Elementary Linear Algebra

Chapter 1:

Systems of Linear Equations & Matrices

Chapter Contents

- Introduction to System of Linear Equations
- Gaussian Elimination
- Matrices and Matrix Operations
- Inverses; Rules of Matrix Arithmetic
- Elementary Matrices and a Method for Finding A^{-1}
- Further Results on Systems of Equations and Invertibility
- Diagonal, Triangular, and Symmetric Matrices

1-1 Linear Equations

- Any straight line in *xy*-plane can be represented algebraically by an equation of the form:

$$a_1x + a_2y = b$$

- General form: Define a linear equation in the n variables $x_1, x_2, \ldots, x_n$:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real constants.

- The variables in a linear equation are sometimes called unknowns.

1-1 Example 1 (Linear Equations)

- The equations $x+3y=7$, $y=\frac{1}{2}x+3z+1$, and $x_1-2x_2-3x_3+x_4=7$ are linear
  - A linear equation does not involve any products or roots of variables
  - All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.
- The equations $x+3\sqrt{y}=5, 3x+2y-z+xz=4,$ and $y=\sin x$ are *not* linear
- A solution of a linear equation is a sequence of n numbers s_1, s_2, …, s_n such that the equation is satisfied.
- The set of all solutions of the equation is called its solution set or general solution of the equation.

1-1 Example 2 (Linear Equations)

- Find the solution of $x_1 - 4x_2 + 7x_3 = 5$
- Solution:
  - We can assign arbitrary values to any two variables and solve for the third variable
  - For example

$$x_1 = 5 + 4s - 7t, \qquad x_2 = s, \qquad x_3 = t$$

where s, t are arbitrary values

1-1 Linear Systems

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{matrix}$$

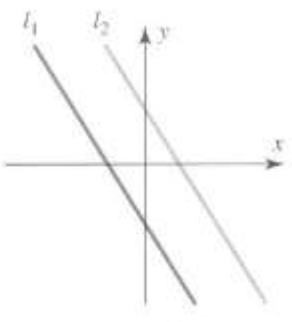
- A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations or a linear system.
- A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a solution of the system
- A system has *no solution* is said to be inconsistent.
- If there is at least one solution of the system, it is called consistent.
- *Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions*

1-1 Linear Systems

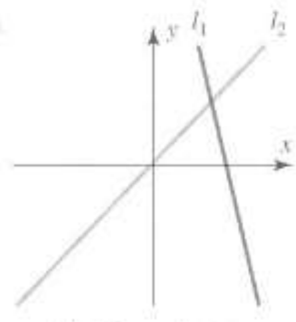
- A general system of two linear equations:

$$a_1x + b_1y = c_1 \ (a_1, b_1 \text{ not both zero})$$
$$a_2x + b_2y = c_2 \ (a_2, b_2 \text{ not both zero})$$

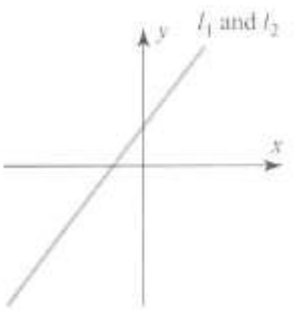
  - Two line may be parallel – no solution
  - Two line may be intersect at only one point – one solution
  - Two line may coincide – infinitely many solutions



(a) No solution



(b) One solution



(c) Infinitely many solutions

1-1 Augmented Matrices

- The location of the $+'$s, the x's, and the $='$s can be abbreviated by writing only the rectangular array of numbers.
- This is called the augmented matrix for the system.
- It must be written in the same order in each equation as the unknowns and the constants must be on the right

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{matrix}$$

1th column ↓

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix} \leftarrow \text{1th row}$$

1-1 Elementary Row Operations

- The basic method for solving a system of linear equations is to replace the given system by **a new system that has the same solution set** but which is **easier** to solve.
- Since the **rows** of an augmented matrix correspond to the **equations** in the associated system, new systems is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically.

