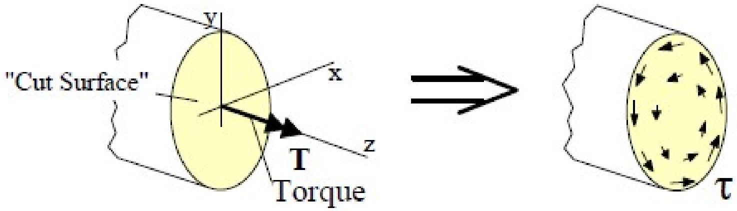
3-12 Torsion



$$\boxed{\tau = \frac{T \cdot r}{J}}$$

r = Distance from shaft axis to point of interest
R = Shaft Radius
D = Shaft Diameter

$$J = \frac{\pi \cdot D^4}{32} = \frac{\pi \cdot R^4}{2} \qquad \text{for solid circular shafts}$$

$$J = \frac{\pi \cdot \left(D_o^{\,4} - D_i^{\,4}\right)}{32} \qquad \text{for hollow shafts}$$

$$\boxed{\begin{array}{ll} \tau_{max} = \dfrac{16 \cdot T}{\pi \cdot D^3} & \text{for solid circular shafts} \\ \tau_{max} = \dfrac{16 \cdot T \cdot D_o}{\pi \cdot \left(D_o^{\,4} - D_i^{\,4}\right)} & \text{for hollow shafts} \end{array}}$$

J = polar second moment of area

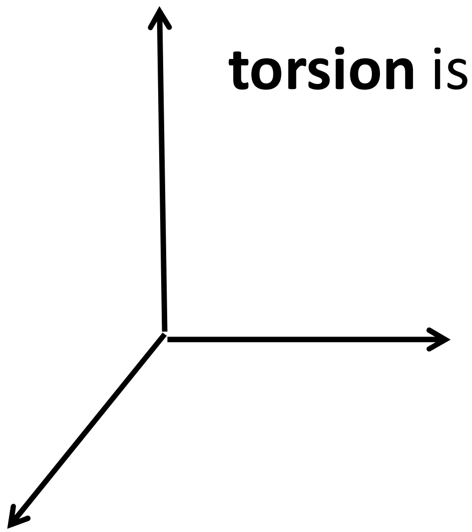
torsion is the twisting of an object due to an applied torque

- We use Hooke's law for a linear elastic material

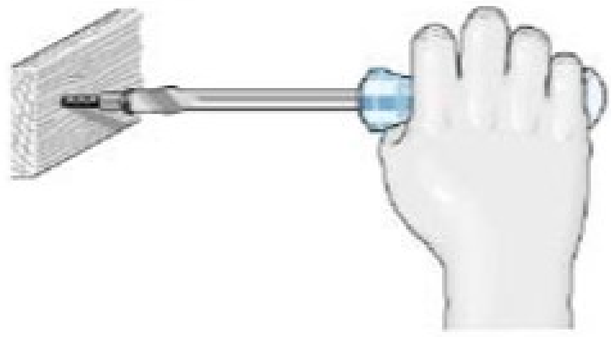
$$\tau = G\gamma$$

3-12 Torsion

torsion is the twisting of an object due to an applied torque



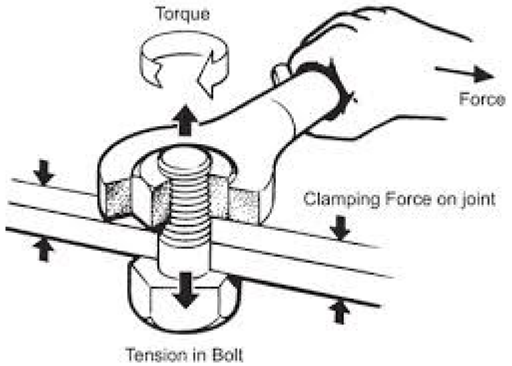
X-axis

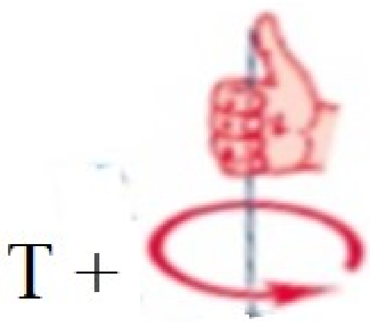


(a)

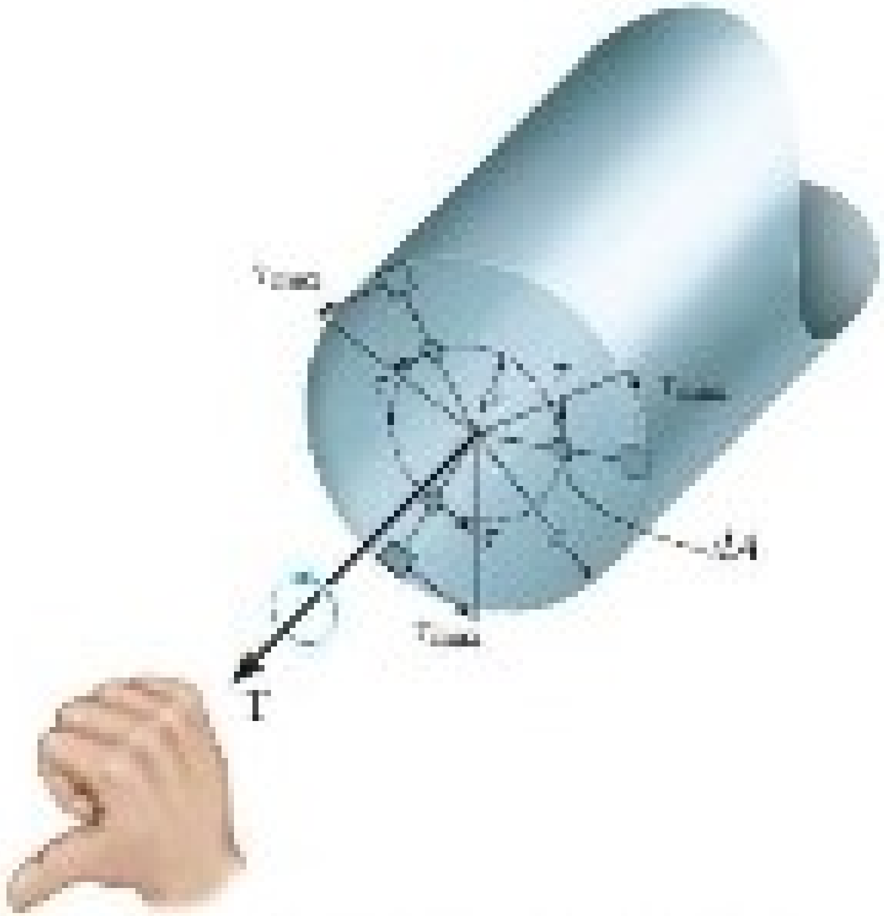
(b)

Torque





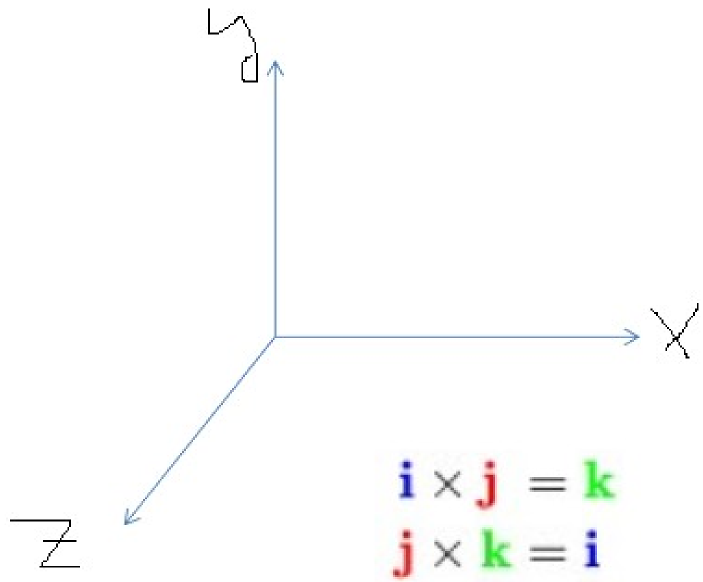
Sign convention –Right Hand Rule


T

The cross product can also be written in determinant form:

$$\vec{T} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (r_y F_z - r_z F_y)\hat{\mathbf{i}} + (r_z F_x - r_x F_z)\hat{\mathbf{j}} + (r_x F_y - r_y F_x)\hat{\mathbf{k}}.$$

