# DETERMINISTIC OPERATIONS RESEARCH

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## Chapter 3

* prototype example " wyndor glass co." construction of mathematical model:

① decision variables: $x_1$ : # of batches that should be produced per week for product 1.
$x_2$ : " for product 2.

② objective : max. profit.

$\Rightarrow$ $\max z = 3000x_1 + 5000x_2$ or $z = 3x_1 + 5x_2$

③ constraints : plant 1 $\Rightarrow$ $1x_1 + 0x_2 \leq 4$ $\Rightarrow$ # of hours

plant 2 $\Rightarrow$ $0x_1 + 2x_2 \leq 12$

plant 3 $\Rightarrow$ $3x_1 + 2x_2 \leq 18$

also $\Rightarrow$ $x_1 \geq 0$ , $x_2 \geq 0$

• graphical method:

constraints:

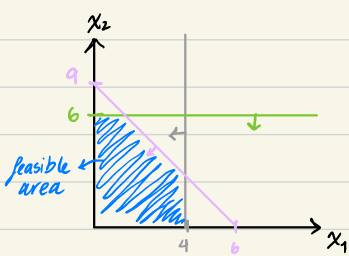
$x_1 \leq 4 \Rightarrow x_1 = 4$

$2x_2 \leq 12 \Rightarrow x_2 = 6$

$3x_1 + 2x_2 \leq 18 \Rightarrow 3x_1 + 2x_2 = 18$

take $x_1 = 0 \rightarrow x_2 = 9$ $(0, 9)$

$x_2 = 0 \rightarrow x_1 = 6$ $(6, 0)$

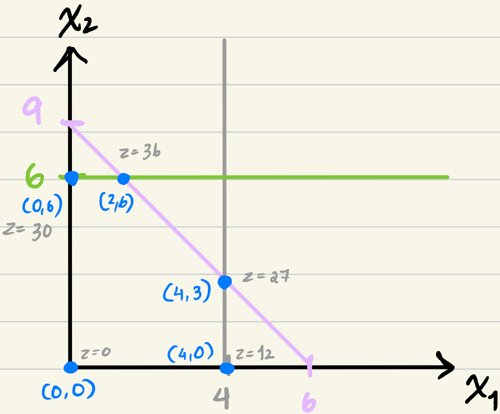



arrows point to feasible area

feasible area : any given point inside works for constraints (general solution)

* to find optimal solution :

if area is convex $\Rightarrow$ take only corner points , one of them is the optimal solution & subst. in mathematical model.




$\Rightarrow$ optimal point : $x_1 = 2$ , $x_2 = 6$

profit = 36,000

example 3.4 " Mary's cancer ":

① DV : $x_1$ : strength of first beam

$x_2$ : " " second "

② objective : minimize $z = 0.4x_1 + 0.5x_2$

③ constraints : $0.3x_1 + 0.1x_2 \leq 2.7$ ①

$0.5x_1 + 0.5x_2 = 6$ ② on line

$0.6x_1 + 0.4x_2 \geq 6$ ③

also , $x_1 , x_2 \geq 0$

$x_1 = 0 \Rightarrow x_2 = 27$

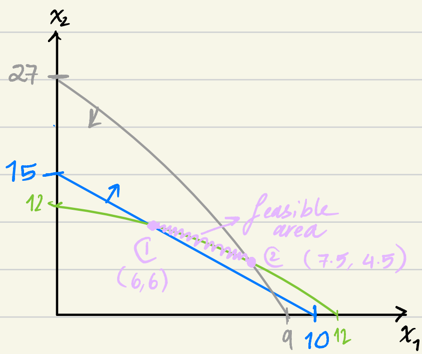
$x_2 = 0 \Rightarrow x_1 = 9$

$x_1 = 0 \Rightarrow x_2 = 12$

$x_2 = 0 \Rightarrow x_1 = 12$

$x_1 = 0 \Rightarrow x_2 = 15$

$x_2 = 0 \Rightarrow x_1 = 10$



to find coordinates.

① $-6(0.5x_1 + 0.5x_2 = 6)$

$5(0.6x_1 + 0.4x_2 = 6)$

$\left.\right\}$ $x_2 = 6$, $x_1 = 6$

② $0.5x_1 + 0.5x_2 = 6$

$-5(0.3x_1 + 0.1x_2 = 2.7)$

$\left.\right\}$ $x_1 = 7.5$, $x_2 = 4.5$

$\Longrightarrow$ $z = 0.4(6) + 0.5(6) = 5.4$

$z = 0.4(7.5) + 0.5(4.5) = 5.25$ $\Rightarrow$ optimal solution (looking for least kilorad).

## Section 3.3

* proportionality : linearity assumption.

* additivity : the sum of individual contributions is the total contribution.

* divisibility : DV are cts. & can take any values.

* certainty : the value assigned to each perimeter is a known constant.

ex. 3.23: DV. $\Rightarrow$ $x_1$ : friend 1 fraction

$x_2$ : friend 2 fraction

Obj. $\Rightarrow$ max. $Z = 9{,}000\,x_1 + 9{,}000\,x_2$

Sub. to $\Rightarrow$ ① $10{,}000\,x_1 + 8{,}000\,x_2 \leq 12{,}000$

② $400\,x_1 + 500\,x_2 \leq 600$

& $x_1, x_2 \geq 0$

## Reclaiming Solid Wastes

The SAVE-IT COMPANY operates a reclamation center that collects four types of solid waste materials and treats them so that they can be amalgamated into a salable product. (Treating and amalgamating are separate processes.) Three different grades of this product can be made (see the first column of Table 3.16), depending upon the mix of the materials used. Although there is some flexibility in the mix for each grade, quality standards may specify the minimum or maximum amount allowed for the proportion of a material in the

[7] An equivalent formulation can express each decision variable in natural units for its abatement method; for example, $x_1$ and $x_2$ could represent the number of *feet* that the heights of the smokestacks are increased.



**TABLE 3.16** Product data for Save-It Co.

| Grade | Specification | Amalgamation Cost per Pound ($) | Selling Price per Pound ($) |
|---|---|---|---|
| *A* | Material 1: Not more than 30% of total<br>Material 2: Not less than 40% of total<br>Material 3: Not more than 50% of total<br>Material 4: Exactly 20% of total | 3.00 | 8.50 |
| *B* | Material 1: Not more than 50% of total<br>Material 2: Not less than 10% of total<br>Material 4: Exactly 10% of total | 2.50 | 7.00 |
| *C* | Material 1: Not more than 70% of total | 2.00 | 5.50 |

product grade. (This proportion is the weight of the material expressed as a percentage of the total weight for the product grade.) For each of the two higher grades, a fixed percentage is specified for one of the materials. These specifications are given in Table 3.16 along with the cost of amalgamation and the selling price for each grade.

The reclamation center collects its solid waste materials from regular sources and so is normally able to maintain a steady rate for treating them. Table 3.17 gives the quantities available for collection and treatment each week, as well as the cost of treatment, for each type of material.

The Save-It Co. is solely owned by Green Earth, an organization devoted to dealing with environmental issues, so Save-It's profits are used to help support Green Earth's activities. Green Earth has raised contributions and grants, amounting to $30,000 per week, to be used exclusively to cover the entire treatment cost for the solid waste materials. The board of directors of Green Earth has instructed the management of Save-It to divide this money among the materials in such a way that *at least half* of the amount available of each material is actually collected and treated. These additional restrictions are listed in Table 3.17.

Within the restrictions specified in Tables 3.16 and 3.17, management wants to determine the *amount* of each product grade to produce *and* the exact *mix* of materials to be used for each grade. The objective is to maximize the net weekly profit (total sales income *minus* total amalgamation cost), exclusive of the fixed treatment cost of $30,000 per week that is being covered by gifts and grants.

$\Rightarrow$

$10{,}000x_1 + 8{,}000x_2 = 12{,}000$

$x_1 = 0 \Rightarrow x_2 = 12/8 = 1.5$

$x_2 = 0 \Rightarrow x_1 = 12/10 = 1.2$

$400x_1 + 500x_2 \leq 600$

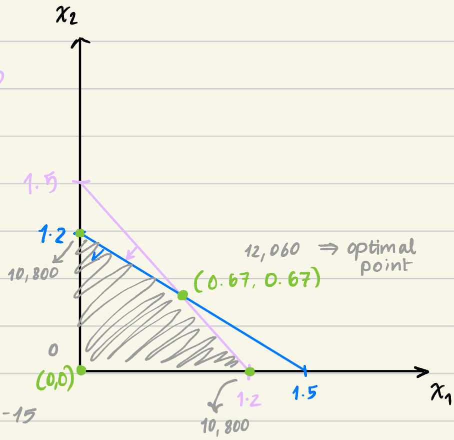
$x_1 = 0 \Rightarrow x_2 = 6/5$

$x_2 = 0 \Rightarrow x_1 = 1.5$

$10x_1 + 8x_2 = 12$

$-2.5(4x_1 + 5x_2 = 6) \Rightarrow -10x_1 - 12.5 = -15$

$\Rightarrow -4.5x_2 = -3 \quad x_2 = 0.67 \rightarrow x_1 = 0.67$




example "Reclaiming Solid Wastes":

① DV: $x_{A1}$ : amount of waste material 1 that should be used to produce grade A. (also $x_{A2}$, $x_{A3}$, $x_{A4}$)
$x_{B1}$, $x_{B2}$, $x_{B3}$, $x_{B4}$, $x_{C1}$, $x_{C2}$, $x_{C3}$, $x_{C4}$

② objective: max. profit

$$z = \underbrace{(8.5-3)}_{\text{grade A}}[x_{A1} + x_{A2} + x_{A3} + x_{A4}]$$
$$\underbrace{(7-2.5)}_{\text{grade B}}[x_{B1} + x_{B2} + x_{B3} + x_{B4}]$$
$$\underbrace{(5.5-2)}_{\text{grade C}}[x_{C1} + x_{C2} + x_{C3} + x_{C4}]$$

③ sub. to: 1)

$$\left.\begin{aligned} x_{A1} &\leq 0.3[x_{A1} + x_{A2} + x_{A3} + x_{A4}] \\ x_{A2} &\geq 0.4[x_{A1} + x_{A2} + x_{A3} + x_{A4}] \\ x_{A3} &\leq 0.5[x_{A1} + x_{A2} + x_{A3} + x_{A4}] \\ x_{A4} &= 0.2[x_{A1} + x_{A2} + x_{A3} + x_{A4}] \end{aligned}\right] \rightarrow \text{grade A}$$

↓

$x_{B1} \leq 0.5 [x_{B1} + x_{B2} + x_{B3} + x_{B4}]$

$x_{B2} \geq 0.1 [x_{B1} + x_{B2} + x_{B3} + x_{B4}]$

$x_{B4} = 0.1 [x_{B1} + x_{B2} + x_{B3} + x_{B4}]$

$x_{C1} \leq 0.7 [x_{C1} + x_{C2} + x_{C3} + x_{C4}]$

2) $x_{A1} + x_{B1} + x_{C1} \leq 3000$
$\geq 1500$

$x_{A2} + x_{B2} + x_{C2} \leq 2000$
$\geq 1000$

$x_{A3} + x_{B3} + x_{C3} \geq 4000$
$\leq 2000$

$x_{A4} + x_{B4} + x_{C4} \geq 1000$
$\leq 500$

3) $3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) + 4(x_{A3} + x_{B3} + x_{C3})$
$+ 5(x_{A4} + x_{B4} + x_{C4}) = 30,000$

* 3.4.10 - imp. question.

3.4.11 solution: ① DV: $x_{11}$ : units shipped from 1st factory to 1st customer.
& $x_{12}$, $x_{13}$, $x_{21}$, $x_{22}$, $x_{23}$

② obj: min. shipping cost $\Rightarrow$ $z = 600x_{11} + 800x_{12} \ldots 600x_{23}$

③ const.: $x_{11} + x_{12} + x_{13} \leq 400$
$x_{21} + x_{22} + x_{23} \leq 500$

$x_{11} + x_{21} = 300$
$x_{12} + x_{22} = 200$
$x_{13} + x_{23} = 400$

$x_{ij} \geq 0 \ \forall i \ \forall j$

**3.4-11.*** The Medequip Company produces precision medical diagnostic equipment at two factories. Three medical centers have placed orders for this month's production output. The table below shows what the cost would be for shipping each unit from each factory to each of these customers. Also shown are the number of units that will be produced at each factory and the number of units ordered by each customer.

| From \ To | Unit Shipping Cost: Customer 1 | Customer 2 | Customer 3 | Output |
|---|---|---|---|---|
| Factory 1 | $600 | $800 | $700 | 400 units |
| Factory 2 | $400 | $900 | $600 | 500 units |
| Order size | 300 units | 200 units | 400 units | |

A decision now needs to be made about the shipping plan for how many units to ship from each factory to each customer.

