# Homework 1 : Generalized finite-Sized Queuing Situation

A store operates with 3 checkout counters. The manager uses the following schedule to determine the number of counters in the operation depending on the number of customers in the store:

| No. of Customers in the Store | No, of Counters in Operation |
|---|---|
| 1-3 | 1 |
| 4-6 | 2 |
| More than 6 | 3 |

Customers arrive in the counter area according to Poisson distribution with a rate of 10 customers per hour. The average checkout time per customer is exponential with mean 12 minutes. If the Maximum number of the customers allowed in the (whole) system are 10. Answer all the following:

1. Construct the state transition diagram
2. Setup the set of possible number of Customers(n) might be exist in the system
3. What is the average customer's arrival rates; $\lambda 0, \lambda 1, \lambda 2, \lambda 3, \ldots \lambda n\text{-}1$.
4. What is the average customer's servicing rates; $\mu 1, \mu 2, \mu 3, \ldots \mu n$.
5. Compute the steady-state probabilities; P0, P1, P2, P3, . . . Pn.
6. What is the steady-state probability of having 0 customer in the store
7. What is the probability that the store is empty of customers
8. What is the probability that the store is full
9. What is the probability that newly arriving customers balk to other store.
10. What is the average number of customers in the system
11. What is the average number of customers Waiting (Queuing) in the system (Lq)
12. Average Expected number of busy counters
13. Average facility utilization

# Steady State Measures of Performance

- Performance measures are derived from the steady state probability of $n$ customers in the system, **($p_n$)** (formulas independent of the queue discipline)

$$L_s = \sum_{n=1}^{\infty} n p_n \qquad L_q = \sum_{n=c+1}^{\infty} (n-c) p_n$$

- Little's Law

$$\lambda_{eff} = \frac{L_s}{W_s} \Rightarrow L_s = \lambda_{eff} W_s \qquad L_q = \lambda_{eff} W_q$$

$\lambda_{eff}$ is the effective arrival rate,

$\lambda_{eff} = \lambda$, when all arriving customers can join the system

$\lambda_{eff} < \lambda$, If Some customers cannot join because the system is full

- [Expected Waiting time in system]

$$W_s = W_q + \frac{1}{\mu} \qquad L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

$$\bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu} \qquad \rho = \left( \begin{matrix} Facility \\ Utilization \end{matrix} \right) = \frac{\bar{c}}{c}$$

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# Steady State Measures of Performance

1. Line length $L_q$ . Expected Number of customers in the line
2. Number of customers in system $L_s$
3. Expected Waiting time in Queue $W_q$
4. Expected Waiting time in system $W_s$
5. Expected number of busy servers
5. Service facility utilization $\rho$ $\bar{c}$

**Calculations of the Above Depends on the Queue Model**

**Independent of the queue discipline**

# Single-Server infinite-Size Poisson Queuing Situation M/M/1:GD/∞ /∞

Poisson arrival rate $\lambda$

Poison Procesing rate $\mu$

Single server

$$\rho = \frac{\lambda}{\mu}$$

$\lambda_{eff} = \lambda$

Average number of customer in system: $L_s = \frac{\rho}{1 - \rho}$

Average number of customer in queue: $L_q = \frac{\rho^2}{1 - \rho}$

Average waiting time in system: $W_s = \frac{L_S}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}$

Average waiting time in queue: $W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)}$

Average number of busy servers: $\bar{c} = L_s - L_q = \rho$

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# Single-Server infinite-Size Poisson Queuing Situation M/M/1:GD/∞ /∞

Automata car wash facility operates with only one bay. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. The time for washing and cleaning a car is exponential, with a mean of 10 minutes. Cars that cannot park in the lot can wait in the street bordering the wash facility. This means that for all practical purposes, there is no limit on the size of the system. The manager of the facility wants to determine the size of the parking lot.

$\lambda = 4$ cars/hour $\quad \mu = \frac{60}{10} = 6$ cars/hour $\quad \rho = \frac{4}{6} = 0.66667$

$$L_s = \frac{\rho}{1-\rho} = \frac{0.66667}{1-0.66667} = 2.0 \text{ Car}$$

$$L_q = \frac{0.6667^2}{1-0.66667} = 1.33333 \text{ car}$$

$$W_s = \frac{L_S}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{6-4} = 0.5 \text{ Hour}$$

$$W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu(1-\rho)} = \frac{0.66667}{6(1-0.66667)} = 0.33333 \text{ Hour}$$

