{jk}^2 \sigma^2(b_k)$

Hence,

$$P\left\{0 \le b_i - \sum_{j=1}^{n} a_{ij} x_j\right\} = P\left\{\frac{-E(b_i) + \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} d_{jk} E(b_k)}{(\sigma^2(b_i) + \sum_{j=1}^{n} a_{ij}^2 \sum_{k=1}^{m} d_{jk}^2 \sigma^2(b_k))^{1/2}} \le z\right\} \ge \alpha_i$$

if and only if

$$-E(b_i) + \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} d_{jk} E(b_k) \le$$

$$K_{\alpha_i} (\sigma^2(b_i) + \sum_{j=1}^{n} a_{ij}^2 \sum_{k=1}^{m} d_{jk}^2 \sigma^2(b_k))^{1/2}$$

Problem Solutions for Chapter 24

(Appears on the book's Web site, www.mhhe.com/hillier)

24-1. (a) the six colored sides: red, white, blue, green, yellow and violet.

(b) $\Pr\{X=0\} = \Pr\{X=1\} = \Pr\{X=2\} = 1/3$

(c) $E[Y] = E[(X+1)^2] = \sum_{k=0}^{\infty} (k+1)^2 \Pr\{X=k\} = 4\ 2/3$.

24-2. (a) $\Pr\{X_1=1\} = \Pr\{\omega_1 \cup \omega_2\} = \Pr\{\omega_1\} + \Pr\{\omega_2\} = 1/3 + 1/5 = 8/15$

$\Pr\{X_1=4\} = \Pr\{\omega_3\} = 3/10$

$\Pr\{X_1=5\} = \Pr\{\omega_4\} = 1/6$

$\Pr\{X_1=k\} = 0$ if $k \notin \{1,4,5\}$

(b) $E[X_1] = 1 \cdot 8/15 + 4 \cdot 3/10 + 5 \cdot 1/6 = 2\ 17/30$

(c) $\Pr\{X_1+X_2=2\} = \Pr\{\omega_1 \cup \omega_2\} = \Pr\{\omega_1\} + \Pr\{\omega_2\} = 1/3 + 1/5 = 8/15$

$\Pr\{X_1+X_2=5\} = \Pr\{\omega_3\} = 3/10$

$\Pr\{X_1+X_2=10\} = \Pr\{\omega_4\} = 1/6$

$\Pr\{X_1+X_2=k\} = 0$ if $k \notin \{2,5,10\}$

(d) $E[(X_1+X_2)] = 2 \cdot 8/15 + 5 \cdot 3/10 + 10 \cdot 1/6 = 4\ 7/30$

$E[X_2] = 1 \cdot 5/6 + 5 \cdot 1/6 = 1\ 2/3$

(e)
$$F_{X_1X_2}(b_1,b_2) = \begin{cases} 0 & \text{for } b_1<1 \text{ or } b_2<1 \\ 8/15 & \text{for } 1 \le b_1 < 4 \text{ and } 1 \le b_2 < \infty \\ 5/6 & \text{for } 4 \le b_1 < 5 \text{ and } 1 \le b_2 < \infty \\ 5/6 & \text{for } 4 \le b_1 < \infty \text{ and } 1 \le b_2 < 5 \\ 1 & \text{for } 5 \le b_1 \text{ and } 5 \le b_2 \end{cases}$$

(f) $$\rho = \frac{\operatorname{cov}[X_1,X_2]}{\sqrt{\operatorname{Var}[X_1]\ \operatorname{Var}[X_2]}} = \frac{E[(X_1-E[X_1])(X_2-E[X_2])]}{\sqrt{E[(X_1-E[X_1])^2]\,E[(X_2-E[X_2])^2]}}$$

$$= \frac{E[X_1X_2]-E[X_1]E[X_2]}{\sqrt{(E[X_1^2]-(E[X_1])^2)(E[X_2^2]-(E[X_2])^2)}}$$

Now $E[X_1] = 77/30$, $E[X_1^2] = 285/30$, $E[X_2] = 50/30$, $E[X_2^2] = 150/30$

and $E[X_1X_2] = 177/30$, and hence $\rho = .64$

(g) $E[2X_1-3X_2] = 2E[X_1] - 3E[X_2] = 2 \cdot 77/30 - 3 \cdot 50/30 = 2/15$

24-3.

	(a)	(b)	(c)
GG	4	1/4
GM	3	1/6
GB	2	1/12
MG	3	1/6
MM	2	1/9
MB	1	1/18
BG	2	1/12
BM	1	1/18
BB	0	1/36

(d) $X \in \{0,1,2,3,4\}$

$\Pr\{X=0\} = 1/36$, $\Pr\{X=1\} = 1/18 + 1/18 = 1/9$, $\Pr\{X=2\} = 1/12 + 1/9 + 1/12 = 5/18$,

$\Pr\{X=3\} = 1/6 + 1/6 = 1/3$, $\Pr\{X=4\} = 1/4$, $\Pr\{X=k\} = 0$ when $k \notin \{0,1,2,3,4\}$

(e) $E[X] = 0 \cdot 1/36 + 1 \cdot 1/9 + 2 \cdot 5/18 + 3 \cdot 1/3 + 4 \cdot 1/4 = 2\ 2/3$