**(a)** Formulate a linear programming model for this problem.

C **(b)** Solve this model by the simplex method.

**3.4-14*** A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both *weight* and *space,* as summarized below:

| Compartment | Weight Capacity (Tons) | Space Capacity (Cubic Feet) |
|---|---|---|
| Front | 12 | 7,000 |
| Center | 18 | 9,000 |
| Back | 10 | 5,000 |

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.

The following four cargoes have been offered for shipment on an upcoming flight as space is available:

| Cargo | Weight (Tons) | Volume (Cubic Feet/Ton) | Profit ($/Ton) |
|---|---|---|---|
| 1 | 20 | 500 | 320 |
| 2 | 16 | 700 | 400 |
| 3 | 25 | 600 | 360 |
| 4 | 13 | 400 | 290 |

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight.

**(a)** Formulate a linear programming model for this problem.

C **(b)** Solve this model by the simplex method to find one of its multiple optimal solutions.

3.4.14 : ① $\frac{F}{12} = \frac{C}{18} = \frac{B}{10}$

② DV: $x_{1f}$ : weight of cargo 1 that will be accepted & stored in the front compartment.

$x_{1C}$ : ... center

$x_{1B}$ : ... back

③ obj: max profit $z = 320(x_{1f} + x_{1C} + x_{1B}) + 400(x_{2f} + x_{2C} + x_{2B})$ ...

④ sub to: $x_{1f} + x_{1C} + x_{1B} \leq 20$ & $x_{1f} + x_{2f} + x_{3f} + x_{4f} \leq 12$

$x_{2f} + x_{2C} + x_{2B} \leq 16$ $\quad x_{1C} + x_{2C} + x_{3C} + x_{4C} \leq 18$

$x_{4f} + x_{4C} + x_{4B} \leq 13$ $\quad x_{1B} + x_{2B} + x_{3B} + x_{4B} \leq 10$

&

$500x_{1f} + 700x_{2f} + 600x_{3f} + 400x_{4f} \leq 7000$

$500x_{1B} + 700x_{2B} + 600x_{3B} + 400x_{4B} \leq 5000$

$500x_{1C} + 700x_{2C}$ ....

&

$$\frac{x_{1f} + x_{2f} + x_{3f} + x_{4f}}{12} = \frac{\Sigma x_{iC}}{18} = \frac{\Sigma x_{iB}}{10}$$

* Simplex Method: like gaussian elimination but can only be used for model written in standard form.

⇒ ① obj. must be maximization.

② all constraints $\leq$ with +ve right hand side.

③ all DV. $\geq 0$

ex: Windor Glass: using simplex method

Initiation Step • $x_3, x_4$ : slack variable

obj: max $z = 3x_1 + 5x_2$ $\Rightarrow$ $z - 3x_1 - 5x_2 = 0$

sub. to: $x_1 \leq 4$ $\Rightarrow$ $x_1 + x_3 = 4$

$2x_2 \leq 12$ $\Rightarrow$ $2x_2 + x_4 = 12$

$3x_1 + 2x_2 \leq 18$ $\Rightarrow$ $3x_1 + 2x_2 + x_5 = 18$

$x_1, x_2 \geq 0$ $x_{3,4,5} \geq 0$

→ initialization

→ identity matrix

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|---|
| obj. | 1 | -3 | -5 | 0 | 0 | 0 | 0 |
| | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | 0 | 3 | 2 | 0 | 0 | 1 | 18 |

Basic Variables:

$x_3 = 4$ $x_4 = 12$ $x_5 = 18$

Non Basic Variables: always 0

$x_1 = 0$ $x_2 = 0$

- note that # of basic variables is same as constraints.
- if we have a -ve value in z row then not optimal solution

Optimality Step

→ most negative

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | -3 | -5 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| 0 | 3 | 2 | 0 | 0 | 1 | 18 |

$x_2$ → entering variable.

Feasibility Step: RHS ÷ coeff of E.V.

$4/0 = \infty$

$12/2 = 6$ $\Rightarrow$ $x_2 = 6$ subject to all constraints

$18/2 = 9$ • always choose min

$x_4$ : leaving variable.

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -5 | 0 | 0 | 0 | 0 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 |

→ min. row

leader

pivot element

**Basic Var.**

$x_3 = 4 \quad x_2 = \quad x_5 = 18$

**Non Basic Var.**

$x_1 = 0 \quad x_4 = 0$

Matrix Operations: ① $R_2/2$ ② $5R_2 + R_0$ ③ $-2R_2 + R_3$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | -3 | 0 | 0 | 5/2 | 0 | 30 |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 0 | 0 | 1 | 0 | 1/2 | 0 | 6 |
| 0 | 3 | 0 | 0 | -1 | 1 | 6 |

**Basic Var.**

$x_3 = 4 \quad x_2 = 6 \quad x_5 = 6$

**Non. Basic Var.**

$x_1 = 0 \quad x_4 = 0$

Optimality Step:

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | |
|---|---|---|---|---|---|---|---|
| 1 | -3 | 0 | 0 | 5/2 | 0 | 30 | |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | 4/1 |
| 0 | 0 | 1 | 0 | 1/2 | 0 | 6 | 6/0 |
| 0 | 3 | 0 | 0 | -1 | 1 | 6 | 6/3 |

pivot element

$x_1$ : entering var.

$x_5$ : leaving var.

$\Rightarrow$ ① $R_3/3$ ② $-R_3+R_1$ ③ $3R_3+R_0$

Basic Var :

$x_3 = 2$ $x_2 = 6$ $x_1 = 2$

Non Basic Var :

$x_5 = 0$ $x_4 = 0$

shadow price

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 3/2 | 1 | 36 |
| 0 | 0 | 0 | 1 | 1/3 | -1/3 | 2 |
| 0 | 0 | 1 | 0 | 1/2 | 0 | 6 |
| 0 | 1 | 0 | 0 | -1/3 | 1/3 | 2 |

shadow price : the multiple of the profit it will become if 1 unit added.

Q 4.3.5 : Obj : max $z = x_1 + 2x_2 + 4x_3$

Sub to : $3x_1 + x_2 + 5x_3 + x_4 = 10$

$x_1 + 4x_2 + x_3 + x_5 = 8$

$2x_1 + 2x_3 + x_6 = 7$

$x_{1,2,3} \geq 0$

func. constraints

-ve

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS | |
|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -2 | -4 | 0 | 0 | 0 | 0 | |
| 0 | 3 | 1 | 5 | 1 | 0 | 0 | 10 | /5 = 2 |
| 0 | 1 | 4 | 1 | 0 | 1 | 0 | 8 | /1 = 8 |
| 0 | 2 | 0 | 2 | 0 | 0 | 1 | 7 | /2 = 3.5 |

Matrix operations: ① $R_1/5$ ② $-R_1+R_2$ ③ $-2R_4+R_3$ ④ $4R_1+R_0$

-ve

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS | |
|---|---|---|---|---|---|---|---|---|
| 1 | 7/5 | -6/5 | 0 | 4/5 | 0 | 0 | 8 | |
| 0 | 3/5 | 1/5 | 1 | 1/5 | 0 | 0 | 2 | / 1/5 = 10 |
| 0 | 2/5 | 19/5 | 0 | -1/5 | 1 | 0 | 6 | / 19/5 = 30/19 |
| 0 | 4/5 | -2/5 | 0 | -2/5 | 0 | 1 | 3 | |

Matrix Operations: ① $\frac{5}{19}R_2$ ② $\frac{6}{5}R_2 + R_0$ ③ $-\frac{1}{5}R_2 + R_1$ ④ $\frac{2}{5}R_2 + R_3$

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS → always +ve |
|---|---|---|---|---|---|---|---|
| 1 | 29/19 | 0 | 0 | 70/95 | 6/19 | 0 | 188/19 |
| 0 | 5/19 | 0 | 1 | 4/19 | -1/19 | 0 | 32/19 |
| 0 | 2/19 | 1 | 0 | -1/19 | 5/19 | 0 | 30/19 |
| 0 | 16/19 | 0 | 0 | -8/19 | 2/19 | 1 | 69/19 |

BV

Optimal Solution.

Q: obj: max $z = 3x_1 + 3x_2 + x_3$

Sub to: $x_1 + 4x_2 + x_3 + x_4 = 12$

$2x_1 + x_2 - x_3 + x_5 = 15$

Tie for EV.

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | -3 | -4 -3 | -1 -1 | 0 | 0 | 0 |
| 0 | 1 | 4 | 1 | 1 | 0 | 12 |
| 0 | 2 | 1 | -1 | 0 | 1 | 15 |

* fraction $\alpha$ $[0, 1]$ ⇒ $P_{new} = \alpha P_1 + (1-\alpha)P_2$ → gives all values of optimal solutions on a line.
* if z // one of functional constraints ⇒ multi-optimal solution
* NBV in an O.S., coefficient in obj. is zero ⇒ we have other O.S.
  ↳ make entering var.

* max $z = c_1 x_1 + 5x_2$ O.P. $(2, 6)$

$x_1 \leq 4$

$2x_2 \leq 12$ $\Rightarrow$ $3/2 \geq c_1/5 \geq 0/2$

$3x_1 + 2x_2 \leq 18$ $7.5 \geq c_1 \geq 0$

if $z = 3x_1 + c_2 x_2$ $\Rightarrow$ $2/3 \leq c_2/3 \leq 2/0$

$2 \leq c_2 \leq \infty$

$c_2 \geq 2$

# * Big M Method :

negative effect (if max $\Rightarrow -Mx_{a_1}$)

• artificial variable

example : obj : min $z = 0.4x_1 + 0.5x_2 + Mx_{a_1} + Mx_{a_2}$

sub to: $0.3x_1 + 0.1x_2 \leq 2.7$ $\Rightarrow$ $0.3x_1 + 0.1x_2 + x_3 = 2.7$

$0.5x_1 + 0.5x_2 = 6$ $\Rightarrow$ $0.5x_1 + 0.5x_2 + x_{a_1} = 6$

$0.6x_1 + 0.4x_2 \geq 6$ $\Rightarrow$ $0.6x_1 + 0.4x_2 - x_4 + x_{a_2} = 6$

$x_1, x_2 \geq 0$ $x_3 \geq 0$ , surplus var $x_4 \geq 0$

• added bc in 3rd constraint if $x_1, x_2 = 0$ then $x_4 = -6$

& obj min $\rightarrow$ max gives:

max $-z = -0.4x_1 - 0.5x_2 - Mx_{a_1} - Mx_{a_2}$

• bc min. z is max of its negative.

to solve $\Rightarrow$

$-z + 0.4x_1 + 0.5x_2 + Mx_{a_1} + Mx_{a_2} = 0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a_1}$ | $x_{a_2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | .4 | .5 | 0 | 0 | M | M | 0 |
| 0 | .3 | .1 | 1 | 0 | 0 | 0 | 2.7 |
| 0 | .5 | .5 | 0 | 0 | 1 | 0 | 6 |
| 0 | .6 | .4 | 0 | -1 | 0 | 1 | 6 |

## ① Restoring Gaussian Form: Getting rid of Ms.

infeasible initial solution.