# Single-Server finite-Size Poisson Queuing Situation M/M/1:GD/N /∞

Poisson arrival rate $\lambda$

Poison Procesing rate $\mu$

[redacted] servers $c = 1$

$$\rho = \frac{\lambda}{\mu}$$

Average number of customer in system : $L_s = \dfrac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})}, \rho \neq 1$

Average waiting time in system : $W_s = \dfrac{L_s}{\lambda_{eff}}$

Average number of customer in queue : $L_q = L_s - \dfrac{\lambda_{eff}}{\mu}$

Average waiting time in queue : $W_q = \dfrac{L_q}{\lambda_{eff}}$

Average number of busy servers : $\bar{c} = L_s - L_q$

$\lambda_{lost} = \lambda p_N$

$\lambda_{eff} = \lambda(1 - p_N)$

$$p_N = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}}$$

N = waiting + getting served

$\lambda_{eff} \neq \lambda_{lost}$

probability of getting full

# Single-Server finite-Size Poisson Queuing Situation M/M/1:GD/N /∞

Consider the car wash facility of last example . Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities. The owner wishes to determine the impact of the limited parking space on losing customers to the competition.

Poisson arrival rate $\lambda = 4$ , Poison Procesing rate $\mu = 6$

multiple servers $c = 1$, Source limit $4 + 1 = 5$ $\quad \rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.666$

$$\lambda_{eff} = \lambda(1 - p_5) = 4(1 - 0.04820) = 3.80752$$

Average number of customer in queue : $L_q = L_s - \frac{\lambda_{eff}}{\mu} = 1.42256 - \frac{3.80752}{6} = 0.78797$

Average waiting time in queue : $W_q = \frac{L_q}{\lambda_{eff}} = \frac{0.78797}{3.80752} = 0.20695$

Average number of busy servers : $\bar{c} = L_s - L_q = 0.666$

# Single-Server finite-Size Poisson Queuing Situation M/M/1:GD/N /∞

Consider the car wash facility of last example . Suppose that the facility has a total of 4 parking spaces. If the parking lot is full, newly arriving cars balk to other facilities. The owner wishes to determine the impact of the limited parking space on losing customers to the competition.

Poisson arrival rate $\lambda = 4$ , Poison Procesing rate $\mu = 6$

multiple servers $c = 1$, Source limit $4 + 1 = 5 = N$ $\rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.6667$

Average number of customer in system : $L_s = \frac{\rho[1-(N+1)\rho^N + N\rho^{N+1}]}{(1-\rho)(1-\rho^{N+1})}, \rho \neq 1$

$$L_s = \frac{0.667[1-(5+1)0.667^5 + 5(0.667)^{5+1}]}{(1-0.667)(1-0.667^{5+1})} = 1.42256\ car$$

$$Ws = \frac{L_s}{\lambda_{eff}} = \frac{1.42256}{\lambda(1-p_5)} = \frac{1.42256}{4(1-0.04820)} = 0.37362 \text{ hour}$$

$$p_N = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}}, p_5 = \frac{(1-0.667)0.667^5}{1-0.667^{5+1}} = 0.04820$$

| | M/M/1:GD/∞/∞ | M/M/1:GD/N/∞ |
|---|---|---|
| Ls | 2 car | 1.423 car |
| Lq | 1.333 car | 0.788 car |
| Ws | 0.5 hour | 0.37 hour |
| Wq | 0.333 hour | 0.2 hour |
| C- bar | 0.667 | 0.667 |

# Multi-Server infinite-Size Poisson Queuing Situation M/M/c:GD/∞ /∞

In this model there are c parallel and identical servers. Assume Poisson arrival rate λ and Poisson service rate μ. There is no limit on the maximum number of customers in the system therefore λ = λeff. For this model $\lambda_n$ , ρ, and $\mu_n$ are defined as:

$$\lambda_n = \{\lambda \quad n \geq 0 \qquad , \mu_n = \begin{cases} n\mu & n < c \\ c\mu & n \geq c \end{cases} \qquad , \rho = \frac{\lambda}{\mu}$$

Substituting $\lambda_n$ , ρ, and $\mu_n$ in the general form of $P_n$

$$P_n = \left( \frac{\lambda_{n-1}\lambda_{n-2}\ldots\lambda_0}{\mu_n\mu_{n-1}\ldots\mu_1} \right) P_0, \text{ for } n = 1,2,\ldots \text{and} \sum_{n=0}^{\infty} P_n = 1$$

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# Multi-Server infinite-Size Poisson Queuing Situation M/M/c:GD/∞ /∞



