Elementary Linear Algebra

Chapter 3:
Euclidean Vector Spaces

Chapter Contents

- Vectors in 2-Space, 3-Space and n-Space
- Norm, Dot Product, and Distance in Rn
- Orthogonality
- The Geometry of Linear Systems
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Section 3.1

Vectors in 2-Space, 3-Space, and n-Space

EXAMPLE 1 Finding the Components of a Vector

The components of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$ are

$$\mathbf{v} = (7-2, 5-(-1), (-8)-4) = (5, 6, -12)$$

DEFINITION 1

If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by R^n.

DEFINITION 2

Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in R^n are said to be ***equivalent*** (also called ***equal***) if

$$v_1 = w_1, \quad v_2 = w_2, \quad \ldots, \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

▶ EXAMPLE 2 **Equality of Vectors**

$$(a, b, c, d) = (1, -4, 2, 7)$$

if and only if $a = 1$, $b = -4$, $c = 2$, and $d = 7$.

DEFINITION 3

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v}+\mathbf{w} = (v_1+w_1, v_2+w_2, \ldots, v_n+w_n) \tag{10}$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \tag{11}$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \tag{12}$$

$$\mathbf{w}-\mathbf{v} = \mathbf{w}+(-\mathbf{v}) = (w_1-v_1, w_2-v_2, \ldots, w_n-v_n) \tag{13}$$

EXAMPLE 3 Algebraic Operations Using Components

If $\mathbf{v} = (1, -3, 2)$ and $\mathbf{w} = (4, 2, 1)$, then

$$\mathbf{v}+\mathbf{w} = (5, -1, 3), \qquad 2\mathbf{v} = (2, -6, 4)$$
$$-\mathbf{w} = (-4, -2, -1), \quad \mathbf{v}-\mathbf{w} = \mathbf{v}+(-\mathbf{w}) = (-3, -5, 1)$$

THEOREM 3.1.1

If $\mathbf{u}, \mathbf{v}$*, and* $\mathbf{w}$ *are vectors in* R^n*, and if* k *and* m *are scalars, then:*

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(c) $\mathbf{u}+0=0+\mathbf{u}=\mathbf{u}$

(d) $\mathbf{u}+(-\mathbf{u})=0$

(e) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

(f) $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$

(g) $k(m\mathbf{u})=(km)\mathbf{u}$

(h) $1\mathbf{u}=\mathbf{u}$

THEOREM 3.1.2

If $\mathbf{v}$ *is a vector in* R^n *and* k *is a scalar, then:*

(a) $0\mathbf{v}=\mathbf{0}$

(b) $k\mathbf{0}=\mathbf{0}$

(c) $(-1)\mathbf{v}=-\mathbf{v}$

DEFINITION 4

If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{14}$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination. In the case where $r = 1$, Formula 14 becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just a scalar multiple of that vector.

Section 3.2 Norm, Dot Product, and Distance in R^n

Norm:

DEFINITION 1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***norm*** of $\mathbf{v}$ (also called the ***length*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

Unit Vectors:

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

EXAMPLE 1 **Calculating Norms**

It follows from Formula 2 that the norm of the vector $\mathbf{v} = (-3, 2, 1)$ in R^3 is

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

and it follows from Formula 3 that the norm of the vector $\mathbf{v} = (2, -1, 3, -5)$ in R^4 is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$

THEOREM 3.2.1

If $\mathbf{v}$ *is a vector in* R^n, *and if* k *is any scalar, then:*

(a) $\|\mathbf{v}\| \geq 0$

(b) $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

(c) $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$

EXAMPLE 2 Normalizing a Vector

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution

The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from 4

$$\mathbf{u} = \frac{1}{3}(2, 2, -1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

As a check, you may want to confirm that $\|\mathbf{u}\| = 1$.

