

Queuing Systems.

* Waiting for service is part of our daily life.

* The principal players in a queuing situation are :-

- ↳ Customer :- person or item to be processed
- ↳ Server :- resource (person, machine, ...) that serves customer.

single	multiple
$s=1$	s
$c=1$	c

- Customer population :- The source of input to the service system.

- ↳ Finite :- ex: students on the register waiting for the entrance service to the lecture.
- ↳ infinite :- ex: Clients waiting for hair cutting at a hair salon.

- Waiting line :- is one or more customers waiting for service.

↳ Forms when ($\text{Capacity} < \text{arrival rate}$).

- Stable System :- when ($\text{Capacity} > \text{arrival rate}$)

- Interarrival time :- The time b/w when one customer arrives at a queue & when the next customer arrives.

- Service Time :- The time it takes for one particular server to complete a customer's service.

- Mean (Avg) time btwn successive arrivals (t_a) :- The avg time btwn the arrivals of a Customer & the arrival of the next Customer.

→ exponential distribution.

$$t_a = \frac{t_1 + t_2 + \dots + t_n}{n}, \quad n: \# \text{ of arrivals}$$

Avg. Arrival Rate (λ) :- # of arrivals per unit time

→ Poisson distribution.

$$\lambda = 1/t_a \rightarrow \text{imp}$$



Note

Arrival activity لا تبدأ بالوصول إلى النظام. Arrival process لا يبدأ بالوصول إلى النظام.
 هذه هي activity التي تستغرق long time duration.
 وبعدها يبدأ الوصول إلى النظام.
 وبعدها يبدأ activity جديد arrival activity (Iterative Process)

⇒ Any queuing system ^{that} provides a service given to all customers by one or more servers → they must be identical & in parallel

- Mean (Avg) servicing time (t_s) :- The avg. time taken to provide the service or perform the required process.

→ exponential distribution

$$t_s = \frac{t_1 + t_2 + \dots + t_n}{n}$$

- Service Rate (μ) :- # of customers completing service per period.

→ Poisson distribution

$$\mu = 1/t_s \rightarrow \text{imp}$$

* If $\lambda < \mu$: This is queuing situation & the sys. is stable.

* If $\lambda \geq \mu$: This is Capacity problem.

arrivals (الوصول) ← λ ← avg. rate

waiting (الانتظار) ← t_w ← servicing (الخدمة) ← t_s

Example: 18.3-1

The time to failure of the machine is: - exponential

- occurs every 5 hours (on avg.) (t)

The machine operator claims that the machine is "in the habit" of breaking down every night around 8:30 p.m.

Solution:

The avg. failure rate $\rightarrow \lambda = 1/5 = 0.2$ Failure per hour

The exp. dis. of the time to failure $\rightarrow P(t) = 0.2 e^{-(0.2)(t)}$

* If the time now is 8:20 p.m.

$\rightarrow T = 8:30 - 8:20 = 10 \text{ min.}$
 $= 10/60 \text{ hr.}$

Failure is expected!
 (10 min) 1 day

$$\rightarrow P(t < \frac{10}{60}) = 1 - e^{-(0.2)(\frac{10}{60})} = 0.03278$$

Example: The clerk at a Customer Complaint desk can serve an avg. of 3 customers per hour.

What is the prob. that a customer will require less than 10 min. of service?

Solution:

$\mu = 3 \text{ Customers/hr.} \rightarrow (\mu \text{ because \# of customers completing service/period})$

$T = 10 \text{ min.} = 10/60 \text{ hr.}$

$$\rightarrow P(t \leq \frac{10}{60}) = 1 - e^{-(3)(\frac{10}{60})} = 0.393 \quad \text{cumulative prob.}$$

$$\rightarrow P(\frac{10}{60}) = 3e^{-(3)(\frac{10}{60})} = 1.819$$

Failure is expected!
 density function: 10 min. 1 day

Example: Customers arrive at a Complaint desk in a large department store at the rate of 2 Customers per hour. What is the prob. that 4 Customers will arrive during the next hour?

Solution:

$$\lambda = 2 \text{ customers/hr}$$

$$T = 1 \text{ hr.}$$

$n = 4 \text{ Customers} \rightarrow \# \text{ of customers/arrivals in } T$

改善 Poisson distribution

$$\rightarrow P_4(1) = \frac{(2 \times 1)^4}{4!} \cdot e^{-2(1)} = 0.09$$

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Relationship between Exponential & Poisson Distribution

	Exponential	Poisson
Random variable	Time b/w successive arrivals (t)	# of arrivals (n) during a specified period (T)
Range	$t \geq 0$	$n = 0, 1, 2, \dots$
Density function	$f(t) = \lambda e^{-\lambda t}$, $t \geq 0$ (μ or λ)	$P_n(T) = \frac{(\lambda T)^n \cdot e^{-\lambda T}}{n!}$
Mean value	$1/\lambda$ time units (μ or λ)	λT arrivals during T
Cumulative probability	$P(t \leq A) = 1 - e^{-\lambda A}$ (μ or λ)	$P_{n \leq N}(T) = P_0(T) + P_1(T) + \dots + P_N(T)$
P{no arrivals during Period A}	$P(t > A) = e^{-\lambda A}$ (μ or λ)	$P_0(A) = e^{-\lambda A}$

* Pure Birth and Death Models

- 2 queuing situations:

→ **Pure birth model:** - In which only arrivals occur
 - Ex: the creation of birth certificates for newly born babies.

- Exponential dis. is used to describe the interarrival time.

→ **Pure death model:** - In which only departures take place.
 - Ex: the random withdrawal of a stocked item in the store.

- Exponential dis. is used to describe the interdeparture time.

Example: 18.4-1 (Pure birth)

The birth rate of babies is one birth every 12 min. ($t=12$)

The time btwn births → exp. dis.

① The avg. # of births (arrivals) per year: arrive → λ arrivals

$$\lambda = \frac{1}{12} \text{ (births/min)}$$

Per year calc dis. (as)

$$\lambda = \frac{1}{12} * 60 \text{ (births/hour)}$$

(min → year is 60)

$$\lambda = \frac{1}{12} * 60 * 24 \text{ (births/day)}$$

$$\lambda = \frac{1}{12} * 60 * 24 * 365 \text{ (births/year)}$$

⑥ Prob. that no births will occur during 1 day :

$$\# \text{ of arrivals} = 0 = n$$

$$T = 1 \text{ day} \rightarrow \lambda = \frac{1}{12} * 60 * 24 = 120 \text{ births/day}$$

$$P_0(1) = \frac{(120 * 1)^0}{0!} \cdot e^{-120(1)}$$

$$= e^{-120(1)} = 7.67 * 10^{-53} = 0$$

$$\text{no arrivals} = P_0(t)$$

$$= e^{-\lambda t}$$

⑦ Prob. of issuing 50 birth certificates in 3 hours, given that 40 certificates were issued during the first 2 hours of 3-hr.

The distribution is poisson $\rightarrow \# \text{ of births} = 50 - 40 = 10 \text{ births} = n$
 in $3 - 2 = 1 \text{ hr.} = T$
 $\lambda = \frac{1}{12} * 60 = 5 \text{ births/hr.}$

$$P_{10}(1) = \frac{(5 * 1)^{10}}{10!} \cdot e^{-5(1)} = 0.01813$$

*** In pure death model, the system starts with N customers at time 0, with no new arrivals.

Departures occur at the rate μ customers per unit time.

Example: 18-4-2 (Pure death)

- The florist stocks 18 dozen roses at the beginning of each week.

- On avg. $\rightarrow$ the florist sells 3 dozens a day. (expected demand)

- Actual demand $\rightarrow$ Poisson distribution.

- When the stock level reaches 5 dozens $\rightarrow$ a new order of 18 new doz. is placed for delivery.

dozens roses ← مستوى المخزون في

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$\Rightarrow$ There's departure $\rightarrow$ service/selling rate $\rightarrow \mu = 3$ dozens/day

← خلال الأسبوع بأكمله يتناقص stock level (مستوى المخزون) فربما يرفع stock level في بداية الأسبوع التالي.

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① The Prob. of placing an order in any one day of the week.

إذا بات يوم من الأيام $n \leq 5$ ← يعني في أي يوم من الأيام $n \leq 5$ ← order.

$P_{n \leq 5}(1) \leftarrow$ في اليوم الأول من الـ Prob. أنه سيتم $n \leq 5$

For day 1: $P_{n \leq 5}(1) = P_0(1) + P_1(1) + P_2(1) + P_3(1) + P_4(1) + P_5(1)$

$\rightarrow 0, 1, 2, 3, 4, 5 \leftarrow n \leq 5$ في أي يوم من الأيام

$18, 17, 16, 15, 14, 13 \leftarrow n \geq 13$ في أي يوم من الأيام

$$= P_0(1) + \frac{(3 \times 1)^{17}}{17!} e^{-(3 \times 1)} + \frac{(3 \times 1)^{16}}{16!} e^{-(3 \times 1)} + \frac{(3 \times 1)^{15}}{15!} e^{-(3 \times 1)} + \frac{(3 \times 1)^{14}}{14!} e^{-(3 \times 1)} + \frac{(3 \times 1)^{13}}{13!} e^{-(3 \times 1)}$$

(Prob. of placing an order in day 1) ↓ ما هي الاحتمالية بقبول

7 days لا يكون في أي يوم واحد في أي يوم واحد in any one day

For day 2 : $P_{n \leq 5}(2) = P_0(2) + P_1(2) + P_2(2) + P_3(2) + P_4(2) + P_5(2)$

$$= P_0(2) + \frac{(3 \times 2)^{17}}{17!} e^{-(3 \times 2)} + \frac{(3 \times 2)^{16}}{16!} e^{-(3 \times 2)} + \frac{(3 \times 2)^{15}}{15!} e^{-(3 \times 2)} + \frac{(3 \times 2)^{14}}{14!} e^{-(3 \times 2)} + \frac{(3 \times 2)^{13}}{13!} e^{-(3 \times 2)}$$

For day 7 : $P_{n \leq 5}(7) = P_0(7) + P_1(7) + P_2(7) + P_3(7) + P_4(7) + P_5(7)$

$$= P_0(7) + \frac{(3 \times 7)^{17}}{17!} e^{-(3 \times 7)} + \frac{(3 \times 7)^{16}}{16!} e^{-(3 \times 7)} + \frac{(3 \times 7)^{15}}{15!} e^{-(3 \times 7)} + \frac{(3 \times 7)^{14}}{14!} e^{-(3 \times 7)} + \frac{(3 \times 7)^{13}}{13!} e^{-(3 \times 7)}$$

التوضيح ← Prob. of departure of 16 dozens in day 1

يعني stock level 2 و 5 (2 dozens)

الاحتمال $P_2(1) = \frac{(3 \times 1)^{16}}{16!} e^{-(3 \times 1)} \leftarrow s.h = 2$ ال Prob.