24-4. (a)

$$1 = \int_0^1 f_X(y)\,dy = \int_0^{\theta} \theta\,dy + \int_{\theta}^{1} K\,dy = \theta y\Big|_0^{\theta} + Ky\Big|_{\theta}^{1} = \theta^2 + K - K\theta$$

So $K = (1-\theta^2)/(1-\theta) = 1+\theta$

(b) For $b<0$, $F_x(b)=0$

For $0 \le b < \theta$, $F_x(b) = \int_0^b \theta\, dy = \theta b$

For $\theta \le b < 1$, $F_x(b) = \theta^2 + \int_\theta^b (1+\theta)\,dy = \theta^2 + (1+\theta)y\Big|_\theta^b$

$= \theta^2 + (1+\theta)b - (1+\theta)\theta = b + \theta b - \theta$

For $1 \le b$, $F_x(b) = 1$

(c) $E[x] = \int_0^\theta y\theta\,dy + \int_\theta^1 y(1+\theta)\,dy = \frac{\theta y^2}{2}\Big|_0^\theta + \frac{(1+\theta)y^2}{2}\Big|_\theta^1 = \frac{(1+\theta-\theta^2)}{2}$

(d) Assume, for example, that $0 \le a \le 1/3$.

$Pr\{X - 1/3 < a\} = Pr\{X < a + 1/3\} = F_x(a+1/3) = a + 1/3 + 1/3(a+1/3) - 1/3$

$= 4/3\, a + 1/9$

$Pr\{-(X - 1/3) < a\} = Pr\{X > 1/3 - a\} = 1 - Pr\{x \le 1/3 - a\} = 1 - 1/3(1/3 - a)$

$= 8/9 + a/3$

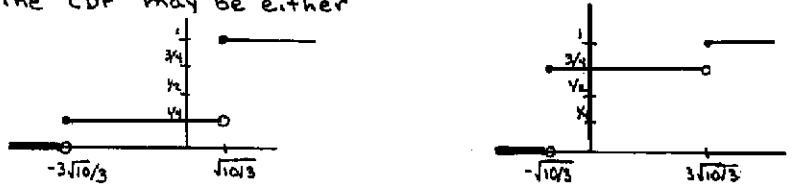
So the equality does not hold.

24-5. (a) $0 = E[x] = x_1 \cdot 1/4 + x_2 \cdot 3/4 \Rightarrow x_1 = -3x_2$

$10 = var[x] = E[X^2] - (E[x])^2 = E[X^2] = x_1^2 \cdot 1/4 + x_2^2 \cdot 3/4$

so $10 = \frac{9x_2^2}{4} + \frac{3x_2^2}{4} = 3x_2^2 \Rightarrow \begin{cases} x_2 = \pm\sqrt{10/3} \\ x_1 = \mp 3\sqrt{10/3} \end{cases}$

(b) The CDF may be either



24-6. (a) $Pr\{\text{survival of} \ge 250 \text{ hours}\} = 1 - Pr\{\text{failure} < 250 \text{ hours}\}$

$= 1 - \int_0^{250} f_x(y)\,dy = 1 - \int_{100}^{250} \frac{100}{y^2}\,dy = 1 - \left(\frac{-100}{y}\right)_{100}^{250} = 1 + \frac{2}{5} - 1 = \frac{2}{5}$

(b) $E[x] = \int_0^\infty y\, f_x(y)\,dy = \int_{100}^\infty \frac{100}{y}\,dy = 100(\ln\infty - \ln 100) = \infty$

24-7. (a) $.4 = Pr\{-1 < x < 2\} = Pr\{x=0\} + Pr\{x=1\} = .3 + Pr\{x=1\} \Rightarrow .1 = Pr\{x=1\}$

$.6 = Pr\{|x| \le 1\} = Pr\{x=-1\} + Pr\{x=0\} + Pr\{x=1\} = Pr\{x=-1\} + .3 + .1 \Rightarrow .2 = Pr\{x=-1\}$

$Pr\{x \ge 2\} = Pr\{x=2\} = Pr\{x=1 \text{ or } -1\} = Pr\{x=1\} + Pr\{x=-1\} = .1 + .2 \Rightarrow .3 = Pr\{x=2\}$

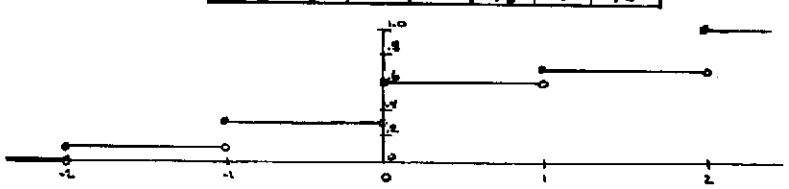
$1 = Pr\{x=-2\} + Pr\{x=-1\} + Pr\{x=0\} + Pr\{x=1\} + Pr\{x=2\}$

$= Pr\{x=-2\} + .2 + .3 + .1 + .3 \Rightarrow .1 = Pr\{x=-2\}$

In summary,

k	-2	-1	0	1	2
$Pr\{x=k\}$	.1	.2	.3	.1	.3

(b)



(c) $E[X] = (-2)\cdot.1 + (-1)\cdot.2 + 0\cdot.3 + 1\cdot.1 + 2\cdot.3 = .3$

24-8. (a) $1 = \int_{-1}^{1} K(1-y^2)\,dy = K(y - y^3/3)\Big|_{-1}^{1} = \frac{4K}{3} \Rightarrow K = 3/4$