① $-MR_2 + R_0$ ② $-MR_3 + R_0$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| $-1$ | $.4 - 1.1M$ | $.5 - .9M$ | $0$ | $M$ | $0$ | $0$ | $-12M$ |
| $0$ | $.3$ | $.1$ | $1$ | $0$ | $0$ | $0$ | $2.7$ |
| $0$ | $.5$ | $.5$ | $0$ | $0$ | $1$ | $0$ | $6$ |
| $0$ | $.6$ | $.4$ | $0$ | $-1$ | $0$ | $1$ | $6$ |

entering

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| $-1$ | $-1.1M$ | $-0.9M$ | $0$ | $M$ | $0$ | $0$ | $-12M$ |
| $0$ | $.3$ | $.1$ | $1$ | $0$ | $0$ | $0$ | $2.7$ |
| $0$ | $.5$ | $.5$ | $0$ | $0$ | $1$ | $0$ | $6$ |
| $0$ | $.6$ | $.4$ | $0$ | $-1$ | $0$ | $1$ | $6$ |

leaving

① $R_1 / .3$ ② $-(.4 - 1.1M)R_1 + R_0$ ③ $-0.4R_1 + 1.1MR_1 + R_0$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| $-1$ | $0$ | $11/3 - 8/15M$ | $-4/3 + 11/3M$ | $M$ | $0$ | $0$ | $-18/5 - 2.1M$ |
| $0$ | $1$ | $1/3$ | $10/3$ | $0$ | $0$ | $0$ | $9$ |
| $0$ | $.5$ | $.5$ | $0$ | $0$ | $1$ | $0$ | $6$ |
| $0$ | $.6$ | $.4$ | $0$ | $-1$ | $0$ | $1$ | $6$ |

④ $-0.3R_1+R_2$ ⑤ $-0.8R_1+R_3$

entering

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| $-1$ | $0$ | $11/3 - 8/15 M$ | $-4/3 + 11/3 M$ | $M$ | $0$ | $0$ | $-18/5 - 2.1M$ |
| $0$ | $1$ | $1/3$ | $10/3$ | $0$ | $0$ | $0$ | $9$ |
| $0$ | $0$ | $1/3$ | $-5/3$ | $0$ | $1$ | $0$ | $3/2$ |
| $0$ | $0$ | $.2$ | $-2$ | $-1$ | $0$ | $1$ | $.6$ |

leaving

① $R_3/.2$ ② $-1/3 R_3 + R_2$ ③ $-1/3 R_3 + R_1$ ④ $(11/3 R_3 + 8/15 M R_3) + R_0$

entering

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| $-1$ | $0$ | $0$ | $106/3 - 5/3 M$ | $55/3 - 5/3 M$ | $-55/3 + 8/3 M$ | | $-73/5 - 0.5M$ |
| $0$ | $1$ | $0$ | $20/3$ | $5/3$ | $0$ | $-5/3$ | $8$ |
| $0$ | $0$ | $0$ | $5/3$ | $5/3$ | $1$ | $-5/3$ | $1/2$ |
| $0$ | $0$ | $1$ | $-10$ | $-5$ | $0$ | $5$ | $3$ |

leaving

① $R_2 \cdot 3/5$ ② $(-55/3 R_2 + 5/3 M) + R_0$ ③ $-5/3 R_2 + R_1$ ④ $5R_2 + R_3$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| $-1$ | $0$ | $0$ | $17$ | $0$ | $-11+M$ | $M$ | $-201/10 - 0.5M$ |
| $0$ | $1$ | $0$ | $5$ | $0$ | $-1$ | $0$ | $7.5$ |
| $0$ | $0$ | $0$ | $1$ | $1$ | $3/5$ | $-1$ | $3/10$ |
| $0$ | $0$ | $1$ | $-5$ | $0$ | $3$ | $0$ | $4.5$ |

feasible & optimal.

5.3.9: $z = 2x_1 + 3x_2 + 2x_3 + Mx_{a1} + Mx_{a2} \Rightarrow \text{Max } -z = -2x_1 - 3x_2 - 2x_3 - Mx_{a1} - Mx_{a2}$

subto: $x_1 + 4x_2 + 2x_3 - x_4 + x_{a1} \geq 8$

$3x_1 + 2x_2 - x_5 + x_{a2} \geq 6$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| -1 | 2 | 3 | 2 | 0 | 0 | M | M | 0 |
| 0 | 1 | 4 | 2 | -1 | 0 | 1 | 0 | 8 |
| 0 | 3 | 2 | 0 | 0 | -1 | 0 | 1 | 6 |

① $-MR_1 + R_0$ ② $-MR_2 + R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| -1 | 2-4M | 3-6M | 2-2M | M | M | 0 | 0 | -14M |
| 0 | 1 | 4 | 2 | -1 | 0 | 1 | 0 | 8 |
| 0 | 3 | 2 | 0 | 0 | -1 | 0 | 1 | 6 |

① $R_1/4$ ② $(-3+6M)R_1 + R_0$ ③ $-2R_1 + R_2$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| -1 | $5/4 - 5/2M$ | 0 | $1/2 + M$ | $3/4 - 1/2M$ | M | $-3/4 + 3/2M$ | 0 | $-6-2M$ |
| 0 | $1/4$ | 1 | $1/2$ | $-1/4$ | 0 | $1/4$ | 0 | 2 |
| 0 | $5/2$ | 0 | -1 | $1/2$ | -1 | $-1/2$ | $-1/2$ | 2 |

① $R_2 \times 2/5$ ② $-1/4 R_2 + R_1$ ③ $(-5/4 + 5/2M) + R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 1 | $1/2$ | $1/2$ | $-1/2 + M$ | $-1/2 + M$ | -7 |
| 0 | 0 | 1 | $3/5$ | $-3/10$ | $1/10$ | $3/10$ | $-1/10$ | $9/5$ |
| 0 | 1 | 0 | $-2/5$ | $1/5$ | $-2/5$ | $-1/5$ | $2/5$ | $4/5$ |

feasible since $x_{a1}, x_{a2} = 0$

Optimal.

# * Two Phase Method

4.6.10

$$\min \quad z = 3x_1 + 2x_2 + 7x_3 + Mx_{a1} + Mx_{a2}$$

phase 1

$\min z = Mx_{a1} + Mx_{a2} \Rightarrow \cancel{max}$

phase 2

$\min z = 3x_1 + 2x_2 + 7x_3$

sub to: $-x_1 + x_2 + x_{a1} = 10$

$2x_1 - x_2 + x_3 - x_4 + x_{a2} \geqq 10$

$x_1, x_2, x_3 \geqslant 0$

phase 1

$\min z = Mx_{a1} + Mx_{a2} \Rightarrow \cancel{max}$ $\Rightarrow$ $\min z = x_{a1} + x_{a2}$ $\Rightarrow$ max $-z + x_{a1} + x_{a2} = 0$

then $z \times M$

phase 2

$\min z = 3x_1 + 2x_2 + 7x_3$ $\Rightarrow$ $\min z = 3x_1 + 2x_2 + 7x_3$

using simplex after restoring gaussian form.

1

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | -1 | 1 | 0 | 0 | 1 | 0 | 10 |
| 0 | 2 | -1 | 1 | -1 | 0 | 1 | 10 |

① $R_0 - R_1$ ② $R_0 - R_2$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | -1 | 0 | -1 | 1 | 0 | 0 | -20 |
| 0 | -1 | 1 | 0 | 0 | 1 | 0 | 10 |
| 0 | 2 | -1 | 1 | -1 | 0 | 1 | 10 |

① $R_2/2$ ② $R_2 + R_1$ ③ $R_2 + R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | -1/2 | -1/2 | 1/2 | 0 | 1/2 | -15 |
| 0 | 0 | 1/2 | 1/2 | -1/2 | 1 | 1/2 | 15 |
| 0 | 1 | -1/2 | 1/2 | -1/2 | 0 | 1/2 | 5 |

① $2R_1$ ② $\frac{1}{2}R_1+R_2$ ③ $\frac{1}{2}R_1+R_0$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | 1 | 1 | $\cancel{0}$ |
| 0 | 0 | 1 | 1 | -1 | 2 | 1 | 30 |
| 0 | 1 | 0 | 1 | -1 | 1 | 1 | 20 |

phase 2 : min. $z = 3x_1 + 2x_2 + 7x_3 \Rightarrow \max -z = -3x_1 - 2x_2 - 7x_3$

& remove artificial var. & change obj.

(Restoring GF → 1) $-2R_1+R_0$ ② $-3R_2+R_0$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| -1 | 0 | 0 | 2 | 5 | -120 |
| 0 | 0 | 1 | 1 | -1 | 30 |
| 0 | 1 | 0 | 1 | -1 | 20 |

ex: max $z = 0.4x_1 + 0.5x_2 - Mx_{a1}$

sub to: $0.3x_1 + 0.1x_2 + x_3 = 2.7$

$0.5x_1 + 0.5x_2 + x_{a1} = 6$

$0.6x_1 + 0.4x_2 - x_4 + x_{a2} = 6$

① min $z = x_{a1} + x_{a2}$

② max $z = 0.4x_1 + 0.5x_2$

second phase objective is always same as Q objective & first is always minimization.

Ch.5

## * General form:

Obj : max $z = c_1 x_1 + c_2 x_2 + \dots c_n x_n$

Sub to : $a_{11} x_1 + a_{12} x_2 \dots a_{1n} x_n \leq b_1$

$a_{21} x_1 \dots a_{2n} x_n \leq b_2$

$\vdots$

$a_{m1} x_1 \dots a_{mn} x_n \leq b_m$

$x \geq 0$

$\Rightarrow$

$c = [c_1 \; c_2 \; \dots \; c_n]$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ $\quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$\Rightarrow$

$z = cx$ & $Ax \leq b$ & $x \geq 0$

© first simplex tableau

$x_b \Rightarrow$ slack variable & $z = cx$

$\Rightarrow [A/I] \times \begin{bmatrix} x \\ x_{sv} \end{bmatrix} = b$

$\Rightarrow$

| $z$ | $\overbrace{x_1 \; \dots \; x_n}^{\text{original var.}}$ | $\overbrace{x_{n+1} \; \dots \; x_{n+m}}^{\text{slack var.}}$ | RHS |
|---|---|---|---|
| 1 | $C_B B^{-1} A - c$ | $C_B B^{-1}$ | $C_B B^{-1} b$ |
| 0 | | | |
| 0 | $B^{-1} A$ | $B^{-1}$ | $B^{-1} b$ |
| $\vdots$ | | | |
| 0 | | | |

where : $C_B$ : coeff of BV in obj

$B$ : " " " constraints

example 5.1.13 :

obj: max $z = 2x_1 + 2x_2 + 3x_3$

Sub to: $2x_1 + x_2 + 2x_3 \leq 4$

$x_1 + x_2 + x_3 \leq 3$

$x_1, x_2, x_3 \geq 0$

① $c = [\,2 \quad 2 \quad 3\,]$ $\quad A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

initial tableau:

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 0 | | | | 1 | 0 | |
| 0 | | | | 0 | 1 | |

$C_B B^{-1} A - C$ $\quad C_B B^{-1}$ $\quad C_B B^{-1} b$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | | 0 | 0 | | | |
| 0 | | 0 | 1 | | | |
| 0 | | 1 | 0 | | | |

$B^{-1}A$ $\quad B^{-1}$ $\quad B^{-1}b$

$x_B = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix}$

$$\begin{array}{cc} x_3 & x_2 \end{array}$$
$C_B = [\,3 \quad 2\,]$

$$B = \begin{array}{c} \begin{array}{cc} x_3 & x_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

also $B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 | 1 | 7 |
| 0 | 1 | 0 | 1 | 1 | -1 | 1 |
| 0 | 0 | 1 | 0 | -1 | 2 | 2 |

• given $B^{-1}$ find BV: find $B$ & search for $A$ same column as $B$.

example : 5.3.2 $\Rightarrow c = [4 \;\; 3 \;\; 1 \;\; 2]$ $A = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\Rightarrow$ $B = \begin{array}{cc} x_2 & x_4 \\ \end{array}$ $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $C_B = [3 \;\; 2]$

① $C_B B^{-1} A - c$

$$[3 \;\; 2]\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix} - [4 \;\; 3 \;\; 1 \;\; 2]$$

$= [3 \;\; 0 \;\; 2 \;\; 0]$

②

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 2 | 0 | 1 | 1 | 9 |
| 0 | | | | | | | |
| 0 | | | | | | | |

example :

obj : $\max\ z = 4x_1 + 3x_2 + x_3 + 2x_4$

sub to : $4x_1 + 2x_2 + x_3 + x_4 + x_5 \leq 5$

$3x_1 + x_2 + 2x_3 + x_4 + x_6 \leq 4$

$c = [4\ \ 3\ \ 1\ \ 2] \qquad A = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

it ① $x_B = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C_B = [0\ \ 0]$

most -ve

$C_B B^{-1} A - C = [-4\ \ -3\ \ -1\ \ -2]$

$C_B B^{-1} = [0\ \ 0] \qquad B^{-1}A = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B^{-1}b = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \qquad c_B B^{-1} b = 0$

entering BV = $x_1$ leaving BV = $x_5$

$x_B = \begin{bmatrix} x_1 \\ x_6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix} \qquad C_B = [4\ \ 0] \qquad B^{-1} = \begin{bmatrix} 1/4 & 0 \\ -3/4 & 1 \end{bmatrix}$

⇒ it ②

$B^{-1}A = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/4 \\ 0 & -1/2 & 5/4 & 1/4 \end{bmatrix} \qquad C_B B^{-1} A - C = [0\ \ -1\ \ 0\ \ -1]$

$C_B B^{-1} = [1\ \ 0] \qquad B^{-1}b = \begin{bmatrix} 5/4 \\ 1/4 \end{bmatrix}$

$C_B B^{-1} b = 5$

it 3 entering BV: $x_4$ leaving BV: $x_6$

$$x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \quad C_B = \begin{bmatrix} 4 & 2 \end{bmatrix}$$

$$\Rightarrow B^{-1}A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 5 & 1 \end{bmatrix} \quad C_B B^{-1} A - C = \begin{bmatrix} 0 & -3 & 5 & 0 \end{bmatrix}$$

$$C_B B^{-1} = \begin{bmatrix} -2 & 4 \end{bmatrix} \quad B^{-1} b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C_B B^{-1} b = 6$$

$\Rightarrow$ it 4 entering BV: $x_2$ leaving BV: $x_1$

$$x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad C_B = \begin{bmatrix} 3 & 2 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow B^{-1}A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \quad C_B B^{-1} A - C = \begin{bmatrix} 3 & 0 & 2 & 0 \end{bmatrix} \quad C_B B^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$B^{-1} b = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad C_B B^{-1} b = 9$$

$\Rightarrow$ optimal & feasible.