بالنسبة لـ $P_0(7), P_0(2), P_0(1)$ ----- $\Leftarrow$
ما نحسبها على معادلة Poisson

P_0 ← موزون (stock level = 0) أي P_0 (System Capacity = 18)
 mean the prob that the system is empty

$\Rightarrow$ So, P is empty calculated from "truncated poisson distribution"

$$P_0(t) = \left[1 - \sum_{n=1}^N P_n(t) \right], t = 1, 2, 3, 4, 5, 6, 7$$

**** طريقة الحل كاملة لـ Day 7** (وهي نفس ما بنسبها سابقاً)

$$\text{For day 7: } P_{n \leq 5}(7) = P_0(7) + \frac{(3 \times 7)^7}{7!} e^{-(3 \times 7)} + \frac{(3 \times 7)^6}{6!} e^{-(3 \times 7)} + \dots$$

$$= P_0(7) + 0.06404 + 0.05184 + 0.03950 + 0.02821 + 0.01880$$

$$P_0(7) = \left[1 - \sum_{n=1}^{18} P_n(7) \right]$$

$$\Rightarrow \sum_{n=1}^{18} P_n(7) = \frac{(3 \times 7)^7}{7!} e^{-(3 \times 7)} + \frac{(3 \times 7)^6}{6!} e^{-(3 \times 7)} + \dots + \frac{(3 \times 7)^0}{0!} e^{-(3 \times 7)} = 0.22696$$

$$\Rightarrow P_0(7) = 1 - 0.22696 = 0.77304$$

$$\Rightarrow P_{n \leq 5}(7) = 0.77304 + 0.06404 + 0.05184 + 0.03950 + 0.02821 + 0.01880$$

$$= \boxed{0.9755}$$

حتى أحصل على آخر جواب باخذ مجموع
 Prob للأيام 7, 6, 5, 4, 3, 2, 1

⑤ The avg. no. of dozen roses discarded at the end of the week (day 7)
 (المتوسط في اليوم السابع قديمه في stock الجاهز)

Day 7 → stock level : 0, 1, 2, 3, 4, ..., 18

(المتوسط في اليوم السابع قديمه في stock الجاهز) Prob. $P_0(7)$ $P_1(7)$ $P_2(7)$ $P_3(7)$ $P_4(7)$ $P_{18}(7)$

$$\Rightarrow \text{So weighted average} = \sum_{n=0}^{18} n p_n(7), \quad n=0, 1, 2, \dots, 18$$

$$= 0(P_0(7)) + 1(P_1(7)) + 2(P_2(7)) + \dots + 18(P_{18}(7))$$

$$\approx 0.6640 \approx \boxed{1 \text{ dozen}} \text{ \#}$$

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* Steady-State Probabilities of Generalized infinite-size Poisson Queuing Situation.

⇒ Queuing System :- is a system where entities (customers) waiting for a service.

⇒ The # of entities as max. → finite ($0 \rightarrow \text{size}$)
 → infinite ($0 \rightarrow \infty$)
 as min. → n. of customers ($0 \rightarrow \text{max}$)

⇒ System state changes randomly → Arrivals. (يكون في الـ 10 sys. ولا يصير 11)
 → Departures. (يكون في الـ 10 sys. ولا يصير 9)
 transition in state ← كل ما يتغير فيه

⇒ Any system goes through 2 phases → Transient Response
 → Steady-State Response

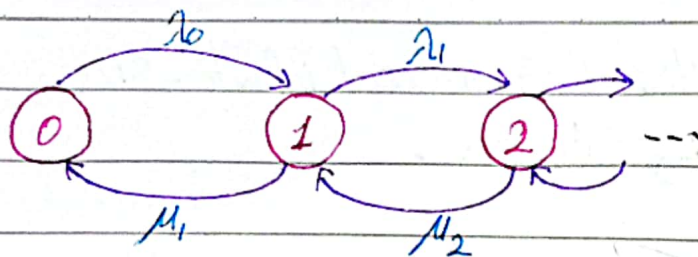
In steady-state we analyze the prob. of states (analyze Q-system)

* Steady-state Prob. of n customers in the system (P_n) function of

- n = # of customers in the system (in-queue + in-service)

λ_n = arrival rate given n customers in the system.

μ_n = departure rate " " " " " "



} Transition-Rate Diagram.

- * State (0) can only change to state (1) when an arrival occurs at λ_0 .
- * μ_0 is undefined because no departures can occur if the system is empty.

⇒ Under steady-state conditions, for $n > 0$

Expected Rate of Flow Into state n = Expected Rate of Flow Out of state n

$$ERFI = ERFO$$

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n$$

for $n=0$ → $\lambda_0 P_0 = \mu_1 P_1 \Rightarrow P_1 = \left[\frac{\lambda_0}{\mu_1} \right] P_0 \#$

for $n=1$ → $\lambda_0 P_0 + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1 \Rightarrow P_2 = \frac{(\lambda_1 + \mu_1) P_1 - \lambda_0 P_0}{\mu_2}$

→ $P_2 = \frac{(\lambda_1 + \mu_1) \left[\frac{\lambda_0}{\mu_1} P_0 \right] - \lambda_0 P_0}{\mu_2} \Rightarrow P_2 = \left[\frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \right] P_0 \#$

for $n=2$ → $P_3 = \left[\frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \right] P_0$

steady-state transition
Probability of having
(n) customers.

⇒ In General :

$$P_n = \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \dots & \lambda_{n-1} \\ \mu_1 & \mu_2 & \mu_3 & \dots & \mu_n \end{bmatrix} P_0$$

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 $\rightarrow$ as min. $\rightarrow$ n. of customers ($0 \rightarrow \text{max}$)

⇒ **System state changes randomly** $\rightarrow$ Arrivals. (يقوم في الـ sys 10 ويطرحوا 11)
 $\rightarrow$ Departures. (يقوم في الـ sys 10 ويطرحوا 9)
 transition in state $\leftarrow$ ما كان في الـ sys

⇒ Any system goes through 2 phases $\rightarrow$ Transient Response
 $\rightarrow$ Steady-State Response

In steady-state we analyze the prob. of states (analyze Q-system)

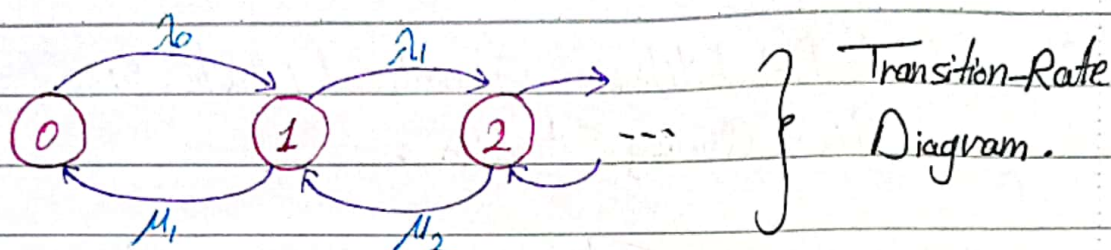
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P_n = Steady-state probability of n customers in the system.



- * State (0) can only change to state (1) when an arrival occurs at λ_0 .
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⇒ Under steady-state conditions, for $n > 0$

Expected Rate of Flow Into state n = Expected Rate of Flow Out of state n

$$ERFI = ERFO$$

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n$$

For $n=0$ → $\lambda_0 P_0 = \mu_1 P_1 \Rightarrow P_1 = \left[\frac{\lambda_0}{\mu_1} \right] P_0 \neq$

For $n=1$ → $\lambda_0 P_0 + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1 \Rightarrow P_2 = \frac{(\lambda_1 + \mu_1) P_1 - \lambda_0 P_0}{\mu_2}$

$$\Rightarrow P_2 = \frac{(\lambda_1 + \mu_1) \left[\frac{\lambda_0}{\mu_1} P_0 \right] - \lambda_0 P_0}{\mu_2} \Rightarrow P_2 = \left[\frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \right] P_0 \neq$$

For $n=2$ → $P_3 = \left[\frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \right] P_0$

steady-state transition
Probability of having
(N) customers.

General: $P_n = \left[\frac{\lambda_0 \lambda_1 \lambda_2 \dots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \dots \mu_n} \right] P_0$

⇒ In General : $P_n = \left[\frac{\lambda_0 \lambda_1 \lambda_2 \dots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \dots \mu_n} \right] P_0$

⇒ The value of P_0 is determined from $\sum_{n=0}^{\infty} P_n = 1$.