(b) For $b < -1$, $F_x(b) = 0$

For $-1 \le b < 1$, $F_x(b) = \int_{-1}^{b} f_x(y)\,dy = \int_{-1}^{b} \frac{3}{4}(1-y^2)\,dy$

$= \frac{3}{4}(y - y^3/3)\Big|_{-1}^{b} = \frac{3}{4}\left(b - \frac{b^3}{3} + 1 - \frac{1}{3}\right) = \frac{3}{4}(b+1) - \frac{1}{4}(b^3+1)$

For $1 \le b$, $F_x(b) = 1$

(c) $E[2X-1] = 2E[X] - 1 = 2\int_{-1}^{1} y\,\frac{3}{4}(1-y^2)\,dy - 1 = 2\cdot\frac{3}{4}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Big|_{-1}^{1} - 1$

$= 0 - 1 = -1$. Note that $E[X] = 0$

(d) $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \int_{-1}^{1} y^2\,\frac{3}{4}(1-y^2)\,dy = \frac{3}{4}\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_{-1}^{1}$

$= 3/4\,(2/15 - (-2/15)) = 1/5$

(e) $\bar{X}$ is approximately normal with mean $E[X]$ and variance $\mathrm{Var}[X]/n$

Therefore, $\dfrac{\bar{X} - E[X]}{\sqrt{\mathrm{Var}[X]/n}} \sim N(0,1)$

and $\Pr\{\bar{X} > 0\} = \Pr\left\{\dfrac{\bar{X} - E[X]}{\sqrt{\mathrm{Var}[X]/n}} > \dfrac{0 - E[X]}{\sqrt{\mathrm{Var}[X]/n}}\right\} = \Pr\{N(0,1) > 0\} = 1/2$

24-9. (a) $1 = \int_0^{1000} \frac{a}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{a}{1000}\left(y - \frac{y^2}{2000}\right)\Big|_0^{1000} = \frac{a}{2} \Rightarrow a = 2$

(b) $E[X] = \int_0^{1000} \frac{y\cdot 2}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{1}{500}\left(\frac{y^2}{2} - \frac{y^3}{3000}\right)\Big|_0^{1000} = \frac{1}{500}\left(500{,}000 - \frac{10^6}{3}\right) = 333\ 1/3$

(c) For $b<0$, $F_x(b) = 0$

For $0 \le b < 1000$, $F_x(b) = \int_0^b \frac{2}{1000}\left(1 - \frac{y}{1000}\right)dy = \frac{1}{500}\left(y - \frac{y^2}{2000}\right)\Big|_0^b = \frac{b}{500} - \frac{b^2}{1{,}000{,}000}$

For $1000 \le b$, $F_x(b) = 1$

(d) Since $F_z(b) = F_x(b/3)$, for $b<0$, $F_z(b) = 0$

for $0 \le b < 3000$, $F_z(b) = b/1500 - b^2/9{,}000{,}000$

for $3000 \le b$, $F_z(b) = 1$

24-10. (a) $\Pr\{X \ge 25\} = 1 - \Pr\{X \le 24\} = 1 - .473 = .527$

$\Pr\{X = 20\} = \Pr\{X \le 20\} - \Pr\{X \le 19\} = .185 - .134 = .051$

(b) $\Pr\{\text{shortage}\} = \Pr\{X > 35\} = 1 - \Pr\{X \le 35\} = 1 - .978 = .022$

24-11. (a) $E[X] = \sum_{n=1}^{\infty} 2^n\,(1/2)^n = \sum_{n=1}^{\infty} 1 = \infty$. Thus the game can never be fair.

(b) The variance is undefined, since the mean is infinite.

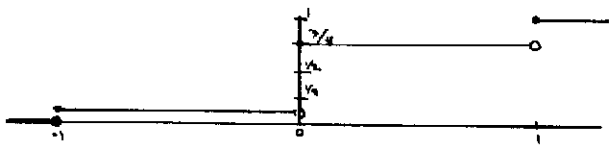
(c) $\Pr\{X \le 8\} = \Pr\{X=2\} + \Pr\{X=4\} + \Pr\{X=8\} = 1/2 + 1/4 + 1/8 = 7/8$

24-12. (a) $1 = \Pr\{D=-1\} + \Pr\{D=0\} + \Pr\{D=1\} = 1/8 + 5/8 + c/8 = 6/8 + c/8 \Rightarrow c = 2$

$E[D] = (-1)\cdot 1/8 + 0\cdot 5/8 + 1\cdot 2/8 = 1/8$

$\mathrm{Var}[D] = E[D^2] - (E[D])^2 = 1\cdot 1/8 + 0\cdot 5/8 + 1\cdot 2/8 - (1/8)^2 = 3/8 - 1/64 = 23/64$

(b) $E[e^D] = e\cdot 1/8 + 1\cdot 5/8 + e\cdot 2/8 = \frac{1}{8}(5 + 3e)$



24-13. (a) Let X_i denote the volume of bottle i, for $i = 1, \ldots, 3$.

Let $Z = X_1 + X_2 + X_3$.