Q: 5.3.1

obj: $z = x_1 - x_2 + 2x_3$

Sub to: $2x_1 - 2x_2 + 3x_3 + x_4 = 5$

$x_1 + x_2 - x_3 + x_5 = 3$

$x_1 - x_2 + x_3 + x_6 = 2$

$$B^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$\rightarrow$ similar 2 B

$$A = \begin{bmatrix} 2 & -2 & 3 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow x_B = \begin{bmatrix} x_2 \\ x_6 \\ x_3 \end{bmatrix}$$

$$C_B = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}$$

# Chapter 6: Duality.

* Primal $\Rightarrow$ std. form

<u>ex:</u> $\max z = 3x_1 + 5x_2$

sub to: $x_1 + 0x_2 \leq 4 \rightarrow y_1$

$0x_1 + 2x_2 \leq 12 \rightarrow y_2$

$3x_1 + 2x_2 \leq 18 \rightarrow y_3$

$x_1, x_2 \geq 0$

$y_1, y_2, y_3$: dual decision variables

in dual:

obj: $\min w = 4y_1 + 12y_2 + 18y_3$

Sub to: $1y_1 + 0y_2 + 3y_3 \geq 3$

$0y_1 + 2y_2 + 2y_3 \geq 5$

$y_{1,2,3} \geq 0$

- RHS of primary = coeff. of y in obj.
- const of dual = # of DV primary
- primary std. form $\Rightarrow$ dual $\geq$
- coeff of 1st const = all coeff of 1st $DV_p$ & so on.
- const. $\geq$ coeff of $DV_p$ in $obj_p$.

<u>ex:</u> P: $\max z = 3x_1 + 4x_2 + 5x_3 + 7x_4$

$x_1 + x_2 - x_3 + x_4 \leq 8$

$x_1 + x_2 + 4x_3 - 2x_4 \leq 7$

$x_3 + x_4 \leq 4$

$x_1, x_2, x_3, x_4 \geq 0$

D: $\min w = 8y_1 + 7y_2 + 4y_3$

~~$y_1 + y_2 + 0y_3 \geq 3$~~

$y_1 + y_2 + 0y_3 \geq 4$

$-y_1 + 4y_2 + y_3 \geq 5$

$y_1 - 2y_2 + y_3 \geq 7$

$y_{1,2,3} \geq 0$.

## * sensible - odd - bizarre :

### ① maximization

1) constraints
- $\leq$ ⇒ sensible
- $=$ ⇒ odd
- $\geq$ ⇒ bizarre

2) Decision variables
- $\geq 0$ ⇒ sensible
- unconstrained ⇒ odd
- $\leq 0$ ⇒ bizarre

### ② minimization:

1) constraints
- $\leq$ ⇒ bizarre
- $=$ ⇒ odd
- $\geq$ ⇒ sensible

2) Decision variables
- $\geq 0$ ⇒ sensible
- unconstrained ⇒ odd
- $\leq 0$ ⇒ bizarre

## * primal problem constraints ⇌ dual decision variable:

- if sensible ⇌ sensible
- if odd ⇌ odd
- if bizarre ⇌ bizarre

## * example :

primal: $\max\ z = 3x_1 + 4x_2 + 5x_3$

sub to:

$x_1 + 2x_2 + x_3 \leq 4$ → S ⇒ $y_1$ is S

$x_1 - x_2 + 2x_3 = 8$ → O ⇒ $y_2$ is O

$2x_1 + 5x_2 - 2x_3 \geq 10$ → B ⇒ $y_3$ is B

$x_1 \geq 0$

$x_2 \leq 0$

$x_3$ is unconstrained

dual : $\min\ w = 4y_1 + 8y_2 + 10y_3$

sub to:

$y_1 + y_2 + 2y_3 \geq 3$

$2y_1 - y_2 + 5y_3 \leq 4$

$y_1 + 2y_2 - 2y_3 = 5$

$y_1 \geq 0$

$y_2$ is unconstrained

$y_3 \leq 0$

• depends on the variable coeff. are taken from. (DV) example: min $z = 3x_1 + 4x_2 + 5x_3$

sub to: $x_1 + 2x_2 + x_3 \leq 4$ → B so $y_1$ B

$x_1 - x_2 + 2x_3 = 8$ → O so $y_2$ O

$2x_1 + 5x_2 - 2x_3 \geq 10$ → S so $y_3$ S

$x_1 \geq 0$ → S

$x_2 \leq 0$ → O

$x_3$ is unconstrained → B

⇒ dual: max $w = 4y_1 + 8y_2 + 10y_3$

sub to: $y_1 + y_2 + 2y_3 \leq 3$

$2y_1 - y_2 + 5y_3 \geq 4$

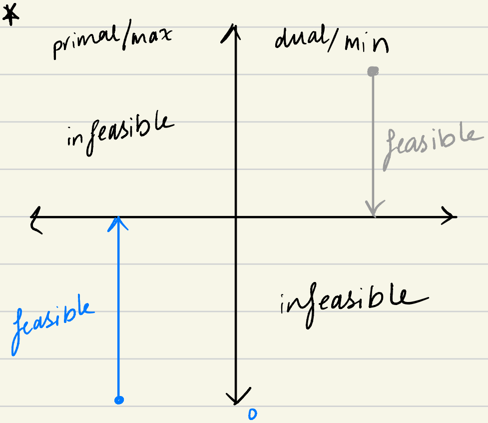
$y_1 + 2y_2 - 2y_3 = 5$

$y_1 \leq 0$

$y_2$ is unconstrained

$y_3 \geq 0$

*




* weak duality property: if $x$ & $y$ both feasible solutions ⇒ $cx \leq yb$

* strong duality property: if $x$ & $y$ optimal soln.s ⇒ $cx = yb$

* complementary soln.s prop: for a solution of $x$ you can find compl. soln. in $y$ that gives same val. but $y$ is infeasible. (coeff of S.V.)
  - applicable to opt. soln. ⇒ comp. optimality.

* symmetry property: all previous rls. are applicable for $y \to x$ & $x \to y$.

  - unbounded & feasible ⇌ unfeasible

* suboptimal : solution on way to most optimal solution.
Superoptimal : a more optimal solution that is infeasible.

* Substitute the coefficients of the optimal tableau of the dual problem in the new constraint from the question if const.
  - → if const is true : not redundant
  - → if const is not true : redundant.

* a feasible solution in:

  ① Big M : when $x_a$ (artificial var) is no longer basic

  ② Two phase: when phase 1 is done

* in comp. property : for opt. soln. $x$ there will be dual soln. $y$ which is the shadow price for $x$

* if we have non basic slack var ⇒ shadow price, RHS increases by coefficient of $x$ in Row 0.

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

- if we inc. 1st const. RHS by 1 → inc by 0
- if we inc. 2nd const. RHS by 1 ⇒ RHS + 3/2
- if we inc. 3rd const. RHS by 1 ⇒ RHS + 1.

three cases of DV:

① DV $\geqslant$ +ve $\Rightarrow$ same

② DV $\geqslant$ −ve :

RHS ①

$$\min \; z = 4x_1 + 7x_2 \quad \Rightarrow \quad 4x_1' - 40 + 7x_2$$

sub to :

$$3x_1 + 2x_2 \leqslant 15 \quad \Rightarrow \quad 3x_1' - 30 + 2x_2 \leqslant 15$$

$$x_1 - x_2 \geqslant 5 \quad \Rightarrow \quad x_1' - 10 - x_2 \geqslant 5$$

$$x_2 \geqslant 0$$

$$x_1 \geqslant -10 \quad \Rightarrow \quad 1)\; x_1 + 10 \geqslant 0 \quad \Rightarrow \quad x_1' \geqslant 0$$

$$x_1' = x_1 + 10$$

$$x_1 = x_1' - 10$$

③ DV unconstrained :

$$\min \; z = 4x_1 + 2x_2 \quad \Rightarrow \quad 4x_1^+ - 4x_1^- + 2x_2$$

sub to :

$$3x_1 + x_2 \geqslant 10 \quad \Rightarrow \quad 3x_1^+ - 3x_1^- + x_2 \geqslant 10$$

$$2x_1 + 5x_2 \leqslant 15 \quad \Rightarrow \quad 2x_1^+ - 2x_1^- + 5x_2 \leqslant 15$$

$$x_2 \geqslant 0$$

$x_1$ unconst. $\Rightarrow$ ① $x_1^+ - x_1^-$ $\Rightarrow$ $x_1^+ \geqslant 0$

$x_1^- \geqslant 0$

if $x_1^+$ BV, RHS = 60 $\Rightarrow$ $x_1 = 60$

if $x_1^-$ BV, RHS = 60 $\Rightarrow$ $x_1 = -60$

* 3 cases for BV :
  1. tie for entering ⇒ choose either
  2. tie for leaving ⇒ degeneracy ⇒ loop
  3. no leaving ⇒ unbounded z

* We get a multioptimal solution if :
  1. objective function is parallel with one of constraints.
  2. if a nonbasic variable has a coefficient of zero in the objective function.

* a constraint is functional if its slack variables $= 0$ & not functional if they $\neq 0$

Chapter 7 ⓒ Changing parameters

* max $z = cx$ $\quad \bar{c}$

sub to: $Ax \leq B$ $\quad \bar{A} \quad \bar{b}$ where: $S^* = B^{-1}$

$x \geq 0$ add DV & const.

$y^* = C_B B^{-1}$

| z | OV | SV | RHS |
|---|---|---|---|
| 1 | $y^*\bar{A} - \bar{c}$ | $y^*$ | $y^* \bar{b}$ |
| 0 | $S^*\bar{A}$ | $S^*$ | $S^* \bar{b}$ |
| 0 | | | |
| 0 | | | |

* recall that:

$C_B B^{-1} A - C$ $\quad C_B B^{-1}$ $\quad C_B B^{-1} b$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | | 0 | 0 | | | |
| 0 | | 0 | 1 | | | |
| 0 | | 1 | 0 | | | |

$B^{-1}A$ $\quad B^{-1}$ $\quad B^{-1}b$

ex. 7.1.1:

ⓐ $x_1 = 5/3$ $\quad x_3 = 3$ $\quad Z = 17$

ⓓ $\bar{c} = [3 \;\; 3 \;\; 4]$ "find revised simplex tableau"

acc. ⓒ

$$\bar{A} = \begin{bmatrix} 6 & 2 & 5 \\ 3 & 3 & 5 \end{bmatrix} \Rightarrow$$

|   | OV | | | SV | | |
|---|---|---|---|---|---|---|
| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
| 1 | 0 | 2 | 0 | $1/5$ | $3/5$ | 17 |
| 0 | 1 | $-1/3$ | 0 | $1/3$ | $-1/3$ | 5 |
| 0 | 0 | 1 | 1 | $-1/5$ | $2/5$ | 3 |

$$\bar{b} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}$$

$\Rightarrow$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | $-4/5$ | 0 | $1/5$ | $3/5$ | 17 |
| 0 | 1 | $-1/3$ | 0 | $1/3$ | $-1/3$ | $5/3$ |
| 0 | 0 | $4/5$ | 1 | $-1/5$ | $2/5$ | 3 |

$\Rightarrow x_2 \to$ entering $\quad x_3 \to$ leaving.

① $R_2 \times 5/4$ ② $1/3\, R_2 + R_1$ ③ $4/5\, R_2 + R_0$

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 1 | 20 |
| 0 | 1 | 0 | $5/12$ | $1/4$ | $-1/6$ | $35/12$ |
| 0 | 0 | 1 | $5/4$ | $-1/4$ | $1/2$ | $15/4$ |

ⓐ adding a const.