Example : 18.5-1

No. of customers

No. of Counters

1-3

1

4-6

2

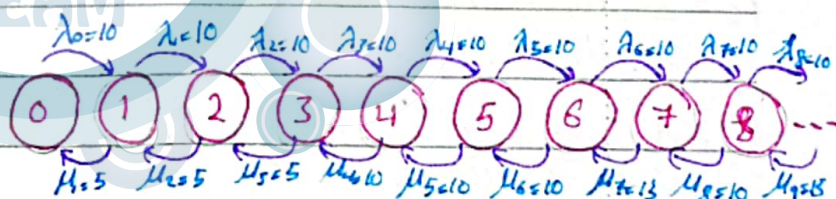
more than 6

3

Arrival rate to the Counters area $\rightarrow \lambda = 10$ Customers/hr

Avg. checkout time $\rightarrow t_s = 12$ min $\Rightarrow \mu = \frac{1}{12} \times 60 = 5$ Customers/hr

n	λ	μ
1-3	10	5
4-6	10	$5+5=10$
>6	10	$5+5+5=15$



⇒ Determine the steady-state prob. of n customers. (P_n)

Solution :

$$P_0 = P_0 = 1/55!$$

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{10}{5} P_0 = 2P_0 = \frac{2}{55}$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \left(\frac{10}{5}\right)^2 P_0 = 4P_0 = \frac{4}{55}$$

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \left(\frac{10}{5}\right)^3 P_0 = 8P_0 = \frac{8}{55}$$

$$P_4 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3 \mu_4} P_0 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right) P_0 = 8P_0 = \frac{8}{55}$$

$$P_5 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} P_0 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^2 P_0 = 8P_0 = \frac{8}{55}$$

$$P_6 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} P_0 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 P_0 = 8P_0 = \frac{8}{55}$$

$$P_7 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7} P_0 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 \left(\frac{2}{3}\right) P_0 = 8\left(\frac{2}{3}\right) P_0 = \frac{8}{55} \left(\frac{2}{3}\right)$$

$$P_8 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6 \lambda_7}{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8} P_0 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 \left(\frac{2}{3}\right)^2 P_0 = 8\left(\frac{2}{3}\right)^2 P_0 = \frac{8}{55} \left(\frac{2}{3}\right)^2$$

For $n \geq 6$ $\rightarrow P_n = 8\left(\frac{2}{3}\right)^{n-6} P_0$

$$\Rightarrow \sum_{n=0}^{\infty} P_n = 1 \rightarrow P_0 + P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 + P_8 + \dots = 1$$

$$\Rightarrow P_0 + 2P_0 + 8P_0 + 8P_0 + 8P_0 + 8\left(\frac{2}{3}\right)^0 P_0 + 8\left(\frac{2}{3}\right)^1 P_0 + 8\left(\frac{2}{3}\right)^2 P_0 + \dots = 1$$

$$31P_0 + 8P_0 \left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \dots \right] = 1$$

to find it

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \rightarrow \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = \frac{1}{1-\frac{2}{3}} = 3$$

$$31P_0 + 8P_0(3) = 1$$

$$55P_0 = 1 \rightarrow$$

$$P_0 = \frac{1}{55} \%$$

Note 3: For the law $\sum_{i=0}^{\infty} x^i$ if $i = \infty$

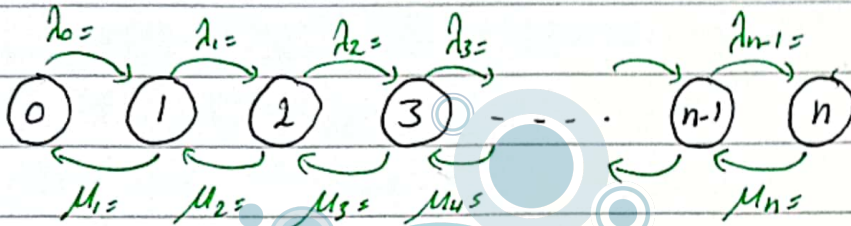
$\left(\frac{1}{1-x}\right)$ بکسر

if $i = n$

Summation ج

← طريقة إجابة السؤال في الامتحان

1. Construct the state transition diagram.



2. Setup the set of possible no. of customers (n) in the system.

$$S = \{0, 1, 2, 3, \dots, n\}$$

3. Compute the steady-state probabilities.

$$P_0, P_1, P_2, P_3, \dots, P_n \leftarrow \text{احتمال وجود}$$

4. What is the probability of having 0 Customers / the sys. is empty?

$$P_0 = (\quad) \% \text{ of the time the sys. has 0 Customer / empty}$$

5. What is the probability that the sys. is full?

$$P_n = (\quad) \% \text{ of the time the sys. is full.}$$

* Steady-State Measures of Performance

1. L_s : Avg. number of entities in the system (System length)

$$L_s = \sum_{n=1}^{\infty} n p_n$$

2. L_q : Avg. number of entities waiting (queuing) (Queue length)

$$L_q = \sum_{n=C+1}^{\infty} (n-C) p_n, \quad C: \# \text{ of servers.}$$

$$\text{Rate} = \frac{\text{length}}{\text{time}} \quad \left\{ \begin{array}{l} \text{Rate} \\ \text{length} \\ \text{time} \end{array} \right\} \text{ (Rate, Length, Time)}$$

$$\lambda_{\text{eff}} = \frac{L_s}{W_s} \Rightarrow W_s = \frac{L_s}{\lambda_{\text{eff}}} \quad (\text{in the system})$$

3. W_s : Avg. expected waiting time in system.

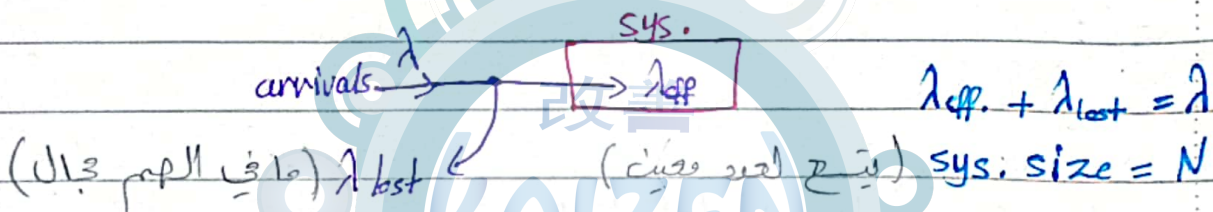
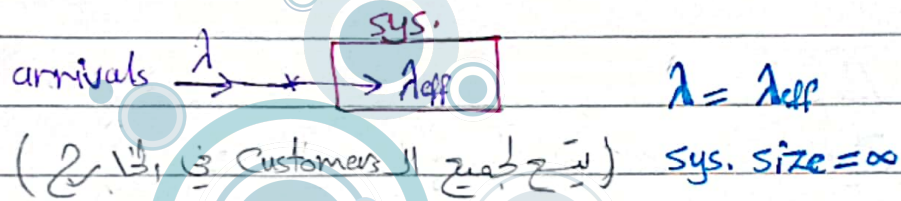
$$\lambda_{\text{eff}} = \frac{L_q}{W_q} \Rightarrow W_q = \frac{L_q}{\lambda_{\text{eff}}} \quad (\text{in the queue})$$

4. W_q : Avg. expected waiting time in queue.

$$W_s > W_q \rightarrow W_s = W_q + \frac{1}{\mu}$$

λ_{eff} = is the effective arrival rate.

- $\lambda_{eff} = \lambda$ when all arriving customers can join the sys.
- $\lambda_{eff} < \lambda$ if some customers cannot join because the sys. is full.



5. C : # of servers.

6. $\bar{C}$: Avg. expected number of busy servers.

$$\bar{C} = L_s - L_q = \frac{\lambda_{eff}}{\mu}$$

7. P : Service facility utilization.

$$P = \left[\begin{array}{c} \text{Facility} \\ \text{Utilization} \end{array} \right] = \frac{\bar{C}}{C}$$

* Specialized Poisson Queues

$(a/b/c) : (d/e/f)$ → 6 characteristics
 6 characteristics
 6 characteristics

a: Arrivals distribution

- اخر { **M:** Arrival rate → poisson dist / interarrival time → exponential dist.
- رفق { **D:** Interarrival time is Constant time.
- a { **E_k:** Interarrival time is Erlang or gamma distribution.
- في اى { **GI:** General distribution of interarrival time.
- الاول

b: Departure (service time) distribution.

- اخر { **M:** Departure rate → poisson dis. / service time → exponential dis.
- رفق { **D:** Service time is Constant time.
- b { **E_k:** Service time is Erlang or gamma dis.
- في اى { **G:** General dis. of service time.
- الاول

c: # of parallel & identical servers.

L : Queue discipline

$\left\{ \begin{array}{l} \text{FCFS : First come, first served .} \\ \text{LCFS : Last come, first served .} \\ \text{SIRO : Service in random order .} \\ \text{GD : general discipline .} \end{array} \right.$

e : Max. number of entities allowed in the sys.

$\rightarrow$ system capacity $\rightarrow$ finite (N)
 $\rightarrow$ infinite (∞)

P : Population size $\rightarrow$ finite (N)
 $\rightarrow$ infinite (∞)

$(M/D/10):(GD/20/\infty)$

Poisson arrival rate $\leftarrow$
 Exp. interarrival time
 Constant service time dis $\leftarrow$
 10 parallel & identical servers $\leftarrow$
 $\rightarrow$ Pop. size is infinite
 $\rightarrow$ 20 customer in the sys.
 $\rightarrow$ The queue discipline is general.

* Single - Server Infinite - Size Poisson Queuing situation.
 $(M/M/1 : GD/\infty/\infty)$

Arrival rate poisson dist. $\leftarrow$ Infinite pop. size
 Service rate poisson dist. $\leftarrow$ Infinite system size
 Single server $\leftarrow$ General discipline

$$\rho = \frac{\lambda}{\mu} \quad / \quad L_s = \frac{\rho}{1-\rho} \quad / \quad L_q = \frac{\rho^2}{1-\rho} \quad / \quad \bar{C} = L_s - L_q = \rho$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu(1-\rho)} \quad / \quad W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$$

$(M/M/1 : GD/\infty/\infty)$ $\downarrow$ $\lambda = \lambda_{eff}$ $\leftarrow$ $\leftarrow$ $\leftarrow$

$\lambda = \lambda_{eff} \leftarrow$
 (∞) $\leftarrow$ population size $\leftarrow$ $\leftarrow$

Example : (M/M/1 : GD ∞ / ∞)

- Car wash facility operates with only one bay $\Rightarrow C = 1$
- Cars arrive according to poisson dist. with mean of 4 cars per hour
 $\Rightarrow \lambda = 4 \text{ cars/hr}$
- The time for washing & cleaning a car is exp. with mean of 10 min.
 $\Rightarrow t_s = 10 \text{ min}$
 $\Rightarrow \mu = \frac{1}{10} \times 60 = 6 \text{ cars/hr}$
- Cars that cannot park in the lot can wait in the street bordering the wash facility $\Rightarrow$ system size is (∞)
- $\Rightarrow$ Determine the size of the parking lot.
Analyze

Solution : $\rho = \frac{\lambda}{\mu} = \frac{4}{6} = \frac{2}{3}$

$L_s = \frac{\rho}{1-\rho} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2 \text{ Car}$

$L_q = \frac{\rho^2}{1-\rho} = \frac{(\frac{2}{3})^2}{1-\frac{2}{3}} = 1.33333 \text{ Car} = \left(\frac{4}{3}\right) \text{ Car}$

$W_s = \frac{L_s}{\lambda} = \frac{2}{4} = \frac{1}{2} \text{ hour} = 0.5 \text{ hour}$

$W_q = \frac{L_q}{\lambda} = \frac{\frac{4}{3}}{4} = \frac{4}{12} = \frac{1}{3} \text{ hour} = 0.33333 \text{ hour}$

$\bar{C} = L_s - L_q = 2 - \frac{4}{3} = \frac{2}{3}$

⇒ Cont.