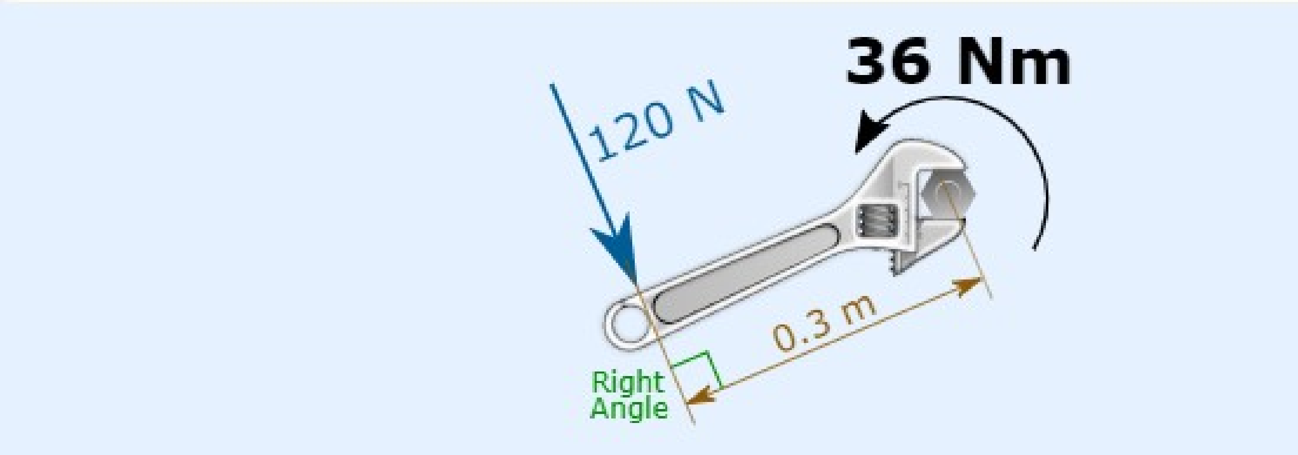


$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\vec{T} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & r_y & 0 \\ 0 & 0 & F_z \end{vmatrix}$$

$$= (r_y F_z - r_z F_y)\hat{\mathbf{i}} + (r_z F_x - r_x F_z)\hat{\mathbf{j}} + (r_x F_y - r_y F_x)\hat{\mathbf{k}}.$$

Example: 120 Newtons of force at 0.3 m



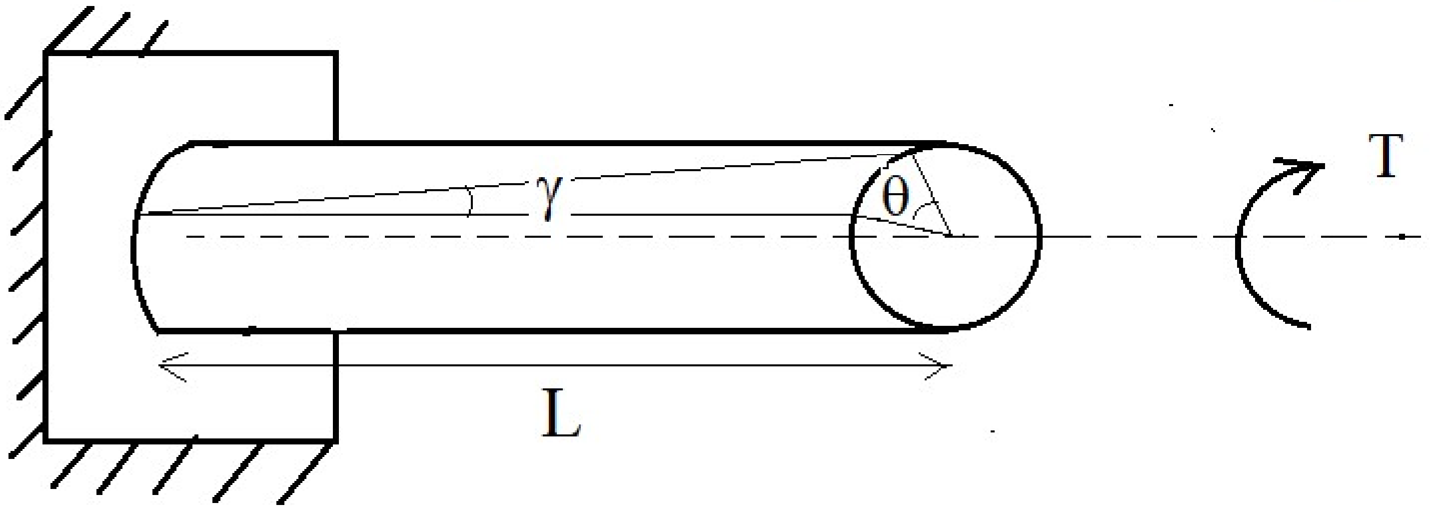
Example Find the Torque value and direction

T= 120 N × 0.3 m × sin(115°) = **32.63... Nm**

If γ is the shear strain θ is the angle of twist in Length L

When a circular shaft is subjected to torque, the shaft will be twisted and the angle of twist is found to be:

$$\theta = \frac{TL}{GJ}$$



T:Torque, L :Length, G: Modulus of rigidity

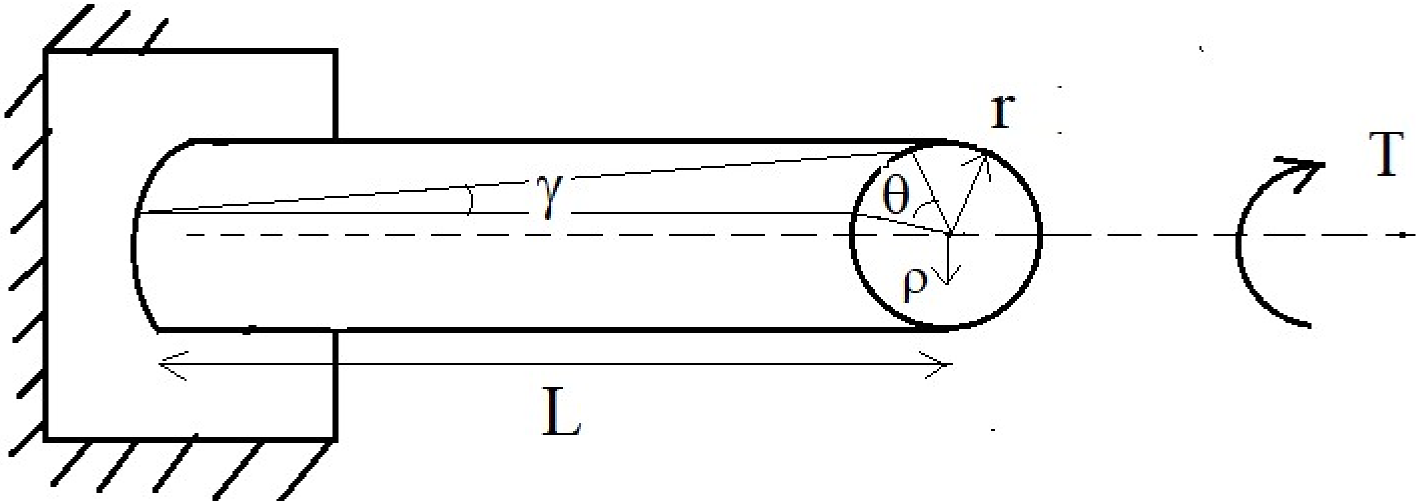
J: Polar moment of inertia.

