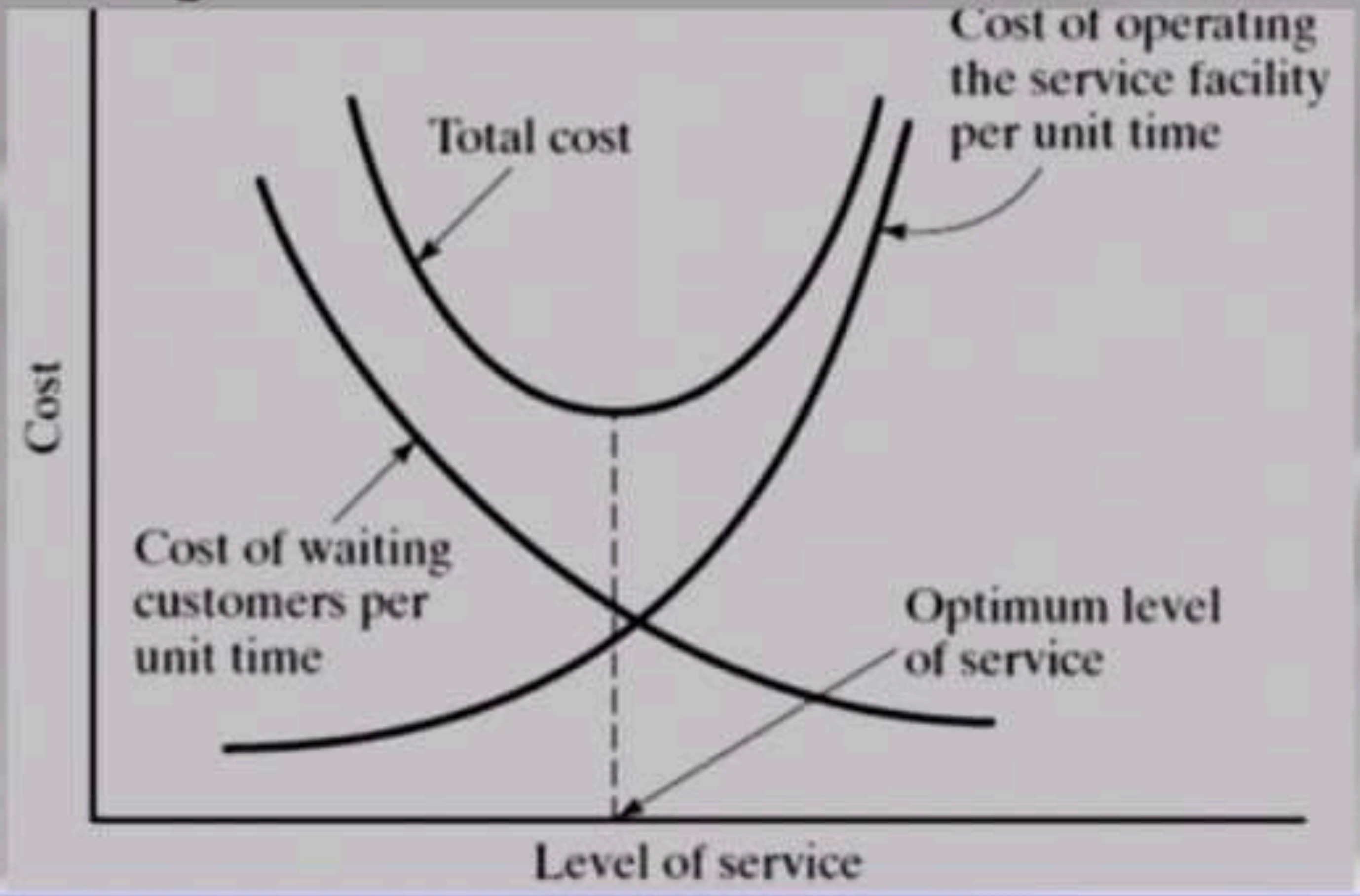
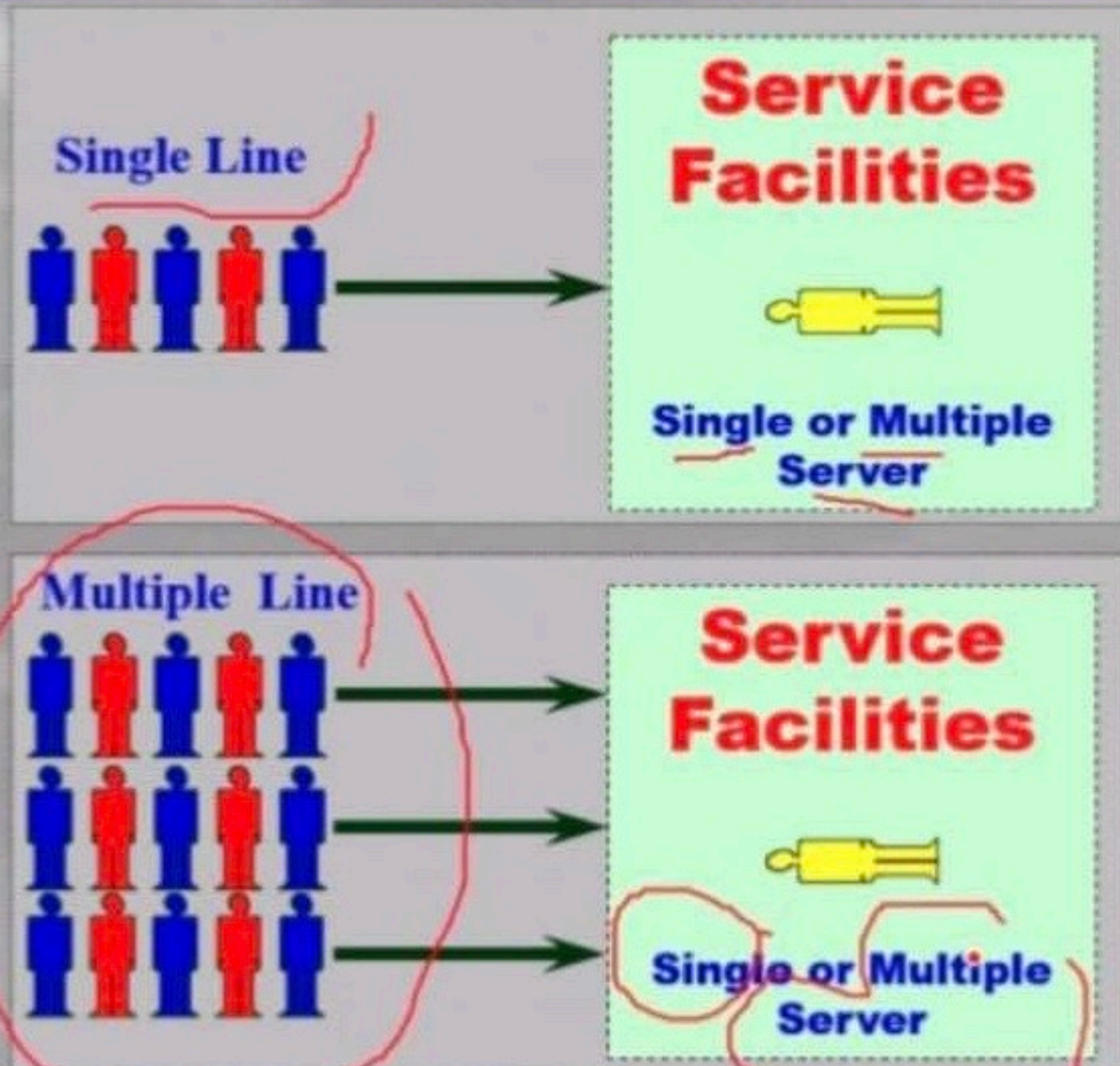
Queuing System