On Avg. the cars in the system = 2 Cars
 $(1\frac{1}{3} \text{ car})$ of them in queue $\rightarrow (2/3 \text{ car})$ under servicing [on avg]

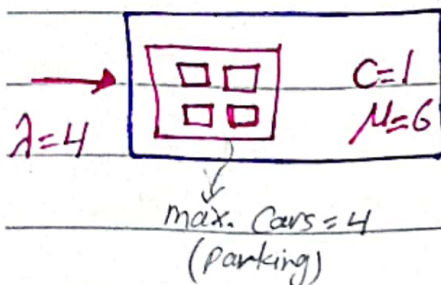
(2/3) في وقت الخدمة (C=1) one server في وقت الخدمة
 (في وقت الخدمة في النظام) في وقت الخدمة

* Single-Server Finite-Size Poisson Queuing Situation
 $(M/M/1 : GD/N/\infty)$

$\rightarrow$ max. no. of customer is N

Consider the last example

- Suppose that the facility has 4 parking spaces
- If parking lot is full, newly arriving cars balk to other facilities



max. no. of cars in the system

$$= 4 + 1 = 5 \text{ Cars}$$

in parking space

under servicing

- Determine the impact of limited parking space on losing customers to the competition.

⇒ Cont.

Finite-size ← infinite-size in new model is
(Given in the exam) نموذج جديد

$$- L_s = \frac{P[1 - (N+1)P^N + NP^{N+1}]}{(1-P)(1-P^{N+1})}, \quad [P \neq 1] = 1.42256 \text{ Car}$$

$$- W_s = \frac{L_s}{\lambda_{\text{eff}}} = \frac{1.42256}{3.8075} = 0.37362 \text{ hour}$$

$$\left\{ \lambda_{\text{eff}} = \lambda(1 - P_N) = 4(1 - 0.0482) = 3.8075 \right.$$

$$P_N = \frac{(1-P)P^N}{1-P^{N+1}} = \frac{(1-0.66667)(0.66667)^5}{1-(0.66667)^6} = P_5 = 0.0482$$

$$- W_q = W_s - \frac{1}{\mu} = 0.37362 - \frac{1}{6} = 0.20695 \text{ hour}$$

$$- L_q = W_q \times \lambda_{\text{eff}} = 0.20695 \times 3.8075 = 0.78797 \text{ Car}$$

$$\bar{C} = L_s - L_q = 1.42256 - 0.78797 = 0.666$$

is empty الوقت الذي لا يوجد فيه أي سيارة في الخادم (2/3) الوقت الذي لا يوجد فيه أي سيارة في الخادم

$$P_0 = \frac{1}{3}$$

* Multi-Server infinite-Size Poisson Queuing Situation ($M/M/C : GD/\infty/\infty$)

In this model :

- C parallel & identical servers
- no limit on the max. no. of customers
- $\lambda = \lambda_{eff}$

$$\lambda_n = \lambda$$

$$\rho = \frac{\lambda}{\mu}$$

$$\mu_n = \begin{cases} n\mu & , n < C \\ C\mu & , n \geq C \end{cases} \quad (\text{B-situation}) \lambda < \mu \quad \checkmark \quad \leftarrow$$

$$\checkmark \quad \rho < 1 \quad \checkmark \quad \leftarrow$$

$$L_q = \frac{\rho^{C+1}}{(C-1)!(C-\rho)^2} P_0 \Rightarrow P_0 = \left[\sum_{n=0}^{C-1} \frac{\rho^n}{n!} + \frac{\rho^C}{C!} \left[\frac{1}{1-\frac{\rho}{C}} \right] \right]^{-1} \cdot \frac{\rho^C}{C!}$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = W_s \lambda$$

$$\bar{c} = L_s - L_q$$

* Example : $(M/M/C : GD/\infty/\infty)$

- A Community is served by 2 Cab Companies (٤ شركات تاكسي)
- Each company owns 2 Cabs
- The Calls arrive at each Company at the rate of 8/hour $\rightarrow \lambda$
- The avg. time per ride is 12 min $\rightarrow t_s$
- The 2 Companies were bought by an investor $\rightarrow$ Consolidated them into a single Company

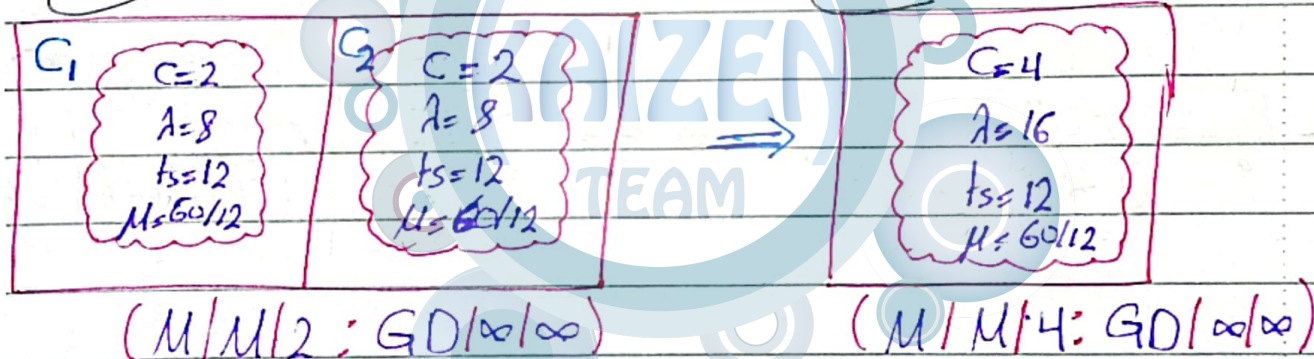


- Analyze the new owner's proposal.

قبل التغيير

تحسين

بعد التغيير



Before

$$P = \frac{8}{5} = 1.6$$

$$L_q = \frac{(1.6)^3}{(2-1)!(2-1.6)^2} P_0 \Rightarrow P_0 = \left[\frac{1.6^0}{0!} + \frac{1.6^1}{1!} + \frac{1.6^2}{2!} \left(\frac{1}{1 - \frac{1.6}{2}} \right) \right]^{-1}$$

$$L_q = 2.844 \text{ Calls}$$

$$P_0 = 0.111$$

$$- W_q = \frac{2.844}{8} = 0.356 \text{ hour}$$

$$- W_s = 0.356 + \frac{1}{5} = 0.556 \text{ hour}$$

$$- L_s = 0.556 \times 8 = 4.444 \text{ Calls}$$

$$- \bar{C} = L_s - L_q = 1.6 = \rho$$

After

$$- \rho = \frac{16}{5} = 3.2$$

$$- L_q = \frac{(3.2)^5}{(4-1)!(4-3.2)^2} P_0 \Rightarrow P_0 = \left[\frac{3.2^0}{0!} + \frac{3.2^1}{1!} + \frac{3.2^2}{2!} + \frac{3.2^3}{3!} \right]$$

$$\left[L_q = 2.386 \text{ Calls} \right] + \frac{3.2^4}{4!} \left[\frac{1}{1 - \frac{3.2}{4}} \right] = 0.027$$

$$- W_q = \frac{2.386}{16} = 0.149 \text{ hour}$$

$$- W_s = 0.149 + \frac{1}{5} = 0.349 \text{ hour}$$

$$- L_s = 0.349(16) = 5.586 \text{ Calls}$$

$$- \bar{C} = L_s - L_q = 3.2 = \rho$$

* Multi-Server Finite-Size Poisson Queuing Situation (M/M/C:GD/N/∞)

In this model :

- The sys. limits is finite & equal to N
- The max. queue size is (N-c)
- $\lambda \neq \lambda_{eff}$ ($\lambda = \lambda_{eff} + \lambda_{lost}$)
- $c \leq N$

$$P_n = \begin{cases} \frac{\rho^n}{n!} P_0, & 0 \leq n < c \\ \frac{\rho}{c!} \frac{\rho^{n-c}}{c^{n-c}} P_0, & c \leq n \leq N \end{cases}$$

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} \right] + \frac{\rho^c \left[1 - \left(\frac{\rho}{c} \right)^{N-c+1} \right]}{c! \left[1 - \frac{\rho}{c} \right]} \right\}^{-1} \quad , \frac{\rho}{c} \neq 1$$