We get

$$L_q = \sum_{n=c}^{\infty} (n-c) p_n \qquad = \frac{\rho^{c+1}}{(c-1)!(c-\rho)^2} p_0$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = W_s \lambda$$

# Multi-Server infinite-Size Poisson Queuing Situation M/M/c:GD/∞ /∞

We get

$$p_n = \begin{cases} \dfrac{\lambda^n}{\mu(2\mu)(3\mu)\cdots(n\mu)} p_0 = \dfrac{\lambda^n}{n!\mu^n} p_0 = \dfrac{\rho^n}{n!} p_0, & n < c \\ \dfrac{\lambda^n}{\left(\prod_{i=1}^{c} i\mu\right)(c\mu)^{n-c}} p_0 = \dfrac{\lambda^n}{c!c^{n-c}\mu^n} p_0 = \dfrac{\rho^n}{c!c^{n-c}} p_0, & n \geq c \end{cases}$$

Next we compute $p_0$ when $\{(\rho/c) < 1\}$

$$p_0 = \left\{\left(\sum_{n=0}^{c-1} \frac{\rho^n}{n!}\right) + \frac{\rho^c}{c!} \sum_{n=c}^{\infty} \left(\frac{\rho}{c}\right)^{n-c}\right\}^{-1}$$

$$= \left\{\left(\sum_{n=0}^{c-1} \frac{\rho^n}{n!}\right) + \frac{\rho^c}{c!}\left(\frac{1}{1-\frac{\rho}{c}}\right)\right\}^{-1}, \quad \frac{\rho}{c} < 1$$

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# Multi-Server infinite-Size Poisson Queuing Situation M/M/c:GD/∞ /∞

A community is served by two cab (Taxi) companies. Each company owns two cabs, and the two companies are known to have equal shares of the market. This is evident by the fact that calls arrive at each company's dispatching office at the rate of eight per hour. The average time per ride is 12 minutes. Calls arrive according to a Poisson distribution, and the ride time is exponential . The two companies recently were bought by an investor who is interested in consolidating them into a single dispatching office to provide better service to customers. Analyze the new owner's proposal.

1. Cabs = servers        cab ride = the service

   each company is (M/M/2: GD/ /∞ /∞)   with $\lambda = 8$ calls/ Hour

   $\mu = 60/12 = 5$ ride per hour

2. Consolidation (M/M/4: GD/ /∞ /∞)   with $\lambda = 2 * 8 = 16$ calls/ Hour

   $\mu = 60/12 = 5$ ride per hour

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# Multi-Server Infinite Size Models
# (M/M/2):(GD/∞ /∞ )Models

Poisson arrival rate $\lambda = 8$ Poison Procesing rate $\mu = 5$

multiple servers $c = 2$ $\rho = \frac{\lambda}{\mu} = \frac{8}{5} = 1.6$

Average number of customer in queue: $L_q = \frac{\rho^{c+1}}{(c-1)!(c-\rho)^2} p_0 = \frac{1.6^3(0.111)}{(2-1)!(2-\rho)^2} = 2.844 \; Calls$

Average waiting time in queue: $W_q = \frac{L_q}{\lambda} = \frac{2.844}{8} = 0.356 \; hour$

Average waiting time in system: $W_s = W_q + \frac{1}{\mu} = 0.356 + \frac{1}{5} = 0.556 \; hour$

Average number of customer in system: $L_s = W_s \lambda = (0.556)(8) = 4.444 \; Calls$

Noted that $L_s = L_q + \rho = 2.844 + 1.6 = 4.444 \; Calls$

and $\bar{c} = L_s - L_q = \rho = 1.6$

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left( \frac{1}{1-\frac{\rho}{c}} \right) \right\}^{-1} = \left\{ \left( \frac{1.6^0}{0!} + \frac{1.6^1}{1!} \right) + \frac{1.6^2}{2!} \left( \frac{1}{1-\frac{1.6}{2}} \right) \right\}^{-1} = 0.111$$

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# Multi-Server Infinite Size Models
# (M/M/c) : (GD/ ∞ /∞ )

| Scenario | Lambda | Mu | Nbr. of Servers | System Limit | Source Limit |
|---|---|---|---|---|---|
| 1 | 8.00 | 5.00 | 2 | infinity | infinity |
| 2 | 16.00 | 5.00 | 4 | infinity | infinity |