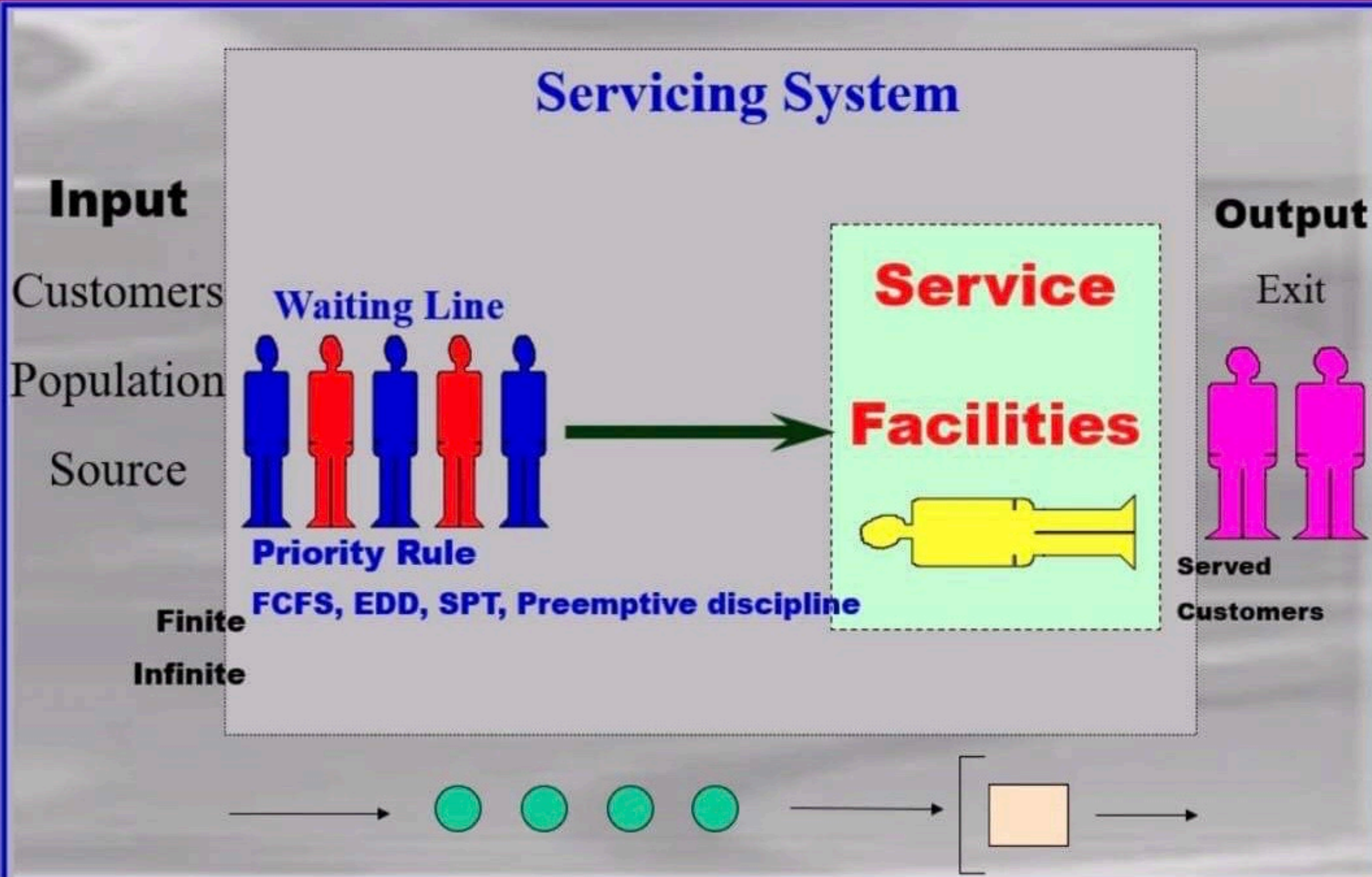
- Sales orders waiting for shipping,
- Clients waiting at a bank,
- Clients waiting for hair cutting at a hair salon
- Products waiting for processing on a certain machine
- Trucks waiting to be unloaded at a warehouse,
- Machines waiting to be repaired by a maintenance crew,
- Patients waiting to be examined by a physician
- Lines of theater goers waiting to purchase tickets
- Or inventory items waiting to be used.



Queuing System

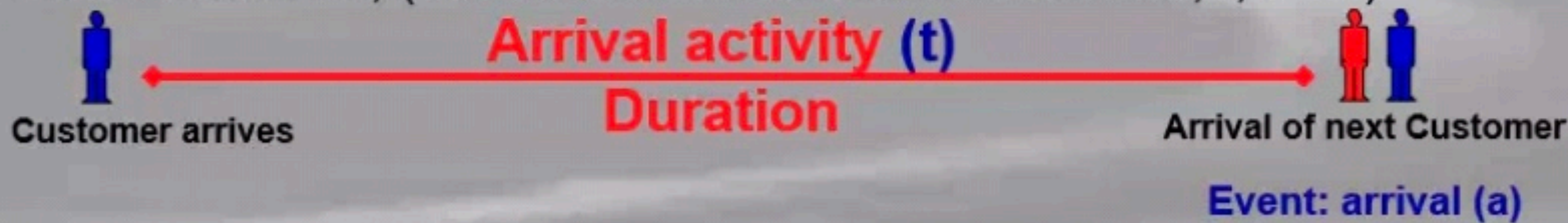


Queuing System

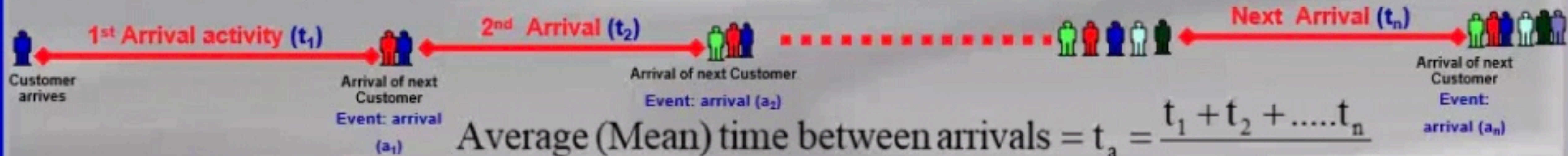


Queuing System

- **Customer**: person, item, or entity to be "processed"
  - Arrivals
  - Number of arrivals, n, during a specified period T, n = 0, 1, 2,3,…..
- **Customer Population:** The source of input to the service system,
  - Finite or
  - Infinite.
- **Interarrival time:** The time between the arrival of a customer and the arrival of the next customer, (Time between successive arrivals, t, $t \ge 0$)



- **Mean (Average) time between successive arrivals (t_a)** : The average time between the arrival of a customer and the arrival of the next customer.
  - Constant Arrival , the same time between arrivals,
  - Variable, according to a certain probability distribution (exponential distribution)

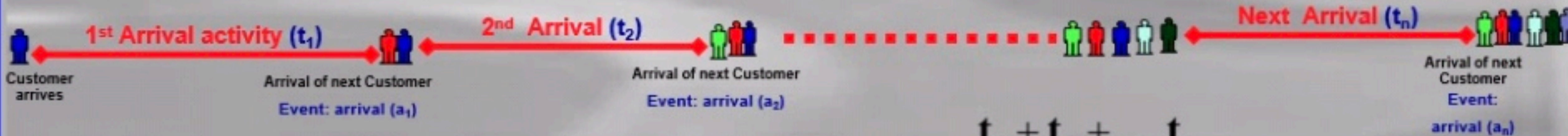


$$\text{Average (Mean) time between arrivals} = t_a = \frac{t_1 + t_2 + \dots .. t_n}{n}$$

Queuing System

- **Average Arrival Rate, (λ)** : Number of entities arriving in the system during unit time (customers per unit time).
  - Constant Arrival , the same time between arrivals,

  Variable, according to a certain probability distribution (Poisson distribution)



$$\textbf{Average (Mean) time between arrivals} = t_a = \frac{t_1 + t_2 + \ldots . t_n}{n}$$

$$\lambda = \frac{1}{t_a}$$

- Random interarrival time (t_a) is described quantitatively in queuing models by the exponential distribution, and arrival rate λ is described by Poisson

	Exponential	Poisson
Random variable	*Time* between successive arrivals, t	*Number* of arrivals, n, during a specified period T
Range	$t \geq 0$	$n = 0, 1, 2, \ldots$
Density function	$f(t) = \lambda e^{-\lambda t}, t \geq 0$	$p_n(T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}, n = 0, 1, 2, \ldots$
Mean value	$\frac{1}{\lambda}$ time units	λT arrivals during T
Cumulative probability	$P\{t \leq A\} = 1 - e^{-\lambda A}$	$p_{n \leq N}(T) = p_0(T) + p_1(T) + \cdots + p_N(T)$
P\{no arrivals during period A\}	$P\{t > A\} = e^{-\lambda A}$	$p_0(A) = e^{-\lambda A}$

Queuing System

- **Server:** resource (machine, person, runway, parking spot) that serves or "processes" customer
  - Single server (s = 1, or c =1)
  - Multiple servers (s, or c)
- **Servicing time(interdeparture time):** Average time taken to provide the service or perform the required process
  - Constant Arrival , the same time between arrivals,
  - Variable, according to a certain probability distribution (exponential distribution)
- **Service rate(Service capacity):** Average time taken to provide the service or perform the required process (service rate).
  - per server (μ)
  - per the whole system ($s\mu$)

$$\text{Average (Mean) servicing time} = t_s = \frac{t_1 + t_2 + \ldots.. t_n}{n}$$

$$\mu = \frac{1}{t_s}$$

Queuing System

- **Queue or Waiting Line:**

Considering the whole system:

- if $\lambda < \mu$: this is queuing situation ($t_a > t_s$) and the system is stable. In this case if arrival rate **in some time during operation** exceeds capacity, then a **queue or (a waiting line)** forms.
- $\lambda \geq \mu$: (Capacity Problem) because the steady-state probabilities do not exist ($t_a \leq t_s$).