Then $E[Z] = E[X_1 + X_2 + X_3] = E[X_1] + E[X_2] + E[X_3] = 3\cdot 15 = 45$

and $\mathrm{Var}[Z] = \mathrm{Var}[X_1 + X_2 + X_3] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \mathrm{Var}[X_3] = 3\cdot(.08)^2 = .0192$

So $\sigma_Z = \sqrt{\mathrm{Var}[Z]} = .139$

(b) $Z \sim N(45, .0192)$, so

$$\Pr\{Z \ge 45.2\} = \Pr\left\{\frac{Z - 45}{.139} \ge \frac{45.2 - 45}{.139}\right\}$$

$$= \Pr\{N(0,1) \ge 1.44\} = .075$$

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24-14. (a) For $b<0$, $F_x(b)=0$

For $0\le b<1$, $F_x(b)=\int_0^b 6y(1-y)\,dy = 6\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big|_0^b = 3b^2-2b^3$

For $1\le b$, $F_x(b)=1$

(b) $E[X]=\int_0^1 y\,6y(1-y)\,dy = 6\left(\frac{y^3}{3}-\frac{y^4}{4}\right)\Big|_0^1 = \frac{1}{2}$

$\mathrm{Var}[X]=E[X^2]-(E[X])^2=\int_0^1 y^2\,6y(1-y)\,dy-\frac{1}{4}=6\left(\frac{y^4}{4}-\frac{y^5}{5}\right)\Big|_0^1-\frac{1}{4}=\frac{1}{20}$

(c) $\Pr\{X>.5\}=1-\Pr\{X\le.5\}=1-[3(1/2)^2-2(1/2)^3]=1/2$

(d) $E\left[\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}\right]=\frac{1}{6}\,6\,E[X_1]=\frac{1}{2}$

(e) $\mathrm{Var}\left[\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}\right]=\frac{1}{36}\,6\,\mathrm{Var}[X_1]=\frac{1}{120}$

24-15. (a) Let X_1 be the voltage of battery 1 and X_2 the voltage of battery 2. Then $X_1\sim N(1\frac{1}{2},.0625)$ and $X_2\sim N(1\frac{1}{2},.0625)$. Let $Z=X_1+X_2$. Then $Z\sim N(3,.1250)$. Therefore,

$\Pr\{\text{failure}\}=\Pr\{Z<2.75\}+\Pr\{Z>3.25\}$

$=2\Pr\{Z>3.25\}$ by symmetry

$=2\Pr\{N(0,1)>(3.25-3)/\sqrt{.1250}\}$

$=2\Pr\{N(0,1)>.707\}$

$=.48$

(b) Chebyshev's Inequality states $\Pr\{|X-\mu|\ge k\sigma\}\le 1/k^2$. We are interested in the case where $|X-\mu|\ge.25$. Since $\sigma\cong.354$, it follows that $k=.25/.354=.707$. But because $1/k^2>1$, the inequality does not give a useful bound.

24-16. $\Pr\{(1000)(1/5000)|\bar{X}-\mu|\le 15\}=.90 \Leftrightarrow \Pr\{|\bar{X}-\mu|\le 75\}=.90$

$\Leftrightarrow \Pr\{|\bar{X}-\mu|>75\}=.10$

$\Leftrightarrow \Pr\{\bar{X}-\mu>75\}=.05$

$\Leftrightarrow \Pr\{(\bar{X}-\mu)/\sigma_{\bar{X}}>75/\sigma_{\bar{X}}\}=.05$

$\Leftrightarrow \Pr\{N(0,1)>75/\sigma_{\bar{X}}\}=.05$

$\Leftrightarrow 75/\sigma_{\bar{X}}=1.645$

$\Leftrightarrow \sigma_{\bar{X}}=45.6$ or $\sigma_{\bar{X}}^2=2079$

Now since $\sigma_{\bar{X}}^2=\sigma_X^2/n$, we have $2079=40000/n \Rightarrow n=19.24$

Hence, choosing $n\ge 20$ would be sufficient.

24-17. (a) $f_{X_1}(s)=\int_{-\infty}^{\infty} f_{X_1,X_2}(s,t)\,dt$

Let $u=\frac{s-\mu_{X_1}}{\sigma_{X_1}}$ and $v=\frac{t-\mu_{X_2}}{\sigma_{X_2}}$ so that $dt=\sigma_{X_2}\,dv$

Then $f_{X_1}(s)=\frac{1}{2\pi\sigma_{X_1}\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(u^2-2\rho uv+v^2)\right\}dv$

$=\frac{1}{2\pi\sigma_{X_1}\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(v^2-2\rho uv+\rho^2u^2-\rho^2u^2+u^2)\right\}dv$

Now let $z=\frac{v-\rho u}{\sqrt{1-\rho^2}}$ so that $dv=\sqrt{1-\rho^2}\,dz$

Then $f_{X_1}(s)=\frac{\exp(-u^2/2)}{2\pi\sigma_{X_1}}\int_{-\infty}^{\infty}\exp(-z^2/2)\,dz$

But $\int_{-\infty}^{\infty}\exp(-z^2/2)\,dz=\sqrt{2\pi}$

So $f_{X_1}(s)=\frac{1}{\sqrt{2\pi}\,\sigma_{X_1}}e^{-u^2/2}=\frac{1}{\sqrt{2\pi}\,\sigma_{X_1}}\exp\left[-\frac{1}{2}\left(\frac{s-\mu_{X_1}}{\sigma_{X_1}}\right)^2\right]$

Hence, $X_1\sim N(\mu_{X_1},\sigma_{X_1}^2)$.