EXAMPLE 3 Linear Combinations of Standard Unit Vectors

$$(2, -3, 4) = 2\mathbf{i} - 3\,\mathbf{j} + 4\mathbf{k}$$
$$(7, 3, -4, 5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4$$

DEFINITION 2

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then we denote the ***distance*** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

EXAMPLE 4 Calculating Distance in R^n

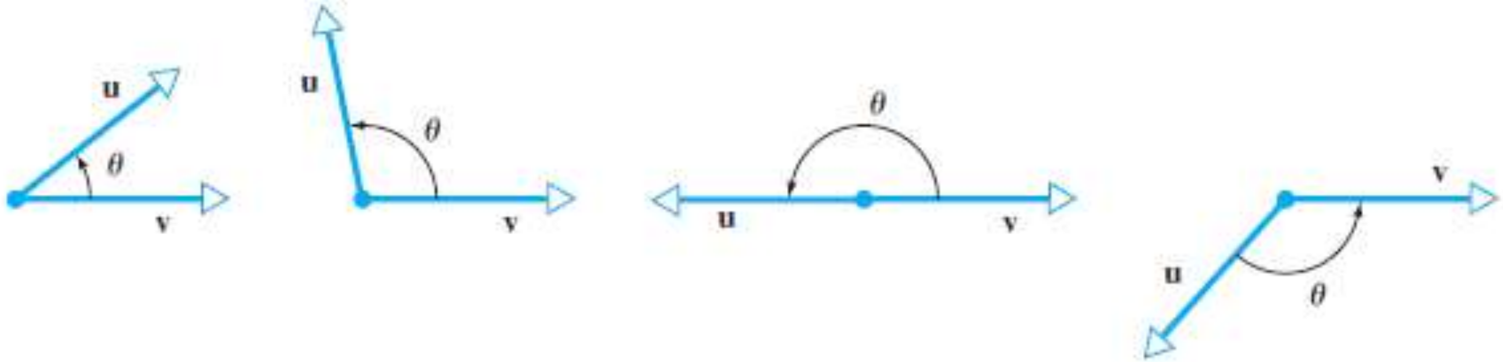
If

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

The Dot Product



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$.

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the *dot product* (also called the *Euclidean inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \tag{13}$$

▶ EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

Solution

The lengths of the vectors are

$$\|\mathbf{u}\| = 1 \quad \text{and} \quad \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$$

and the cosine of the angle θ between them is

$$\cos(45°) = 1/\sqrt{2}$$

Thus, it follows from Formula 12 that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta = (1)\left(2\sqrt{2}\right)\left(1/\sqrt{2}\right) = 2$$