- **Queue size** may be finite or, infinite
- **Queue discipline**: FCFS, SPT, EDD, SIRO, GD
- **Customer behavior:** *patient (JOCKEY ينتظر)*or (*impatient* balking يحبط , reneging يخرج عن الدور)

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Specialized Poisson Queues

- **(a/b/c):(d/e/f)** (Kendall): (lee(d/e)/taha)
  - a: Arrivals distribution
  - b: Departure (service time) distribution.
  - c: Number of parallel servers
  - d: Queue discipline.
  - e: Maximum number allowed in the system.
  - f: Size of the calling source (finite or infinite)

FCFS	First come, first served
LCFS	Last come, first served
SIRO	Service in random order
GD	General discipline

a,b	
M	Poisson(Markovian) arrivals or departures distribution (or equivalently exponential interarrival or service time distribution)
D	Constant (deterministic) time
E_k	Erlang or gamma distribution of time (or equivalently the sum of independent exponential distributions)
GI	General distribution of interarrival time
G	General distribution of service time

Notation Interpretation

(a/b/c):(d/e/f)

(M/D/10):(GD/20/∞)

- M: Poisson arrivals time, Exponential interarrival
- D: Constant Service time distribution
- 10: 10 Identical servers
- GD: The queue discipline is general (any type)
- 20: 20 Customer in the entire system
- ∞: Size of the source is infinite

(a/b/c):(d/e/f)

(GI/G/R):(SIRO/N/k)

- GI: General distribution of interarrival time
- G: General distribution of service time
- R: R Available servers
- SIRO: The queue discipline is service in random order
- N: There are N Customers in system
- k: Size of the source is finite

Generalized Poisson queuing Model

- Combining arrivals and departures based on exponential distribution.

  Poisson distributed arrivals = exponential distributed interarrival.

  Exponential distributed departures = Poisson distributed interdeparture.

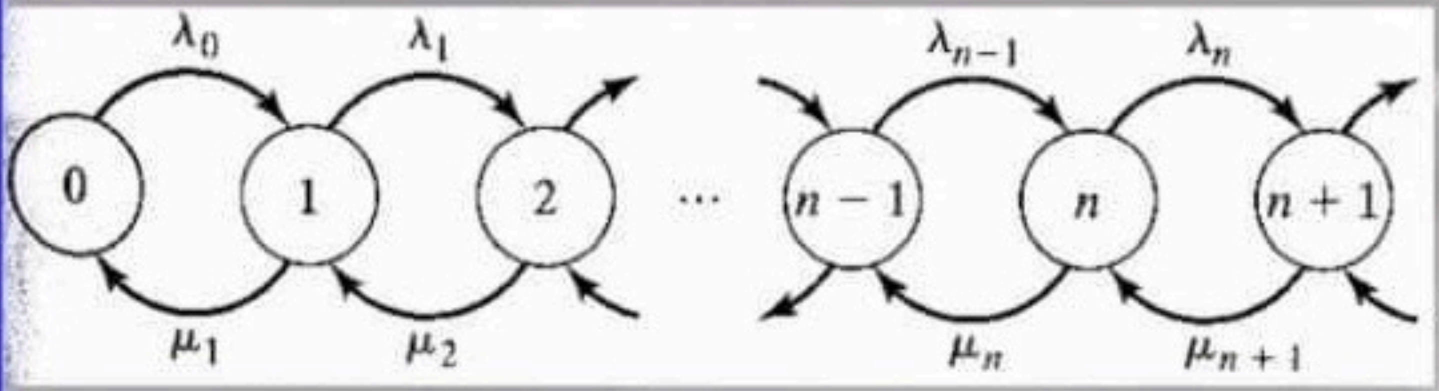
- Steady-state probability of *n* customers in the system (p_n) function of

- *n* = Number of customers in the system (in-queue plus in-service)

  λ_n = Arrival rate given *n* customers in the system

  μ_n = Departure rate given *n* customers in the system

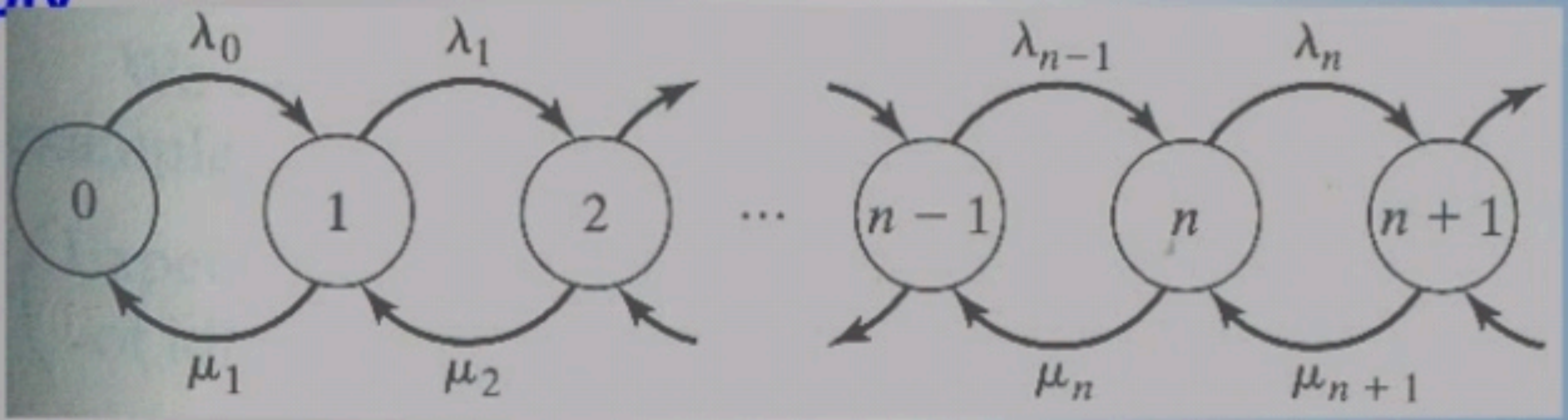
- Using the transition-rate diagram, the steady state is n when No. of customers in the system *n*.



state n can change only to two possible states: n-1 when a departure occurs at the rate μ_n and n+1 when an arrival occurs at the rate λ_n

Generalized Poisson queuing Model

- State 0 can only change to state 1 when an arrival occurs at the rate λ_0. Notice that μ_0 is undefined because no departures can occur if the system is empty



- Expected rate of flow into state n ERFI= $\lambda_{n-1}P_{n-1}+\mu_{n+1}P_{n+1}$
- Expected rate of flow out of state n ERFO= $(\lambda_n+\mu_n)P_n$
- Under steady-state conditions , for n>0 ERFO=ERFI

$$\lambda_{n-1}P_{n-1}+\mu_{n+1}P_{n+1}=(\lambda_n+\mu_n)P_n,\ \text{for } n=1,2,\ldots$$

$$\lambda_0 P_0=\mu_1 P_1,\ \text{for } n=0 \Rightarrow P_1=\left(\frac{\lambda_0}{\mu_1}\right)P_0$$

- In General $P_n=\left(\frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_1}\right)P_0,\ \text{for } n=1,2,\ldots \text{and } \sum_{n=0}^{\infty}P_n=1$

Probabilistic Operations Research

Generalized Poisson queuing Model

A store operates with 3 checkout counters. The manager uses the following schedule to determine the number of counters in the operation depending on the number of customers in the store:

No. of Customers in the Store	No, of Counters in Operation
1-3	1
4-6	2
More than 6	3