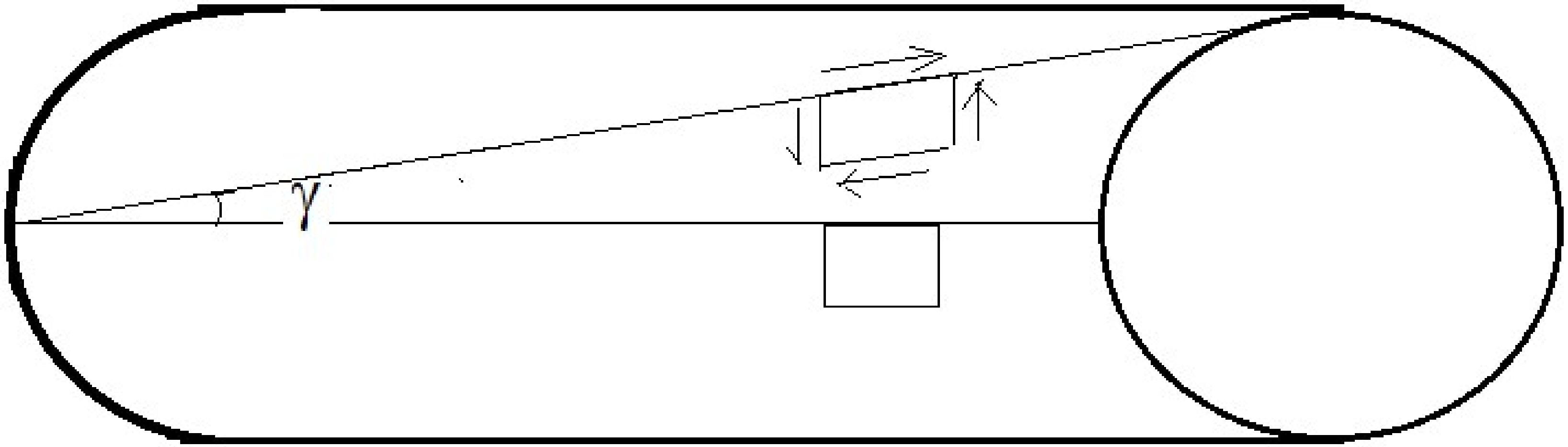
$$G = \frac{E}{2(1+v)}$$

the shear strain Y at any distance ρ from the center is:

$$\gamma = \frac{\rho\theta}{L}$$

The maximum shear strain occurs at the outer surface when ρ= r





$$\tau = \frac{T\rho}{J}$$

$$\tau_{max} = \frac{Tr}{J}$$

Ex: what is the maximum diameter of a solid shaft which will not twist more than 3° in a length of 6 m when subjected to a torque of 12 kN-m. What is the maximum shear stress induced in the shaft. Use G=82 GPa

$$\tau = \frac{T\rho}{J} \qquad \theta = \frac{TL}{GJ}$$

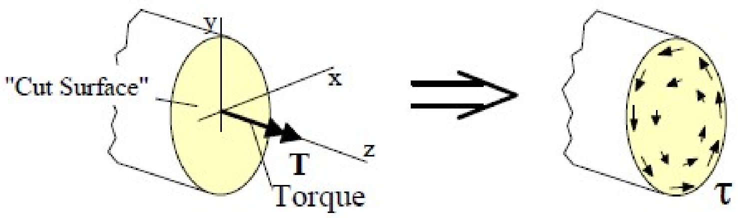
$$\frac{T}{J} = \frac{\tau}{r} = \frac{G \times \theta}{L}$$

$$J = \frac{\pi \cdot D^4}{32} = \frac{\pi \cdot R^4}{2} \qquad J = \frac{12 \times 10^6 \times 6000}{82 \times 10^3 \times 3\pi/180}$$

D= 114.32 mm

$$\tau = \frac{T.R}{J}$$

3-12 Torsion



$$\tau = \frac{T \cdot r}{J}$$

r = Distance from shaft axis to point of interest
R = Shaft Radius
D = Shaft Diameter

$$J = \frac{\pi \cdot D^4}{32} = \frac{\pi \cdot R^4}{2} \quad \text{for solid circular shafts}$$

$$J = \frac{\pi \cdot \left(D_o{}^4 - D_i{}^4\right)}{32} \quad \text{for hollow shafts}$$

$$\tau_{max} = \frac{16 \cdot T}{\pi \cdot D^3} \quad \text{for solid circular shafts}$$

$$\tau_{max} = \frac{16 \cdot T \cdot D_o}{\pi \cdot \left(D_o{}^4 - D_i{}^4\right)} \quad \text{for hollow shafts}$$

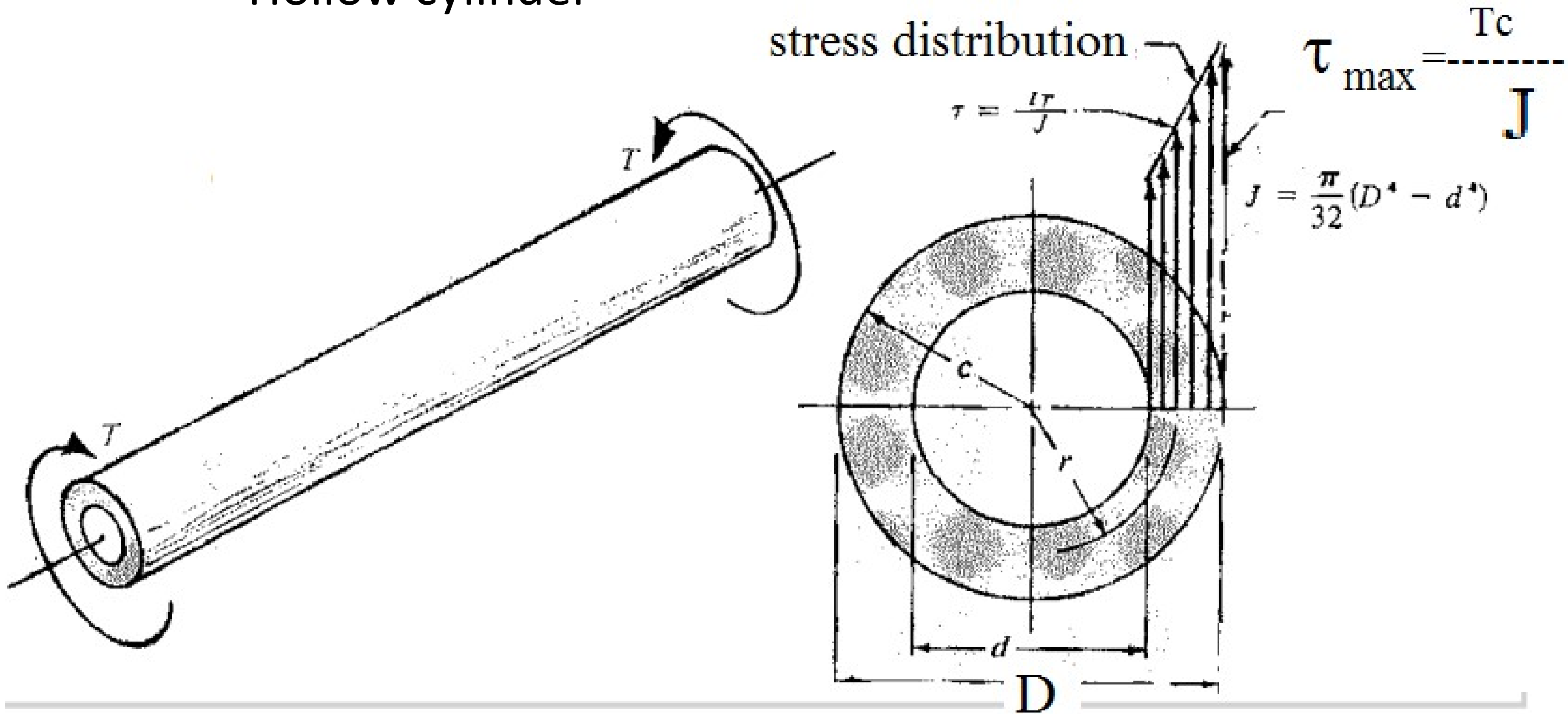
J = polar second moment of area

torsion is the twisting of an object due to an applied torque

- We use Hooke's law for a linear elastic material

$$\tau = G\gamma$$

Hollow cylinder



For rectangular cross-sections, the maximum shear stress is found as:

The maximum shearing stress in a rectangular *b × c section bar* occurs in the middle of the *longest side b and is of the magnitude*

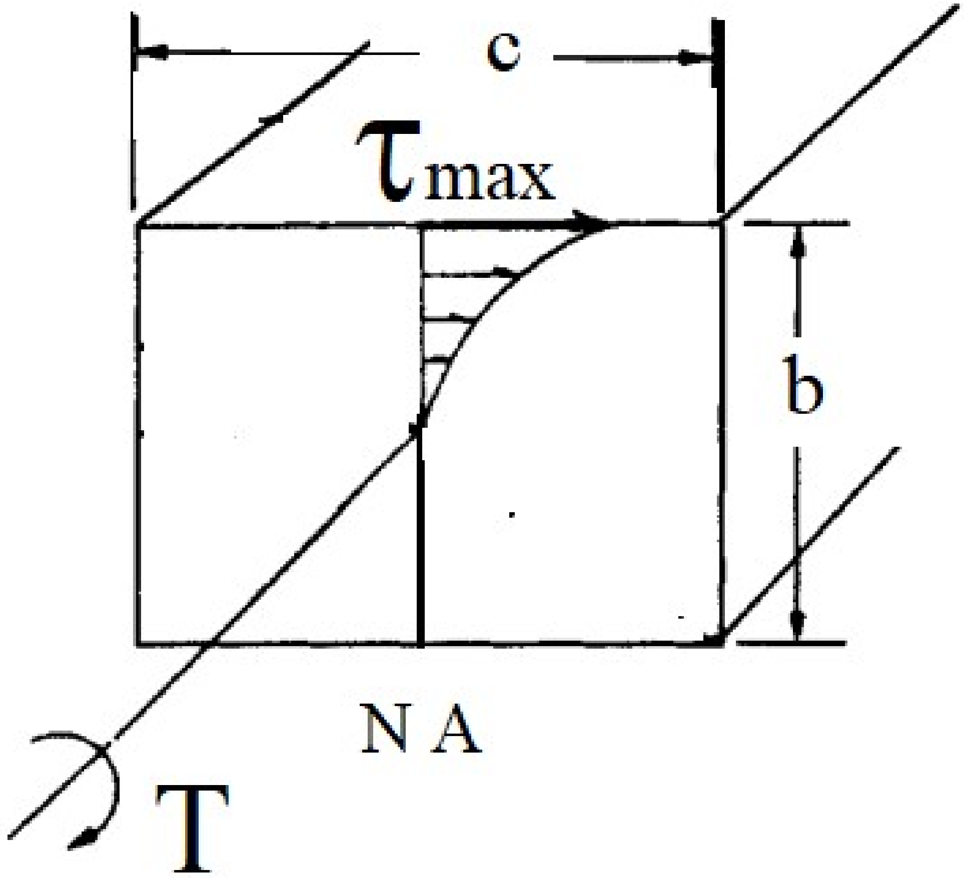
$$\tau_{\max} = \frac{T}{\alpha b c^2} \doteq \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) \qquad (3\text{–}40)$$

The parameter α is a factor that is a function of the ratio b/c as shown in the following table.

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

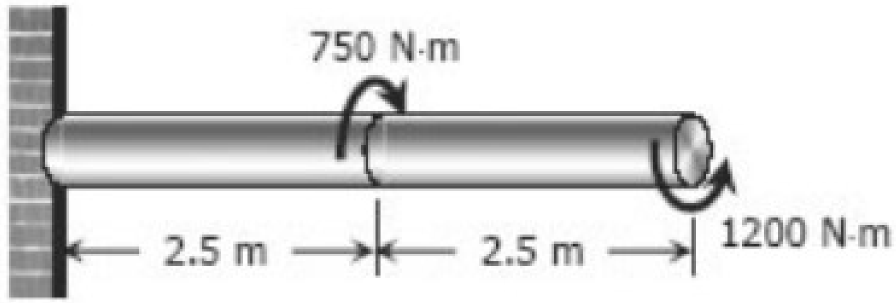
$$\frac{\theta}{L} = \frac{T}{\beta\, d b^3 G}$$

For torsion of rectangular sections the maximum shear stress tmax and angle of twist 0 are given by

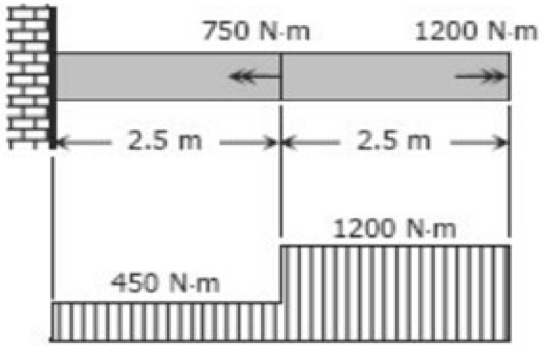


where “b” is the longest side

Ex: A solid steel shaft is loaded as shown. Using G = 83 GPa, determine the required diameter of the shaft if the shearing stress is limited to 60 MPa and the angle of rotation at the free end is not to exceed 4 deg



$$\tau_{max} = \frac{16T}{\pi D^3}$$



$$60000000 = \frac{32 * 1200}{\pi D^3}$$

D= 58.8 mm

$$60000000 = \frac{32 * 450}{\pi D^3}$$

D= 42.6 mm

finding D based on the maximum angle of twist

$$\theta_{total} = \theta_{450} + \theta_{1200}$$

$$\theta = \frac{TL}{JG}$$

$$\theta = \frac{1}{JG}\sum TL$$

$$4^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{1}{\frac{1}{32}\pi D^4 \; 83*10^9}(450*2.5+1200*2.5)$$

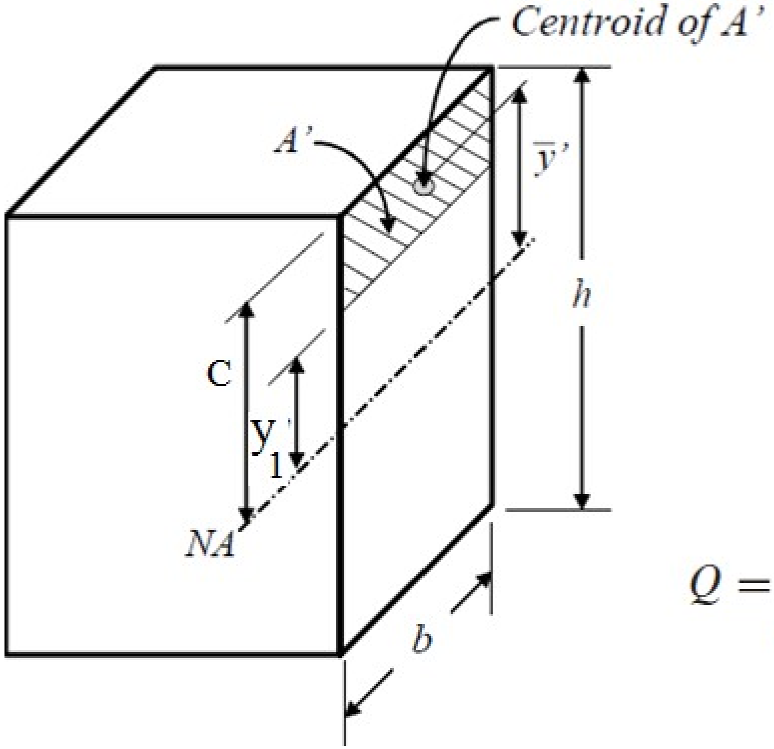
D= 51.9 mm

3–11 Shear Stresses for Beams in Bending

$$\tau = \frac{V}{Ib}\int_{y_1}^{c} y\,dA \qquad (3\text{–}29)$$

the integral is the first moment of the area A' with respect to the neutral axis

This integral is usually designated as Q



$$\tau = \frac{VQ}{Ib}$$

$$Q = \int_{y_1}^{c} y\,dA = b\int_{y_1}^{c} y\,dy = \left.\frac{by^2}{2}\right|_{y_1}^{c}$$