By symmetry, $X_2\sim N(\mu_{X_2},\sigma_{X_2}^2)$

(b) $\text{Corr} = \dfrac{E[(X_1 - E[X_1])(X_2 - E[X_2])]}{\sigma_{X_1}\sigma_{X_2}}$

$$= \frac{1}{\sigma_{X_1}\sigma_{X_2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (s-\mu_{X_1})(t-\mu_{X_2})\, f_{X_1,X_2}(s,t)\, ds\,dt$$

Let $u = \dfrac{s-\mu_{X_1}}{\sigma_{X_1}}$ and $v = \dfrac{t-\mu_{X_2}}{\sigma_{X_2}}$

Then $\text{Corr} = \dfrac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv \exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)\left(u^2 - 2\rho uv + v^2\right)\right\} du\,dv$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} du\, u e^{-u^2/2} \int_{-\infty}^{\infty} dv\, v \exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)(v-\rho u)^2\right\}$$

Now let $z = \dfrac{v-\rho u}{\sqrt{1-\rho^2}}$

Then $\text{Corr} = \dfrac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} du\, u e^{-u^2/2} \left[0 + \rho u\sqrt{1-\rho^2}\sqrt{2\pi}\right]$

$$= \frac{\rho}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, u^2 e^{-u^2/2}$$

$$= \rho$$

(c) see part (a)

(d) Let $u = \dfrac{X_1 - \mu_{X_1}}{\sigma_{X_1}}$ and $v = \dfrac{X_2 - \mu_{X_2}}{\sigma_{X_2}}$

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}$$

$$= \frac{\left(\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{(1-\rho^2)}}\right) \exp\left\{\left(\frac{-1}{2(1-\rho^2)}\right)\left(u^2 - 2\rho uv + v^2\right)\right\}}{\left(\frac{1}{\sqrt{2\pi}\,\sigma_{X_2}}\right) e^{-v^2/2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{X_1}\sqrt{1-\rho^2}} \exp\left\{(-1/2)\left[\frac{x_2 - \mu_{X_1} - \rho\frac{\sigma_{X_1}}{\sigma_{X_2}}(x_2-\mu_{X_2})}{\sigma_{X_1}\sqrt{1-\rho^2}}\right]^2\right\}$$

24- 18. (a) $1 = \int_{100}^{150}\int_{50}^{100} c\, ds\,dt = 2500c \Rightarrow c = \dfrac{1}{2500}$

(b)

$$F_{X_1,X_2}(b_1,b_2) = \begin{cases} 0 & \text{for } b_1 < 100 \text{ or } b_2 < 50 \\ \int_{100}^{b_1}\int_{50}^{b_2} \frac{1}{2500}\, ds\,dt = \frac{1}{2500}(b_1-100)(b_2-50) & \text{for } 100 \le b_1 < 150 \text{ and } 50 \le b_2 < 100 \\ \int_{100}^{150}\int_{50}^{b_2} \frac{1}{2500}\, ds\,dt = \frac{50}{2500}(b_2-50) = \frac{(b_2-50)}{50} & \text{for } 150 \le b_1 \text{ and } 50 \le b_2 < 100 \\ \int_{100}^{b_1}\int_{50}^{100} \frac{1}{2500}\, ds\,dt = \frac{50}{2500}(b_1-100) = \frac{(b_1-100)}{50} & \text{for } 100 \le b_1 < 150 \text{ and } 100 \le b_2 \\ 1 & \text{for } 150 \le b_1 \text{ and } 100 \le b_2 \end{cases}$$

$$F_{X_1}(b_1) = \begin{cases} 0 & \text{for } b_1 < 100 \\ \int_{100}^{b_1}\int_{50}^{100} \frac{ds\,dt}{2500} = \frac{50}{2500}(b_1-100) = \frac{(b_1-100)}{50} & \text{for } 100 \le b_1 < 150 \\ 1 & \text{for } 150 \le b_1 \end{cases}$$

$$F_{X_2}(b_2) = \begin{cases} 0 & \text{for } b_2 < 50 \\ \int_{100}^{150}\int_{50}^{b_2} \frac{ds\,dt}{2500} = \frac{50}{2500}(b_2-50) = \frac{(b_2-50)}{50} & \text{for } 50 \le b_2 < 100 \\ 1 & \text{for } 100 \le b_2 \end{cases}$$

(c) Observe $f_{X_1}(s) = 1/50$, $100 \le s < 150$. Hence,

$$f_{X_2|X_1=s}(t) = \frac{f_{X_1,X_2}(s,t)}{f_{X_1}(s)} = \frac{(1/2500)}{(1/50)} = \frac{1}{50}$$

when $100 \le s < 150$ and $50 \le t < 100$

$f_{X_2|X_1=s}(t) = 0$ for other values of t

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24-19. (a) $P_{X_1}(0) = \sum_{k=0}^{3} P_{X_1,X_2}(0,k) = \frac{1}{2}$