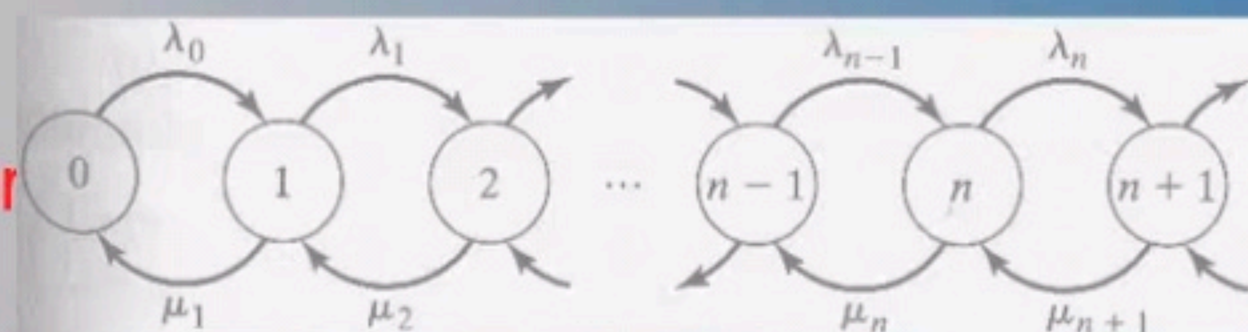
Customers arrive in the counter area according to Poisson distribution with a rate of 10 customers per hour. The average checkout time per customer is exponential with mean 12 minutes. Determine the steady state probability of n customers in the checkout area.

$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_n = 10$ Cus./hr

$\mu_1 = \mu_2 = \mu_3 = 60/12 = 5$Cus./hr.

$\mu_4 = \mu_5 = \mu_6 = (2)\ 60/12 = 10$ Cus./hr

$\mu_7 = \ldots = \mu_n = (3)\ 60/12 = 15$Cus./hr.



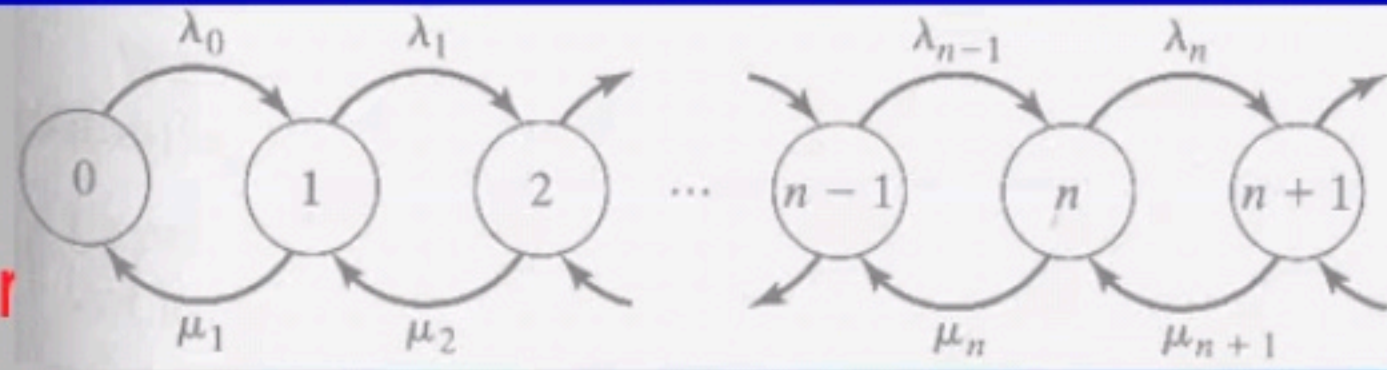
Generalized Poisson queuing Model

$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_n = 10$ Cus./hr

$\mu_1 = \mu_2 = \mu_3 = 60/12 = 5$Cus./hr.

$\mu_4 = \mu_5 = \mu_6 = (2)\ 60/12 = 10$ Cus./hr

$\mu_7 = \ldots = \mu_n = (3)\ 60/12 = 15$Cus./hr.



$$P_n = \left(\frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1}\right)P_0, \text{ for n} = 1,2,\ldots \text{and} \sum_{n=0}^{\infty} P_n = 1$$

$$\lambda_0 P_0 = \mu_1 P_1, \text{for n} = 0 \Rightarrow P_1 = \left(\frac{\lambda_0}{\mu_1}\right)P_0$$

$$\lambda_0 P_0 = \mu_1 P_1, \Rightarrow P_1 = \left(\frac{\lambda_0}{\mu_1}\right)P_0 = \left(\frac{10}{5}\right)P_0 = 2P_0 = 2/55$$

$$\lambda_1 P_1 \cdot \mu_1 P_1 = \lambda_0 P_0 \cdot \mu_2 P_2, \Rightarrow P_2 = \left(\frac{\lambda_1\lambda_0}{\mu_2\mu_1}\right)P_0 = \left(\frac{10}{5}\right)^2 P_0 = 4P_0 = 4/55$$

$$\lambda_2 P_2 \cdot \mu_2 P_2 = \lambda_1 P_1 \cdot \mu_3 P_3, \Rightarrow P_3 = \left(\frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}\right)P_0 = \left(\frac{10}{5}\right)^3 P_0 = 8P_0 = 8/55$$

$$\Rightarrow P_4 = \left(\frac{\lambda_3\lambda_2\lambda_1\lambda_0}{\mu_4\mu_3\mu_2\mu_1}\right)P_0 = \left(\frac{10}{5}\right)^3\left(\frac{10}{10}\right)P_0 = 8P_0 = 8/55$$

$$\Rightarrow P_5 = \left(\frac{\lambda_4\lambda_3\lambda_2\lambda_1\lambda_0}{\mu_5\mu_4\mu_3\mu_2\mu_1}\right)P_0 = \left(\frac{10}{5}\right)^3\left(\frac{10}{10}\right)^2 P_0 = 8P_0 = 8/55$$

$$\Rightarrow P_6 = \left(\frac{\lambda_5\lambda_4\lambda_3\lambda_2\lambda_1\lambda_0}{\mu_6\mu_5\mu_4\mu_3\mu_2\mu_1}\right)P_0 = \left(\frac{10}{5}\right)^3\left(\frac{10}{10}\right)^3 P_0 = 8P_0 = 8/55$$

$$\Rightarrow P_n = \left(\frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1}\right)P_0 = \left[\left(\frac{10}{5}\right)^3\left(\frac{10}{10}\right)^3\left(\frac{10}{15}\right)^{n-6}P_0\right] = \left[8\left(\frac{2}{3}\right)^{n-6}P_0\right] = 8\left(\frac{2}{3}\right)^{n-6}/55, n = 7,8\ldots$$

$$P_0 + P_0\left\{\begin{array}{l}2+4+8+8+8+8+8\left(\frac{2}{3}\right)\\ +8\left(\frac{2}{3}\right)^2+8\left(\frac{2}{3}\right)^3+\ldots\ldots\end{array}\right\} = 1$$

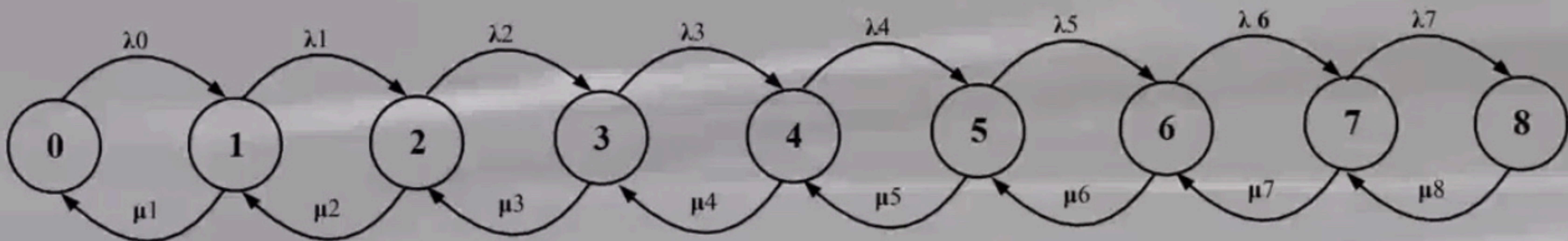
$$\Rightarrow P_0 = \frac{1}{55}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

A car wash facility operates with 4 bays (Servers). Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if all the servers are busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes.
Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