$$\tau = \frac{1}{It}\left(\frac{dM}{dx}\right)\int_{A'} y\,dA$$

$= Q$

$dM/dx = V$

Internal Shear (lb)

$$\tau = \frac{VQ}{It}$$

First Moment of area at point of interest

$$Q = \bar{y}' \cdot A'$$

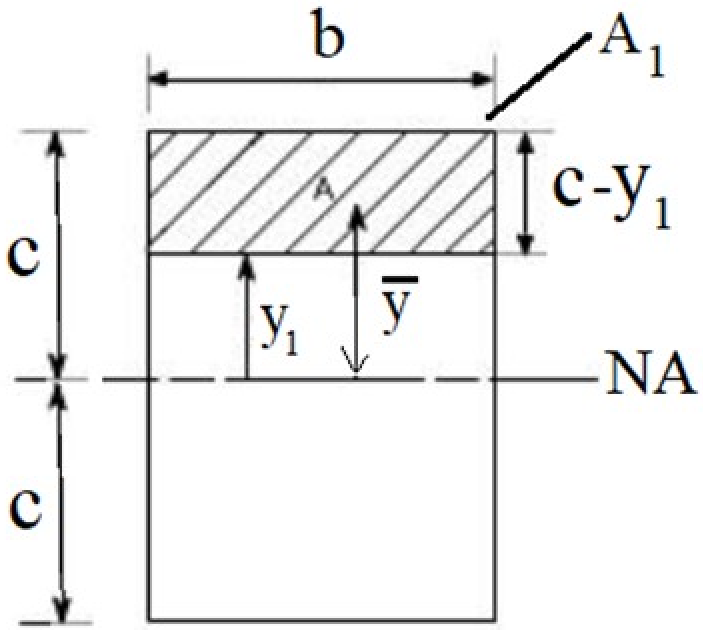
Moment of inertia of entire cross section

Thickness of cross-section at point of interest

The average shear stress over the cross section is given by: $\tau = \dfrac{V}{A}$

Shearing stress distribution in typical cross-sections:

1-For the rectangular section



Shear stress at any point y1 from the neutral axis is

$$\tau = \frac{V}{2I}\left(c^2 - y_1^2\right)$$

The maximum shear stress occurs at the neutral axis and is zero at both the top and bottom surface of the beam

$$\tau_{max} = \frac{3V}{2A}$$

the shaded area A 1 is the area of interes

The first moment of A1 with respect to the NA is :

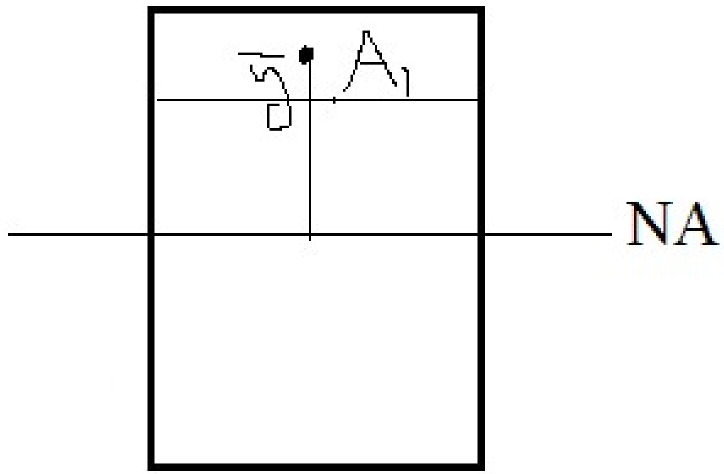
$Q_{NA} = y\,A_1 = (y1+(c-y1)/2)*(b*(c-y1)$

$$Q_{NA} = \frac{c^2 b}{2} - \frac{b\,y_1^2}{2}$$

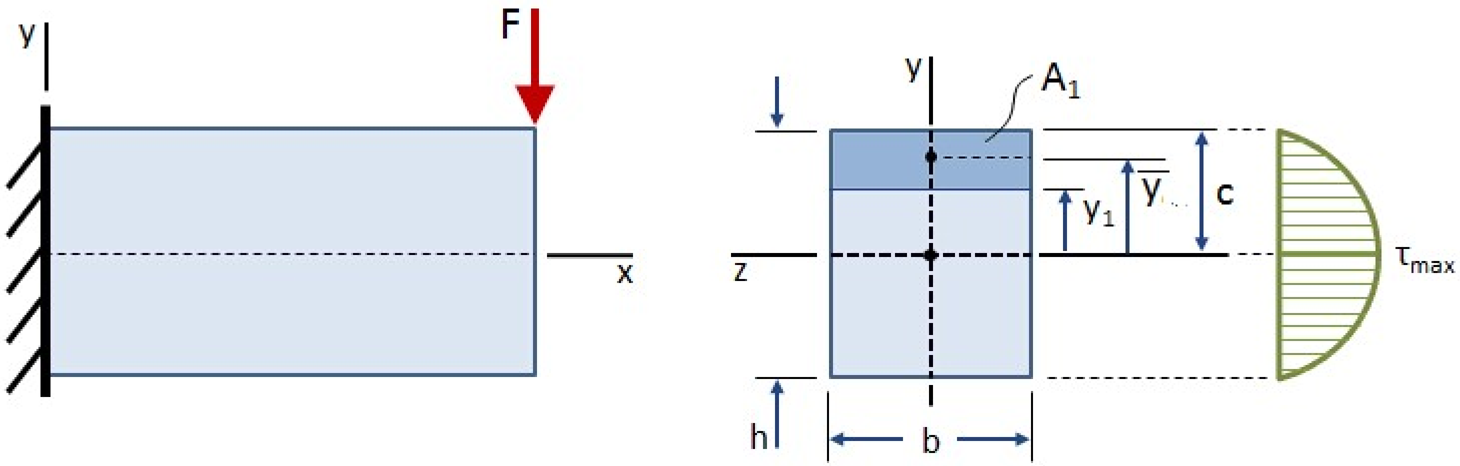
- First moment of area Q

$$Q_{NA} = \int_A y \, dA$$

$$\bar{y}_c = Q/A$$



The shear stress varies over the height of the cross section



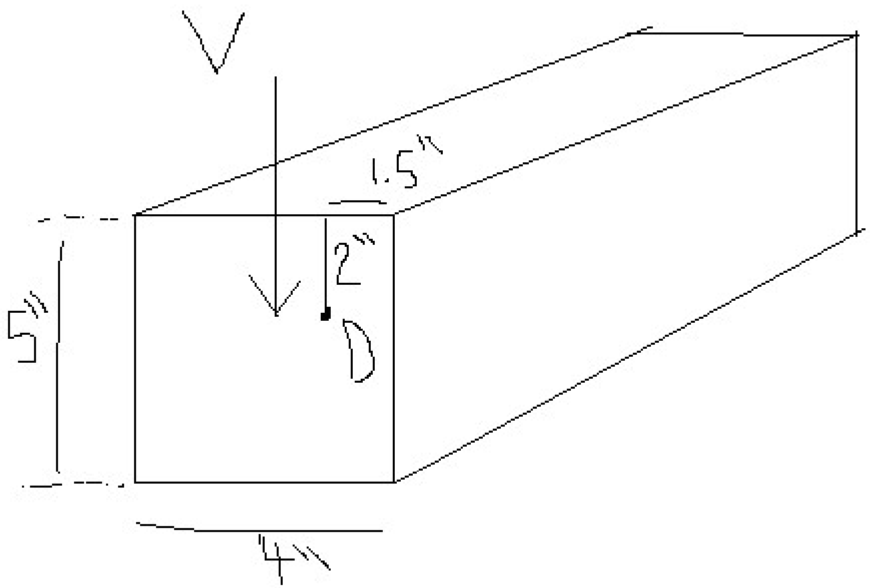
The shear stress is zero at the free surfaces (the top and bottom of the beam), and it is maximum at the centroid.