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} \right] + \frac{\rho^c (N-c+1)}{c!} \right\}^{-1} \quad , \frac{\rho}{c} = 1$$

$$- L_q = \left[\frac{P^{c+1}}{(c-1)!(c-p)^2} \left\{ 1 - \left[\frac{P}{c} \right]^{N-c+1} - (N-c+1) \left[1 - \frac{P}{c} \right] \left[\frac{P}{c} \right]^{N-c} \right\} P_0 \right] \quad \frac{P}{c} \neq 1$$

$$\left[\frac{P^c (N-c)(N-c+1) P_0}{2c!} \right] \quad , \frac{P}{c} = 1$$

$$- \lambda_{\text{eff}} = \lambda(1 - P_N) \Rightarrow P_N = \frac{P^N}{c! c^{N-c}} P_0 \quad \left\{ P_N \rightarrow \text{Prob. that the sys. is full} \right\}$$

$$- \lambda_{\text{lost}} = \lambda P_N$$

$$- W_q = \frac{L_q}{\lambda_{\text{eff}}}$$

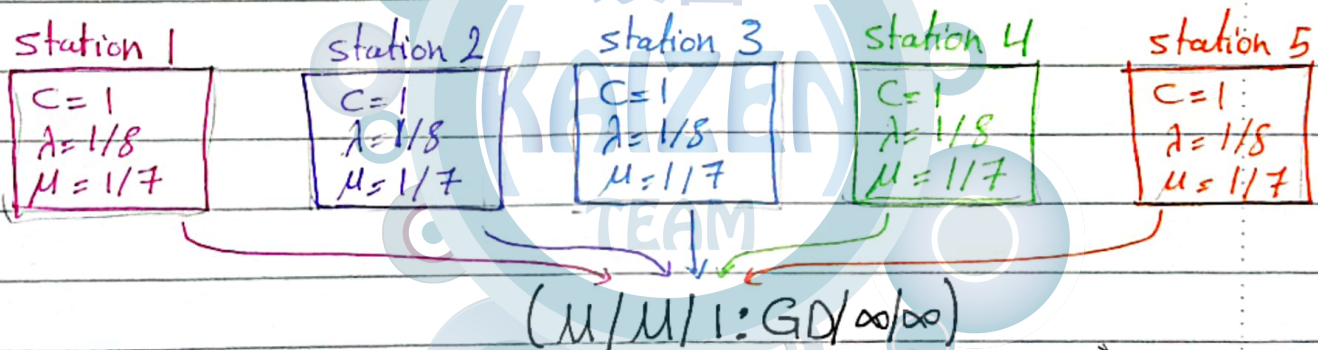
$$- W_s = W_q + \frac{1}{\mu}$$

$$- L_s = W_s (\lambda_{\text{eff}})$$

$$- \bar{C} = L_s - L_q$$

* Queuing Network Example 1.

- An assembly line has 5 stations $\rightarrow$ every unit being produced must pass in the same fixed order.
- The arrival of units $\rightarrow$ Poisson process for all.
- The mean time b/w arrivals is 8 min $\rightarrow t_a = 8 \text{ min}$
- The stations operate independently. $\lambda = 1/8$
- All stations have same mean processing time of 7 min $\rightarrow t_s = 7 \text{ min}$
 $\mu = 1/7$
- Analyze $\Downarrow$ this situation.



(فوق البسيط! في ال models في Q-Network متوفا identical)

Solution:

بما! انه كل station is independent كل station
 $\left(\mu/\mu/1:GD/\infty/\infty \right)$ في كل وحدة $(t_s, t_a, w_s, \dots)$
 في وجود المتطلبات لكل station في وجود ال sys. كامل.

For entire assembly line

$$Ls_{total} = Ls① + Ls② + Ls③ + Ls④ + Ls⑤$$

$$Lq_{total} = Lq① + Lq② + \dots + Lq⑤$$

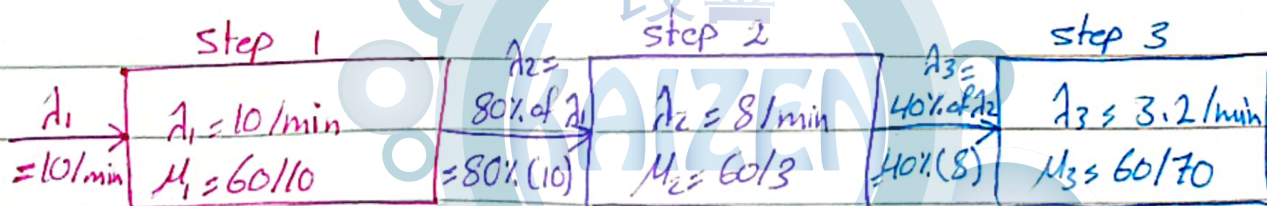
$$Ws_{total} = Ws① + Ws② + \dots + Ws⑤$$

$$Wq_{total} = Wq① + Wq② + \dots + Wq⑤$$

**** $\bar{C}$ total $\rightarrow$ Utilization bottleneck station $\cup$ 4 $\rightarrow$ $\bar{C}$ $\rightarrow$ Stations $\cup$ utilization $\bar{C}$ $\rightarrow$**

Queuing Network Example 2

- Batch jobs submitted to a Computer Center pass through 3 steps.
- Because of errors $\rightarrow$ only 80% of jobs go through step ②
- 40% of jobs that pass through step ② $\rightarrow$ are routed to step ③
- Jobs arrive to the Computer Center with avg. rate of 10/min $\rightarrow \lambda$
- Processing time of each step is:
 - step ① $\rightarrow t_s = 10s$
 - step ② $\rightarrow t_s = 3s$
 - step ③ $\rightarrow t_s = 70s$
- Unlimited queue for each processor.



$(C_i = \lambda_i / \mu_i)$ $\leftarrow$ job arrival rate at step i divided by the service rate at step i (integer) $\leftarrow$ the number of jobs in the queue at step i (integer)

$$C_1 = \frac{\lambda_1}{\mu_1} = \frac{10}{6} = 2, \quad C_2 = \frac{\lambda_2}{\mu_2} = 1, \quad C_3 = \frac{\lambda_3}{\mu_3} = 4$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$(M/M/2:GD/\infty/\infty) \quad (M/M/1:GD/\infty/\infty) \quad (M/M/4:GD/\infty/\infty)$$

queue station $\leftarrow$ model of the queueing network $\leftarrow$ jobs queuing network

Mean no. at station $\rightarrow L_s$

Mean no. in queue $\rightarrow L_q$

Mean time at station $\rightarrow W_s$

Mean time in queue $\rightarrow W_q$

Mean no. in service $\rightarrow \bar{C}$

Mean time in service $\rightarrow W_s - W_q$

Efficiency $\rightarrow \bar{C}/C$

mean no. of jobs in service } متوسط عدد

avg. no. of busy service } متوسط

jobs arrive to the step 2 } لوف كى فى الـ step 2

with avg. rate of 2/min.

step 1 step total arrival rate 11 } لوف كى



80% (16) $\rightarrow$ 2

$$\lambda_2 = 8 + 2 = 10/\text{min.}$$

Computer centre 11 } لوف كى $\lambda = 2/\text{min}$

* Definition of a Markov Chain

$X_t \rightarrow$ a random variable that characterizes the state of the system at discrete points in time $t = 1, 2, \dots$

The Family of random variables $\{X_t\}$ forms a Stochastic process with no. of states, either $\rightarrow$ finite, $\rightarrow$ infinite.

Example: 17.1 The Condition of a machine at the time of the monthly preventive maintenance is $\rightarrow$ poor / fair / good

The stochastic process for this situation :-

$$X_t = \begin{cases} 0, & \text{If the Condition is poor} \\ 1, & \text{If the Condition is fair} \\ 2, & \text{If the Condition is good} \end{cases}, t = 1, 2, \dots$$

$\rightarrow$ X_t is finite because it represents 3 states $\rightarrow$ poor(0)
 $\rightarrow$ fair(1)
 $\rightarrow$ good(2)

⇒ State Space (S) :- possible values of the state of the machine over time

belongs to $S = \{0, 1, 2\}$ → it is a set of all possible states of the machine
 $S = \{0, 0, 1, 2, 2, 0, 1\}$

Definitions

$$X_t = \{X_1, X_2, X_3, \dots, X_t\}$$

⇒ The relationship b/w the values of (X_t) is **Stochastic**

⇒ The process that generated the values of (X_t) is **Stochastic Process**
 (A collection of random variables X_t).

⇒ **Markovian Property** is satisfied when next (future) state depends only on the current state & no way on the past.

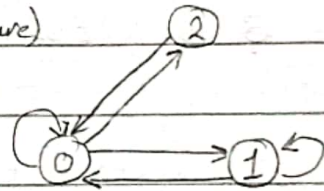
⇒ **Markov Process** is a stochastic process which has Markov prop.

⇒ A discrete-time Markov Chain is a stochastic process with the following characteristics: (DTMC)

- 1- A discrete state space.
- 2- Markov property.
- 3- The one-step transition probabilities, P_{ij} , from time n to $n+1$ remain constant overtime.

Transition Matrix $\Rightarrow P =$

	0	1	2 (future)
0 (current)	0.6	0.3	0.1
1	0.8	0.2	0
2	1	0	0



- discrete state space $\rightarrow S = \{0, 1, 2\}$ ✓

- Markov property $\rightarrow$ next state depends on current state ✓

0 either $\rightarrow 1$
or $\rightarrow 2$

- The one-step transition prob. $\rightarrow$ the prob. of going from state $1 \rightarrow 0$ is $\boxed{0.8}$ always

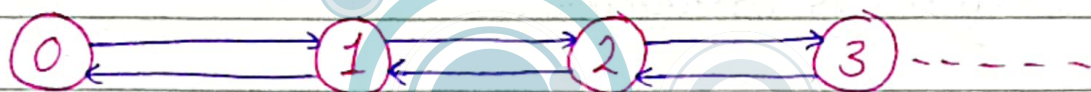
The prob. of

Classification of the states in Markov chain

↳ based on the transition prob. of the transition matrix $[P]$

1. Accessible & Inaccessible

A state j is to be accessible from state i $\Rightarrow$ If it is possible to pass from $i \rightarrow j$

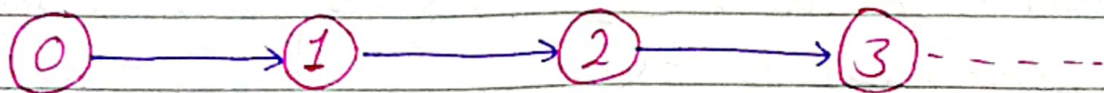


state 3 is accessible from state 0 $\Rightarrow$ because it's possible to pass from $0 \rightarrow 3$

state 2 is accessible from state 1 $\Rightarrow$ because it's possible to pass from $1 \rightarrow 2$

**** Every state is accessible from every other state.**

A state j is inaccessible from state i $\Rightarrow$ If there is no path from state $i \rightarrow j$ ($i > j$)



state 3 is accessible from state 0, but state 0 inaccessible from 3

state 2 is accessible from state 1, but state 1 inaccessible from 2

2. Communicate states & irreducible system.

2 states communicate $\rightarrow$ If both are accessible from each other.
 A system is irreducible $\rightarrow$ If all states communicate.