1. Construct the state transition diagram



2. Setup the set of possible number of cars (n) might be exist in the system

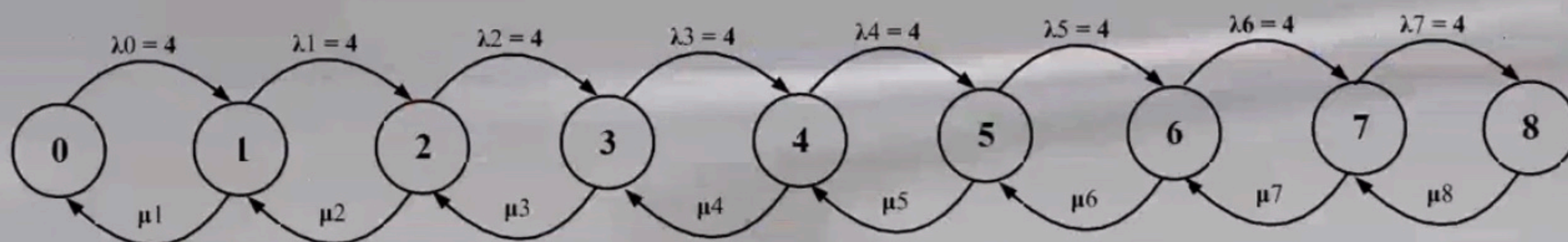
$$n = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

A car wash facility operates with 4 bays (Servers). Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if all the servers are busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes.
Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

1. Construct the state transition diagram
2. Setup the set of possible number of cars (n) might be exist in the system
3. What is the average car's arrival rates; $\lambda 0, \lambda 1, \lambda 2, \lambda 3, \dots \lambda n\text{-}1$.

Cars arrive according to a Poisson distribution with a mean of 4 cars per hour = $\lambda 0, \lambda 1, \lambda 2, \lambda 3, \dots \lambda 7$.



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Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

4. What is the average car's servicing rates; μ0, μ1, μ2, μ3, . . . μn.

The time for washing and cleaning a car is exponential, with a mean of 10 minutes, this is the service time per car per working bay it is not the rate.

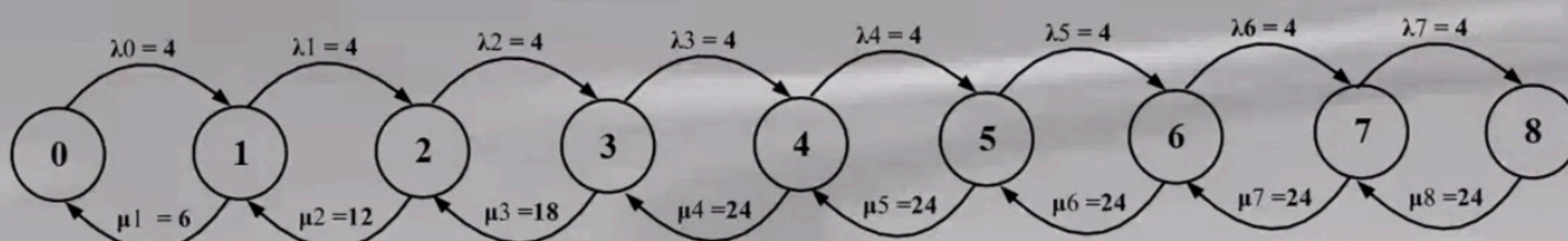
$$\text{Service rate } \mu \text{ per single bay (Cars/Hour)} = \frac{\text{Available minutes per hour (Minutes/ Hour)}}{\text{Servicing time per car (Minutes/ Car)}}$$

$$\text{Service rate } \mu \text{ per single bay (Cars/Hour)} = \frac{60}{10} = 6 \text{ Cars/Hour} \quad \rightarrow \text{Applied when } n = 1$$

$$\text{Service rate } \mu \text{ per two bays (Cars/Hour)} = 2\left(\frac{60}{10}\right) = 12 \text{ Cars/Hour} \quad \rightarrow \text{Applied when } n = 2$$

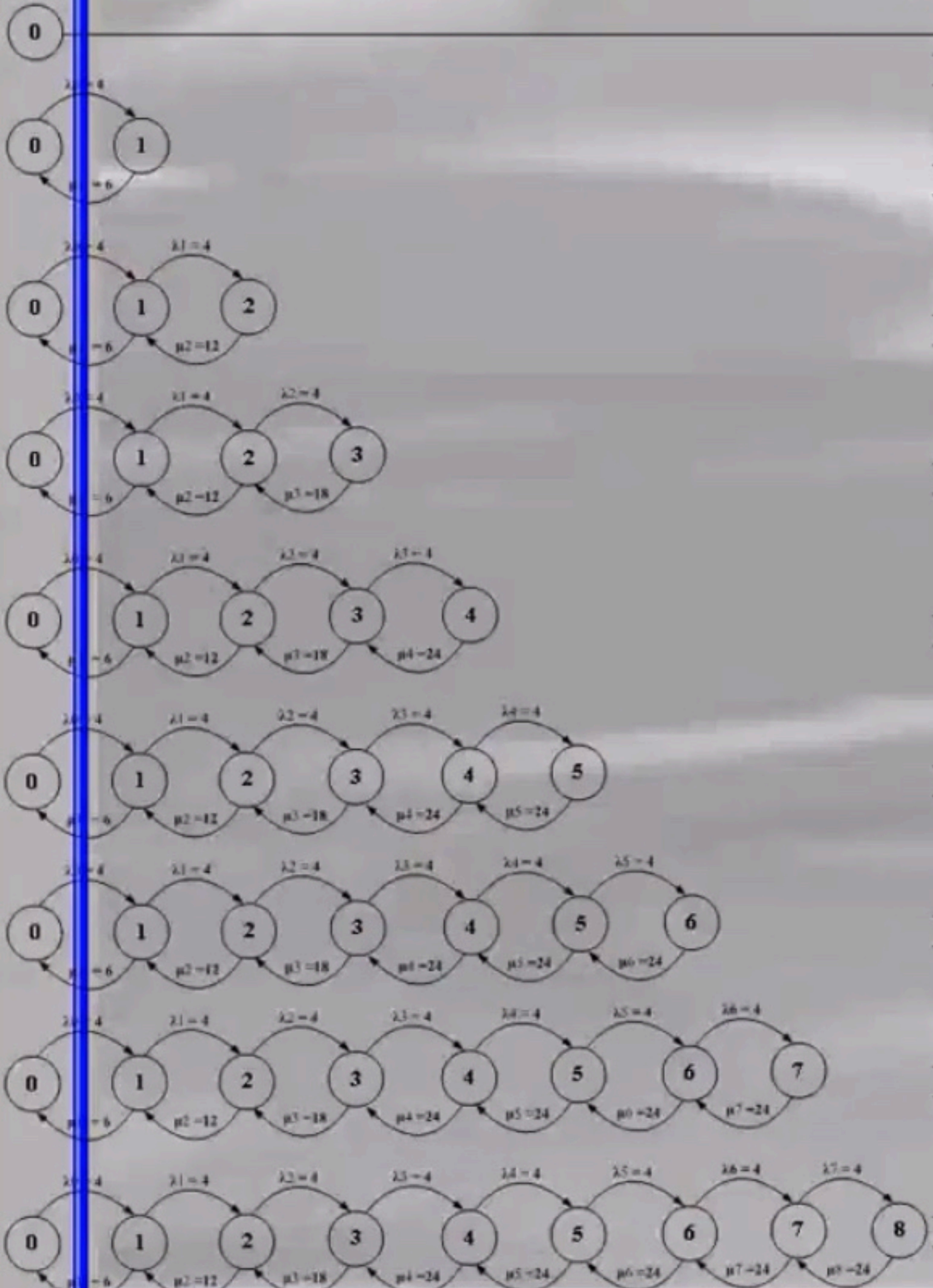
$$\text{Service rate } \mu \text{ per three bays (Cars/Hour)} = 3\left(\frac{60}{10}\right) = 18 \text{ Cars/Hour} \quad \rightarrow \text{Applied when } n = 3$$

$$\text{Service rate } \mu \text{ per four bays (Cars/Hour)} = 4\left(\frac{60}{10}\right) = 24 \text{ Cars/Hour} \quad \rightarrow \text{Applied when } n \geq 4$$



Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.



$$P_0 = P_0$$

$$P_1 = \left(\frac{\lambda_0}{\mu_1}\right)P_0 = \left(\frac{4}{6}\right)P_0$$

$$P_2 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)P_0$$

$$P_3 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)\left(\frac{4}{18}\right)P_0$$

$$P_4 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)\left(\frac{4}{18}\right)\left(\frac{4}{24}\right)P_0$$

$$P_5 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)\left(\frac{4}{18}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)P_0$$