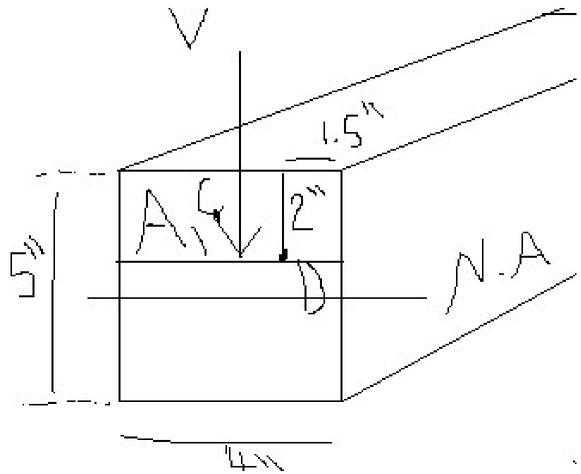
The equation for shear stress at any point located a distance y_1 from the centroid of the cross section is given by:

$$\tau = \frac{VQ}{Ib}$$

V : shear force acting at the cross section,
I_c is the second moment of area of the cross section,
b: is the width of the cross section. These terms are all constants. The
Q : first moment of area

Ex: the beam is subjected to a vertical shear force V= 3 kip.
1- determine the shear stress at point D. 2- calculate the maximum shear stress in the beam



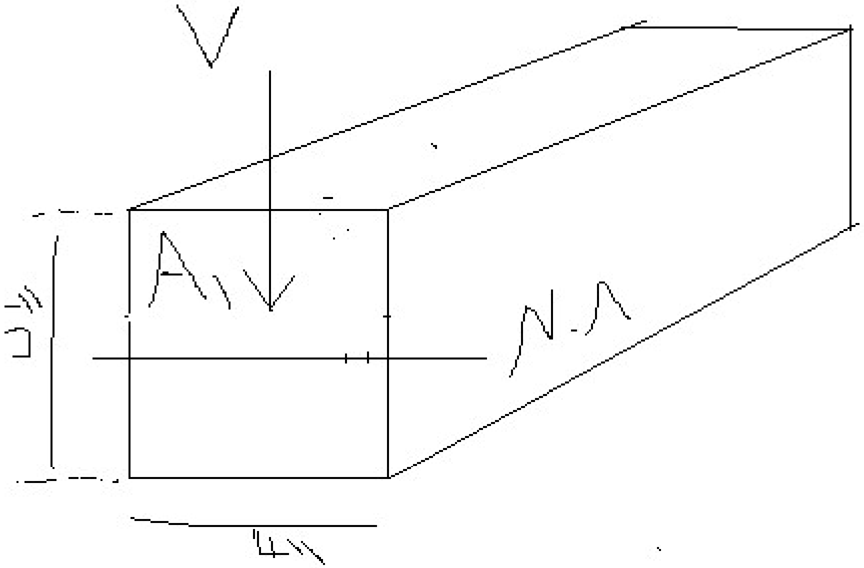


$$Q = \bar{y}\,A_1 = (0.5+0.5*2)(2*4) = 12\ in^3$$

$$I = \frac{1}{12}\,b\,h^3 = \frac{1}{12}\,4*5 = 41.7\ in^3$$

$$\tau = \frac{VQ}{Ib} = \frac{3*12}{41.7*4} = 0.216\ ksi$$

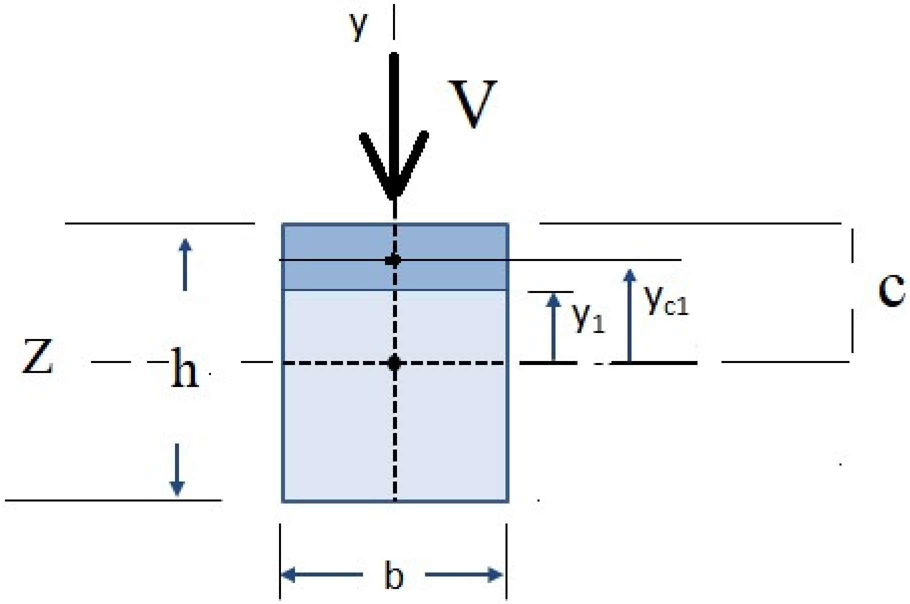
$$\tau_{\max} = \frac{3V}{2A}$$



$$\tau_{\max} = \frac{3V}{2A} = \frac{3*3}{2*(5*4)} = 0.225 \text{ ksi}$$

The integral represented by Q is the *first moment of the shaded area A^** with respect to the neutral axis z:

$$\tau = \frac{VQ}{Ib}$$



$$Q = \int_{y_1}^{c} y\,dA = b\int_{y_1}^{c} y\,dy = \frac{by^2}{2}\Big|_{y_1}^{c}$$

C= h/2

where $I_c = b{\cdot}h^3/12$ is the centroidal moment of inertia of the cross section.

$$Q = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

The maximum value of Q occurs at the neutral axis of the beam (where y_1 = 0):

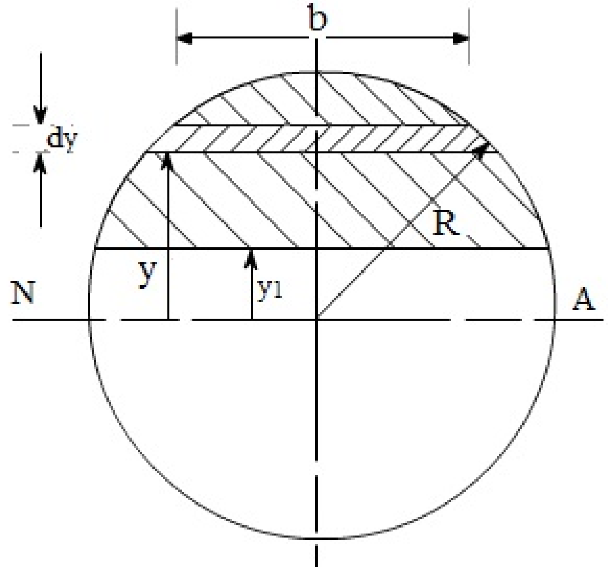
$$Q_{max} = \frac{bh^2}{8}$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

