Elementary Linear Algebra

Chapter 2:
Determinants

Chapter Content

- Determinants by Cofactor Expansion
- Evaluating Determinants by Row Reduction
- Properties of the Determinant Function
- A Combinatorial Approach to Determinants

2-1 Minor and Cofactor

- Definition
  - Let A be $m \times n$
    - The (i,j)-minor of A, denoted M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column from A
    - The (i,j)-cofactor of A, denoted C_{ij}, is $(-1)^{i+j} M_{ij}$
- Remark
  - Note that $C_{ij} = \pm M_{ij}$ and the signs $(-1)^{i+j}$ in the definition of cofactor form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

2-1 Example 1

- Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$
- The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$
- The cofactor of a_{11} is $C_{11} = (-1)^{1+1}M_{11} = M_{11} = 16$
- Similarly, the minor of entry a_{32} is $M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$
- The cofactor of a_{32} is $C_{32} = (-1)^{3+2}M_{32} = -M_{32} = -26$

2-1 Cofactor Expansion

- The definition of a 3×3 determinant in terms of minors and cofactors
  - $\det(A) = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13}$
    $= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
  - this method is called cofactor expansion along the first row of A
- Example 2

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = 3(-4) - (1)(-11) + 0 = -1$$

2-1 Cofactor Expansion

- $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$
  $= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$
  $= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

- Theorem 2.1.1 (Expansions by Cofactors)
  - The determinant of an $n\times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \le i, j \le n$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion along the *j*th column)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion along the *i*th row)

2-1 Example 3 & 4

- Example 3
  - cofactor expansion along the first column of A

$$\det(A)=\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3(-4)-(-2)(-2)+5(3)=-1$$

- Example 4
  - smart choice of row or column

$$A=\begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

  - det(A) = ?

2-1 Adjoint of a Matrix

- If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

  is called the *matrix of cofactors from A*. The transpose of this matrix is called the *adjoint of A* and is denoted by *adj(A)*

- Remarks
  - If one multiplies the entries in any row by the corresponding cofactors from a different row, the sum of these products is always zero.

2-1 Example 5

- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  - $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = ?$

  - *Let*

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

2-1 Example 6 & 7

- Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$
- The cofactors of A are: $C_{11} = 12$, $C_{12} = 6$, $C_{13} = -16$, $C_{21} = 4$, $C_{22} = 2$, $C_{23} = 16$, $C_{31} = 12$, $C_{32} = -10$, $C_{33} = 16$
- The matrix of cofactor and adjoint of A are

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

- The inverse (see below) is

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorem 2.1.2
(Inverse of a Matrix using its Adjoint)

- If A is an invertible matrix, then $$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$
  - Show first that

$$A \operatorname{adj}(A) = \det(A)\, I$$

$$A \operatorname{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

$$\boxed{a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \det(A)}$$

$$\boxed{a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0 (i \neq j)}$$

Theorem 2.1.3

- If A is an n × n triangular matrix (upper triangular, lower triangular, or diagonal), then det(A) is the product of the entries on the main diagonal of the matrix;
  - $\det(A) = a_{11}a_{22}\ldots a_{nn}$
  - E.g.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = ?$$

2-1 Prove Theorem 1.7.1c

- A triangular matrix is invertible if and only if its diagonal entries are all nonzero

2-1 Prove Theorem 1.7.1d

- The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular

Theorem 2.1.4 (Cramer's Rule)

- If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(\lambda I - A) \neq 0$, then the system has a unique solution. This solution is $$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

  where A_j is the matrix obtained by replacing the entries in the column of A by the entries in the matrix $\mathbf{b} = [b_1 \ \ b_2 \ \ \cdots \ \ b_n]^T$

2-1 Example 9

- Use Cramer's rule to solve

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

- Since

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

- Thus,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}, x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Exercise Set 2.1
Question 13

In Exercises 11–14 find A^{-1} using Theorem 2.1.2.

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise Set 2.1
Question 23

Use Cramer's rule to solve for y without solving for x, z, and w.

$$\begin{aligned} 4x + y + z + w &= 6 \\ 3x + 7y - z + w &= 1 \\ 7x + 3y - 5z + 8w &= -3 \\ x + y + z + 2w &= 3 \end{aligned}$$

Exercise Set 2.1
Question 2 7

Prove that if A is an invertible lower triangular matrix, then A^{-1} is lower triangular.