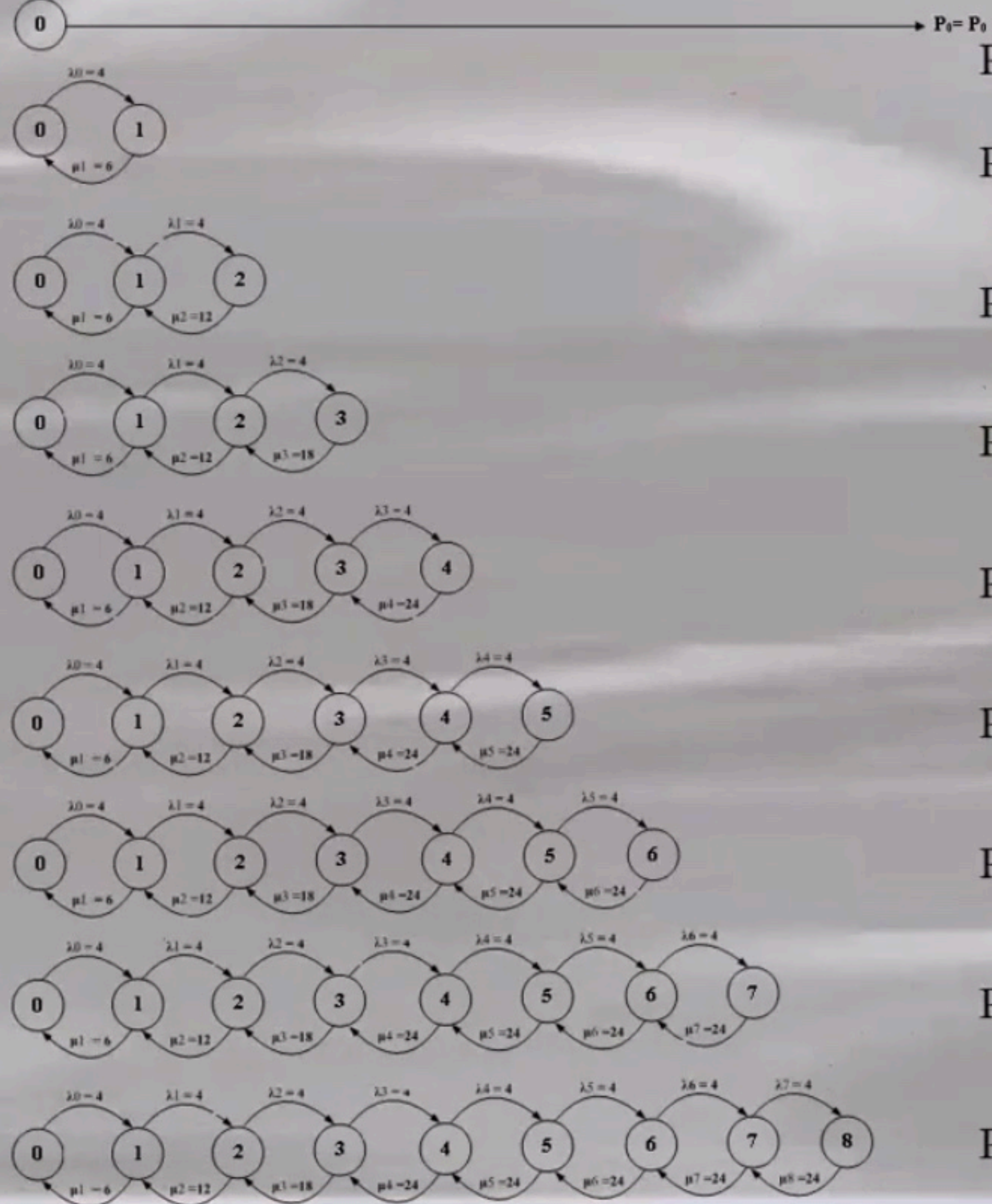
$$P_6 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)\left(\frac{4}{18}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)P_0$$

$$P_7 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\frac{\lambda_6}{\mu_7}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)\left(\frac{4}{18}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)P_0$$

$$P_8 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\frac{\lambda_6}{\mu_7}\frac{\lambda_6}{\mu_7}\right)P_0 = \left(\frac{4}{6}\right)\left(\frac{4}{12}\right)\left(\frac{4}{18}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)\left(\frac{4}{24}\right)P_0$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.



$$P_0 = P_0$$

$$P_1 = \left(\frac{\lambda_0}{\mu_1}\right)P_0 = \left(\frac{4}{6}\right)P_0 = \left(\frac{2}{3}\right)P_0$$

$$P_2 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\right)P_0 = \left(\frac{2}{9}\right)P_0$$

$$P_3 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\right)P_0 = \left(\frac{4}{81}\right)P_0$$

$$P_4 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\right)P_0 = \left(\frac{2}{243}\right)P_0$$

$$P_5 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\right)P_0 = \left(\frac{1}{729}\right)P_0$$

$$P_6 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\right)P_0 = \left(\frac{1}{4{,}374}\right)P_0$$

$$P_7 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\frac{\lambda_6}{\mu_7}\right)P_0 = \left(\frac{1}{26{,}244}\right)P_0$$

$$P_8 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\frac{\lambda_6}{\mu_7}\frac{\lambda_6}{\mu_7}\right)P_0 = \left(\frac{1}{157{,}464}\right)P_0$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1}{6}\right)+\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^3+\left(\frac{1}{6}\right)^4+\left(\frac{1}{6}\right)^5\right)\right)=1$$

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1}{6}\right)+\left(\frac{1}{36}\right)+\left(\frac{1}{216}\right)+\left(\frac{1}{1296}\right)+\left(\frac{1}{7776}\right)\right)\right)=1$$

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1296+216+36+6+1}{7776}\right)\right)\right)=1$$

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1555}{7776}\right)\right)\right)=1$$

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{1555}{157{,}464}\right)\right)=1$$

$$P_0\left(1+\left(\frac{104{,}976+34{,}992+7{,}776+1{,}555}{157{,}464}\right)\right)=1$$

$$P_0\left(1+\left(\frac{149{,}299}{157{,}464}\right)\right)=1$$

$$P_0(1+(0.94814))=1 \Rightarrow P_0=\frac{1}{1.94815}=0.5133=51.33\%$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1}{6}\right)+\left(\frac{1}{36}\right)+\left(\frac{1}{216}\right)+\left(\frac{1}{1296}\right)+\left(\frac{1}{7776}\right)\right)\right)=1$$

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1296+216+36+6+1}{7776}\right)\right)\right)=1$$

$$P_0\left(1+\left(\frac{2}{3}\right)+\left(\frac{2}{9}\right)+\left(\frac{4}{81}\right)+\left(\frac{4}{81}\right)\left(\left(\frac{1555}{7776}\right)\right)\right)=1$$