$P_{X_1}(1) = 1 - P_{X_1}(0) = 1/2$

$P_{X_2}(0) = \sum_{k=0}^{1} P_{X_1,X_2}(k,0) = \frac{1}{8}$

$P_{X_2}(1) = \sum_{k=0}^{1} P_{X_1,X_2}(k,1) = \frac{3}{8}$

$P_{X_2}(2) = 1 - P_{X_2}(0) - P_{X_2}(1) = 1/2$

(b) $P_{X_1|X_2=1}(0) = \frac{P_{X_1,X_2}(0,1)}{P_{X_2}(1)} = \frac{(1/4)}{(3/8)} = \frac{2}{3}$

$P_{X_1|X_2=1}(1) = \frac{P_{X_1,X_2}(1,1)}{P_{X_2}(1)} = \frac{(1/8)}{(3/8)} = \frac{1}{3}$

(c) No! For example, from (a) and (b)

$$P_{X_1|X_2=1}(0) = \frac{2}{3} \neq \frac{1}{2} = P_{X_1}(0)$$

(d) $E[X_1] = 1/2$ and $\mathrm{Var}[X_1] = 1/4$

$E[X_2] = 11/8$ and $\mathrm{Var}[X_2] = 31/64$

(e) $P_{X_1+X_2}(0) = 1/8$

$P_{X_1+X_2}(1) = 1/4 + 0 = 1/4$

$P_{X_1+X_2}(2) = 1/8 + 1/8 = 1/4$

$P_{X_1+X_2}(3) = 3/8$

24-20. (a) $\Pr\{F\} = \Pr\{F \cap \Omega\}$

$= \Pr\{F \cap (E_1 \cup E_2 \cup \ldots \cup E_m)\}$

$= \Pr\{\bigcup_i (F \cap E_i)\}$

$= \sum_{i=1}^{\infty} \Pr\{F \cap E_i\}$ since $\Pr\{E_i \cap E_j\} = 0$ for $i \neq j$

$= \sum_{i=1}^{\infty} \Pr\{F|E_i\}\Pr\{E_i\}$ since $\Pr\{F|E_i\} = \frac{\Pr\{F \cap E_i\}}{\Pr\{E_i\}}$

(b) $\Pr\{E_i|F\} = \frac{\Pr\{E_i \cap F\}}{\Pr\{F\}}$

$= \frac{\Pr\{E_i \cap F\}}{\sum_{i=1}^{\infty} \Pr\{F|E_i\}\Pr\{E_i\}}$ by part (a)

$= \frac{\Pr\{F|E_i\}\Pr\{E_i\}}{\sum_{i=1}^{\infty} \Pr\{F|E_i\}\Pr\{E_i\}}$

Problem Solutions for Chapter 25

(Appears on the book's Web site, www.mhhe.com/hillier)

25.1-1. The minimal paths for the system are x_1x_2 and x_1x_3.
Hence $\phi(x_1,x_2,x_3) = \max[x_1x_2, x_1x_3] = x_1 \max[x_2,x_3] =$
$= x_1[1-(1-x_2)(1-x_3)]$

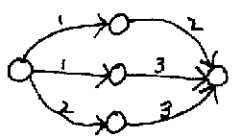
25.1-2. The minimal paths for the system are $x_1x_2x_3$ and $x_1x_2x_4$.
Hence $\phi(x_1,x_2,x_3,x_4) = \max[x_1x_2x_3, x_1x_2x_4] = x_1x_2 \max[x_3,x_4] =$
$= x_1x_2[1-(1-x_3)(1-x_4)]$

25.2-1. Note: we assume throughout this chapter that the component reliabilities are independent.
$R(p_1,p_2,p_3) = E[\phi(x_1,x_2,x_3)] = p_1[1-(1-p_2)(1-p_3)]$

25.2-2. $R(p_1,p_2,p_3,p_4) = E[\phi(x_1,x_2,x_3,x_4)] = p_1p_2[1-(1-p_3)(1-p_4)]$

25.3-1. (a) yes, $k=2$, $n=3$

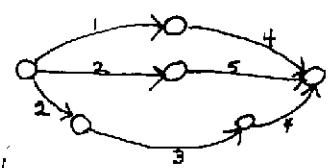
(b)



(c) $\phi(x_1,x_2,x_3) = 1-(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)$
$= x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 - x_1x_2 - x_1x_3 - x_2x_3 - x_1^2x_2^2x_3^2$

(d) $R(p_1,p_2,p_3) = 1-(1-p_1p_2)(1-p_1p_3)(1-p_2p_3)$

25.3-2. (a)

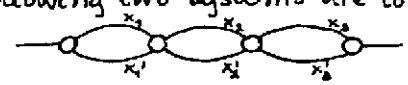


(b) $\phi(x_1,x_2,x_3,x_4,x_5) = 1-(1-x_1x_4)(1-x_2x_5)(1-x_2x_3x_4)$

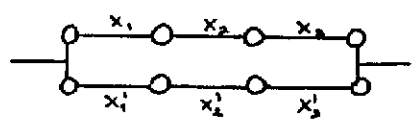
(c) $R(t) = 1-(1-R_1(t)R_4(t))(1-R_2(t)R_5(t))(1-R_2(t)R_3(t)R_4(t))$

25.3-3. Let x_i and x_i' (for $i=1,2,3$) denote the two units of type i.
Then the following two systems are to be compared:

System A



System B



25.3-3 (continued)

$\phi_A(x_1,x_2,x_3,x_1',x_2',x_3') = [\max(x_1,x_1')][\max(x_2,x_2')][\max(x_3,x_3')]$
$\phi_B(x_1,x_2,x_3,x_1',x_2',x_3') = \max[x_1x_2x_3, x_1'x_2'x_3']$
Now $[\max(x_1,x_1')][\max(x_2,x_2')][\max(x_3,x_3')] \geq x_1x_2x_3$ and
$[\max(x_1,x_1')][\max(x_2,x_2')][\max(x_3,x_3')] \geq x_1'x_2'x_3'$
So $\phi_A(x,x') \geq \max[x_1x_2x_3, x_1'x_2'x_3'] = \phi_B(x,x')$
Therefore system A is more reliable.