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

1-1 Elementary Row Operations

- Elementary row operations
  - Multiply an equation through by a nonzero constant
  - Interchange two equation
  - Add a multiple of one equation to another

1-1 Example 3 (Using Elementary Row Operations)

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\tfrac{7}{2} & -\tfrac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \Rightarrow \begin{aligned} x + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \Rightarrow \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\tfrac{7}{2} & -\tfrac{17}{2} \\ 0 & 0 & -\tfrac{1}{2} & -\tfrac{3}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\tfrac{7}{2} & -\tfrac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \tfrac{11}{2} & \tfrac{35}{2} \\ 0 & 1 & -\tfrac{7}{2} & -\tfrac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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1-2 Echelon Forms

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- A matrix is in reduced row-echelon form
  - If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leader 1.
  - If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
  - In any two successive rows that do not consist entirely of zeros, the leader 1 in the lower row occurs farther to the right than the leader 1 in the higher row.
  - Each *column* that contains a leader 1 has zeros everywhere else.
- A matrix that has the *first three properties* is said to be in row-echelon form.

1-2 Example 1

- Reduce row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Row-echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1-2 Example 2

- Matrices in **row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix}1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & *\end{bmatrix}$$

- Matrices in **reduced row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & *\end{bmatrix}$$

1-2 Example 3

- Solutions of linear systems

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1-2 Elimination Methods

- A step-by-step elimination procedure that can be used to reduce any matrix to reduced row-echelon form

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

1-2 Elimination Methods

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1th and 2th rows in the preceding matrix were interchanged.

1-2 Elimination Methods

- Step3. If the entry that is now at the top of the column found in Step1 is a, multiply the first row by 1/a in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

← **The 1st row of the preceding matrix was multiplied by 1/2.**

- Step4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

← **-2 times the 1st row of the preceding matrix was added to the 3rd row.**

1-2 Elimination Methods

- Step5. Now cover the top row in the matrix and begin again with Step1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & -5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The 1st row in the submatrix was multiplied by -1/2 to introduce a leading 1.

1-2 Elimination Methods

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

-5 times the 1st row of the submatrix was added to the 2nd row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again Step1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submetrix was multiplied by 2 to introduce a leading 1.

1-2 Elimination Methods

- Step1~Step5: the above procedure produces a row-echelon form and is called Gaussian elimination
- Step1~Step6: the above procedure produces a reduced row-echelon form and is called Gaussian-Jordan elimination
- Every matrix has **a unique reduced row-echelon** form but a row-echelon form of a given matrix is not unique
- Back-Substitution
  - To solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form **without continuing all the way to the reduced row-echelon form**.
  - When this is done, the corresponding system of equations can be solved by a technique called back-substitution

1-2 Example 4

- Solve by Gauss-Jordan elimination

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

1-2 Example 5

- From the computations in example 4 , a row-echelon form of the augmented matrix is given.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- To solve the system of equations:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad + 3x_6 &= 1 \\ x_6 &= 1/3 \end{aligned}$$

1-2 Example 6

- Solve the system of equations by Gaussian elimination and back-substitution.

$$\begin{aligned} x + \quad y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

1-2 Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if the constant terms are all zero.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\quad \vdots \qquad & \;\;\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}
$$

- Every homogeneous system of linear equation is **consistent**, since all such system have $x_1 = 0$, $x_2 = 0$, …, $x_n = 0$ as a solution.
  - This solution is called the trivial solution.
  - If there are another solutions, they are called nontrivial solutions.
- There are *only two possibilities* for its solutions:
  - There is **only** the trivial solution
  - There are **infinitely** many solutions in addition to the trivial solution

1-2 Example 7

- Solve the homogeneous system of linear equations by Gauss-Jordan elimination

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \qquad\quad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \qquad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

- The augmented matrix

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- Reducing this matrix to reduced row-echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The general solution is

$$x_1 = -s - t, x_2 = s$$

$$x_3 = -t, x_4 = 0, x_5 = t$$

- Note: the trivial solution is obtained when $s = t = 0$

1-2 Example 7 (Gauss-Jordan Elimination)