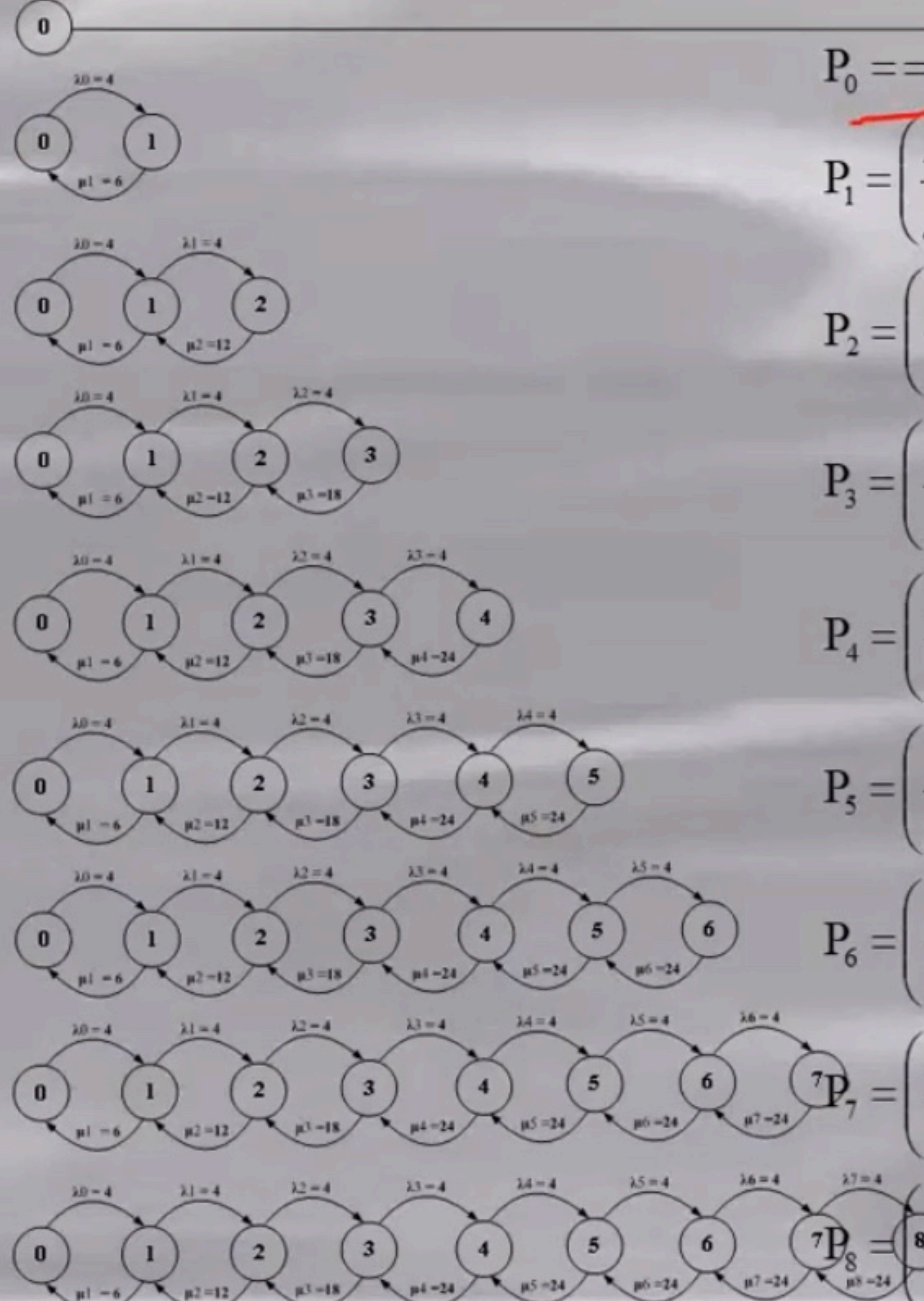
$$P_0\left(1+\left(\frac{104{,}976+34{,}992+7{,}776+1{,}555}{157{,}464}\right)\right)=1$$

$$P_0\left(1+\left(\frac{149{,}299}{157{,}464}\right)\right)=1$$

$$P_0(1+(0.94814))=1 \Rightarrow P_0=\frac{1}{1.94815}=0.5133=51.33\%$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.



$$P_0 = = 0.513 = 51.3\%$$

$$P_1 = \left(\frac{\lambda_0}{\mu_1}\right)P_0 = \left(\frac{4}{6}\right)P_0 = \left(\frac{2}{3}\right)P_0 = 0.342 = 34.2\%$$

$$P_2 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\right)P_0 = \left(\frac{2}{9}\right)P_0 = 0.114 = 11.04\%$$

$$P_3 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\right)P_0 = \left(\frac{4}{81}\right)P_0 = 0.0254 = 2.54\%$$

$$P_4 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\right)P_0 = \left(\frac{2}{243}\right)P_0 = 0.00423 = 0.423\%$$

$$P_5 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\right)P_0 = \left(\frac{1}{729}\right)P_0 = 7.03\times10^{-4} = 7.03\times10^{-2}\%$$

$$P_6 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\right)P_0 = \left(\frac{1}{4{,}374}\right)P_0 = 1.17\times10^{-4} = 1.17\times10^{-2}\%$$

$$P_7 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\frac{\lambda_6}{\mu_7}\right)P_0 = \left(\frac{1}{26{,}244}\right)P_0 = 1.95\times10^{-5} = 1.95\times10^{-3}\%$$

$$P_8 = \left(\frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4}\frac{\lambda_4}{\mu_5}\frac{\lambda_5}{\mu_6}\frac{\lambda_6}{\mu_7}\frac{\lambda_6}{\mu_7}\right)P_0 = \left(\frac{1}{157{,}464}\right)P_0 = 3.26\times10^{-6} = 3.26\times10^{-4}\%$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

A car wash facility operates with 4 bays (Servers). Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if all the servers are busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes.
Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

1. Construct the state transition diagram
2. Setup the set of possible number of cars (n) might be exist in the system
3. What is the average car's arrival rates; $\lambda 0$, $\lambda 1$, $\lambda 2$, $\lambda 3$, . . . $\lambda n-1$.
4. What is the average car's servicing rates; $\mu 1$, $\mu 2$, $\mu 3$, . . . μn.
5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.
6. What is the steady-state probability of having 0 car in the washing facility

$P_0 = 0.513 = 51.3\%$ of the time the system has zero car

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

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6. What is the steady-state probability of having 0 car in the washing facility
7. What is the probability that the washing facility is empty of cars

$P_0 = 0.513 = 51.3\%$ of the time the system is empty

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

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8. What is the probability that the washing facility is full

$P_8 = 3.26 \times 10^{-4}$ % of the time the system is full

0916555 Probabilistic Operation Research

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

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Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

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5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.
6. What is the steady-state probability of having 0 car in the washing facility
7. What is the probability that the washing facility is empty of cars
8. What is the probability that the washing facility is full
9. What is the probability that newly arriving cars balk to other facilities.
   $P_8 = 3.26 \times 10^{-4}$ % of the time newly arriving cars balk to other facilities

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

A car wash facility operates with 4 bays (Servers). Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if all the servers are busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes.
Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

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5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.
6. What is the steady-state probability of having 0 car in the washing facility
7. What is the probability that the washing facility is empty of cars
8. What is the probability that the washing facility is full
9. What is the probability that newly arriving cars balk to other facilities.
10. What is the average number of cars in the system

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

10. What is the average number of cars in the system (Ls)

Expected Number of cars (n)	Probability of having (n) cars Pn
0	0.513
1	0.342
2	0.114
3	0.0254
4	0.00423
5	7.03×10^{-4}
6	1.17×10^{-4}
7	1.95×10^{-5}
8	3.26×10^{-6}

$$L_s = \sum_{n=1}^{8} np_n = 0.67$$

Steady-State Transition Probabilities of A Generalized finite-Sized Queuing Situation

11. What is the average number of cars Waiting (Queuing) in the system (Lq)

Expected Number of cars (n)	Probability of having (n) cars Pn	Number of cars waiting (queuing)
0	0.513	-
1	0.342	-
2	0.114	-
3	0.0254	-
4	0.00423	-
5	7.03×10^{-4}	1
6	1.17×10^{-4}	2
7	1.95×10^{-5}	3
8	3.26×10^{-6}	4

$$L_q = \sum_{n=c+1=5}^{8} (n-c)\, p_n = 0.001$$

Probabilistic Operations Research

Role of Exponential Distribution

- Random interarrival and service times are described quantitatively in queuing models by the exponential distribution, which is defined as

$$f(t) = \lambda e^{-\lambda T}$$

A service machine always has a standby unit for immediate replacement upon failure. The time to failure of the machine (or its standby unit) is exponential and occurs every 5 hours, on the average. The machine operator claims that the machine is "in the habit" of breaking down every night around 8:30 p.m. Analyze the operator's claim. The average failure rate of the machine is λ= 1/5 = 0.2 failure per hour. Thus, the exponential distribution of the time to failure is

$$f(t) = 0.2 e^{-(0.2)(t)} \qquad t > 0$$

if the time now is 8:20 p.m.,

T = 8:30 – 8:20 = 10 Min. = 10/60 hr.

$$p(t < \tfrac{10}{60}) = 1 - 0.2 e^{-(0.2)\left(\frac{10}{60}\right)} = 0.03278$$

Probabilistic Operations Research

Role of Exponential Distribution

- Random interarrival and service times are described quantitatively in queuing models by the exponential distribution, which is defined as

$$f(t) = \lambda e^{-\lambda T}$$

- exponential distribution describes the probability that t_a, and t_s at a particular facility will be no more than *T* time periods.

$$P(t_a \le T) = 1 - e^{-\lambda T} \quad Mean = \frac{1}{\lambda} \quad Variance = \left(\frac{1}{\lambda}\right)^2$$

$$P(t_s \le T) = 1 - e^{-\mu T} \quad Mean = \frac{1}{\mu} \quad Variance = \left(\frac{1}{\mu}\right)^2$$