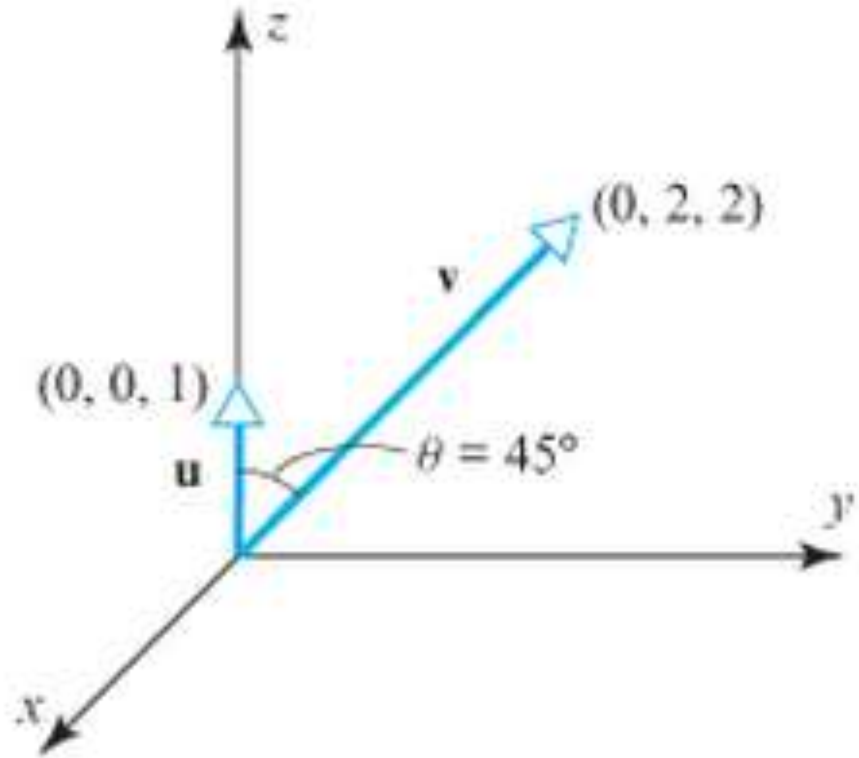


Figure 3.2.5

▶ EXAMPLE 6 A Geometry Problem Solved Using Dot Product

Find the angle between a diagonal of a cube and one of its edges.

Solution

Let k be the length of an edge and introduce a coordinate system as shown in Figure 3.2.7. If we let $\mathbf{u}_1 = (k, 0, 0)$, $\mathbf{u}_2 = (0, k, 0)$, and $\mathbf{u}_3 = (0, 0, k)$, then the vector

$$\mathbf{d} = (k, k, k) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. It follows from Formula 13 that the angle θ between $\mathbf{d}$ and the edge $\mathbf{u}_1$ satisfies

$$\cos\theta = \frac{\mathbf{u}_1 \cdot \mathbf{d}}{\|\mathbf{u}_1\| \, \|\mathbf{d}\|} = \frac{k^2}{(k)\left(\sqrt{3k^2}\right)} = \frac{1}{\sqrt{3}}$$

With the help of a calculator we obtain

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74°$$

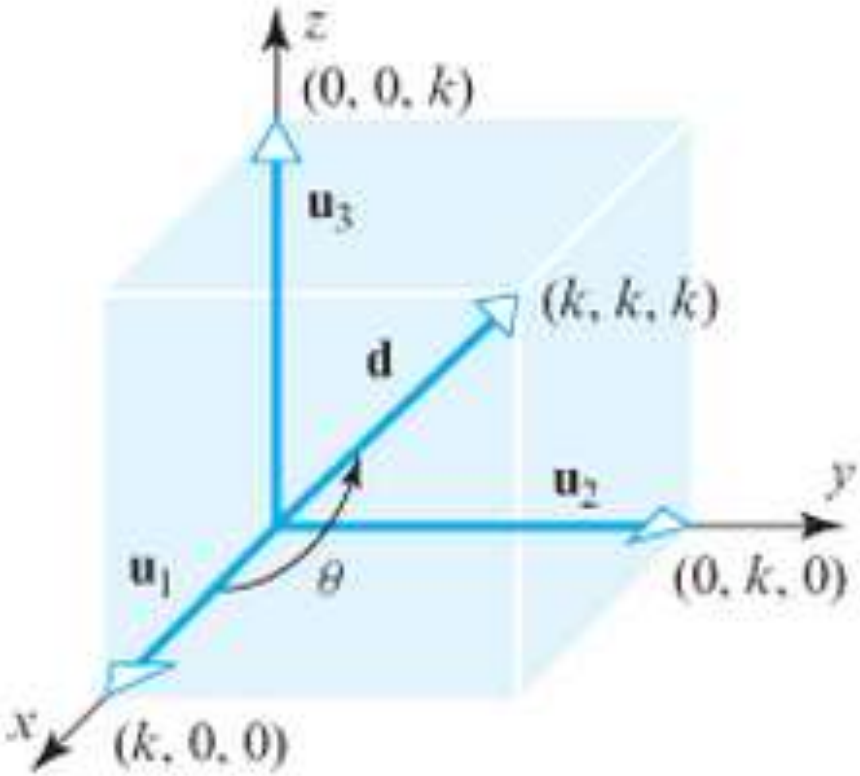


Figure 3.2.7

DEFINITION 4

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n} \tag{17}$$

EXAMPLE 7 Calculating Dot Products Using Components

(a) Use Formula 15 to compute the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ in Example 5.