ⓑ dual :- min $w = 25y_1 + 20y_2$

sub to:- $6y_1 + 3y_2 \geq 3$

$3y_1 + 4y_2 \geqslant 1$

$5y_1 + 5y_2 \geqslant 4$

$y_1 , y_2 \geqslant 0$

Optimal

$$\boxed{\begin{aligned} w &= 17 \\ y_1 &= 1/5 \\ y_2 &= 3/5 \end{aligned}}$$

$\Rightarrow$ ⓓ $2y_1 + 3y_2 \geqslant 3$

$2(1/5) + 3(3/5) = 11/5 = 2.2$ not $\geqslant 3 \Rightarrow$ const. not redundant $\rightarrow$ makes obj. worse.

ⓕ dual :- min $w = 25y_1 + 20y_2$

sub to:- $6y_1 + 3y_2 \geqslant 3$

$3y_1 + 4y_2 \geqslant 1$

$5y_1 + 5y_2 \geqslant 4$

$3y_1 + 2y_2 \geqslant 2 \implies$ not redundant , opt. worse $\geqslant 17$.

$y_1 , y_2 \geqslant 0$

Q 7.2.2 :

" revise " → revised simplex tableau

ⓐ $b_1 = 30$

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

→ revised bc include $\bar{b}$ so will change

so,

$$\bar{b} = \begin{bmatrix} 30 \\ 90 \end{bmatrix}$$

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 150 |
| 0 | -1 | 1 | 3 | 1 | 0 | 30 |
| 0 | 16 | 0 | -2 | -4 | 1 | -30 → infeasible, superoptimal. |

to make it feasible ⇒ dual simplex method

① take most -ve from RHS.

② BV in it is leaving var.

③ divide obj. by row.

④ take min after abs. value

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 150 |
| 0 | -1 | 1 | 3 | 1 | 0 | 30 |
| 0 | 16 | 0 | -2 | -4 | 1 | -30 ① |
| | ignore | ignore | 2/-2 = -1 | 5/-4 = -1.25 | ignore | |

• $x_5$ leaving

• $x_3$ entering

① $R_2/-2$ ② $-3R_2+R_1$ ③ $-2R_2+R_0$

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 16 | 0 | 0 | 1 | 1 | 120 |
| 0 | 23 | 1 | 0 | -5 | 3/2 | -15 → still not feasible |
| 0 | -8 | 0 | 1 | 2 | -1/2 | 15 |

• $x_4$ ent

• $x_2$ leav

① $R_1/-5$ ② $-2R_1+R_2$ ③ $-R_1+R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | $16+\frac{23}{5}$ | $\frac{1}{5}$ | 0 | 0 | $\frac{13}{10}$ | 117 |
| 0 | $-\frac{23}{5}$ | $-\frac{1}{5}$ | 0 | 1 | $-\frac{3}{10}$ | 3 |
| 0 | $\frac{6}{5}$ | $\frac{2}{5}$ | 1 | 0 | $\frac{1}{10}$ | 9 |

feasible & optimal

ⓑ $b_2 = 70$

$$\bar{b} = \begin{bmatrix} 20 \\ 70 \end{bmatrix}$$

OG tab.

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

change

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | -10 |

→ superopt. & infeasible

① $R_2/-2$ ② $-3R_2+R_1$ ③ $-2R_2+R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 16 | 0 | 0 | 1 | 1 | 90 |
| 0 | 23 | 1 | 0 | -5 | $\frac{3}{2}$ | 5 |
| 0 | -8 | 0 | 1 | 2 | $-\frac{1}{2}$ | 5 |

optimal & feasible

(c) OG tab.

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

$$\bar{b} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}$$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 50 |
| 0 | -1 | 1 | 3 | 1 | 0 | 10 |
| 0 | 16 | 0 | -2 | -4 | 1 | 60 |

feasible & optimal.

(d) $c_3 = 8$ $\bar{c} = [-5 \quad 5 \quad 8]$

depends on $\bar{c}$ so will change

$y^* \bar{A} - \bar{c}$

OG tab.

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

$$\bar{A} = \begin{bmatrix} -1 & 1 & 3 \\ 12 & 4 & 10 \end{bmatrix}$$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 7 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

(e) $\bar{c} = [-2 \quad \overbrace{5 \quad 13}^{\text{changed}}]$ $\bar{A} = \begin{bmatrix} 0 & 1 & 3 \\ 5 & 4 & 10 \end{bmatrix}$ $\bar{b} = \begin{bmatrix} 20 \\ 90 \end{bmatrix}$

OG tab

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 2 | 0 | 2 | 5 | 0 | 100 |
| 0 | 0 | 1 | 3 | 1 | 0 | 20 |
| 0 | 5 | 0 | -2 | -4 | 1 | 10 |

(f) $x_2 \rightarrow$ B.V.

$$\bar{A} = \begin{bmatrix} -1 & 1 & 3 \\ 12 & 4 & 10 \end{bmatrix} \qquad \bar{c} = \begin{bmatrix} -5 & 5 & 13 \end{bmatrix} \qquad \Rightarrow \bar{A} = \begin{bmatrix} -1 & 2 & 3 \\ 12 & 5 & 10 \end{bmatrix} \quad \bar{c} = \begin{bmatrix} -5 & 6 & 13 \end{bmatrix}$$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 4 | 2 | 5 | 0 | 100 |
| 0 | -1 | 2 | 3 | 1 | 0 | 20 |
| 0 | 16 | -3 | -2 | -4 | 1 | 10 |

to restore gaussian form bc $x_2 \rightarrow$ BV by first row

① $R_1/2$ ② $3R_1 + R_2$ ③ $-4R_1 + R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 2 | 0 | -4 | 3 | 0 | 60 |
| 0 | $-1/2$ | 1 | $3/2$ | $1/2$ | 0 | 10 |
| 0 | 14.5 | 0 | $5/2$ | $-5/2$ | 1 | 40 |

feasible but not optimal.
(revised tableau)

① $2/3 R_1$ ② $-5/2 R_1 + R_2$ ③ $4R_1 + R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | $2+4/3$ | $8/3$ | 0 | $13/3$ | 0 | $260/3$ |
| 0 | $-1/3$ | $2/3$ | 1 | $1/3$ | 0 | $20/3$ |
| 0 | $14.5+5/6$ | $-5/3$ | 0 | -3.3 | 1 | $70/3$ |

(g)

OG tab

| z | $x_1$ | $x_2$ | $x_3$ | $x_6$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 0 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 0 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | 0 | -4 | 1 | 10 |

$x_6 \rightarrow$ w/ o.v.

$$\bar{c} = [-5 \quad 5 \quad 13 \quad 0] \qquad \bar{b} = \begin{bmatrix} 20 \\ 90 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 12 & 4 & 10 & 0 \end{bmatrix}$$

| z | $x_1$ | $x_2$ | $x_3$ | $x_6$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -7 | -4 | 1 | 10 |

$$\bar{c} = [-5 \quad 5 \quad 13 \quad 10] \qquad \bar{b} = \begin{bmatrix} 20 \\ 90 \\ 50 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 12 & 4 & 10 & 5 \end{bmatrix}$$

Optimal, feasible, proper gaussian form $\Rightarrow$ doesn't affect optimal value, corresponding dual constraint is redundant.

(h)

OG tab

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 10 |

new const

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 0 | 10 |
| 0 | 2 | 3 | 5 | 0 | 0 | 1 | 50 |

① $-3R_1 + R_3$ ② $x_6$ leaving (using dual simplex) $x_3$ entering

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 5 | 0 | 0 | 100 |
| 0 | -1 | 1 | 3 | 1 | 0 | 0 | 20 |
| 0 | 16 | 0 | -2 | -4 | 1 | 0 | 10 |
| 0 | 5 | 0 | -4 | -3 | 0 | 1 | -10 |

① $R_3/-4$ ② $2R_3+R_2$ ③ $-3R_3+R_1$ ④ $-2R_3+R_0$

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | $5/2$ | 0 | 0 | 3.5 | 0 | $1/2$ | 95 |
| 0 | $11/4$ | 1 | 0 | $-5/4$ | 0 | $3/4$ | $50/4$ |
| 0 | 14.5 | 0 | 0 | -2.5 | 1 | $-1/2$ | 15 |
| 0 | $-5/4$ | 0 | 1 | $3/4$ | 0 | $-1/4$ | $10/4$ |

feasible & optimal

# Chapter 9

★ Prototype example :

① DV: $x_{11}$ : # of trucks that will be shipped from cannery 1 to WH 1

$x_{12}$ : " " C 1 to WH 2

$\Rightarrow$ $x_{ij}$ : # of trucks shipped from $i^{th}$ cannery to WH $j$

$i : \{1, 2, 3\}$ $\quad$ $j : \{1, 2, 3, 4\}$

② Obj : $\min z = 464x_{11} + 513x_{12} + 654x_{13} \cdots 685x_{34}$

or

$\min z = \sum_{i=1}^{3}\sum_{j=1}^{4} S_{ij} x_{ij}$ where $S_{ij}$ is shipping cost for TL

③ const :

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 75$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 125$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 100$$

$\rightarrow$ can be $\sum_{j=1}^{4} x_{ij} = O_i$ where $O_i$ is output.

$$x_{11} + x_{21} + x_{31} = 80$$
$$x_{12} + x_{22} + x_{32} = 65$$
$$x_{13} + x_{23} + x_{33} = 70$$
$$x_{14} + x_{24} + x_{34} = 85$$

$\rightarrow$ $\sum_{i=1}^{3} x_{ij} = A_j$ where $A_j$ is allocation.

$$x_{ij} \geq 0 \quad \forall i \ \forall j$$

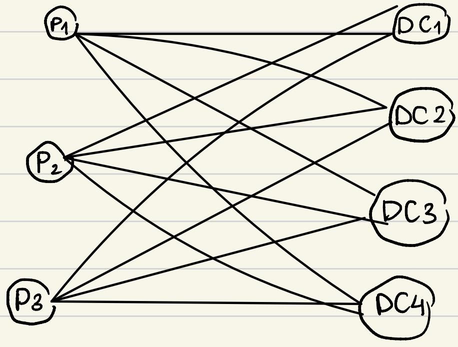
★ to make sure you get int. solution :

since output sum = allocation sum $\rightarrow$ 1st group of const. can be $=$

★

**9.1-2.** The Childfair Company has three plants producing child push chairs that are to be shipped to four distribution centers. Plants 1, 2, and 3 produce 12, 17, and 11 shipments per month, respectively. Each distribution center needs to receive 10 shipments per month. The distance from each plant to the respective distributing centers is given below:

| | | cost × 100.5 **Distance** | | | | |
|---|---|---|---|---|---|---|
| | | **Distribution Center** | | | | |
| | | **1** | **2** | **3** | **4** | output |
| | 1 | 800 miles | 1,300 miles | 400 miles | 700 miles | 12 |
| Plant | 2 | 1,100 miles | 1,400 miles | 600 miles | 1,000 miles | 17 |
| | 3 | 600 miles | 1,200 miles | 800 miles | 900 miles | 11 |
| allocated | | 10 | 10 | 10 | 10 | |




⇒ Quiz

**9.1-6.** The Onenote Co. produces a single product at three plants for four customers. The three plants will produce 60, 80, and 40 units, respectively, during the next time period. The firm has made a commitment to sell 40 units to customer 1, 60 units to customer 2, and at least 20 units to customer 3. Both customers 3 and 4 also want to buy as many of the remaining units as possible. The net profit associated with shipping a unit from plant $i$ for sale to customer $j$ is given by the following table:

| | | **Customer** | | | | |
|---|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** | $O_i$ |
| | 1 | $800 | $700 | $500 | $200 | 60 |
| Plant | 2 | 500 | 200 | 100 | 300 | 80 |
| | 3 | 600 | 400 | 300 | 500 | 40 |
| Q | | 40 | 60 | 20 | 40 | 180 |

$\underbrace{40 \quad 60 \quad 20 \quad 40}_{160}$

Management wishes to know how many units to sell to customers 3 and 4 and how many units to ship from each of the plants to each of the customers to maximize profit.

**(a)** Formulate this problem as a transportation problem where the objective function is to be maximized by constructing the appropriate parameter table that gives unit profits.

**(b)** Now formulate this transportation problem with the usual objective of minimizing total cost by converting the parameter table from part (*a*) into one that gives unit costs instead of unit profits.

**(c)** Display the formulation in part (*a*) on an Excel spreadsheet.

C **(d)** Use this information and the Excel Solver to obtain an optimal solution.