- Two important points:
  - None of the three row operations alters the final column of zeros, so the system of equations corresponding to the reduced row-echelon form of the augmented matrix must also be a homogeneous system.
  - If the given homogeneous system has *m* equations in *n* unknowns with $m < n$, and there are *r* nonzero rows in reduced row-echelon form of the augmented matrix, we will have $r < n$. It will have the form:

$$\begin{array}{ll}
\cdots x_{k1} \qquad\qquad + \sum () = 0 & x_{k1} = -\sum () \\
\qquad \cdots x_{k2} \qquad + \sum () = 0 & x_{k2} = -\sum () \\
\qquad\qquad \ddots \qquad \vdots & \vdots \\
\qquad\qquad\qquad x_{kr} + \sum () = 0 & x_{kr} = -\sum ()
\end{array}$$

$$\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{array}$$

  - (Theorem 1.2.1)

Theorem 1.2.1

- A homogeneous system of linear equations with more unknowns than equations has **infinitely many solutions**.

- Remark
  - This theorem applies only to homogeneous system!
  - A nonhomogeneous system with more unknowns than equations need not be consistent; however, if the system is consistent, it will have infinitely many solutions.
  - e.g., two parallel planes in 3-space

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1-3 Definition and Notation

- A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix
- A general $m \times n$ matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} or $\langle A \rangle_{ij}$. If a_{ij} is real number, it is common to be referred as **scalars**
- The preceding matrix can be written as $[a_{ij}]_{m \times n}$ or $[a_{ij}]$

1-3 Definition

- Two matrices are defined to be equal if they have the same size and their corresponding entries are equal
  - If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j

- If A and B are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A.

1-3 Definition

- The difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A

- If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be the scalar multiple of A
  - If $A = [a_{ij}]$, then $\langle cA \rangle_{ij} = c\langle A \rangle_{ij} = ca_{ij}$

1-3 Definitions

- If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows.

  - $(AB)_{m \times n} = A_{m \times r} B_{r \times n}$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$\langle AB \rangle_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{ir}b_{rj}$$

1-3 Example 5

- Multiplying matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

1-3 Example 6

- Determine whether a product is defined

  Matrices A: 3×4, B: 4×7, C: 7×3

1-3 Partitioned Matrices

- A matrix can be partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns
- For example, three possible partitions of a 3×4 matrix A:
  - The partition of A into four submatrices A_{11}, A_{12}, A_{21}, and A_{22}
  - The partition of A into its row matrices $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$
  - The partition of A into its column matrices $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{bmatrix}$$

1-3 Multiplication by Columns and by Rows

- It is possible to compute a particular row or column of a matrix product AB without computing the entire product:

  jth column matrix of AB = A[jth column matrix of B]

  ith row matrix of AB = [ith row matrix of A]B

- If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ denote the row matrices of A and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ denote the column matrices of B,then

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n\end{bmatrix}$$

$$AB = \begin{bmatrix}\mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m\end{bmatrix} B = \begin{bmatrix}\mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B\end{bmatrix}$$

1-3 Example 7

- Multiplying matrices by rows and by columns

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

1-3 Matrix Products as Linear Combinations

- Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Then

$$A\mathbf{x}=\begin{bmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{bmatrix}=x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}+x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}+\cdots+x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- *The product A**x** of a matrix A with a column matrix **x** is a linear combination of the column matrices of A with the coefficients coming from the matrix **x***

1-3 Example 8

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the linear combination of column matrices

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

The matrix product

$$[1 \quad -9 \quad -3]\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} = [-16 \quad -18 \quad 35]$$

can be written as the linear combination of row matrices

$$1[-1 \quad 3 \quad 2] - 9[1 \quad 2 \quad -3] - 3[2 \quad 1 \quad -2] = [-16 \quad -18 \quad 35]$$

1-3 Example 9

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of AB can be expressed as linear combinations of the column matrices of A as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

♦

1-3 Matrix Form of a Linear System

- Consider any system of *m* linear equations in *n* unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned} \Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow A\mathbf{x} = \mathbf{b}$$

- The matrix *A* is called the coefficient matrix of the system
- The augmented matrix of the system is given by