25.4-1. (a) Minimal paths: x_1x_3 and x_2x_4
Minimal cuts: x_1x_2, x_1x_4, x_2x_3 and x_3x_4.

(b) From the minimal path representation
$\phi(x_1,x_2,x_3,x_4) = \max[x_1x_3, x_2x_4] = 1-(1-x_1x_3)(1-x_2x_4)$
$R(p_1,p_2,p_3,p_4) = 1-(1-p_1p_3)(1-p_2p_4)$
and when $p_i = p = 0.90$ for all i
$R(p) = 1-(1-(0.9)^2)(1-(0.9)^2) = .9639$

(c) Upper bound $= 1-(1-p_1p_3)(1-p_2p_4)$
Lower bound $= (1-q_1q_2)(1-q_1q_4)(1-q_2q_3)(1-q_3q_4)$
where $q_i = 1-p_i$.
When $p_i = p = 0.90$ for all i,
Upper bound $= 1-(1-(0.9)^2)(1-(0.9)^2) = .9639$
Lower bound $= (1-(0.1)^2)^4 = .96060$

25.4-2. (a) Minimal paths: x_1x_5, $x_1x_3x_4$, $x_2x_3x_5$, x_2x_4
Minimal cuts: x_1x_2, $x_1x_3x_4$, $x_2x_3x_5$, x_4x_5

(b) $R(p_1,p_2,p_3,p_4,p_5) = P\{(X_1X_5=1)\cup(X_1X_3X_4=1)\cup(X_2X_3X_5=1)\cup(X_2X_4=1)\}$
$= P(X_1X_5=1) + P(X_1X_3X_4=1) + P(X_2X_3X_5=1) + P(X_2X_4=1)$
$- P(X_1X_3X_4X_5=1) - P(X_1X_2X_3X_5=1) - P(X_1X_2X_4X_5=1)$
$- P(X_1X_2X_3X_4X_5=1) - P(X_1X_2X_3X_4=1) - P(X_1X_2X_3X_4X_5=1)$
$+ P(X_1X_2X_3X_4X_5=1) + P(X_1X_2X_3X_4X_5=1) + P(X_1X_2X_3X_4X_5=1) +$
$+ P(X_1X_2X_3X_4X_5=1) - P(X_1X_2X_3X_4X_5=1) =$
$= p_1p_5 + p_1p_3p_4 + p_2p_3p_5 + p_2p_4 - p_1p_3p_4p_5$
$- p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 - p_2p_3p_4p_5 +$
$+ 2p_1p_2p_3p_4p_5$
When $p_i = p = 0.90$ for all i
$R(p) = 2(0.9)^2 + 2(0.9)^3 - 5(0.9)^4 + 2(0.9)^5 = .97848$

(c) Upper bound $= 1-(1-p_1p_5)(1-p_1p_3p_4)(1-p_2p_3p_5)(1-p_2p_4)$
Lower bound $= (1-q_1q_2)(1-q_1q_3q_4)(1-q_2q_3q_5)(1-q_4q_5)$
when $p_i = p = 0.90$ for all i
Upper bound $= 1-(1-(0.9)^2)^2(1-(0.9)^3)^2 = .99735$
Lower bound $= (1-(0.1)^2)^2(1-(0.1)^3)^2 = .97814$

25.4-3. (a) Minimal paths: x_1x_2 and x_2x_3
Minimal cuts: x_1x_3 and x_2

(b) From the minimal path representation
$\phi(x_1,x_2,x_3) = \max[x_1x_2, x_2x_3] = x_2\max[x_1,x_3] =$
$= x_2[1-(1-x_1)(1-x_3)]$
Hence $R(p_1,p_2,p_3) = p_2[1-(1-p_1)(1-p_3)] = p_2p_1 + p_2p_3 - p_1p_2p_3$
$= .891$ when $p_i = p = 0.90$ for all i

25.4-3 (continued)

(c) Upper bound $= 1-(1-p_1p_2)(1-p_2p_3)$
$= .9639$ when $p_i = p = 0.90$ for all i

Lower bound $= (1-q_1q_3)(1-q_2)$
$= .891$ when $p_i = p = 0.90$ for all i

25.4-4. (a) Minimal paths: x_1x_5, $x_1x_3x_6$, x_2x_6 and $x_2x_4x_5$
Minimal cuts: x_1x_2, $x_1x_4x_6$, $x_2x_3x_5$ and x_5x_6