(b) Calculate $\mathbf{u} \cdot \mathbf{v}$ for the following vectors in R^4:

$$\mathbf{u} = (-1, 3, 5, 7), \quad \mathbf{v} = (-3, -4, 1, 0)$$

Solution (a) The component forms of the vectors are $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = (0)\ (0) + (0)\ (2) + (1)\ (2) = 2$$

which agrees with the result obtained geometrically in Example 5.

Solution (b)

$$\mathbf{u} \cdot \mathbf{v} = (-1)\ (-3) + (3)\ (-4) + (5)\ (1) + (7)\ (0) = -4$$

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetry property]**

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ **[Homogeneity property]**

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ **[Positivity property]**

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(*b*) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(*c*) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(*d*) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(*e*) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

EXAMPLE 8 Calculating with Dot Products

$$\begin{aligned}(\mathbf{u}-2\mathbf{v})\cdot(3\mathbf{u}+4\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+4\mathbf{v})-2\mathbf{v}\cdot(3\mathbf{u}+4\mathbf{v})\\ &= 3(\mathbf{u}\cdot\mathbf{u})+4(\mathbf{u}\cdot\mathbf{v})-6(\mathbf{v}\cdot\mathbf{u})-8(\mathbf{v}\cdot\mathbf{v})\\ &= 3\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})-8\|\mathbf{v}\|^2\end{aligned}$$

Dot Products and Matrices

Table 1

Form	Dot Product	Example	
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}$	$\mathbf{v}\mathbf{u}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$
$\mathbf{u}$ a row matrix and $\mathbf{v}$ a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=\begin{bmatrix}1 & -3 & 5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5 & 4 & 0\end{bmatrix}$	$\mathbf{u}\mathbf{v}^T=\begin{bmatrix}1 & -3 & 5\end{bmatrix}\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=\begin{bmatrix}5 & 4 & 0\end{bmatrix}\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

THEOREM 3.2.4 Cauchy-Schwarz Inequality

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$\| \mathbf{u} \cdot \mathbf{v} \| \leq \| \mathbf{u} \| \, \| \mathbf{v} \| \tag{22}$$

or in terms of components

$$| u_1 v_1 + u_2 v_2 + \cdots + u_n v_n | \leq \left(u_1^2 + u_2^2 + \cdots + u_n^2 \right)^{1/2} \left(v_1^2 + v_2^2 + \cdots + v_n^2 \right)^{1/2} \tag{23}$$

THEOREM 3.2.5

If $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *then:*

(a) $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$ [Triangle inequality for vectors]

(b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ [Triangle inequality for distances]

THEOREM 3.2.6 Parallelogram Equation for Vectors

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n, *then*

$$\| \mathbf{u} + \mathbf{v} \|^2 + \| \mathbf{u} - \mathbf{v} \|^2 = 2 \left(\| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 \right) \tag{24}$$

THEOREM 3.2.7

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n *with the Euclidean inner product, then*

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 \tag{25}$$

EXAMPLE 9 Verifying that $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$

Suppose that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

from which we obtain

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10(0) + 5(5) = 11$$

$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

Thus, $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$ as guaranteed by Formula 26. We leave it for you to verify that Formula 27 also holds.

Section 3.3
Orthogonality

DEFINITION 1 Two nonzero vectors **u** and **v** in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n. A nonempty set of vectors in R^n is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

EXAMPLE 1 Orthogonal Vectors

(a) Show that **u** = (-2, 3, 1, 4) and **v** = (1, 2, 0, -1) are orthogonal vectors in $\boldsymbol{R}^4$.

(b) Show that the set **S** = {**i**, **j** ,**k**} of standard unit vectors is an orthogonal set in $\boldsymbol{R}^3$.