**Comparative analysis**

| Scenario | c | .ambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 8.000 | 5.000 | 8.000 | 0.111 | 4.444 | 2.844 | 0.556 | 0.356 |
| 2 | 4 | 16.000 | 5.000 | 16.000 | 0.027 | 5.586 | 2.386 | 0.349 | 0.149 |

| | M/M/2:GD:∞/∞ | M/M/4:GD:∞/∞ |
|---|---|---|
| Ls | 4.4 | 5.58 |
| Lq | 2.8 | 2.38 |
| Ws | 0.556 | 0.349 |
| Wq | 0.356 | 0.149 |

# Multi-Server Infinite Size Models (M/M/4):(GD/∞ /∞)

Poisson arrival rate $\lambda = 8 \times 2 = 16$ Poison Procesing rate $\mu = 5$

multiple servers $c = 4$ $\rho = \dfrac{\lambda}{\mu} = \dfrac{16}{5} = 3.2$

Average number of customer in queue: $L_q = \dfrac{\rho^{c+1}}{(c-1)!(c-\rho)^2} p_0 = \dfrac{3.2^5(0.027)}{(4-1)!(4-3.2)^2} = 2.386\ Calls$

Average waiting time in queue: $W_q = \dfrac{L_q}{\lambda} = \dfrac{2.386}{16} = 0.149\ hour$

Average waiting time in system: $W_s = W_q + \dfrac{1}{\mu} = 0.149 + \dfrac{1}{5} = 0.349\ hour$

Average number of customer in system: $L_s = W_s \lambda = (0.349)(16) = 5.586\ Calls$

Noted that $L_s = L_q + \rho = 2.386 + 3.2 = 5.586\ Calls$

and $\bar{c} = L_s - L_q = \rho = 3.2$

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}\left(\frac{1}{1-\frac{\rho}{c}}\right)\right\}^{-1} = \left\{\left(\frac{3.2^0}{0!} + \frac{3.2^1}{1!} + \frac{3.2^2}{2!} + \frac{3.2^3}{3!}\right) + \frac{3.2^4}{4!}\left(\frac{1}{1-\frac{3.2}{4}}\right)\right\}^{-1} = 0.027$$

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# Multi-Server finite-Size Poisson Queuing Situation M/M/c:GD/N /∞

(M/M/c: GD/ N /∞), $c \leq N$. This model differs from that of (M/M/c: GD/ ∞ /∞) model in that the system limits is finite and equal to N. this means that the maximum queue size is (N-c).

$\lambda_n$ , ρ, $\mu_n$ and for this model are defined as:

$$\lambda_n = \begin{cases} \lambda & 0 \leq n \leq N \\ 0 & n \geq N \end{cases} \quad , \mu_n = \begin{cases} n\mu & 0 \leq n \leq N \\ c\mu & n \geq N \end{cases} \quad , \rho = \frac{\lambda}{\mu}$$

Substituting $\lambda n$ , ρ, $\mu n$ and in the general form of $P_n$

$$P_n = \left( \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} \right) P_0, \text{ for } n = 1,2,\ldots \text{and} \sum_{n=0}^{\infty} P_n = 1$$

# Multi-Server finite-Size Poisson Queuing Situation M/M/c:GD/N /∞

We get

$$p_n = \begin{cases} \dfrac{\rho^n}{n!} p_0, & 0 \le n < c \\ \dfrac{\rho^n}{c!\,c^{n-c}} p_0, & c \le n \le N \end{cases} \quad \text{where, } p_0 = \begin{cases} \left( \left( \sum_{n=0}^{c-1} \dfrac{\rho^n}{n!} \right) + \dfrac{\rho^c \left( 1 - \left( \frac{\rho}{c} \right)^{N-c+1} \right)}{c!\left( 1 - \frac{\rho}{c} \right)} \right)^{-1} & , \dfrac{\rho}{c} \ne 1 \\ \left( \left( \sum_{n=0}^{c-1} \dfrac{\rho^n}{n!} \right) + \dfrac{\rho^c}{c!} (N - c + 1) \right)^{-1} & , \dfrac{\rho}{c} = 1 \end{cases}$$

Next we compute $L_q = \sum_{n=c+1}^{\infty} (n-c) p_n$

$$L_q = \begin{cases} \dfrac{\rho^{c+1}}{(c-1)!(c-\rho)^2} \left\{ 1 - \left( \dfrac{\rho}{c} \right)^{N-c+1} - (N-c+1)\left( 1 - \dfrac{\rho}{c} \right)\left( \dfrac{\rho}{c} \right)^{N-c} \right\} p_0 & , \dfrac{\rho}{c} \ne 1 \\ \dfrac{\rho^c (N-c)(N-c+1)}{2c!} p_0 & , \dfrac{\rho}{c} = 1 \end{cases}$$