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

1-3 Example 10

- A function using matrices
  - Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

  - The product y = Ax is
  - The product y = Bx is

1-3 Definitions

- If A is any $m \times n$ matrix, then the transpose of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A
  - That is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth

- If A is a square matrix, then the trace of A , denoted by tr(A), is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.
  - For an $n \times n$ matrix $A = [a_{ij}]$, $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$

1-3 Example 11 & 12

- Transpose: $(A^T)_{ij} = (A)_{ij}$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$$

- Trace of matrix:

$$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

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1-4 Properties of Matrix Operations

- For real numbers a and b ,we always have $ab = ba$, which is called the ***commutative law for multiplication***. For matrices, however, AB and BA need not be equal.

- Equality can fail to hold for three reasons:
  - The product AB is defined but BA is undefined.
  - AB and BA are both defined but have different sizes.
  - It is possible to have $AB \neq BA$ even if both AB and BA are defined and have the same size.

Theorem 1.4.1 (Properties of Matrix Arithmetic)

- Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid:
  - $A + B = B + A$ (commutative law for addition)
  - $A + (B + C) = (A + B) + C$ (associative law for addition)
  - $A(BC) = (AB)C$ (associative law for multiplication)
  - $A(B + C) = AB + AC$ (left distributive law)
  - $(B + C)A = BA + CA$ (right distributive law)
  - $A(B - C) = AB - AC,$ $(B - C)A = BA - CA$
  - $a(B + C) = aB + aC,$ $a(B - C) = aB - aC$
  - $(a+b)C = aC + bC,$ $(a-b)C = aC - bC$
  - $a(bC) = (ab)C,$ $a(BC) = (aB)C = B(aC)$
- Note: the cancellation law is not valid for matrix multiplication!

1-4 Proof of $A(B + C) = AB + AC$

- show the same size

- show the corresponding entries are equal

1-4 Example 2

As an illustration of the associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus,

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so $(AB)C = A(BC)$, as guaranteed by Theorem 1.4.1c.

1-4 Zero Matrices

- A matrix, all of whose entries are zero, is called a **zero matrix**
- A zero matrix will be denoted by *0*
- If it is important to emphasize the size, we shall write $0_{m\times n}$ for the $m\times n$ zero matrix.
- In keeping with our convention of using **boldface symbols** for matrices with one column, we will denote a zero matrix with one column by **0**

1-4 Example 3

- The cancellation law does not hold

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

- AB=AC = $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

- AD = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Theorem 1.4.2 (Properties of Zero Matrices)

- Assuming that the sizes of the matrices are such that the indicated operations can be performed ,the following rules of matrix arithmetic are valid
  - $A + 0 = 0 + A = A$
  - $A - A = 0$
  - $0 - A = -A$
  - $A0 = 0; \quad 0A = 0$

1-4 Identity Matrices

- A square matrix with 1′s on the main diagonal and 0′s off the main diagonal is called an **identity matrix** and is denoted by I, or I_n for the $n \times n$ identity matrix
- If A is an $m \times n$ matrix, then $AI_n = A$ and $I_m A = A$
  - Example 4

- An identity matrix plays the same role in matrix arithmetic as the number 1 plays in the numerical relationships $a \cdot 1 = 1 \cdot a = a$

Theorem 1.4.3

- If R is the reduced row-echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n

1-4 Invertible

- If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* and B is called an *inverse* of A. If no such matrix B can be found, then A is said to be *singular*.