Solution (a) The vectors are orthogonal since

$$\mathbf{u}\cdot\mathbf{v} = (-2)\ (1) + (3)\ (2) + (1)\ (0) + (4)\ (-1) = 0$$

Solution (b) We must show that all pairs of distinct vectors are orthogonal, that is ,

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$$

This is evident geometrically (Figure 3.2.2), but it can be seen seen as well from the computations

$$\mathbf{i}\cdot\mathbf{j} = (1, 0, 0)\cdot(0, 1, 0) = 0$$
$$\mathbf{i}\cdot\mathbf{k} = (1, 0, 0)\cdot(0, 0, 1) = 0$$
$$\mathbf{j}\cdot\mathbf{k} = (0, 1, 0)\cdot(0, 0, 1) = 0$$

THEOREM 3.3.2 Projection Theorem

If $\mathbf{u}$ *and* $\mathbf{a}$ *are vectors in* R^n, *and if* $\mathbf{a} \neq 0$, *then* $\mathbf{u}$ *can be expressed in exactly one way in the form* $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, *where* $\mathbf{w}_1$ *is a scalar multiple of* $\mathbf{a}$ *and* $\mathbf{w}_2$ *is orthogonal to* $\mathbf{a}$.

▶ EXAMPLE 5 Vector Component of $\mathbf{u}$ Along $\mathbf{a}$

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

Solution

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$

$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus the vector component of $\mathbf{u}$ along $\mathbf{a}$ is

$$\operatorname{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{15}{21}(4, -1, 2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \operatorname{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$$

As a check, you may wish to verify that the vectors $\mathbf{u} - \operatorname{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{a}$ are perpendicular by showing that their dot product is zero.

THEOREM 3.3.3 Theorem of Pythagoras in R^n

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal vectors in* R^n *with the Euclidean inner product, then*

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2 \tag{14}$$

EXAMPLE 6 Theorem of Pythagoras in R^4

We showed in Example 1 that the vectors

$$\mathbf{u}=(-2, 3, 1, 4) \quad \text{and} \quad \mathbf{v}=(1, 2, 0, -1)$$

are orthogonal. Verify the Theorem of Pythagoras for these vectors.

Solution

We leave it for you to confirm that

$$\mathbf{u}+\mathbf{v}=(-1, 5, 1, 3)$$
$$\|\mathbf{u}+\mathbf{v}\|^2=36$$
$$\|\mathbf{u}\|^2+\|\mathbf{v}\|^2=30+6$$

Thus, $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$

THEOREM 3.4.3

If $\mathbf{A}$ is an $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of $\mathbf{A}$.

EXAMPLE 6 Orthogonality of Row Vectors and Solution Vectors

We showed in Example 6 of Section 1.2 that the general solution of the homogeneous linear system

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which we can rewrite in vector form as

$$\mathbf{x} = (-3r - 4s - 2t,\ r,\ -2s,\ s,\ t,\ 0)$$

According to Theorem 3.4.3, the vector $\mathbf{x}$ must be orthogonal to each of the row vectors

$$\begin{aligned}
\mathbf{r}_1 &= (1,\ 3,\ -2,\ 0,\ 2,\ 0) \\
\mathbf{r}_2 &= (2,\ 6,\ -5,\ -2,\ 4,\ -3) \\
\mathbf{r}_3 &= (0,\ 0,\ 5,\ 10,\ 0,\ 15) \\
\mathbf{r}_4 &= (2,\ 6,\ 0,\ 8,\ 4,\ 18)
\end{aligned}$$

We will confirm that $\mathbf{x}$ is orthogonal to $\mathbf{r}_1$, and leave it for you to verify that $\mathbf{x}$ is orthogonal to the other three row vectors as well. The dot product of $\mathbf{r}_1$ and $\mathbf{x}$ is

$$\mathbf{r}_1 \cdot \mathbf{x} = 1(-3r - 4s - 2t) + 3(r) + (-2)\ (-2s) + 0(s) + 2(t) + 0(0) = 0$$

which establishes the orthogonality.

THEOREM 3.4.4

The general solution of a consistent linear system $\mathbf{Ax} = \mathbf{b}$ *can be obtained by adding any specific solution of* $\mathbf{Ax} = \mathbf{b}$ *to the general solution of* $\mathbf{Ax} = \mathbf{0}$.