Every state is accessible from each other $\rightarrow$ all states communicate
 $\rightarrow$ the system is irreducible.

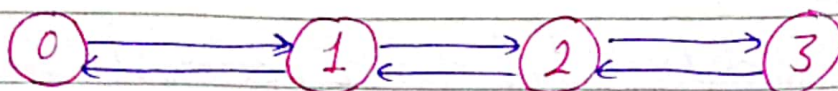


Not all states accessible from each other $\rightarrow$ no state communicate.

3. Recurrent & transient states.

State i is said to be recurrent $\rightarrow$ If the system leaves the state but it will definitely return to it.

State i is said to be transient $\rightarrow$ If the system leaves the state and may never return to it.



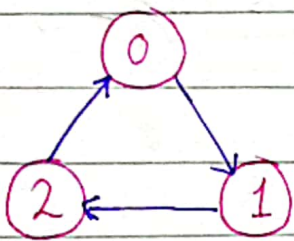
All states are recurrent

4. Periodic, aperiodic states & ergodic system.

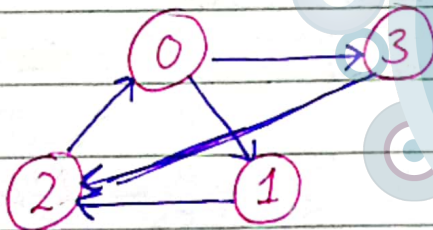
A state is periodic $\rightarrow$ If it can return to itself after a fixed no. of transitions > 1 .

The no. of transitions is the period of the state.

* A state that is not periodic is aperiodic.



$\Rightarrow$ All states periodic with period 3
 3 steps in 1 cycle (state 0)
 (state 1 & 2 also)

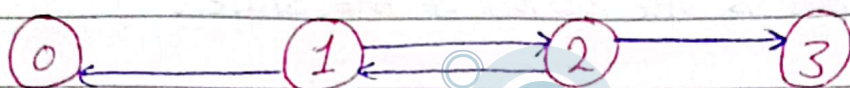


$\Rightarrow$ States 0 & 2 periodic with period 3
 States 1 & 3 aperiodic

A Markov chain said to be ergodic $\rightarrow$ If all its states are ergodic $\rightarrow$ a state is recurrent & aperiodic

5. Absorbing state

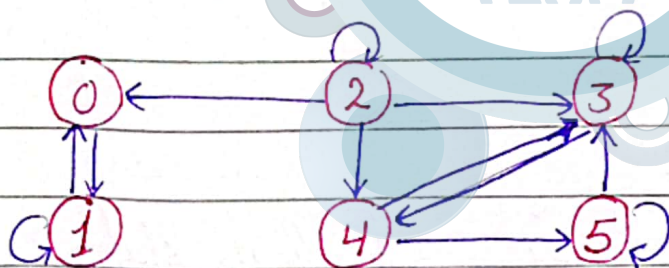
An absorbing state $\rightarrow$ is a state that, once entered, cannot be left.



States 0 & 3 are absorbing states, because once the sys. reaches one of them, it stays there (that is, $P_{ii} = 1$)

6. A class of states

A class is a group/set of states, the classes are either recurrent, or transient, or periodic, or aperiodic.



3 classes
 $\{0, 1\}$ $\{2\}$ $\{3, 4, 5\}$

States 0 & 1 can reach each other but no other state $\rightarrow$ Form a class.

State 2 can only be reached from 2 $\rightarrow$ is in a class of its own.

States 3, 4 & 5 all communicate with each other $\rightarrow$ Form a class

Absolute and n-step transition probabilities.

* Chapman-Kolmogorov equations:

Matrix $P^0 \rightarrow$ Initial prob. matrix $\rightarrow$ Identity matrix
 $P^1 \rightarrow$ One-step transition prob. matrix $\rightarrow P^1 = P^0 P$
 $P^2 \rightarrow$ 2-step transition prob. matrix $\rightarrow P^2 = P^1 P$
 $\vdots$
 $P^n \rightarrow$ n-step transition prob. matrix $\rightarrow P^n = P^{n-1} P$

vector $a^0 \rightarrow$ transition prob. vector from state to all other possible states (after 0 steps) $\rightarrow$ Identity vector / row
 $a^1 \rightarrow$ transition prob. vector from state to all other possible states after 1 step $\rightarrow [P]$ السطر الأول في $a^0 P^1$

$a^2 \rightarrow$ " " " " " after 2 steps $\rightarrow a^1 P^1$
 $a^3 \rightarrow$ " " " " " after 3 steps $\rightarrow a^2 P^1$
 $a^4 \rightarrow$ " " " " " after 4 steps $\rightarrow a^3 P^1$
 $\rightarrow a^2 P^2$

من أجل أن نحصل على الطريقة العامة من Prob. السطر الأول في

From transient behavior $\rightarrow$ steady-state behavior.

After many steps, the system reaches the steady-state Condition

$$\pi P = \pi \quad \leftarrow \text{أي vector لا يتغير بـ } [P] \text{ في النهاية}$$

Example : The Gardener Problem.

Compute the steady-state transition prob. vector from good state to all other possible states.

vector	matrix			same vector:
$[\pi_1, \pi_2, \pi_3]$	0.3	0.6	0.1	$= [\pi_1, \pi_2, \pi_3]$
	0.1	0.6	0.3	
	0.05	0.4	0.55	

السطر × العمود الأول $\rightarrow 0.3\pi_1 + 0.1\pi_2 + 0.05\pi_3 = \pi_1$

السطر × العمود الثاني $\rightarrow 0.6\pi_1 + 0.6\pi_2 + 0.4\pi_3 = \pi_2$

السطر × العمود الثالث $\rightarrow 0.1\pi_1 + 0.3\pi_2 + 0.55\pi_3 = \pi_3$

↓

$$\pi_1 + \pi_2 + \pi_3 = 1$$

*** أي transition prob. vector مجموع العناصر فيه يساوي 1

*** $[P]$ مجموع أي سطر فيها يساوي 1

⇒ Cont.

جواب 3 و 4 ←

$$0.1017 + 0.5254 + 0.3729 = 1$$

$\pi_1 = 0.1017$
 $\pi_2 = 0.5254$
 $\pi_3 = 0.3729$

$\pi_1 \rightarrow$ steady-state transition prob. from good $\rightarrow$ good.

$\pi_2 \rightarrow$ steady-state transition prob. from good $\rightarrow$ fair.

$\pi_3 \rightarrow$ steady-state transition prob. from good $\rightarrow$ poor.

↓

$$\pi_{\text{good}} = \lim_{n \rightarrow \infty} a_{\text{good}}^{(n)}$$

SO, the steady-state probabilities are defined as:

$$\pi_j = \lim_{n \rightarrow \infty} a_j^{(n)}, \quad j = 0, 1, 2, \dots$$

These prob. can be determined from:

$$\pi = \pi P$$

$$\sum \pi_j = 1$$

* First return time : # of steps / transitions it takes to return to the original state for the first time.

دوره اولی که از یک step می آید به حالت خوب می آید.

* Mean First return time (mean recurrence time) : the expected # of transitions before the system returns to a state j for the first time.

↓

$i \neq j$

↓

$$|M_{ij}| = (I - N_j)^{-1} \mathbf{1}$$

↓

$i = j$

↓

$$M_{ji} = \frac{1}{\pi_j}, j = 0, 1, 2, \dots$$

I : $(m-1)$ - Identity matrix.

N_j : trans. matrix P less its j th row & j th col. of target state j .

$\mathbf{1}$: $(m-1)$ Col. vector with all element = 1.

π_j : steady-state

transition prob.

From $j \rightarrow j$.

Q1) How many seasons it takes on avg. to pass from (Fair $\rightarrow$ good) & to pass from (poor $\rightarrow$ good) For the first time? (Gardener example)

$\Rightarrow$ mean recurrence time / $j = \text{good} (1)$

$i = \text{Fair} (2), \text{Poor} (3)$

$[N_j] \leftarrow [P]$ is good & Col. + row 1 is 1 ①
 $I_{2 \times 2} \leftarrow [N_j]$ is 0 & is 1 identity mat. ②

$$\begin{bmatrix} \mu_{21} \\ \mu_{31} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.55 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7.5 & 5 \\ 6.6666 & 6.6666 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12.50 \\ 13.33 \end{bmatrix}$$

$\mu_{F \rightarrow g} = 12.5 \text{ season.}$

$\mu_{P \rightarrow g} = 13.3 \text{ Season.}$

Q2) How many seasons it takes on avg. to pass from $(g \rightarrow f)$ & to pass from $(p \rightarrow f)$ for the first time?

$j = \text{Fair (2)}$

$i = \text{good (1), Poor (3)}$

$[N_j] \leftarrow [P]$ // is Fair // Col. + row // $\leftarrow$

$$\begin{bmatrix} \mu_{12} \\ \mu_{32} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.1 \\ 0.05 & 0.55 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4516 & 0.3225 \\ 0.1612 & 2.258 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.77 \\ 2.42 \end{bmatrix} \text{ season.}$$

Q3) How many seasons it takes on avg. to pass from $(g \rightarrow p)$ & to pass from $(f \rightarrow p)$ for the first time?

$j = \text{Poor (3)} / i = \text{good (1), Fair (2)}$

$[N_j] \leftarrow [P]$ // is Poor // Col. + row // $\leftarrow$

$$\begin{bmatrix} \mu_{13} \\ \mu_{23} \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.6 \\ 0.1 & 0.6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8181 & 2.7272 \\ 0.4545 & 3.1818 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.55 \\ 3.64 \end{bmatrix} \text{ season.}$$

The first passage time: is the # of transitions to reach state j from state i for the first time.

Mean first passage time: the expected # of transitions to reach state j from state i for the first time.