C **(e)** Repeat parts (*c*) and (*d*) for the formulation in part (*b*). Compare the optimal solutions for the two formulations.

appropriate parameter table :

| | customer 1 | 2 | 3 | 4 | Dummy | output |
|---|---|---|---|---|---|---|
| 1 | 800 | . . | . | . . | 0 | 60 |
| 2 | | $P_{ij}$ | | | 0 | 80 |
| 3 | | | | | 0 | 40 |
| Q | 40 | 60 | 20 | 40 | 20 | 180 |

$\underbrace{40 \quad 60 \quad 20 \quad 40 \quad 20}_{180}$

⇒

ⓒ DV:- $x_{ij}$ : # units sold from plant $i$ to customer $j$

$i = \{1 \ 2 \ 3\}$

$j = \{1 \ 2 \ 3 \ 4 \ 5\}$

ⓞ obj: $\max z = 800\, x_{11} + 700\, x_{12} \ldots$

$= \sum_i \sum_j P_{ij} x_{ij}$

→ Sub to :- $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 60$

⇒ $\sum_{j=1}^{5} x_{ij} = O_i \quad \forall i$

&

$x_{11} + x_{21} + x_{31} = 40$

⇒ $\sum_{i=1}^{3} x_{ij} = Q_j \quad \forall j$

&

$x_{ij} \geq 0 \quad \forall i, \forall j$

→ solution if we remove "at least"

Solution if we keep "at least":

⇒

| | | Customer | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | $O_i$ |
| Plant | 1 | \$800 | \$700 | \$500 | \$200 | 60 |
| | 2 | 500 | 200 | 100 | 300 | 80 |
| | 3 | 600 | 400 | 300 | 500 | 40 |
| Q | | 40 | 60 | ≥20 | ≥0 | |
| max | | 40 | 60 | 80 | 60 | |

$\underbrace{40 \quad 60 \quad 80 \quad 60}_{240}$

• 40+60+20 = 120

180 − 120 = 60 units → cust. 3 or 4

• split cust. 3 → 20 definitely
→ 60 → yes
→ no

• cust 4 will either recive 60 or not

⇒ appropriate parameter table:

customers

| plants | 1 | 2 | 3 | 3extra | 4 | $O_i$ |
|---|---|---|---|---|---|---|
| 1 | 800 | 700 | 500 | 500 | 200 | 60 |
| 2 | 500 | 200 | 100 | 100 | 300 | 80 |
| 3 | 600 | 400 | 300 | 300 | 500 | 40 |
| 4 | −M | −M | −M | 0 | 0 | 60 |
| Q | 40 | 60 | 20 | 60 | 60 | 240 |

dummy plant

• −M bc negative effect for max profit.

• 60 → max extra value.

① DV :- $x_{ij}$ : # units sold from plant $i$ to customer $j$.

$i = \{1 \ 2 \ 3 \ 4\}$

$j = \{1 \ 2 \ 3 \ 4 \ 5\}$

② obj: $\max z = \sum_j \sum_i P_{ij} x_{ij}$

③ Subto: $\sum_j x_{ij} = O_i \quad \forall i$

$\sum_i x_{ij} = Q_j$

**9.1-9.** The MJK Manufacturing Company must produce two products in sufficient quantity to meet contracted sales in each of the next three months. The two products share the same production facilities, and each unit of both products requires the same amount of production capacity. The available production and storage facilities are changing month by month, so the production capacities, unit production costs, and unit storage costs vary by month. Therefore, it may be worthwhile to overproduce one or both products in some months and store them until needed.

For each of the three months, the second column of the following table gives the maximum number of units of the two products combined that can be produced on Regular Time (RT) and on Overtime (O). For each of the two products, the subsequent columns give (1) the number of units needed for the contracted sales, (2) the cost (in thousands of dollars) per unit produced on Regular Time, (3) the cost (in thousands of dollars) per unit produced on Overtime, and (4) the cost (in thousands of dollars) of storing each extra unit that is held over into the next month. In each case, the numbers for the two products are separated by a slash /, with the number for Product 1 on the left and the number for Product 2 on the right.

| | Maximum Combined Production | | Product 1/Product 2 | | | |
|---|---|---|---|---|---|---|
| | | | | Unit Cost of Production ($1,000's) | | Unit Cost of Storage |
| Month | RT | OT | Sales (1st prod. / 2nd prod.) | RT | OT | ($1,000's) |
| 1 | 10 | 3 | 5/3 | 15/16 | 18/20 | 1/2 |
| 2 | 8 | 2 | 3/5 | 17/15 | 20/18 | 2/1 |
| 3 | 10 | 3 | 4/4 | 19/17 | 22/22 | |

The production manager wants a schedule developed for the number of units of each of the two products to be produced on Regular Time and (if Regular Time production capacity is used up) on Overtime in each of the three months. The objective is to minimize the total of the production and storage costs while meeting the contracted sales for each month. There is no initial inventory, and no final inventory is desired after the three months.

(a) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

C (b) Obtain an optimal solution.

• if cost ⇒ M

if profit ⇒ −M

app. parameter table

usage months

| production months | 1: (1) $P_1$ | 1: (2) $P_2$ | 2: (3) $P_1$ | 2: (4) $P_2$ | 3: (5) $P_1$ | 3: (6) $P_2$ | Dummy | production capacity $P_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 (1) RT | 15 | 16 | 15+1 | 16+2 | 15+1+2 | 16+2+1 | 0 | 10 |
| (2) OT | 18 | 20 | 18+1 | 20+2 | 18+1+2 | 20+2+1 | 0 | 3 |
| 2 (3) RT | M | M | 17 | 15 | 17+2 | 15+1 | 0 | 8 |
| (4) OT | M | M | 20 | 18 | 20+2 | 18+1 | 0 | 2 |
| 3 (5) RT | M | M | M | M | 19 | 17 | 0 | 10 |
| (6) OT | M | M | M | M | 22 | 22 | 0 | 3 |
| $S_j$ Sales | 5 | 3 | 3 | 5 | 4 | 4 | 12 | total = 36 |

$C_{ij}$

since sum of production > sales ⇒ add dummy month

Mathematical Model:

• production time periods • usage time periods

(1) DV:- $x_{ij}$ : # of products produced in time period $i$ to be used in time period $j$.

$i = \{1 \dots 6\}$ $j = \{1 \dots 7\}$

(2) obj:- min cost $z = \sum_{j=1}^{7} \sum_{i=1}^{6} C_{ij} x_{ij}$

③ sub to :- $\sum_{j=1}^{7} x_{ij} = P_i \quad \forall i \rightarrow x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} + x_{37} = 8$

$$\sum_{i=1}^{6} x_{ij} = S_j \quad \forall j \rightarrow x_{15} + x_{25} + x_{35} + x_{45} + x_{55} + x_{65} = 4$$

$$x_{ij} \geq 0 \quad \forall i \quad \forall j$$

* Prototype example : "Job shop company"

**Prototype Example**

The JOB SHOP COMPANY has purchased three new machines of different types. There are four available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centers that will have a heavy work flow to and from these machines. (There will be no work flow *between* the new machines.) Therefore, the objective is to assign the new machines to the available locations to minimize the total cost of materials handling. The estimated cost in dollars per hour of materials handling involving each of the machines is given in Table 9.24 for the respective locations. Location 2 is not considered suitable for machine 2, so no cost is given for this case.

To formulate this problem as an assignment problem, we must introduce a *dummy machine* for the extra location. Also, an extremely large cost $M$ should be attached to the assignment of machine 2 to location 2 to prevent this assignment in the optimal solution. The resulting assignment problem *cost table* is shown in Table 9.25. This cost table contains all the necessary data for solving the problem. The optimal solution is to assign machine 1 to location 4, machine 2 to location 3, and machine 3 to location 1, for a total cost of \$29 per hour. The dummy machine is assigned to location 2, so this location is available for some future real machine.

[10]For example, see L. J. LeBlanc, D. Randels, Jr., and T. K. Swann: "Heery International's Spreadsheet Optimization Model for Assigning Managers to Construction Projects," *Interfaces*, **30**(6): 95–106, Nov.–Dec. 2000. Page 98 of this article also cites seven other applications of the assignment problem.



**TABLE 9.24** Materials-handling cost data ($) for Job Shop Co.

| | | Location | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | |
| Machine | 1 | 13 | 16 | 12 | 11 | 1 |
| | 2 | 15 | M | 13 | 20 | 1 |
| | 3 | 5 | 7 | 10 | 6 | 1 |
| | D | 0 | 0 | 0 | 0 | 1 |

① DV :- $x_{ij} = \begin{cases} 1 & \text{if machine } i \text{ located at } j \\ 0 & \text{otherwise} \end{cases}$

$i = 1, 2, 3, 4 \qquad j = 1, 2, 3, 4$

② obj :- min cost $z = \sum_{j=1}^{4} \sum_{i=1}^{4} C_{ij} x_{ij}$

③ Sub to :- $\sum_{j=1}^{4} x_{ij} = 1 \quad \forall i$

$$\Rightarrow x_{31} + x_{32} + x_{33} + x_{34} = 1$$

$$\sum_{i=1}^{4} x_{ij} = 1 \quad \forall j$$

$$\Rightarrow x_{12} + x_{22} + x_{32} + x_{42} = 1$$

$$x_{ij} \geq 0 \quad \forall i, \forall j$$

**9.3-4.*** The coach of an age group swim team needs to assign swimmers to a 200-yard medley relay team to send to the Junior Olympics. Since most of his best swimmers are very fast in more than one stroke, it is not clear which swimmer should be assigned to each of the four strokes. The five fastest swimmers and the best times (in seconds) they have achieved in each of the strokes (for 50 yards) are

$i$

| Stroke ($j$) | Carl | Chris | David | Tony | Ken | |
|---|---|---|---|---|---|---|
| Backstroke | 37.7 | 32.9 | 33.8 | 37.0 | 35.4 | 1 |
| Breaststroke | 43.4 | 33.1 | 42.2 | 34.7 | 41.8 | 1 |
| Butterfly | 33.3 | 28.5 | 38.9 | 30.4 | 33.6 | 1 |
| Freestyle | 29.2 | 26.4 | 29.6 | 28.5 | 31.1 | 1 |
| D | 0 | 0 | 0 | 0 | 0 | 1 |
| | 1 | 1 | 1 | 1 | 1 | |

The coach wishes to determine how to assign four swimmers to the four different strokes to minimize the sum of the corresponding best times.

① DV :- $x_{ij} = \begin{cases} 1 & \text{if swimmer } i \text{ is assigned to stroke } j \\ 0 & \text{otherwice} \end{cases}$

② obj :- $\min z = \sum_{j}^{5} \sum_{i}^{5} T_{ij} \, x_{ij}$

③ Sub to :- $\sum_{i}^{5} x_{ij} = 1 \quad \forall j$

$\sum_{j}^{5} x_{ij} = 1 \quad \forall i$

**9.1-7.** The Move-It Company has two plants producing forklift trucks that then are shipped to three distribution centers. The production costs are the same at the two plants, and the cost of shipping for each truck is shown for each combination of plant and distribution center:

| | | Distribution Center | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | |
| Plant | A | $800 | $700 | $400 | $\leqslant 50$ |
| 60 | B | $600 | $800 | $500 | $\leqslant 50$ |
| | units | 20 | 20 | 20 | |

A total of 60 forklift trucks are produced and shipped per week. Each plant can produce and ship any amount up to a maximum of 50 trucks per week, so there is considerable flexibility on how to divide the total production between the two plants so as to reduce shipping costs. However, each distribution center must receive exactly 20 trucks per week.

Management's objective is to determine how many forklift trucks should be produced at each plant, and then what the overall shipping pattern should be to minimize total shipping cost.

**(a)** Formulate this problem as a transportation problem by constructing the appropriate parameter table.

**(b)** Display the transportation problem on an Excel spreadsheet.

C **(c)** Use Solver to obtain an optimal solution.

**9.3-7.** Reconsider Prob. 9.1-7. Now assume that distribution centers 1, 2, and 3 must receive exactly 10, 20, and 30 units per week, respectively. For administrative convenience, management has decided that each distribution center will be supplied totally by a single plant, so that one plant will supply one distribution center and the other plant will supply the other two distribution centers. The choice of these assignments of plants to distribution centers is to be made solely on the basis of minimizing total shipping cost.

**(a)** Formulate this problem as an assignment problem by constructing the appropriate cost table, including identifying the corresponding assignees and tasks.

C **(b)** Obtain an optimal solution.

**(c)** Reformulate this assignment problem as an equivalent transportation problem (with four sources) by constructing the appropriate parameter table.

C **(d)** Solve the problem as formulated in part (*c*).

**(e)** Repeat part (*c*) with just two sources.

C **(f)** Solve the problem as formulated in part (*e*).

9.1.7.

Ⓒ transportation solution:

- DV :- $x_{ij}$ : # FL that should be produced at plant $i$ & shipped to DC $j$

- obj :- $\min \sum_{i=1}^{2} \sum_{j=1}^{3} C_{ij} X_{ij}$

- sub to :-

$x_{11} + x_{21} = 20$ $\qquad x_{11} + x_{12} + x_{13} \leqslant \boxed{50}$

$x_{12} + x_{22} = 20$ $\qquad x_{21} + x_{22} + x_{23} \leqslant \boxed{50}$ } b on matlab

$x_{13} + x_{23} = 20$ $\qquad x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = 60$

On matlab : A is coeff of ineq. matrix, define as empty matrix then edit from workspace.