Point-line and point-plane Distance formulas

THEOREM 3.3.4

(*a*) *In* R^2 *the distance* D *between the point* $P_0(x_0, y_0)$ *and the line* $ax + by + c = 0$ *is*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}} \tag{15}$$

(*b*) *In* R^3 *the distance* D *between the point* $P_0(x_0, y_0, z_0)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{16}$$

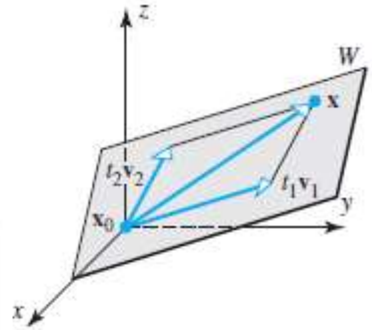
Section 3.4
The Geometry of Linear Systems

THEOREM 3.4.1 *Let L be the line in R^2 or R^3 that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is*

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{1}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v} \tag{2}$$

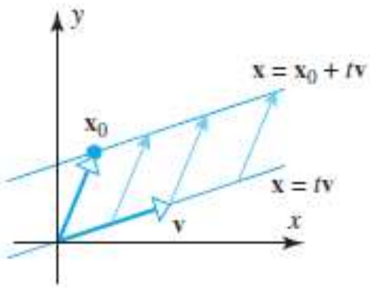


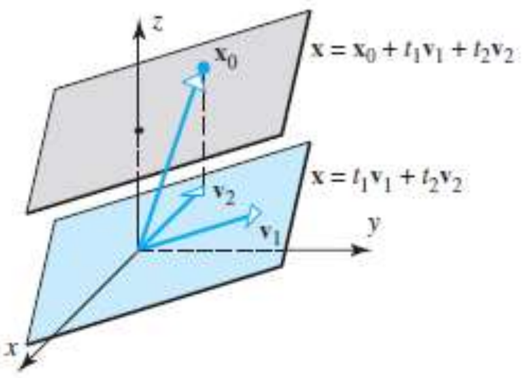
THEOREM 3.4.2 *Let W be the plane in R^3 that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by*

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{3}$$

If $\mathbf{x}_0 = \mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{4}$$





DEFINITION 1 If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{5}$$

defines the ***line through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to ***pass through the origin***.

DEFINITION 2 If $\mathbf{x}_0$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{6}$$

defines the ***plane through*** $\mathbf{x}_0$ ***that is parallel to*** $\mathbf{v}_1$ ***and*** $\mathbf{v}_2$. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the plane is said to ***pass through the origin***.

Section 3.5 Cross Product

DEFINITION 1 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{1}$$

Cross Products and Dot Products

THEOREM 3.5.1 Relationships Involving Cross Product and Dot Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 3-space, then*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ *(*$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$*)*

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ *(*$\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$*)*

(*c*) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ *(Lagrange's identity)*

(*d*) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ *(relationship between cross and dot products)*

(*e*) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ *(relationship between cross and dot products)*

Properties of Cross Product

THEOREM 3.5.2 Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

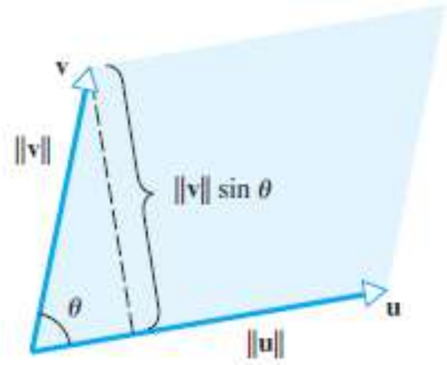
(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Geometry of the Cross Product



$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$$

THEOREM 3.5.3 Area of a Parallelogram

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in 3-space, then* $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.

Geometry of Determinants

THEOREM 3.5.4

(*a*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$. *(See Figure 3.5.7a.)*

(*b*) *The absolute value of the determinant*

$$\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, *and* $\mathbf{w} = (w_1, w_2, w_3)$. *(See Figure 3.5.7b.)*