- Remark:
  - The inverse of A is denoted as A^{-1}
  - Not every (square) matrix has an inverse
  - An inverse matrix has exactly one inverse

1-4 Example 5 & 6

- Verify the inverse requirements

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

- A matrix with no inverse

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} \quad \text{is singular}$$

1-4 Theorems

- Theorem 1.4.4
  - If B and C are both inverses of the matrix A, then $B = C$

- Theorem 1.4.5
  - The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

    is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 1.4.6

- If A and B are invertible matrices of the same size ,then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

- Example 7

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

1-4 Powers of a Matrix

- If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad A^n = \underbrace{AA\cdots A}_{n \text{ factors}} \quad (n > 0)$$

- If A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}} \quad (n > 0)$$

- Theorem 1.4.7 (Laws of Exponents)
  - If A is a square matrix and r and s are integers, then $A^rA^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Theorem 1.4.8 (Laws of Exponents)

- If A is an invertible matrix, then:
  - A^{-1} is invertible and $(A^{-1})^{-1} = A$
  - A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, \ldots$
  - For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = (1/k)A^{-1}$

1-4 Example 8

- Powers of matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

- $A^3 = ?$

- $A^{-3} = ?$

1-4 Polynomial Expressions Involving Matrices

- If A is a square matrix, say $m \times m$, and if

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

  is any polynomial, then we define

$$p(A) = a_0 I + a_1 A + \dots + a_n A^n$$

  where I is the $m \times m$ identity matrix.
- That is, $p(A)$ is the $m \times m$ matrix that results when A is substituted for x in the above equation and a_0 is replaced by $a_0 I$

1-4 Example 9 (Matrix Polynomial)

If

$$p(x) = 2x^2 - 3x + 4 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

then

$$p(A) = 2A^2 - 3A + 4I = 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

Theorems 1.4.9 (Properties of the Transpose)

- If the sizes of the matrices are such that the stated operations can be performed, then
  - $((A^T)^T = A$
  - $(A + B)^T = A^T + B^T$ and $(A - B)^T = A^T - B^T$
  - $(kA)^T = kA^T$, where k is any scalar
  - $(AB)^T = B^T A^T$

Theorem 1.4.10 (Invertibility of a Transpose)

- If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

- Example 10

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

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1-5 Elementary Row Operation

- An elementary row operation (sometimes called just a row operation) on a matrix A is any one of the following three types of operations:
  - Interchange of two rows of A
  - Replacement of a row $\mathbf{r}$ of A by $c\mathbf{r}$ for some number $c \neq 0$
  - Replacement of a row $\mathbf{r}_1$ of A by the sum $\mathbf{r}_1 + c\mathbf{r}_2$ of that row and a multiple of another row $\mathbf{r}_2$ of A

1-5 Elementary Matrix

- An $n \times n$ elementary matrix is a matrix produced by applying exactly one elementary row operation to I_n
  - E_{ij} is the elementary matrix obtained by interchanging the i-th and j-th rows of I_n
  - $E_i(c)$ is the elementary matrix obtained by multiplying the i-th row of I_n by $c \neq 0$
  - $E_{ij}(c)$ is the elementary matrix obtained by adding c times the j-th row to the i-th row of I_n, where $i \neq j$

1-5 Example 1

- Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.	Interchange the second and fourth rows of I_4.	Add 3 times the third row of I_3 to the first row.	Multiply the first row of I_3 by 1.

1-5 Elementary Matrices and Row Operations

- Theorem 1.5.1
  - Suppose that E is an $m \times m$ elementary matrix produced by applying a particular elementary row operation to I_m, and that A is an $m \times n$ matrix. Then EA is the matrix that results from applying that same elementary row operation to A

1-5 Example 2 (Using Elementary Matrices)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the same matrix that results when we add 3 times the first row of A to the third row. ♦

1-5 Inverse Operations

- If an elementary row operation is applied to an identity matrix I to produce an elementary matrix E, then there is a second row operation that, when applied to E, produces I back again

Row operation on I That produces E	**Row operation on E That produces I**
Multiply row i by c≠0	Multiply row i by 1/c
Interchange row i and j	Interchange row i and j
Add c times row i to row j	Add -c times row i to row j

1-5 Inverse Operations

- Examples

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&7\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Multiply the second row by 7.

Multiply the second row by $\frac{1}{7}$.

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}0&1\\1&0\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Interchange the first and second rows.

Interchange the first and second rows.

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&5\\0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Add 5 times the second row to the first.

Add -5 times the second row to the first.