9.3.7.

② Assignment Solution:

| | | Distribution Center | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | |
| Plant | A | $800 | $700 | $400 | $\leq 50$ |
| | B | $600 | $800 | $500 | $\leq 50$ |
| | unit | 10 | 20 | 30 | |

Dist center.

| | | 10<br>1 | 20<br>2 | 30<br>3 |
|---|---|---|---|---|
| A | 10 | 800(10) | M | M |
| | 20 | 800 (10) | 700( 20) | M |
| | 30 | 800(10) | 700(20) | 400( 30) |
| B | 10 | 600 (10) | M | M |
| | 20 | 600 (10) | 800( 20) | M |
| | 30 | 600 (10) | 800 (20) | 500 (30) |

- DV:- $x_{ij} = \begin{cases} 1 & \text{if branch } i \text{ assigned to DC. } j \\ 0 & \text{otherwise.} \end{cases}$

- obj:- $\min\ z = \sum_{j=1}^{3} \sum_{i=1}^{6} c_{ij}\, x_{ij}$

- Sub to:- $x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} = 1$

$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} = 1$

$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} = 1$

$x_{11} + x_{12} + x_{13} \leq 2$

$\sum_{j=1}^{3} x_{ij} \leq 2 \quad \forall i$

$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} \leq 2$

$x_{41} + x_{42} + x_{43} + x_{51} + x_{52} + x_{53} + x_{61} + x_{62} + x_{63} \leq 2$

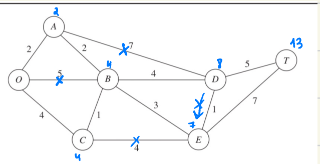
# Chapter 10

* Prototype example "Seervada Park":

T → final entrance.

shortest path between O & T :

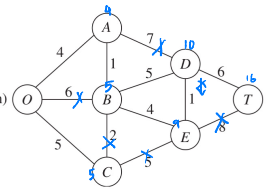
**FIGURE 10.1**
The road system for Seervada Park.



*

**10.3-4.*** Use the algorithm described in Sec. 10.3 to find the *shortest path* through each of the following networks, where the numbers represent actual distances between the corresponding nodes.

(a)



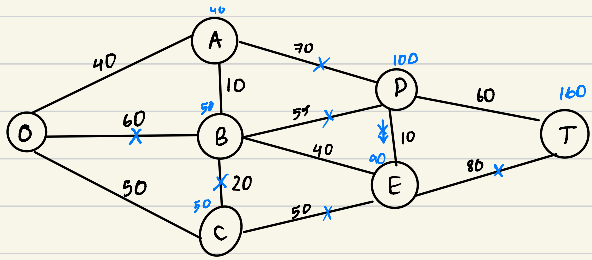

OABDT & OABEDT = 16

**10.3-2.** You need to take a trip by car to another town that you have never visited before. Therefore, you are studying a map to determine the shortest route to your destination. Depending on which route you choose, there are five other towns (call them A, B, C, D, E) that you might pass through on the way. The map shows the mileage along each road that directly connects two towns without any intervening towns. These numbers are summarized in the following table, where a dash indicates that there is no road directly connecting these two towns without going through any other towns.

| | Miles between Adjacent Towns | | | | | |
|---|---|---|---|---|---|---|
| **Town** | **A** | **B** | **C** | **D** | **E** | **Destination** |
| Origin | 40 | 60 | 50 | — | — | — |
| A | | 10 | — | 70 | — | — |
| B | | | 20 | 55 | 40 | — |
| C | | | | — | 50 | — |
| D | | | | | 10 | 60 |
| E | | | | | | 80 |

(a) Formulate this problem as a shortest-path problem by drawing a network where nodes represent towns, links represent roads, and numbers indicate the length of each link in miles.
(b) Use the algorithm described in Sec. 10.3 to solve this shortest-path problem.
C (c) Formulate and solve a spreadsheet model for this problem.
(d) If each number in the table represented your *cost* (in dollars) for driving your car from one town to the next, would the answer in part (*b*) or (*c*) now give your minimum cost route?
(e) If each number in the table represented your *time* (in minutes) for driving your car from one town to the next, would the answer in part (*b*) or (*c*) now give your minimum time route?

⇒



⇒ OABEDT = 160

* seervada park second issue: "Minimum Spanning Tree"

② all units connected through telephone lines → all nodes must be connected

Min spanning tree algorithm:

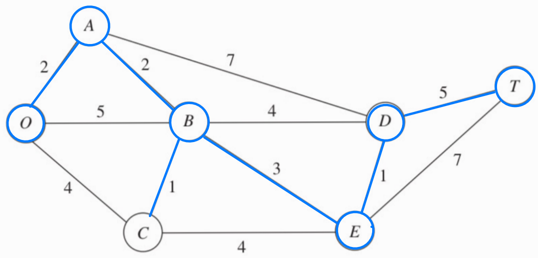
① choose any node

② see # of lines & choose min

③ treat entire as 1 node & repeat

• if 2 lines are equal choose either → 2 opt. solutions.

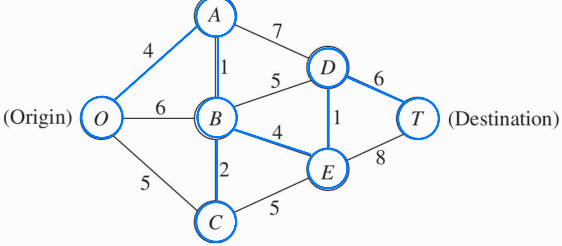
**FIGURE 10.1**
The road system for Seervada Park.



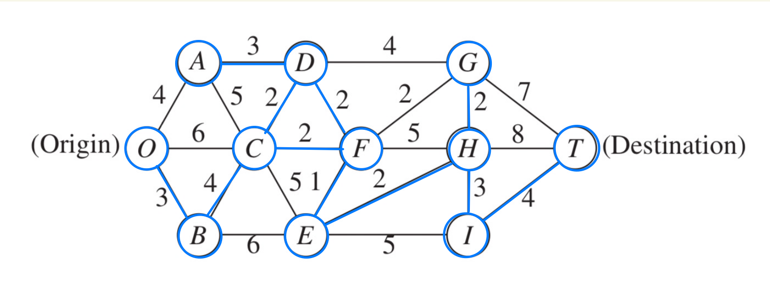

starting with D:
= 14 ⇒ min.

**10.3-4.*** Use the algorithm described in Sec. 10.3 to find the *shortest path* through each of the following networks, where the numbers represent actual distances between the corresponding nodes.

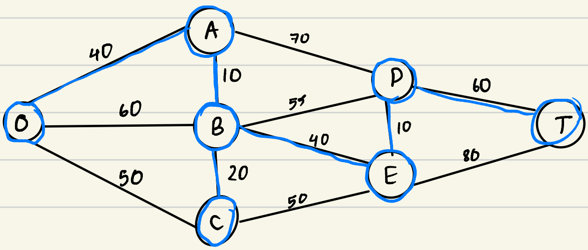
**(a)**




starting with B:
⇒ 18




= 26




* Seervada Park 3rd Problem :

③ Max # of sightseers.

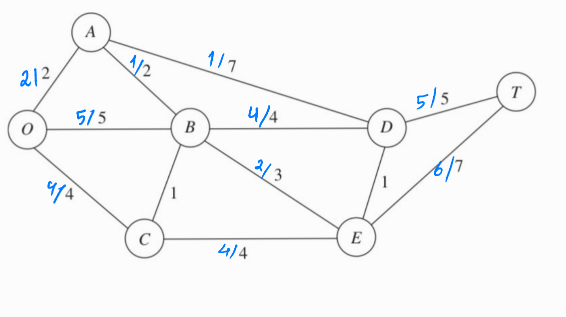



**FIGURE 10.1**
The road system for Seervada Park.

→ #s now are max that can be accomodated.

⇒ O is source node

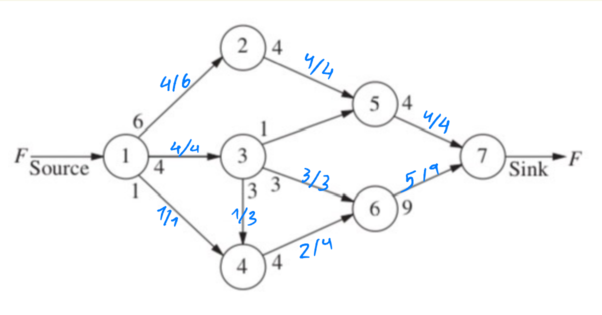
⇒ T is demand node

⇒ ABCDE are transshipment nodes.

Algorithm : ① choose any path
② find max flow.

ex: OBDT = (5, 4, 5) = 4
OADT = (2, 7, 1) = 1
OCET = (4, 4, 7) = 4
OABET = (1, 2, 3, 3) = 1
OBET = (1, 2, 2) = 1
→ max flow = 4+1+4+1+1 = 11

## * 10.5.1




① 1 2 5 7 = (6, 4, 4) = 4

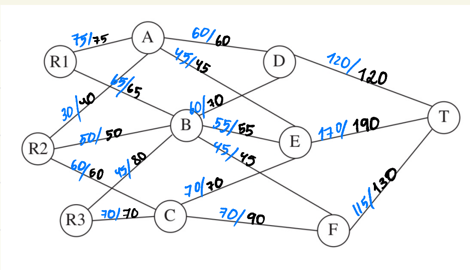
② 1 3 6 7 = (4, 3, 9) = 3

③ 1 4 6 7 = (1, 4, 6) = 1

④ 1 3 4 6 7 = (1, 3, 3, 5) = 1

max flow = 9

## * 10.5.3




| To / From | A | B | C | To / From | D | E | F | To / From | T |
|---|---|---|---|---|---|---|---|---|---|
| R1 | 75 | 65 | — | A | 60 | 45 | — | D | 120 |
| R2 | 40 | 50 | 60 | B | 70 | 55 | 45 | E | 190 |
| R3 | — | 80 | 70 | C | — | 70 | 90 | F | 130 |

$R_1$ ADT = (75, 60, 120) = 60

$R_1$ BDT = (65, 70, 60) = 60

$R_1$ AET = (15, 45, 190) = 15

$R_1$ BET = (5, 5, 55, 175) = 5

$R_2$ AET = (40, 30, 170) = 30

$R_2$ BET = (50, 50, 140) = 50

$R_2$ CET = (60, 70, 90) = 60

max flow = 395

$R_3$ CET = (70, 10, 30) = 10

$R_3$ BFT = (80, 45, 130) = 45

$R_3$ CFT = (60, 90, 85) = 60

9.2:

* $\min \ z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Sub to: $\sum_{j=1}^{n} x_{ij} = s_i \Rightarrow$ m sources $u_i$

$\sum_{i=1}^{m} x_{ij} = d_j \Rightarrow$ n destinations $v_j$

trans. problem model:

$$\min \ z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + M(z_1 + \dots z_{m+n})$$

Sub to: $\sum_{j=1}^{n} x_{ij} + z_i = s_i$

$\sum_{i=1}^{m} x_{ij} + z_j = d_j$

| Line | $z$ | $x_{ij}$ | $z_i$ | $z_{m+j}$ | RHS |
|---|---|---|---|---|---|
| 0 | -1 | $c_{ij}$ | M | M | 0 |
| 1 | 0 | | | | |
| $\vdots$ $i$ | 0 | 1 | 1 | 0 | $s_i$ |
| $\vdots$ | 0 | | | | |
| $m+j$ $\vdots$ | 0 | 1 | 0 | 1 | $d_j$ |
| $m+n$ | 0 | 0 | 0 | 0 | |

everything else $= 0$.

iteration 1 :

| line | $Z$ | $x_{ij}$ | $z_i$ | $z_{m+j}$ | RHS |
|---|---|---|---|---|---|
| 0 | $-1$ | $c_{ij}-u_i-v_j$ | $M-u_i$ | $M-v_j$ | $-\sum_{i=1}^{m} s_i u_i - \sum_{j=1}^{n} d_j v_j$ |

- where $u$ & $v$ are the dual variables

- $x_{ij}$ nonbasic $\Rightarrow$ $c_{ij}-u_i-v_j$ is the rate at which $Z$ will change as $x_{ij}$ is increased

* in transp. simplex $\Rightarrow$ 1 no artificial var needed.