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# Multi-Server finite-Size Poisson Queuing Situation M/M/c:GD/N /∞

$Arrival\ rate = \lambda \quad Servecing\ rate = \mu$

$servers = c \qquad \rho = \frac{\lambda}{\mu}$

$$p_N = \frac{\rho^N}{c!c^{N-c}} p_0 \qquad \lambda_{eff} = \lambda(1 - p_N) \qquad \lambda_{lost} = \lambda p_N$$

$$W_q = \frac{L_q}{\lambda_{eff}} \qquad W_s = W_q + \frac{1}{\mu} \qquad L_S = W_s \lambda_{eff}$$

# Multi-Server finite-Size Poisson Queuing Situation M/M/c:GD/N /∞



In the Consolidated cab company (M/M/4: GD/ ∞ /∞), suppose that the new funds cannot be secured to purchase additional cabs. The owner was advised by a friend that one way to reduce the waiting time is for the dispatching office to inform new customers of potential excessive delay once the waiting list reaches 6 customers. This move is certain to get new customers to seek service elsewhere but will reduce the waiting time for those on the waiting list. Investigate the plausibility of the friend's advise

**(M/M/4: GD/ N /∞), with $\lambda = 2 * 8 = 16$ calls per hour**

**$\mu = 60/12 = 5$ ride per hour**

**$C = 4$ Cabs**

**$N = 6 + 4 = 10$**

**(M/M/4: GD/ N /∞),**

# Multi-Server finite-Size Poisson Queuing Situation M/M/c:GD/N /∞

$\lambda = 16$ Calls/Hr. $\mu = 5$ Calls/Hr. servers = 4, $\rho = \frac{16}{5} = 3.2$

$$L_q = \frac{3.2^5}{(4-1)!(4-3.2)^2}\left\{1-\left(\frac{3.2}{4}\right)^{10-4+1} - (10-4+1)\left(1-\frac{3.2}{4}\right)\left(\frac{3.2}{4}\right)^{10-4}\right\}(0.03121)$$

$= 1.15421$ calls

$$p_0 = \left(\sum_{n=0}^{4-1}\frac{3.2^n}{n!} + \frac{3.2^4\left(1-\left(\frac{3.2}{4}\right)^{10-4+1}\right)}{4!\left(1-\frac{3.2}{4}\right)}\right)^{-1} = 0.03121,\quad W_q = \frac{L_q}{\lambda_{eff}} = \frac{1.15421}{15.42815} = 0.07481\,hr,$$

$$\lambda_{eff} = 16(1-0.03574) = 15.42815,\quad p_{10} = \begin{cases} \frac{\rho^n}{n!}(p_0), & 0 \le n < 4 \\ \frac{3.2^{10}}{4!4^{10-4}}(0.03121) = 0.03574, & 4 \le 10 \le N = 10 \end{cases}$$

$$W_s = 0.07481 + \frac{1}{5} = 0.27481\,hr,\qquad L_S = (0.27481)15.42815 = 4.23984\,calls,$$

$$\lambda_{lost} = \lambda\, p_N = 0.57184,\quad \bar{c} = L_s - L_q = 3.08562,\quad L_s = L_q + \rho_{eff} = 1.15421 + \frac{\lambda_{eff}}{5} = 4.23984$$

# Multi-Server finite-Size Poisson Queuing Situation M/M/c:GD/N /∞

QUEUEING OUTPUT ANALYSIS

Select Output Option: Scenario2

Next Iteration | All Iterations | Write to Printer

Title: example 15 6 6 page 588

Scenario 2:(M/M/4):(GD/10/infinity)

| | | | |
|---|---|---|---|
| Lambda = | 16.000 | Mu = | 5.000 |
| L'da eff = | 15.428 | Rho/c = | 0.800 |
| Ls = | 4.240 | Lq = | 1.154 |
| Ws = | 0.275 | Wq = | 0.075 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.031 | 0.031 | 6 | 0.087 | 0.794 |
| 1 | 0.100 | 0.131 | 7 | 0.070 | 0.864 |
| 2 | 0.160 | 0.291 | 8 | 0.056 | 0.920 |
| 3 | 0.170 | 0.461 | 9 | 0.045 | 0.964 |
| 4 | 0.136 | 0.598 | 10 | 0.036 | 1.000 |
| 5 | 0.109 | 0.707 | | | |

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

- A shop with **K** machines. When a machine breaks down, one of **R** repairpersons is called upon to do the repair.
- The rate of breakdown per machine is $\lambda$ breakdowns per unit time, and

$$\lambda_n = \begin{cases} (K-n)\lambda, & 0 \le n \le K \\ 0, & n \ge K \end{cases}$$

- A repairperson will service broken machines at the rate of $\mu$ machines per unit time.

$$\mu_n = \begin{cases} n\mu, & 0 \le n \le R \\ R\mu, & R \le n \le K \end{cases}$$

- All breakdowns and services follow the Poisson distribution.
- The source in this model is finite (**K**) because only machines in working order can break down. Once all machines are broken, no new calls for service can occur.
- In terms of the queuing model, having **n** machines in the system signifies that **n** machines are broken out of **K**.