Theorem 1.5.2

- Elementary Matrices and Nonsingularity
  - Each elementary matrix is nonsingular, and its inverse is itself an elementary matrix. More precisely,
  - $E_{ij}^{-1} = E_{ji}$ $(= E_{ij})$
  - $E_i(c)^{-1} = E_i(1/c)$ with $c \neq 0$
  - $E_{ij}(c)^{-1} = E_{ij}(-c)$ with $i \neq j$

Theorem 1.5.3(Equivalent Statements)

- If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false
  - A is invertible
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A is I_n
  - A is expressible as a product of elementary matrices

1-5 A Method for Inverting Matrices

- To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}

- Remark
  - Suppose we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \ldots E_2 E_1 A = I_n$$

then

$$A^{-1} = E_k \ldots E_2 E_1 I_n$$

1-5 Example 4
(Using Row Operations to Find A^{-1})

- Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

- Solution:
  - To accomplish this we shall adjoin the identity matrix to the right side of A, thereby producing a matrix of the form $[A \mid I]$
  - We shall apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so that the final matrix will have the form $[I \mid A^{-1}]$

1-5 Example 4

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← We added −2 times the first row to the second and −1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← We added 2 times the second row to the third.

1-5 Example 4 (continue)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We multiplied the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

1-5 Example 5

- Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Apply the procedure of example 4 to find A^{-1}

1-5 Example 6

- According to example 4, A is an invertible matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

- $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + \quad\quad + 8x_3 &= 0 \end{aligned}$$ has only trivial solution

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Theorems 1.6.1

- Every system of linear equations has either no solutions, exactly one solution, or in finitely many solutions.

Theorem 1.6.2

- If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

1-6 Example 1

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2 the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

1-6 Linear Systems with a Common Coefficient Matrix

- To solve a sequence of linear systems, $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_1$, …, $A\mathbf{x} = \mathbf{b}_k$, with common coefficient matrix A
- If A is invertible, then the solutions $\mathbf{x}_1 = A^{-1}\mathbf{b}_1$, $\mathbf{x}_2 = A^{-1}\mathbf{b}_2$, …, $\mathbf{x}_k = A^{-1}\mathbf{b}_k$
- A more efficient method is to form the matrix $[A|\mathbf{b}_1|\mathbf{b}_2|\ldots|\mathbf{b}_k]$
- By reducing it to reduced row-echelon form we can **solve all *k* systems at once** by Gauss-Jordan elimination.

1-6 Example 2

- Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 + \quad\quad + 8x_3 &= 9 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 + \quad\quad + 8x_3 &= -6 \end{aligned}$$

Theorems 1.6.3

- Let A be a square matrix
  - If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$
  - If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$

Theorem 1.6.4 (Equivalent Statements)

- If A is an $n \times n$ matrix, then the following statements are equivalent
  - A is invertible
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A is I_n
  - A is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$

Theorem 1.6.5

- Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

- Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices b such that the system of equations $Ax=b$ is consistent.

1-6 Example 3

- Find b_1, b_2, and b_3 such that the system of equations is consistent.

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= b_1 \\
x_1 + \quad\ + x_3 &= b_2 \\
2x_1 + x_2 + 3x_3 &= b_3
\end{aligned}
$$

1-6 Example 4

- Find b_1, b_2, and b_3 such that the system of equations is consistent.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + \qquad + 8x_3 &= b_3 \end{aligned}$$

Chapter Contents

- Introduction to System of Linear Equations
- Gaussian Elimination
- Matrices and Matrix Operations
- Inverses; Rules of Matrix Arithmetic
- Elementary Matrices and a Method for Finding A^{-1}
- Further Results on Systems of Equations and Invertibility
- Diagonal, Triangular, and Symmetric Matrices

1-7 Diagonal Matrix

- A square matrix A is $m \times n$ with $m = n$; the (i,j)-entries for $1 \leq i \leq m$ form the main diagonal of A

- A diagonal matrix is a square matrix all of whose entries *not* on the main diagonal equal zero. By $\text{diag}(d_1, \ldots, d_m)$ is meant the $m \times m$ diagonal matrix whose (i,i)-entry equals d_i for $1 \leq i \leq m$