2 row 0 can be obtained without using other rows $\Rightarrow$ $c_{ij}-u_i-v_j=0$

3

**TABLE 9.15** Format of a transportation simplex tableau

| | | Destination | | | | | |
|---|---|---|---|---|---|---|---|
| | | **1** | **2** | ··· | ***n*** | **Supply** | $u_i$ |
| Source | 1 | $c_{11}$ | $c_{12}$ | ··· | $c_{1n}$ | $s_1$ | |
| | 2 | $c_{21}$ | $c_{22}$ | ··· | $c_{2n}$ | $s_2$ | |
| | ⋮ | ··· | ··· | ··· | ··· | ⋮ | |
| | $m$ | $c_{m1}$ | $c_{m2}$ | ··· | $c_{mn}$ | $s_m$ | |
| Demand | | $d_1$ | $d_2$ | ··· | $d_n$ | $Z =$ | |
| | $v_j$ | | | | | | |

Additional information to be added to each cell:

| *If $x_{ij}$ is a basic variable* | *If $x_{ij}$ is a nonbasic variable* |
|---|---|
| $c_{ij}$   $x_{ij}$ | $c_{ij}$   $c_{ij}-u_i-v_j$ |

[4]Since nonbasic variables are automatically zero, the current BF solution is fully identified by recording just the values of the basic variables. We shall use this convention from now on.

4 iterations $\rightarrow$ 1 initialization

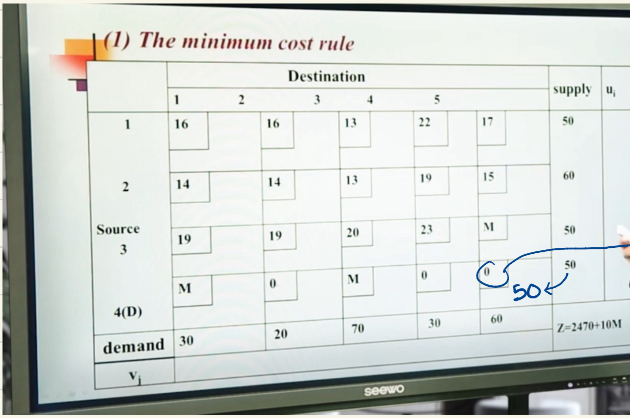
$\rightarrow$ 2 optimality test

$\rightarrow$ 3 iteration.

\# of func. const $= m+n$

* initialization : to select BV : ① Northwest corner rule
② Vogel's approx. method
③ Russel's approx. method

① min cost : give priority to mode with least cost / unit.




| Source | 1 | 2 | 3 | 4 | 5 | supply |
|---|---|---|---|---|---|---|
| 1 | 16 | 16 | 13 | 22 | 17 | 50 |
| 2 | 14 | 14 | 13 | 19 | 15 | 60 |
| 3 | 19 | 19 | 20 | 23 | M | 50 |
| 4(D) | M | 0 | M | 0 | 0 | 50 |
| demand | 30 | 20 | 70 | 30 | 60 | Z=2470+10M |

since source 4 min 0 can supply only 50 we give it 50
& eliminate the row

| Source | Destination 1 | 2 | 3 | 4 | 5 | supply | $u_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 16 | 16 | 13 / 50 | 22 | 17 | 50 | |
| 2 | 14 / 30 | 14 / 10 | 13 / 20 | 19 | 15 | 60 | |
| 3 | 19 | 19 / 10 | 20 | 23 / 30 | M / 10 | 50 | |
| 4(D) | M | 0 | M | 0 | 0 / 50 | 50 | |
| demand | 30 | 20 | 70 20 | 30 | 60 | Z=2470+10M | |
| $v_j$ | | | | | | | |

satisfied
so eliminate

$m+n-1 = 8$ ✓
initial BF soln.

② Northwest corner rule : ① select $x_{11}$

② if $x_{ij}$ was the last BV selected , select $x_{i,j+1}$ (column)

if source $i$ has any supply remaining

③ if not, select $x_{i+1,j}$. → row

| | Destination | | | | | supply | $u_i$ |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | | |
| 1 | 16 / 30 | 16 / 20 | 13 | 22 | 17 | 50 | |
| Source 2 | 14 | 14 / 0 | 13 / 60 | 19 | 15 | 60 | |
| 3 | 19 | 19 | 20 / 10 | 23 / 30 | M / 10 | 50 | |
| 4(D) | M | 0 | M | 0 | 0 / 50 | 50 | |
| demand | 30 | 20 | 70 | 30 | 60 | Z=2470+10M | |
| $v_j$ | | | | | | | |

→ 30 eliminates D1 so eliminate column

$z = \Sigma(\text{cost} \times \text{unit})$

③ Vogel's approx. method : for each row & column find diff between smallest & next to the smallest unit cost $c_{ij}$ still remaining in that row/column → having largest diff → select the var. having smallest remaining unit cost.

| | | Destination | | | | | Supply | Row Difference |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | | |
| | 1 | 16 | 16 | 13 | 22 | 17 | 50 | 3 |
| Source | 2 | 14 | 14 | 13 | 19 | 15 | 60 | 1 |
| | 3 | 19 | 19 | 20 | 23 | M | 50 | 0 |
| | 4(D) | M | 0 | M | 0 | 0 | 50 | 0 |
| Demand | | 30 | 20 | 70 | 30 | 60 | | |
| Column difference | | 2 | 14 | 0 | 19 | 15 | | |

① 16 − 14 = 2

② largest row diff ⇒ 3

column ⇒ 19

⇒ smallest cost in column 4 is 0 → 30

③ eliminate column

| | | Destination | | | | Row | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 5 | Supply | Difference |
| | 1 | 16 | 16 | 13 | 17 | 50 | 3 |
| Source | 2 | 14 | 14 | 13 | 15 | 60 | 1 |
| | 3 | 19 | 19 | 20 | M | 50 | 0 |
| | 4(D) | M | 0 | M | 0 | 20 | 0 |
| Demand | | 30 | 20 | 70 | 60 | | |
| Column difference | | 2 | 14 | 0 | 15 | | |

| | | Destination | | | | Row | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 5 | Supply | Difference |
| | 1 | 16 | 16 | 13 | 17 | 50 | 3 |
| Source | 2 | 14 | 14 | 13 | 15 | 60 | 1 |
| | 3 | 19 | 19 | 20 | M | 50 | 0 |
| | ~~4(D)~~ | ~~M~~ | ~~0~~ | ~~M~~ | ~~0~~ 20 | ~~20~~ | 0 |
| Demand | | 30 | 20 | 70 | 60 | Select $x_{45}=20$ | |
| Column difference | | 2 | 14 | 0 | 15 | Eliminate row 4(D) | |

| | | Destination | | | | Row | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 5 | Supply | Difference |
| | ~~1~~ | ~~16~~ | ~~16~~ | ~~13~~ 50 | ~~17~~ | ~~50~~ | 3 |
| Source | 2 | 14 | 14 | 13 | 15 | 60 | |
| | 3 | 19 | 19 | 20 | M | 50 | |
| Demand | | 30 | 20 | 70 | 40 | Select $x_{13}$=50 | |
| Column difference | | 2 | 2 | 0 | 2 | Eliminate row | |

| | | Destination | | | | Row | |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 5 | Supply | Difference |
| Source | 2 | 14 | 14 | 13 | 15 40 | 60 | 1 |
| | 3 | 19 | 19 | 20 | M | 50 | 0 |
| Demand | | 30 | 20 | 20 | 40 | Select $x_{25}=40$ | |
| Column difference | | 5 | 5 | 7 | M-15 | Eliminate column 5 | |

| | | Destination | | | Row | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | Supply | Difference |
| Source | ~~2~~ | ~~14~~ | ~~14~~ | ~~13~~ 20 | ~~20~~ | ~~1~~ |
| | 3 | 19 | 19 | 20 0 | 50 | 0 |
| Demand | | 30 | 20 | 20 | Select $x_{23}=20$, $x_{33}=0$ | |
| Column difference | | 5 | 5 | 7 | Eliminate row 2 | |

- Note that the allocation of $x_{23}=20$ **exhausts both the remaining supply** in its row **and the remaining demand** in its column.
- Eliminate both the row and column, setting $x_{33}$ to zero to provide a degenerate basic variable.

| | | Destination | | |
|---|---|---|---|---|
| | | 1 | 2 | Supply |
| Source | 3 | 19 | 19 | 50 |
| Demand | | 30 30 | 20 20 | Select $x_{31}=30$ $x_{32}=20$ |

| | | Destination | | | | | Row | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | Supply | Difference |
| | 1 | 16 | 16 | 13 50 | 22 | 17 | 50 | 3 |
| Source | 2 | 14 | 14 | 13 20 | 19 | 15 40 | 60 | 1 |
| | 3 | 19 30 | 19 20 | 20 0 | 23 | M | 50 | 0 |
| | 4(D) | M | 0 | M | 0 30 | 0 20 | 50 | 0 |
| Demand | | 30 | 20 | 70 | 30 | 60 | Z=2460 | |
| Column difference | | 2 | 14 | 0 | 19 | 15 | | |

* Russel's approx method : ① find highest cost value for each row & column
② subtract cost at highest cost.
③ select most -ve & allocate

①

| | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| S1 | 4 | 5 | 7 | 3 | 4 |
| S2 | 1 | 5 | 6 | 2 | 1 |
| S3 | 2 | 2 | 3 | 4 | 5 |
| Demand | 2 | 4 | 3 | 4 | |

②

| | V | 4 | 5 | 7 | 4 | |
|---|---|---|---|---|---|---|
| U | Iteration-1 | D1 | D2 | D3 | D4 | Supply |
| 7 | S1 | 4 | 5 | 7 | 3 | 4 |
| 6 | S2 | 1 | 5 | 6 | 2 | 1 |
| 5 | S3 | 2 | 2 | 3 | 4 | 5 |
| | Demand | 2 | 4 | 3 | 4 | |

③

| | V | 4 | 5 | 7 | 4 | |
|---|---|---|---|---|---|---|
| U | Iteration-1 | D1 | D2 | D3 | D4 | Supply |
| 7 | S1 | 4 $\Delta = (4-(7+4)) = -7$ | 5 $\Delta = (5-(7+5)) = -7$ | 7 $\Delta = (7-(7+7)) = -7$ | $\boxed{3}$ $\Delta = (3-(7+4)) = -8$ | 4 |
| 6 | S2 | 1 $\Delta = (1-(6+4)) = -9$ | 5 $\Delta = (5-(6+5)) = -6$ | 6 $\Delta = (6-(6+7)) = -7$ | 2 $\Delta = (2-(6+4)) = -8$ | 1 |
| 5 | S3 | 2 $\Delta = (2-(5+4)) = -7$ | 2 $\Delta = (2-(5+5)) = -8$ | 3 $\Delta = (3-(5+7)) = -9$ | 4 $\Delta = (4-(5+4)) = -5$ | 5 |
| | Demand | 2 | 4 | 3 | 4 | |

④

| | D1 | D2 | D3 | D4 | Supply |
|---|---|---|---|---|---|
| S1 | 4 | 5 | 7 | 3 $\boxed{4}$ | 4 |
| S2 | 1 $\boxed{1}$ | 5 | 6 | 2 | 1 |
| S3 | 2 | 2 $\boxed{2}$ | 3 $\boxed{3}$ | 4 | 5 |
| Demand | 2 | 4 | 3 | 4 | |

Total Transportation Cost = (1x1) + (2x2) + (3x3) + (3x4) = 26

5. Repeat step 2 until step 4 until all products are distributed.

| | V | 4 | 5 | 4 | |
|---|---|---|---|---|---|
| U | Iteration-2 | D1 | D2 | D4 | Supply |
| 5 | S1 | 4 $\Delta = (4-(5+4)) = -5$ | 5 $\Delta = (5-(5+5)) = -5$ | 3 $\Delta = (3-(5+4)) = -6$ | 4 |
| 4 | S3 | 2 $\Delta = (2-(4+4)) = -6$ | 2 $\boxed{2}$ $\Delta = (2-(5+4)) = -7$ | 4 $\Delta = (4-(4+4)) = -4$ | 2-2=0 |
| | Demand | 1 | 4-2=2 | 4 | |

- Next , eliminating Row S3 (Supply = 0)

6. If the allocation has been completed then calculate the distribution cost.

| | V | 4 | 5 | 3 | |
|---|---|---|---|---|---|
| U | Iteration-3 | D1 | D2 | D4 | Supply |
| 5 | S1 | 4 $\Delta = (4-(5+4)) = -5$ | 5 $\Delta = (5-(5+5)) = -5$ | 3 $\boxed{4}$ $\Delta = (3-(5+3)) = -5$ | 4-4=0 |
| | Demand | 1 | 2 | 4-4=0 | |