at NA

2-Shear Stresses in Circular Sections

$$\tau = \frac{4V(R^2 - y_1^2)}{3\pi R^4}$$

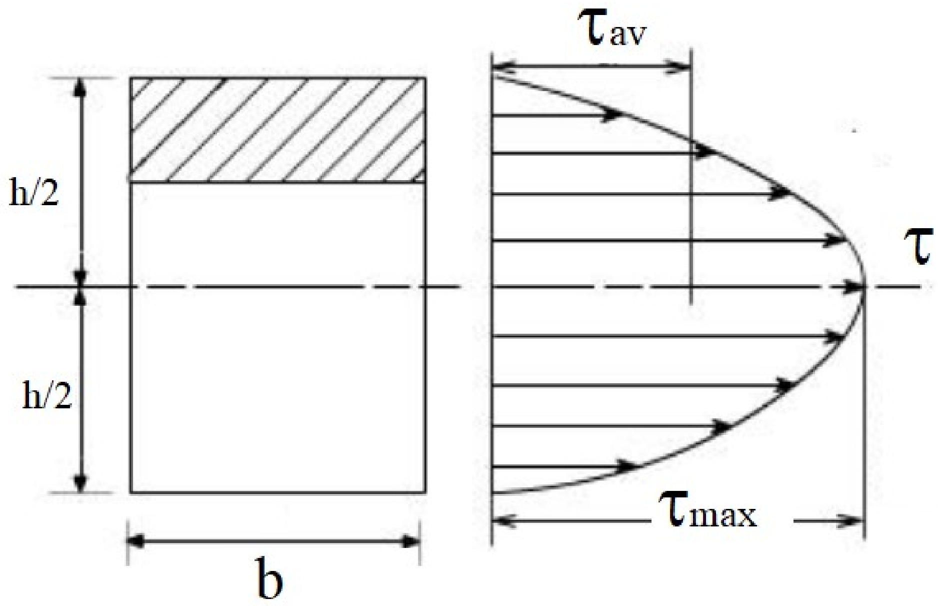


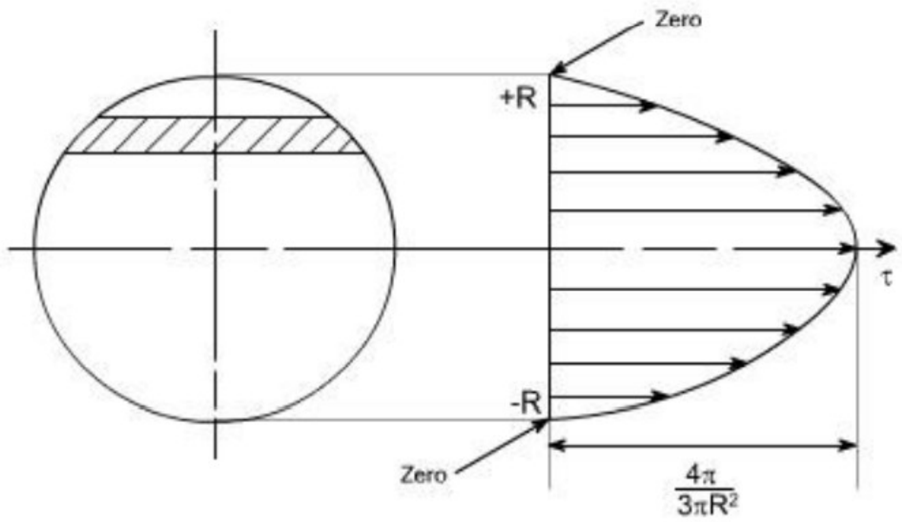
The maximum value of first moment, Q, occurring at the centroid, is given by:

$$Q_{max} = \frac{2R^3}{3}$$

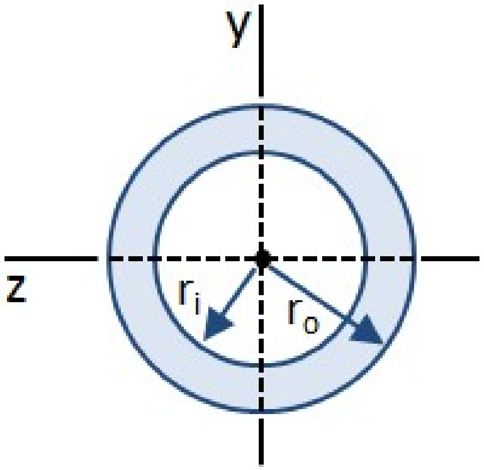
$$\tau_{max} = \frac{4V}{3\pi R^4} = \frac{4V}{3A}$$

Shearing stress distribution in typical cross-sections:





3-Shear Stresses in Circular Tube Sections



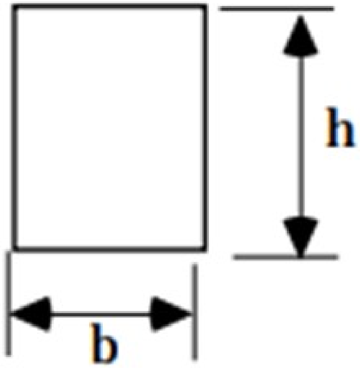
$$A = \pi\,(r_o^{\,2} - r_i^{\,2})$$

$$Q_{max} = \frac{2}{3}\left(r_o^3 - r_i^3\right)$$

$$I_c = \pi\,(r_o^{\,4} - r_i^{\,4})\,/\,4$$

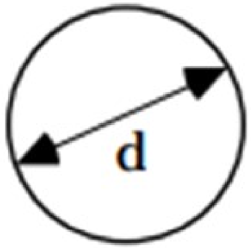
$$\tau_{max} = \frac{VQ}{It} = \frac{4V}{3\pi}\;\frac{r_o^2 + r_i r_o + r_i}{r_o^{\,4} - r_i^{\,4}}$$