1-7 Properties of Diagonal Matrices

- A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

- Powers of diagonal matrices are easy to compute

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

1-7 Properties of Diagonal Matrices

- Matrix products that involve diagonal factors are especially easy to compute

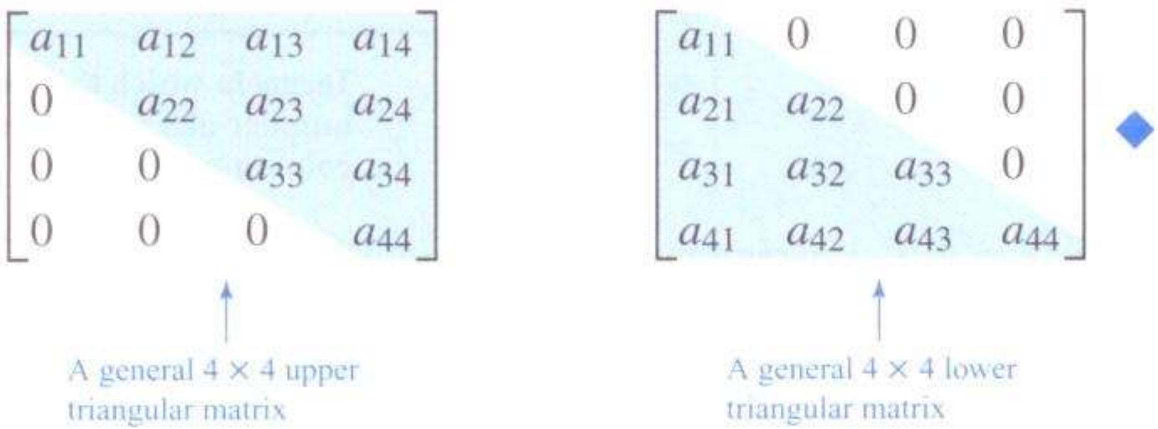
$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

1-7 Triangular Matrices

- A $m \times n$ lower-triangular matrix L satisfies $(L)_{ij} = 0$ if $i < j$, for $1 \leq i \leq m$ and $1 \leq j \leq n$
- A $m \times n$ upper-triangular matrix U satisfies $(U)_{ij} = 0$ if $i > j$, for $1 \leq i \leq m$ and $1 \leq j \leq n$
- A unit-lower (or –upper)-triangular matrix T is a lower (or upper)-triangular matrix satisfying $(T)_{ii} = 1$ for $1 \leq i \leq \min(m,n)$

1-7 Example 2 (Triangular Matrices)



- The diagonal matrix
  - both upper triangular and lower triangular
- A square matrix in row-echelon form is upper triangular

Theorem 1.7.1

- The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular
- The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular
- A triangular matrix is invertible if and only if its diagonal entries are all nonzero
- The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular

1-7 Example 3

- Consider the upper triangular matrices

$$A=\begin{bmatrix}1 & 3 & -1\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix} \qquad B=\begin{bmatrix}3 & -2 & 2\\ 0 & 0 & -1\\ 0 & 0 & 1\end{bmatrix}$$

1-7 Symmetric Matrices

- A (square) matrix A for which $A^T = A$, so that $\langle A \rangle_{ij} = \langle A \rangle_{ji}$ for all i and j, is said to be symmetric.

- Example 4

$$\begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

Theorem 1.7.2

- If A and B are symmetric matrices with the same size, and if k is any scalar, then
  - A^T is symmetric
  - $A + B$ and $A - B$ are symmetric
  - kA is symmetric

- Remark
  - The product of two symmetric matrices is symmetric if and only if the matrices commute, i.e., $AB = BA$
- Example 5

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix}$$

Theorem 1.7.3

- If A is an invertible symmetric matrix, then A^{-1} is symmetric.

- Remark:
  - In general, a symmetric matrix needs not be invertible.
  - The products AA^T and A^TA are always symmetric

1-7 Example 6

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that A^TA and AA^T are symmetric as expected.

Theorem 1.7.4

- If A is an invertible matrix, then AA^T and A^TA are also invertible