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

$$p_n = \begin{cases} C_n^K \rho^n p_o, & o \le n \le R \\ C_n^K \dfrac{n!\rho^n}{R!R^{n-R}} p_o, & R \le n \le K \end{cases}$$

$$C_n^K = \frac{K!}{n!(K-n)!}$$

$$p_o = \left( \sum_{n=o}^{R} C_n^K \rho^n + \sum_{n=R+1}^{K} C_n^K \frac{n!\rho^n}{R!R^{n-R}} \right)^{-1}$$

$$L_s = \sum_{n=o}^{K} n p_n$$

$$\lambda_{eff} = \lambda(K - L_s)$$

Little's formula $Ws = \dfrac{L_s}{\lambda_{eff}}$

$Wq = Ws - \dfrac{1}{\mu}$, and Little's formula $Lq = \lambda_{eff} Wq$

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2023-04

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K



Toolco operates a machine shop with 22 machines. On the average, a machine breaks down every 2 hours. It takes an average of 12 minutes to complete a repair. When a machine breaks down, one of four repairpersons is called upon to do the repair. Both the time between breakdowns and the repair time are exponential. Toolco is interested in analyzing this situation and knowing the number of repairpersons needed to keep the shop running "smoothly."

**$\lambda = \frac{1}{2} = 0.5$ breaks down per hour**

**$\mu = 60/12 = 5$ breaks down per hour**

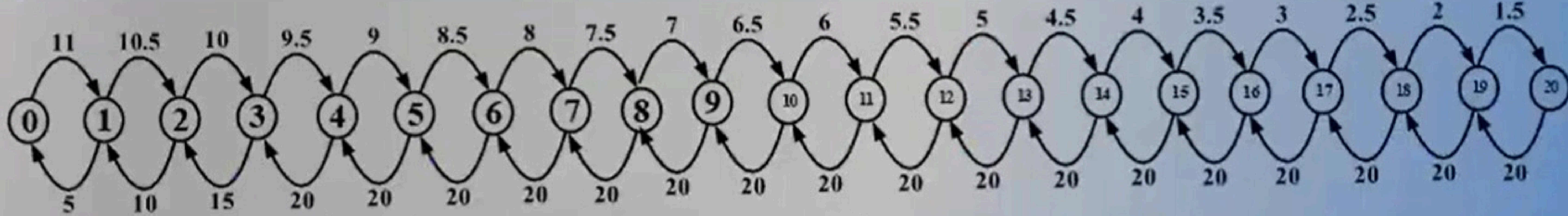
**System limit = K = 22 machines**

**Source limit = K = 22 machines**

**R = 1, 2, 3, 4.**

**(M/M/R: GD/ K /K),**

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K




When R = 4 $\lambda_n = \begin{cases} (22-n)0.5, & 0 \le n \le 22 \\ 0, & n \ge 22 \end{cases}$ , and $\mu_n = \begin{cases} 5n, & 0 \le n \le 4 \\ 4\mu, & 4 \le n \le 22 \end{cases}$

$P_0 = P_0$

$P_1 = \left(\frac{11}{5}\right)P_0$

$P_2 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)P_0$

$P_3 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)P_0$

$P_4 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)\left(\frac{9.5}{20}\right)P_0$ $\Rightarrow P_0 = 0.1199$

$P_5 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)\left(\frac{9.5}{20}\right)\left(\frac{9.0}{20}\right)P_0$

