Question 3 (4'marks): Evaluate the determinant of the given matrix by the cofactor expansion then evaluate the inverse.

$$\begin{bmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{bmatrix}$$

$$3\begin{bmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix} - 6\begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{bmatrix} + 0 + \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{bmatrix}$$

$$3\left[3\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} - 0 + 2\right.$$

0

Question 4 (6 marks): solve the system by inverting the coefficient matrix, by and using $x = A^{-1}b$

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 5 \end{aligned}$$

2

$[A/I] \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$

① $R_2 \to -2R_1 + R_2$
$R_3 \to -2R_1 + R_3$

② $R_2/-4$

$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$

1/2

③ $R_1 \to -3R_2 + R_1$
$R_3 \to 3R_2 + R_3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{-1}{2} & \frac{3}{4} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & 0 & \frac{7}{4} & \frac{-1}{2} & \frac{-3}{4} & 1 \end{array}\right]$

④ $R_3 \times \frac{4}{7}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/4 & -\frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 1 & & \frac{1}{2} & \frac{-1}{4} & 0 \\ 0 & 0 & 1 & \frac{-2}{7} & \frac{-3}{7} & \frac{4}{7} \end{array}\right]$

⑤ $-R_3 + R_1 \leftarrow R_1$
$-\frac{1}{4}R_3 + R_2 \leftarrow R_2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{14} & \frac{33}{28} & \frac{-4}{7} \\ 0 & 1 & 0 & \frac{4}{7} & \frac{-1}{7} & \frac{-1}{7} \\ 0 & 0 & 1 & \frac{-2}{7} & \frac{-3}{7} & \frac{4}{7} \end{array}\right] = [I \mid A^{-1}]$

$\begin{bmatrix} \frac{-3}{14} & \frac{33}{28} & \frac{-4}{7} \\ \frac{4}{7} & \frac{-1}{7} & \frac{-1}{7} \\ \frac{-2}{7} & \frac{-3}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ $3\times3 \quad 3\times1$ $= 3\times1$

$\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$

$\begin{bmatrix} \frac{-3}{14} & \frac{33}{28} & \frac{-4}{7} \\ \frac{4}{7} & \frac{-1}{7} & \frac{-1}{7} \\ \frac{-2}{7} & \frac{-3}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = 3$

Question 1 (16 marks): Answer the following questions:

A) For:

$$A = \begin{bmatrix} (a+b-1) & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 2a-3b-7 \end{bmatrix}$$

Find all values of a and b for which A and B are both invertible:

$5a+5b=7$

$5a-9b=22$

$AB = I$

$$\begin{bmatrix} a+b-1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & (2a-3b-7) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5(a+b-1) & 0 \\ 0 & 3(2a-3b-7) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} 5(a+b-1)=1 \\ 3(2a-3b-7)=1 \end{matrix}$$

$$\boxed{\begin{matrix} a = 2.31 \\ b = -0.91 \end{matrix}}$$

- Let A be an $n \times n$ symmetric matrix

let $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$

B) Show that A^2 is symmetric.

$$A^2 = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} (a^2+c^2) & (ac+bc) \\ (ac+bc) & (c^2+b^2) \end{bmatrix}$$

So A^2 is symmetric

$2(ac+bc)-3c$

C) Show that $2A^2 - 3A + I$ is symmetric.

$$2\begin{bmatrix} (a^2+c^2) & (ac+bc) \\ (ac+bc) & (c^2+b^2) \end{bmatrix} - 3\begin{bmatrix} a & c \\ c & b \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ((2a^2-3a)+(2c^2-3c)) & \\ (2(ac+bc)-3c) & (2(c^2+b^2)-3b) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(a^2+c^2)-3a+1 & (2(ac+bc)-3c) \\ (2(ac+bc)-3c) & (2(c^2+b^2)-3b) \end{bmatrix}$$

the diagonal is the same

D) Find a diagonal matrix A that satisfies

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$I \to E$

$\boxed{-2R_1 + R_3}$

E) Consider the matrices:

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matric such that: $EA = B$

$\cancel{E = AB}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

F) Determine conditions on the bi's, if any, in order to guarantee that the linear system is consistent.

$$x_1 + 3x_2 = b_1$$
$$-2x_1 + x_2 = b_2$$

consistent $\to$ at least one solution

① $Ax = b$, A should be invertable

$A \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \to A^{-} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$

$\frac{1}{7} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$

$\det(A) \neq 0$, so it's invertable, consistent

4

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G) Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$, and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

$Ax - x = 0$

$(A - I)x = 0 \to \boxed{(A - B)C = (AC - BC)}$

$\left(\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & \cancel{0} & \cancel{2} \\ 2 & 1 & -2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 = 0$ $\boxed{x_1 = 0}$

$2x_1 + x_2 - 2x_3 = 0$

$x_2 - 2x_3 = 0$

$\boxed{x_2 = 0}$

$x_1 + x_2 + 2x_3 = 0$

$x_2 + 2x_3 = 0$

$\boxed{x_3 = 0}$

Question 2 (4 marks): Show that the given matrix is invertible for all values of θ, and find (A^{-1}).

invertable : $\det(A) \neq 0$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cos\theta(\cos\theta - 0) - \sin\theta(-\sin\theta) + 0$

$\boxed{\cos\theta^2 + \sin\theta^2 = 1}$ it's invertable for all θ.

$$[A/I] = \left[\begin{array}{ccc|ccc} \cos\theta & \sin\theta & 0 & 1 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \tan\theta & 0 & \sec\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & \sec\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_2 / \cos\theta$$

$$\left[\begin{array}{ccc|ccc} 1 & \tan\theta & 0 & \sec & 0 & 0 \\ 0 & (-\sin\theta(\tan + \cos\theta) & 0 & -\sin\sec\theta & \sec\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] -\sin\theta R_1 + R_2$$

Question 2 (4 marks): Show that the given matrix is invertible for all values of θ, and find (A^{-1}).

invertable: det(A) ≠ 0

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2

$\cos\theta(\cos\theta - 0) - \sin\theta(-\sin\theta) + 0$

1.5

$\boxed{\cos\theta^2 + \sin\theta^2 = 1}$ it's invertable for all θ.

$$[A/I] = \left[\begin{array}{ccc|ccc} \cos\theta & \sin\theta & 0 & 1 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \tan\theta & 0 & \sec\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & \sec\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] R_1/\cos\theta$$

$$\left[\begin{array}{ccc|ccc} 1 & \tan\theta & 0 & \sec & 0 & 0 \\ 0 & (-\sin\theta\tan+\cos\theta) & 0 & -\sin\sec\theta & \sec\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] -\sin\theta R_1+$$

0