The determination of the probability defined as:

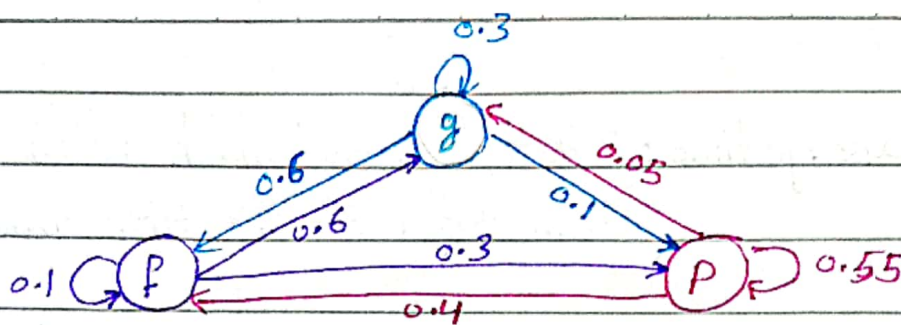
$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \rightarrow f_{ij}^{(n)}: \text{the prob. of a first passage from state } i \rightarrow j \text{ in } (n) \text{ transitions.}$$

$$\rightarrow \text{If } f_{ij} < 1 \rightarrow \mu_{ij} = \infty$$

$$\rightarrow \text{If } f_{ij} \geq 1 \rightarrow \mu_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)} \quad (\text{Another way to find } \mu_{ij}, i \neq j)$$

First passage time
First return time
Recurrence time

} same meaning



- 1) Compute the prob. that the first passage From $(g \rightarrow P)$ will take one season.

$0.6 \Leftarrow$ From $g \rightarrow P$ in 1 season $\Leftarrow$ diagram $\Leftarrow$

$$f_{g \rightarrow P}^{(1)} = 0.6$$

- 2) Compute the prob. that the first passage From $(g \rightarrow P)$ will take 2 seasons.

$$\begin{aligned} \text{From } g \rightarrow g \rightarrow P &= 0.3 \times 0.6 & \textcircled{1} \\ \text{From } g \rightarrow P \rightarrow P &= 0.6 \times 0.1 & \textcircled{2} \\ \text{From } g \rightarrow P \rightarrow P &= 0.1 \times 0.4 & \textcircled{3} \end{aligned}$$

$$(0.3)(0.6) + (0.6)(0.1) + (0.1)(0.4) = 0.28 = f_{g \rightarrow P}^{(2)}$$

3) Compute the prob. that the first passage from $(g \rightarrow f)$ will take 3 seasons.

$$g \rightarrow g \rightarrow g \rightarrow f = 0.3 \times 0.3 \times 0.6$$

①: $g \rightarrow g \rightarrow g \rightarrow f$

$$g \rightarrow g \rightarrow f \rightarrow f = 0.3 \times 0.6 \times 0.1$$

②

$$g \rightarrow g \rightarrow p \rightarrow f = 0.3 \times 0.1 \times 0.4$$

③

$$g \rightarrow f \rightarrow g \rightarrow f = 0.6 \times 0.6 \times 0.6$$

④

$$g \rightarrow f \rightarrow f \rightarrow f = 0.6 \times 0.1 \times 0.1$$

⑤

$$g \rightarrow f \rightarrow p \rightarrow f = 0.6 \times 0.3 \times 0.4$$

⑥

$$g \rightarrow p \rightarrow g \rightarrow f = 0.1 \times 0.05 \times 0.6$$

⑦

$$g \rightarrow p \rightarrow p \rightarrow f = 0.1 \times 0.55 \times 0.4$$

⑧

$$g \rightarrow p \rightarrow f \rightarrow f = 0.1 \times 0.4 \times 0.1$$

⑨

↓

The summation = $0.4070 = f_{g \rightarrow f}^{(3)}$

$$\Rightarrow \mu_{g \rightarrow f} = 1(f_{g \rightarrow f}^{(1)}) + 2(f_{g \rightarrow f}^{(2)}) + 3(f_{g \rightarrow f}^{(3)})$$

$$= 1(0.6) + 2(0.28) + 3(0.4070) = 2.381. \#$$

- 1) Compute the prob. that the first passage from $(g \rightarrow p)$ will take one season. $(n=1)$

$$P_{g \rightarrow p}^{(1)} = 0.1$$

- 2) Compute the prob. that the first passage from $(g \rightarrow p)$ will take 2 seasons. $(n=2)$

$$\left. \begin{array}{l} \textcircled{1} g \rightarrow g \rightarrow p = 0.3 \times 0.1 \\ \textcircled{2} g \rightarrow f \rightarrow p = 0.6 \times 0.3 \\ \textcircled{3} g \rightarrow p \rightarrow p = 0.1 \times 0.55 \end{array} \right\} (0.3)(0.1) + (0.6)(0.3) + (0.1)(0.55) = 0.265 = P_{g \rightarrow p}^{(2)}$$

- 3) Compute the prob. that the first passage from $(g \rightarrow p)$ will take 3 seasons. $(n=3)$

$$\begin{array}{ll} \textcircled{1} g \rightarrow g \rightarrow g \rightarrow p = 0.3 \times 0.3 \times 0.1 & \textcircled{6} g \rightarrow f \rightarrow p \rightarrow p = 0.6 \times 0.3 \times 0.55 \\ \textcircled{2} g \rightarrow g \rightarrow f \rightarrow p = 0.3 \times 0.6 \times 0.3 & \textcircled{7} g \rightarrow p \rightarrow g \rightarrow p = 0.1 \times 0.05 \times 0.1 \\ \textcircled{3} g \rightarrow g \rightarrow p \rightarrow p = 0.3 \times 0.1 \times 0.55 & \textcircled{8} g \rightarrow p \rightarrow f \rightarrow p = 0.1 \times 0.4 \times 0.3 \\ \textcircled{4} g \rightarrow f \rightarrow g \rightarrow p = 0.6 \times 0.6 \times 0.1 & \textcircled{9} g \rightarrow p \rightarrow p \rightarrow p = 0.1 \times 0.55 \times 0.55 \\ \textcircled{5} g \rightarrow f \rightarrow f \rightarrow p = 0.6 \times 0.1 \times 0.3 & \end{array}$$

* The sum $= 0.2753 = P_{g \rightarrow p}^{(3)}$

$$\Rightarrow M_{g \rightarrow p} = 1(0.1) + 2(0.265) + 3(0.2753) = 1.4559 \neq$$

**** CTMC** is like DTMC, but it moves states continuously through time rather than as discrete time steps.
 ↓ ال State بتغير بوقت مستمر الزمان

ATM example

In ATM model there are 2 stochastic processes:

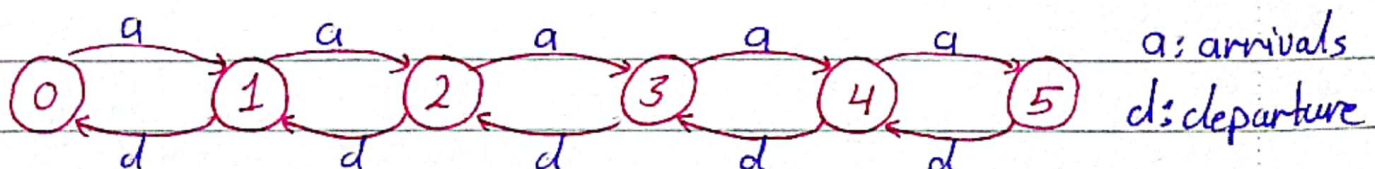
→ Arrival process & Service process
 so, it's a stochastic model.

← transition prob. matrix P ←

state transition diagram / network ①

of states is limited → max. queue size 4
 → under servicing 1

$$S = \{0, 1, 2, 3, 4, 5\}$$



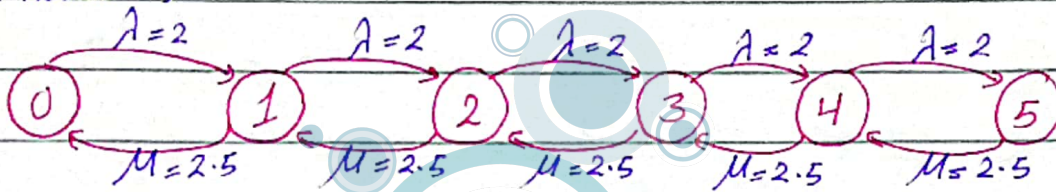
(A) arrival rate & event (a) ←

(μ) service rate & event (d) ←

rate network ← state transition network (2) ②

Given : - mean time btwn arrivals = $1/\lambda = 1/2 \rightarrow \lambda = 2$ per min.
 - mean time for service activity = $1/\mu = 0.4 \rightarrow \mu = 2.5$ /min.

Rate network:



Rate matrix $R \leftarrow$ Rate network II is given (3) ③

	to	0	1	2	3	4	5		0	1	2	3	4	5
From	0	0	λ	0	0	0	0	0	0	2	0	0	0	0
	1	μ	0	λ	0	0	0	1	2.5	0	2	0	0	0
$R =$	2	0	μ	0	λ	0	0	=2	0	2.5	0	2	0	0
	3	0	0	μ	0	λ	0	3	0	0	2.5	0	2	0
	4	0	0	0	μ	0	λ	4	0	0	0	2.5	0	2
	5	0	0	0	0	μ	0	5	0	0	0	0	2.5	0

Zero λ rate, state $0 \rightarrow 1$ is given
 Zero μ rate, state $4 \rightarrow 1$ is given

④ $\rightarrow$ تحول ال Rate matrix R $\leftarrow$ Transition prob. matrix P

Δ (دلتا)

$\leftarrow$ بضرب كل الأرقام ال داخل $[R]$ فاعلى القطر $\leftarrow$ ولحتى
يكون مجموع كل سطري $[P]$ يساوي 1 : القطر = المجموع - 1

Let $\Delta = 0.05$ For $t = 1 \text{ min} \rightarrow n = 1 \text{ min} / 0.05 = 20 \text{ steps}$

(ال 1 min تنقسم ل 20 جزء Δ وال Δ جزء واحد من ال min)

$$t = n \cdot \Delta$$

	0	1	2	3	4	5	
0	0.9	0.1	0	0	0	0	= 1
1	0.125	0.775	0.1	0	0	0	= 1
2	0	0.125	0.775	0.1	0	0	= 1
3	0	0	0.125	0.775	0.1	0	= 1
4	0	0	0	0.125	0.775	0.1	= 1
5	0	0	0	0	0.125	0.875	= 1

* Initial transition prob. vector From 0 to all other possible states

$$\rightarrow a_{0 \rightarrow j}^0 = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