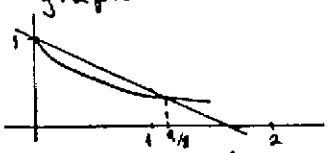
(b) $R(p_1,p_2,p_3,p_4,p_5,p_6) = P\{(X_1X_5=1)\cup(X_1X_3X_6=1)\cup(X_2X_6=1)\cup(X_2X_4X_5=1)\}$
$= p_1p_5 + p_1p_3p_6 + p_2p_6 + p_2p_4p_5 - p_1p_3p_5p_6 - p_1p_2p_5p_6$
$- p_1p_2p_4p_5 - p_1p_2p_3p_6 - p_2p_4p_5p_6 + p_1p_2p_3p_5p_6$
$+ p_1p_2p_4p_5p_6$
$= 2p^2 + 2p^3 - 5p^4 + 2p^5$ when $p_i = p$ for all i
$= .97848$ when $p_i = p = 0.90$ for all i

(c) Upper bound $= 1-(1-p_1p_5)(1-p_1p_3p_6)(1-p_2p_6)(1-p_2p_4p_5)$
$= .99735$ when $p_i = p = 0.90$ for all i

Lower bound $= (1-q_1q_2)(1-q_1q_4q_6)(1-q_2q_3q_5)(1-q_5q_6)$
$= .97814$ when $p_i = p = 0.90$ for all i

25.5-1. (a) $R(t) \geq e^{-t/\mu}$ for $t \leq \mu \Rightarrow R(1/4) \geq e^{-(1/4)/.6} \approx .659$
Therefore $.659 \approx e^{-.25/.6} \leq R(1/4) \leq 1$

(b) $R(t) \leq e^{-wt}$ for $t > \mu$ where $1-\mu w = e^{-wt}$. So we must find w such that $1-.6w = e^{-w}$. Drawing the graph



So $w \approx 9/8$
Hence $0 \leq R(1) \leq e^{-9/8} \approx .325$

25.5-2. $f(t) = \frac{\beta}{\eta} t^{\beta-1} e^{-t^\beta/\eta}$ and $R(t) = e^{-t^\beta/\eta}$

So $r(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\eta} t^{\beta-1}$ which is $\begin{cases} \text{non decreasing for } \beta \geq 1 \\ \text{non increasing for } \beta \leq 1 \end{cases}$

So the Weibull is IFR for $\beta \geq 1$ and DFR for $\beta \leq 1$.

25.5-3. $R(t) = P\{T_1 > t \text{ and } T_2 > t\} = e^{-\frac{t}{\theta_1}} e^{-\frac{t}{\theta_2}} = e^{-t\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)}$

Thus, the failure rate of the system is exponential with parameter $\frac{1}{\theta_1} + \frac{1}{\theta_2}$, and as noted in Section 21.6 the exponential distribution is both IFR & DFR

25.5-4. Let X_i denote the failure time of component i
Let X denote the failure time of the system
Also let $\lambda_i = 1/\mu_i$
Then $F(t) = P\{X \leq t\} = P\{X_1 \leq t, X_2 \leq t\} = (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})$

So $r(t) = \frac{f(t)}{1-F(t)} = \frac{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}}$

Note $r(0) = 0$

$$\frac{dr(t)}{dt} = \frac{\lambda_1^2 e^{-(\lambda_1+2\lambda_2)t} + \lambda_2^2 e^{-(2\lambda_1+\lambda_2)t} - (\lambda_1-\lambda_2)^2 e^{-(\lambda_1+\lambda_2)t}}{\left[e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}\right]^2}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}\left[\lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t} - (\lambda_1-\lambda_2)^2\right]}{\left[e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}\right]^2}$$

25.5-4 (Continued)

Let $K(t) = \lambda_1^2 e^{-\lambda_2 t} + \lambda_2^2 e^{-\lambda_1 t} - (\lambda_1 - \lambda_2)^2$

Note that $K(0) = 2\lambda_1\lambda_2 > 0$

and $K(\infty) = -(\lambda_1 - \lambda_2)^2 < 0$ since $\lambda_1 < \lambda_2$

and $\frac{dK(t)}{dt} = -\lambda_1^2\lambda_2 e^{-\lambda_2 t} - \lambda_2^2\lambda_1 e^{-\lambda_1 t} < 0$

Hence, $K(t)$ is a strictly decreasing function that is positive at $t=0$ and negative at $t=\infty$.
So $K(t)=0$ has one and only one solution.
Suppose then that $K(t_0)=0$ for some $0<t_0<\infty$.
That is

$$K(t)\begin{cases} >0 & \text{for } t<t_0 \\ =0 & \text{for } t=t_0 \\ <0 & \text{for } t>t_0 \end{cases}$$

Then $\frac{dr(t)}{dt}\begin{cases} >0 & \text{for } t<t_0 \\ =0 & \text{for } t=t_0 \\ <0 & \text{for } t>t_0 \end{cases}$

Then $r(t)$ is increasing for $t \le t_0$ and decreasing for $t \ge t_0$.
Thus the system can be IFR if and only if $t_0 = \infty$.
But since $K(\infty) = -(\lambda_1-\lambda_2)^2$, this can occur if and only if $\lambda_1 = \lambda_2$. This contradiction implies the system cannot be IFR if $\lambda_1 \neq \lambda_2$ (that is $\mu_1 \neq \mu_2$).

25.5-5. Exponential components are IFR and hence IFRA, so the system of problem 12 is composed of IFRA components. Since systems of IFRA components have IFRA life distributions (see the final subsection of section 21.6), the desired result follows.