- λ mean number of customers arriving per period,
- μ mean number of customers completing service per period,
- **T** target time.
- **The clerk at a customer complaint desk can serve an average of three customers per hour. What is the probability that a customer will require less than 10 minutes of service?**

μ = 3 customers / hour, *T* = 10 minute = 10/60 hour=0.167 hours

the probability that the service time of the customer will be no more than 0.167 hours is

$$P(t \le 0.167) = 1 - e^{-(3)(0.167)} = 0.39$$

Probabilistic Operations Research

Role of Poisson Distribution

- Arrival and service Rate are described quantitatively in queuing models by the Poisson distribution, which is defined as

$$P_n(T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T} \quad for\ n = 0,1,2,....$$

- p_n (T) = probability of n arrivals in T time periods,
- λ average number of customer arrivals per period,
- The mean of the Poisson distribution is λT,
- The variance also is λT.
- Poisson distribution is a discrete distribution;
- **Example**
- **Customers arrive at a complaint desk in a large department store at the rate of two customers per hour. What is the probability that four customers will arrive during the next hour?**

λ = 2 customers / hour, *T* = 1 hour, and *n* = 4 customers

The probability that four customers will arrive in the next hour is

$$P_4(1) = \frac{(2(1))^4}{4!} e^{-2\ (1)} = 0.09$$

Probabilistic Operations Research

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

- **Pure birth model:**
  - Arrivals only are allowed,
  - Ex. Creation of birth certificates for newly born babies.
  - Exponential distributed interarrival time (Poisson distributed arrival rate)
- **Pure death model:**
  - Departure only are allowed, system starts with N customers at time 0
  - Ex. Random withdrawal of a stocked item in a store.
  - Exponential distributed inter-departure time (service time)

	Exponential	Poisson
Random Variable	interarrival time, Service time)	Number of arrivals, n, during a specified period T (# of served)
Range	$t \geq 0$	$n = 0, 1, 2, 3, \ldots..$
Density function	$f(t) = \mu e^{-\mu t}, t \geq 0$	$P_n(T) = \frac{(\lambda T)^n e^{-\lambda t}}{n!}, n = 0, 1, 2, \ldots$
Mean value	$(1/\mu)$ *time* units	λT arrivals during T
Cumulative probability	$P\{t \leq A\} = 1 - e^{-\mu A}$	$P_{n \leq N}(T) = P_0(T) + P_1(T) + \ldots\ldots + P_N(T)$
p{no arrivals during period A}	$P\{t > A\} = e^{-\mu A}$	$P_0(A) = e^{-\lambda A}$

Probabilistic Operations Research

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

Pure birth model

Example 18.4-1

Babies are born (arrive) in a large city at the rate of one birth every 12 minutes. The time between births follows an exponential distribution. Find the following:

(a) The average number of births per year.

(b) The probability that no births will occur during1 day.

(c) The probability of issuing 50 birth certificates in 3 hours, given that 40 certificates were issued during the first 2 hours of the 3-hr period.

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

Pure birth model

Example 18.4-1

Babies are born (arrive) in a large city at the rate of one birth every 12 minutes. The time between births follows an exponential distribution. Find the following:

(a) The average number of births per year.

$$\lambda = \frac{1}{12} = \text{births/Minutes}$$

$$\lambda = \frac{24 \times 60}{12} = 120 \text{ births/day}$$

$$\lambda = \frac{24 \times 60 \times 365}{12} = 43{,}800 \text{ births/year}$$

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

Pure birth model

Example 18.4-1

Babies are born (arrive) in a large city at the rate of one birth every 12 minutes. The time between births follows an exponential distribution. Find the following:

(a) The average number of births per year.

(b) The probability that no births will occur during 1 day.

$$\lambda = 120 \text{ births/day}$$

$$P\{\text{no arrivals during period } A\} = e^{-\lambda A}$$

$$P\{\text{no arrivals during year } 1\} = e^{-120(1)} = 0$$

time between successive births exceeds one day

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

Pure birth model

Example 18.4-1

Babies are born (arrive) in a large city at the rate of one birth every 12 minutes. The time between births follows an exponential distribution. Find the following:

(c) The probability of issuing 50 birth certificates in 3 hours, given that 40 certificates were issued during the first 2 hours of the 3-hr period.

Because the distribution of the number of births is Poisson, the probability of issuing 50 certificates in 3 hours, given that 40 certificates were issued during the first 2 hours, is equivalent to having 10(= 50 − 40) births in one (= 3 − 2) hr—that is,

$$p_{10}(1) = \frac{\left(\frac{60}{12} \times 1\right)^{10} e^{-5\times 1}}{10!} = .01813$$

Probabilistic Operations Research

Pure death Model

Example 18.4-2

The florist section in a grocery store stocks **18** dozen roses at the beginning of each week. On the average, the florist sells **3 dozens a day** (one dozen at a time), but the actual demand follows a Poisson distribution. Whenever the stock level reaches **5** dozens, a new order of **18** new dozens is placed for delivery at the beginning of the following week. Because of the nature of the item, all roses left at the end of the week are disposed of. Determine the following:

(a) The probability of placing an order in any one day of the week.

(b) The average number of dozen roses discarded at the end of the week.

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

Pure death Model

Example 18.4-2

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(a) The probability of placing an order in any one day of the week.

$\mu = 3$ dozens per day

$p(n \leq 5)$ in any one day $= p(n \leq 5)$ day $1 + p(n \leq 5)$ day $2 + \ldots + p(n \leq 5)$ day 7

$$p_{n\leq 5}(t) = p_0(t) + p_1(t) + p_3(t) + p_4(t) + p_5(t)$$

Pure birth and pure death model
(Relationship between the exponential and the Poisson distributions)

Pure death Model
Example 18.4-2

The florist stocks **18** dozen roses at the beginning of each week. the florist sells **3 dozens a day** (one dozen at a time). Whenever the stock level reaches **5** dozens, a new order of **18** new dozens is placed for delivery at the beginning of the following week.

The probability of placing an order in any one day of the week.

$\mu = 3$ dozens per day

$$p(n \le 5) \text{ in any one day} = p(n \le 5) \text{ day } 1 + p(n \le 5) \text{ day } 2 + \ldots + p(n \le 5) \text{ day } 7$$

$$p_{n\le5} \text{day } t = p_0(t) + p_1(t) + p_2(t) + p_3(t) + p_4(t) + p_5(t) \quad t = 1, 2, 3, 4, 5, 6, 7$$

$$\text{day } 1: p_{n\le5}(1) = p_0(1) + p_1(1) + p_3(1) + p_4(1) + p_5(1)$$

$$= p_0(1) + \frac{3(1)^{19}}{19!} e^{-3(1)} + \frac{3(1)^{18}}{18!} e^{-3(1)} + \frac{3(1)^{17}}{17!} e^{-3(1)} + \frac{3(1)^{16}}{16!} e^{-3(1)} + \frac{3(1)^{15}}{15!} e^{-3(1)}$$

Pure birth and pure death model

(Relationship between the exponential and the Poisson distributions)

Pure death Model

Example 18.4-2

The probability of placing an order in any one day of the week.

$\mu = 3$ dozens per day

$$p(n \le 5) \text{ in any one day} = p(n \le 5) \text{ day } 1 + p(n \le 5) \text{ day } 2 + \ldots + p(n \le 5) \text{ day } 7$$

$$p_{n\le5} \text{day } t = p_0(t) + p_1(t) + p_2(t) + p_3(t) + p_4(t) + p_5(t) \quad t = 1, 2, 3, 4, 5, 6, 7$$

$$\text{day } 1: p_{n\le5}(1) = p_0(1) + p_1(1) + p_3(1) + p_4(1) + p_5(1)$$

$$= p_0(1) + \frac{3(1)^{19}}{19!} e^{-3(1)} + \frac{3(1)^{18}}{18!} e^{-3(1)} + \frac{3(1)^{17}}{17!} e^{-3(1)} + \frac{3(1)^{16}}{16!} e^{-3(1)} + \frac{3(1)^{15}}{15!} e^{-3(1)}$$

$$\text{day } t: p_{n\le5}(t) = p_0(t) + \sum_{n=1}^{5} \frac{(3t)^{18-n} e^{-3t}}{(18-n)!}, t = 1, 2, 3, 4, 5, 6, 7$$

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