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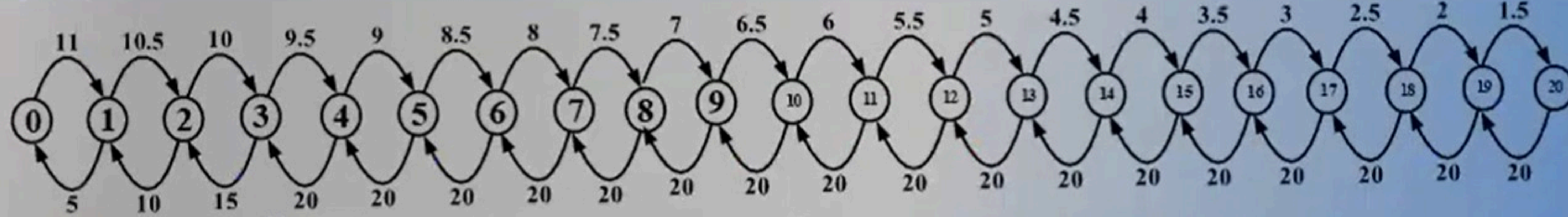
$P_{22} = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)\left(\frac{9.5}{20}\right)\left(\frac{9.0}{20}\right)\left(\frac{8.5}{20}\right)\cdots\cdots\left(\frac{2.0}{20}\right)\left(\frac{1.5}{20}\right)P_0$

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K



When R = 4 $\lambda_n = \begin{cases} (22-n)0.5, & 0 \le n \le 22 \\ 0, & n \ge 22 \end{cases}$, and $\mu_n = \begin{cases} 5n, & 0 \le n \le 4 \\ 4\mu, & 4 \le n \le 22 \end{cases}$

$$P_0 = P_0 = (0.1199)$$

$$P_1 = \left(\frac{11}{5}\right)(0.1199) = 0.2638$$

$$P_2 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)(0.1199) = 0.2770$$

$$P_3 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)(0.1199) = 0.1847$$

$$P_4 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)\left(\frac{9.5}{20}\right)(0.1199) = 0.0877$$

$$\sum_{n=0}^{K} np_n = \sum_{n=0}^{22} np_n = 1$$

$$P_5 = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)\left(\frac{9.5}{20}\right)\left(\frac{9.0}{20}\right)(0.1199) = 0.0395$$

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$$P_{22} = \left(\frac{11}{5}\right)\left(\frac{10.5}{10}\right)\left(\frac{10}{15}\right)\left(\frac{9.5}{20}\right)\left(\frac{9.0}{20}\right)\left(\frac{8.5}{20}\right)\cdots\cdots\left(\frac{2.0}{20}\right)\left(\frac{1.5}{20}\right)(0.1199) \approx 0.0000$$

Probabilistic Operations Research

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

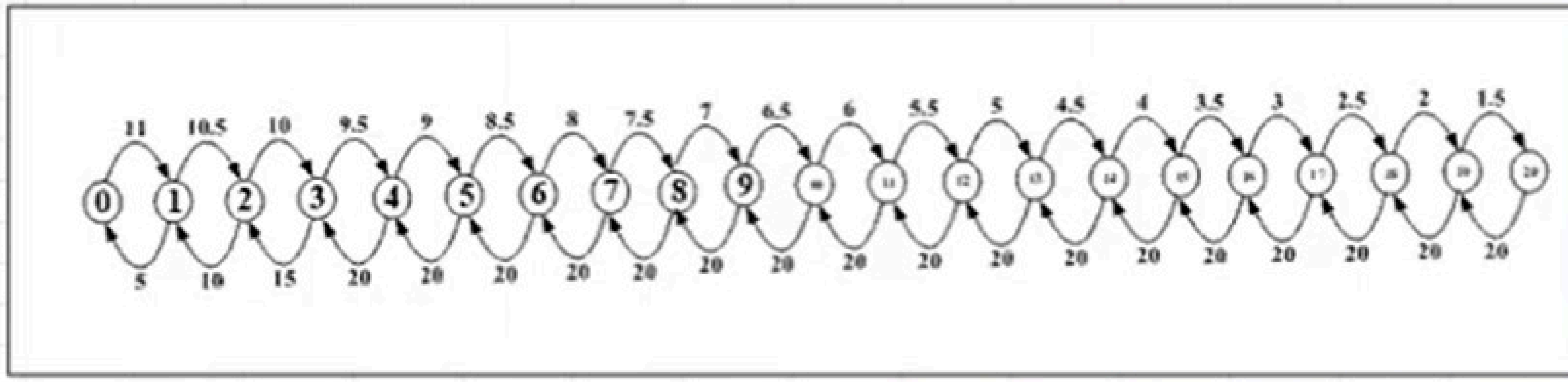
| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_n$ | 11 | 10.5 | 10 | 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 |
| $\mu_n$ | / | 5 | 10 | 15 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| $p_n$ | 0.1199 | 0.2638 | 0.2770 | 0.1847 | 0.0877 | 0.0395 | 0.0168 | 0.0067 | 0.0025 | 0.0009 | 0.0003 |
| | 1.0000 | 2.2000 | 2.3100 | 1.5400 | 0.7315 | 0.3292 | 0.1399 | 0.0560 | 0.0210 | 0.0073 | 0.0024 |
| # waiting | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $\lambda_{eff}$ | 9.9499 | | | | | | | | | | |
| $L_s$ | 2.1002 | | $W_s$ | 0.2111 | | | | | | | |
| Lq | 0.1102 | | Wq | 0.0111 | | | | | | | |