$b = 2\,(r_o - r_i)$ is the effective width of the cross section,



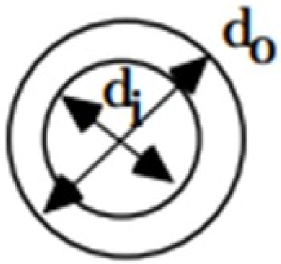
$$\tau_{max} = \frac{3V}{2A}$$

$$A = bh$$



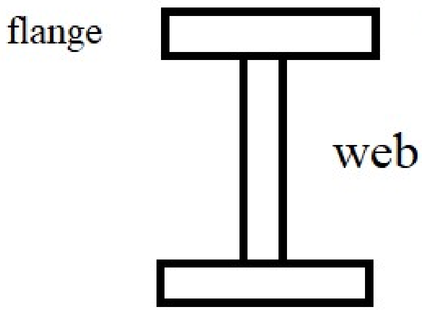
$$\tau_{max} = \frac{4V}{3A}$$

$$A = (\pi/4)\, d^2$$



$$\tau_{max} = \frac{2V}{A}$$

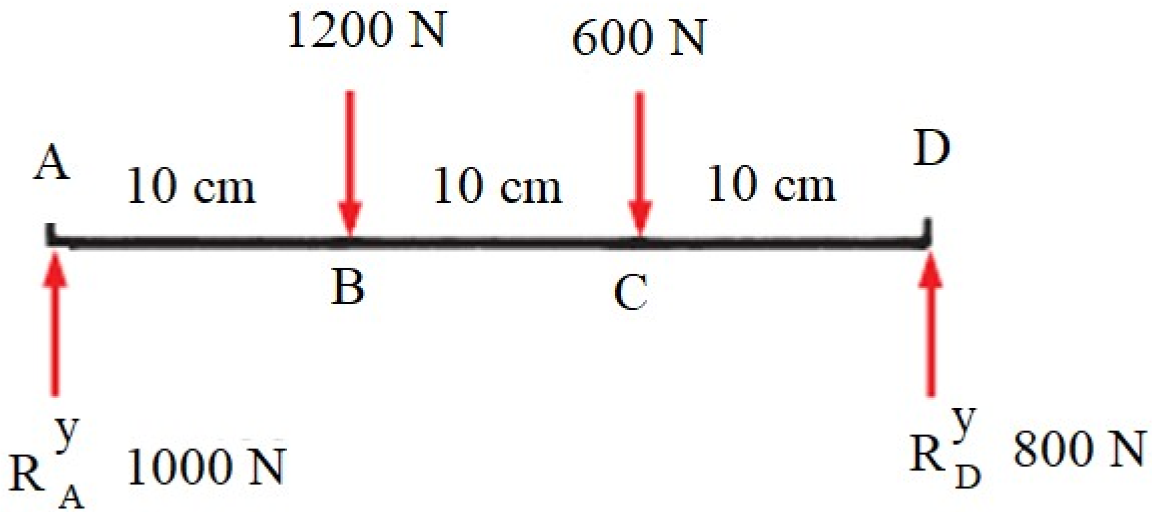
$$A = (\pi/4)\,(d_o^2 - d_i^2)$$

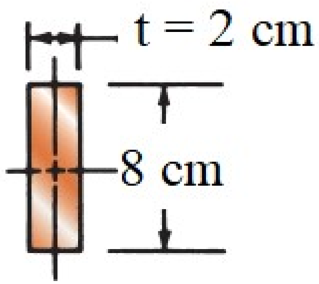


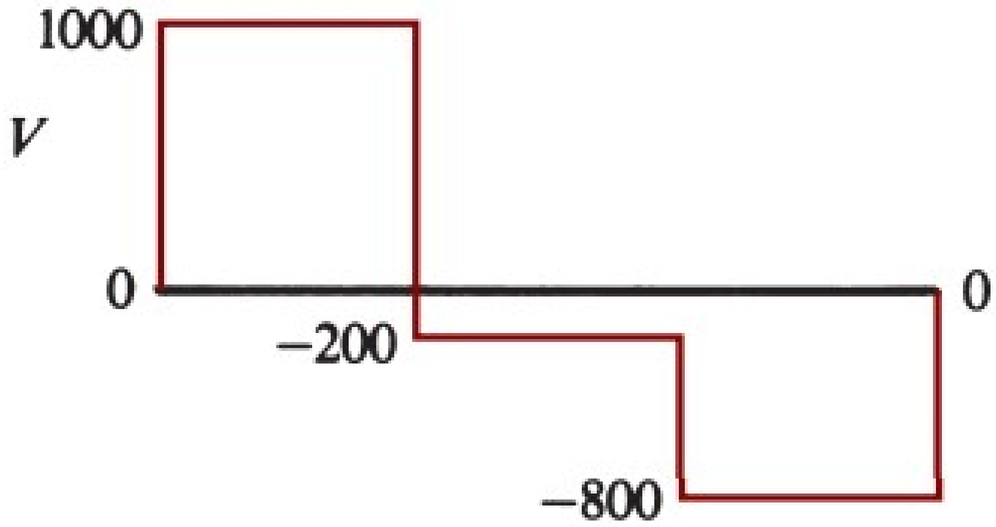
- I-beam (*thin welled*)

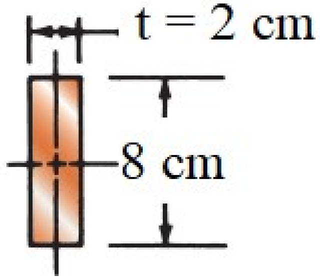
$$\tau_{max} = \frac{V}{A_{web}}$$

Ex: A simply supported beam carrying two concentrated loads. The shearing force diagram is shown, along with the rectangular shape and size of the cross section of the beam. The stress distribution is parabolic, with the maximum stress occurring at the neutral axis. Use Equation (3–16) to compute the maximum shearing stress in the beam.









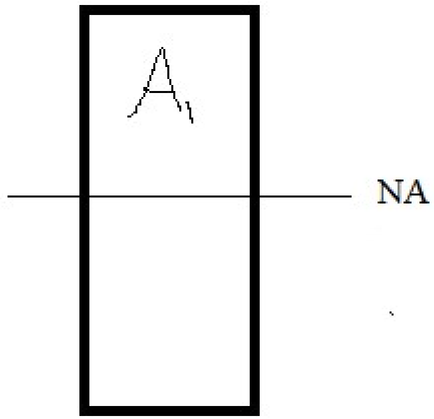
$$I = bh^3/12$$

$$= 2(8)^3/12 = 85.3\ \text{cm}^4$$

$$Q = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

$$Q = \frac{2}{2}\left(\frac{8^2}{4} - 0\right)$$

$$= 16\ \text{cm}^3$$

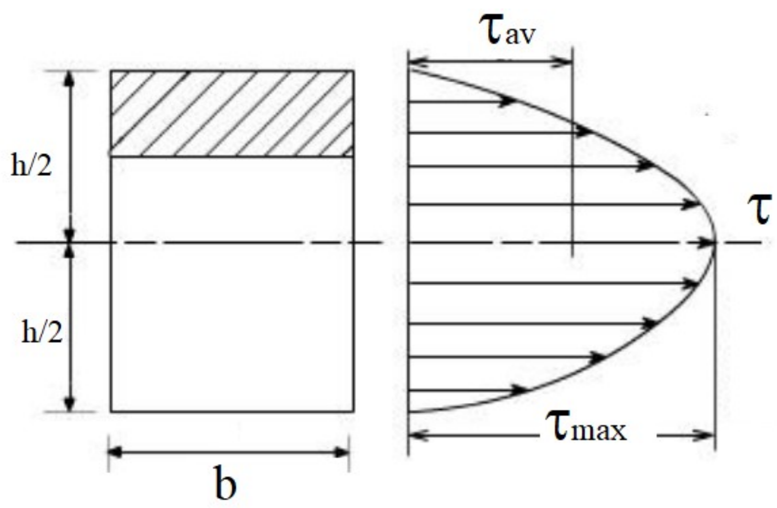


the maximum shearing stress is

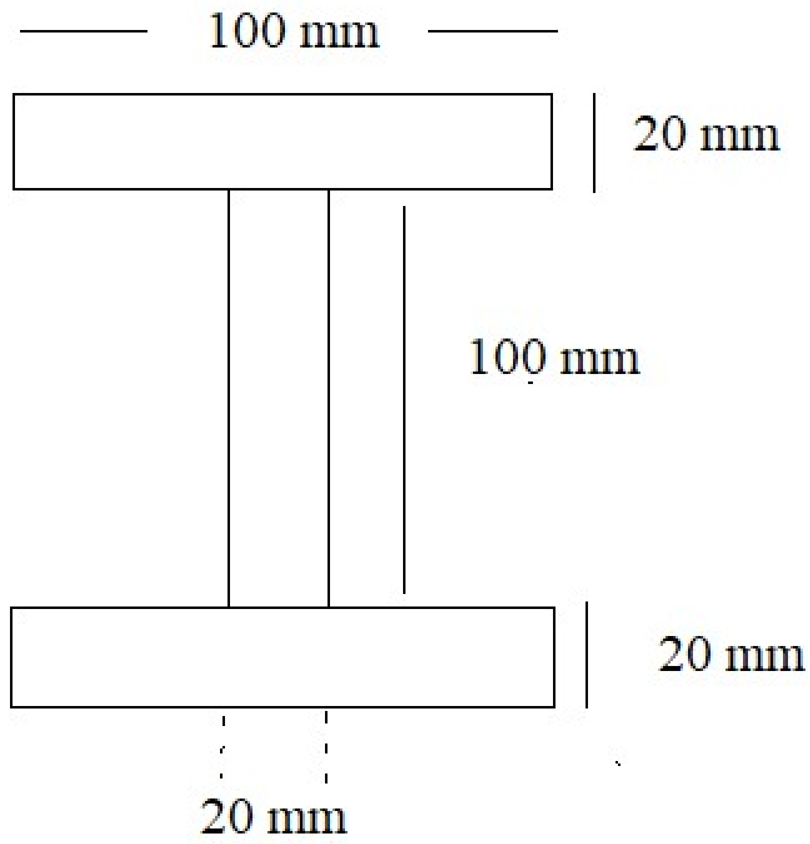
$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A} = \frac{3(1000)}{2(8*2)*10^{-4}} = 93.75*10^{4}\ N/m^{2}$$

The maximum shearing stress occurs at the neutral axis of the rectangular section



Ex: Determine the shear stress on the lower surface of the upper flange . The vertical shear V= 500 N.



$$I = \frac{1}{12}(0.020\,\text{m})(0.100\,\text{m})^3$$
$$+ 2[\frac{1}{12}(0.100\,\text{m})(0.020\,\text{m})^3$$
$$+ (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})^2]$$
$$= 16.20 \times 10^{-6}\,\text{m}^4$$

$$Q = A\bar{y}$$
$$= (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})$$
$$= 120 \times 10^{-6}\,\text{m}^3$$

$$\tau = \frac{V\,Q}{I\,b} = \frac{500 * 120 * 10^{-6}}{16.2 * 10^{-6} * 0.1}$$

$$= 37040 \ \mathrm{N/m^2}$$