* One-step transition prob. vector $\rightarrow a_{0 \rightarrow j}^1 = [0.9 \ 0.1 \ 0 \ 0 \ 0 \ 0]$

* 2-step transition prob. vector $\rightarrow a_{0 \rightarrow j}^2 = a^1 \cdot P$

Steady-state trans. $\rightarrow [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5 \ \pi_6] [P]$
 $= [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6]$

Let $\Delta = 0.025$ for $t = 1 \text{ min}$ $\rightarrow n = 1/0.025 = 40 \text{ steps}$
 steps increase, more accuracy

	0	1	2	3	4	5
0	0.95	$(2\Delta) \rightarrow 0.05$	0	0	0	0
1	$(2.5\Delta) \rightarrow 0.0625$	0.875	0.05	0	0	0
2	0	0.0625	0.875	0.05	0	0
3	0	0	0.0625	0.875	0.05	0
4	0	0	0	0.0625	0.875	0.05
5	0	0	0	0	0.0625	0.875

Transient Computations for the ATM Ex. with $\Delta = 0.025$

Steps(n)	t(min)	States						
		q_0	q_1	q_2	q_3	q_4	q_5	
0 (0.025) →	0	1	0	0	0	0	0	①
40 (0.025) →	1	0.435	0.291	0.160	0.073	0.029	0.011	②
80 (0.025) →	2	0.348	0.258	0.175	0.110	0.066	0.042	③
120 (0.025) →	3							
160 (0.025) →	4							

after ∞ steps

$$\text{Steady state}(SS_{0 \rightarrow j}) = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \pi_5] = [0.271 \quad 0.217 \quad 0.173 \quad 0.139 \quad 0.111 \quad 0.089] \quad ***$$

$$= 1.$$

- ① transition prob. vector From (0 → j) after 0 step.
- ② transition prob. vector From (0 → j) after 40 step (1 min)
- ③ transition prob. vector From (0 → j) after 80 step (2 min)

* The ATM is idle (ATM ni chhita) → $\pi_0 = 0.271$ of the time

* The efficiency of the machine (mach ni chhita) → $1 - \pi_0 = 0.729$

* The proportion of customers obtaining immediate service (Zero cust. ni chhita prob.) → $\pi_0 = 0.271$

* The proportion of customers who arrive and find the system is full (5 cust. at ATM at prob 1) $\rightarrow \pi_5 = 0.089$.

* The proportion of customers who wait (1, 2, 3, 4 cust. at ATM) $\rightarrow 1 - \pi_0 - \pi_5 = 1 - 0.271 - 0.089 = 0.640$

* The expected avg. number in the system $\rightarrow L_s = \sum i \pi_i$
 $= 0(0.271) + 1(0.217) + 2(0.173) + 3(0.139) + 4(0.111) + 5(0.089) = 1.868$

* Throughput rate $\rightarrow \lambda_{\text{eff}} = \lambda (1 - \pi_5) = 1.822 \text{ cust./min}$

* Balking rate (avg. cust. lost to the system) $\rightarrow \lambda \pi_5 = 0.178$
 $\rightarrow \lambda - \lambda_{\text{eff}} = 2 - 1.822 = 0.178$

* The avg. time in system $\rightarrow W_s = \frac{L_s}{\lambda_{\text{eff}}} = \frac{1.868}{1.822} = 1.025$

- The player rolls a pair of dice and sums the no. showing.

رقعة الحجر الأول	1	2	3	4	5	6
رقعة الحجر الثاني	1,2,3,4,5,6	1,2,3,4,5,6	1,2,3,4,5,6	1,2,3,4,5,6	1 → 6	1 → 6
مجموع الوجوه	2,3,4,5,6,7	3,4,5,6,7,8	4,5,6,7,8,9	5,6,7,8,9,10	6 → 11	7 → 12 } Total = 36

Possible outcomes = $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \rightarrow \text{Sum}$
 $\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 \end{matrix} \rightarrow \text{Freq (التكرار)}$

- If the total of 7, 11 on the first roll $\rightarrow$ the player wins.

- If the total of 2, 3, 12 on the first roll $\rightarrow$ the player loses.

- Any other no. is called the (point) $\rightarrow P_4, P_5, P_6, P_8, P_9, P_{10}$
 Total

- Then, the player rolls the pair of dice again:

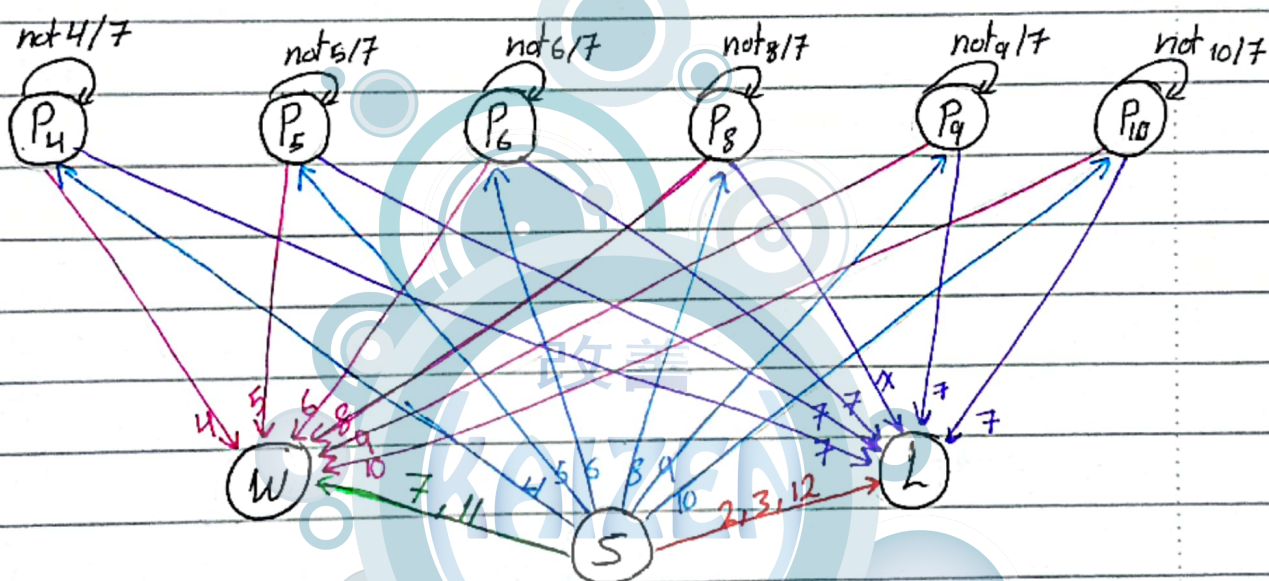
$\rightarrow$ If he at a point and the total is the same point $\rightarrow$ the player wins
 $\rightarrow$ If he at a point and the total is 7 $\rightarrow$ the player loses

$\Downarrow$
 Otherwise, he stays at the point and continues rolling until he wins or loses.

① The state space: $S_{\text{player}} = \{S, W, L, P_4, P_5, P_6, P_8, P_9, P_{10}\}$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{start} & \text{win} & \text{lose} \end{matrix}$

② The state transition network:



③ The transition prob. matrix:

The sum showing	2	3	4	5	6	7	8	9	10	11	12
Prob. of sum	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

From $S \rightarrow W$ (7, 11) = $(6+2)/36 = 0.222$

$S \rightarrow L$ (2, 3, 12) = $(1+2+1)/36 = 0.111$

$S \rightarrow P_4 = 3/36 = 0.083$

From $P_4 \rightarrow W = 3/36 = 0.083$

$P_4 \rightarrow L = 6/36 = 0.167$

$1 - \sum \text{out} = \text{القيمة}$

vector 1، مجموع

1، 1، 1، 1

1، 1، 1، 1

	S	w	L	P ₄	P ₅	P ₆	P ₇	P ₉	P ₁₀	
S	0	0.222	0.111	0.083	0.111	0.139	0.139	0.111	0.083	= 1
w	0	1	0	0	0	0	0	0	0	= 1
L	0	0	1	0	0	0	0	0	0	= 1
P ₄	0	0.083	0.167	0.75	0	0	0	0	0	= 1
P ₅	0	0.111	0.167	0	0.722	0	0	0	0	= 1
P ₆	0	0.139	0.167	0	0	0.694	0	0	0	= 1
P ₇	0	0.139	0.167	0	0	0	0.694	0	0	= 1
P ₉	0	0.111	0.167	0	0	0	0	0.722	0	= 1
P ₁₀	0	0.083	0.167	0	0	0	0	0	0.750	= 1

If the current state is Start (S):

* The initial transition prob. vector from (S → j)

$$a_{S \rightarrow j}^0 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

* The one-step transition prob. vector from (S → j) (الخطوة الأولى في [P])

$$a_{S \rightarrow j}^1 = [0 \quad 0.222 \quad 0.111 \quad 0.083 \quad 0.111 \quad 0.139 \quad 0.139 \quad 0.111 \quad 0.083]$$

* The 2-step transition prob. vector from (S → j) = $a_{S \rightarrow j}^1 \cdot [P]$

$$a_{S \rightarrow j}^2 = [0 \quad 0.299 \quad 0.222 \quad 0.063 \quad 0.080 \quad 0.096 \quad 0.096 \quad 0.080 \quad 0.063]$$

* The 3-step transition prob. vector from (S → j) = $a_{S \rightarrow j}^2 \cdot [P]$

$$a_{S \rightarrow j}^3 = [0 \quad 0.354 \quad 0.302 \quad 0.047 \quad 0.058 \quad 0.067 \quad 0.067 \quad 0.058 \quad 0.047]$$

→ After many steps, the system reaches the steady-state condition.

* The steady-state transition prob. vector from $(s \rightarrow j)$

$$[\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5 \ \pi_6 \ \pi_7 \ \pi_8 \ \pi_9] \cdot [P] =$$

$$[0 \quad 0.4928 \quad 0.5071 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

جواب 9 : Subst 10 is 10

There are 2 types of states:

→ Absorbing states = $\{W, L\}$ → indicate that the game is over

→ Transient states = $\{S, P_4, P_5, P_6, P_8, P_9, P_{10}\}$ → transient

Subject

Date

No.

Good Luck