| n | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_n$ | 5.5 | 5 | 4.5 | 4 | 3.5 | 3 | 2.5 | 2 | 1.5 | | |
| $\mu_n$ | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | |
| $p_n$ | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | |
| | 0.0007 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.1199 |
| # waiting | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |

Toolco operates a machine shop with 22 machines. On the average, a machine breaks down every 2 hours. It takes an average of 12 minutes to complete a repair. When a machine breaks down, one of four repairpersons is called upon to do the repair. Both the time between breakdowns and the repair time are exponential. Toolco is interested in analyzing this situation and knowing the number of repairpersons needed to keep the shop running "smoothly."

$$\lambda_n = \begin{cases} (K-n)\lambda, & 0 \le n \le K \\ 0, & n \ge K \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 \le n \le R \\ R\mu, & R \le n \le K \end{cases}$$



*Excel file: Computing the steady state transition probabilities: MM R=4*

| c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|
| 4 | 0.500 | 5.00 | 9.9500 | 0.1199 | 2.1001 | 0.1102 | 0.2111 | 0.0111 |

Probabilistic Operations Research

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_n$ | 11 | 10.5 | 10 | 9.5 | 9 | 8.5 | 8 | 7.5 | 7 | 6.5 | 6 | 5.5 | 5 | 4.5 | 4 | 3.5 | 3 | 2.5 | 2 | 1.5 | | |
| $\mu_n$ | / | 5 | 10 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | |
| $p_n$ | 0.1078 | 0.2371 | 0.2490 | 0.1660 | 0.1051 | 0.0631 | 0.0357 | 0.0191 | 0.0095 | 0.0044 | 0.0019 | 0.0008 | 0.0003 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | |
| | 1.0000 | 2.2000 | 2.3100 | 1.5400 | 0.9753 | 0.5852 | 0.3316 | 0.1769 | 0.0884 | 0.0413 | 0.0179 | 0.0072 | 0.0026 | 0.0006 | 0.0003 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.1078 |
| # waiting | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | |

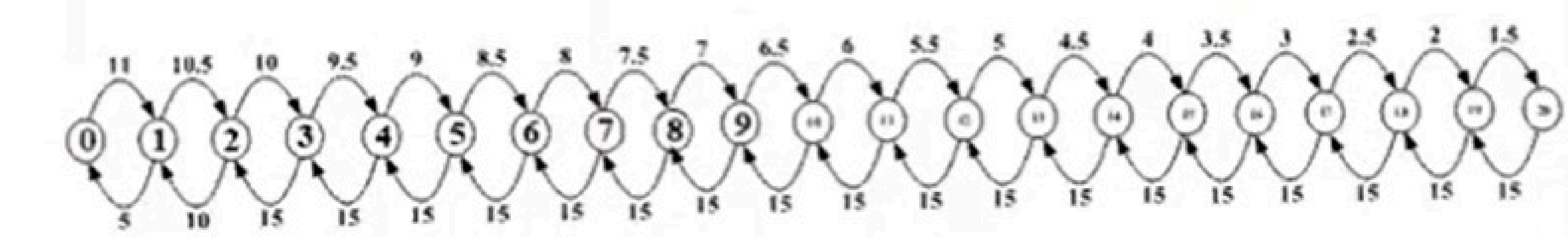
| | | | |
|---|---|---|---|
| $\lambda_{eff}$ | 9.7666 | | |
| $L_s$ | 2.4668 | $W_s$ | 0.2526 |
| Lq | 0.5132 | Wq | 0.0525 |

**R = 3**

Toolco operates a machine shop with 22 machines. On the average, a machine breaks down every 2 hours. It takes an average of 12 minutes to complete a repair. When a machine breaks down, one of four repairpersons is called upon to do the repair. Both the time between breakdowns and the repair time are exponential. Toolco is interested in analyzing this situation and knowing the number of repairpersons needed to keep the shop running "smoothly."

$$\lambda_n = \begin{cases} (K-n)\lambda, & 0 \le n \le K \\ 0, & n \ge K \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 \le n \le R \\ R\mu, & R \le n \le K \end{cases}$$



## *Excel file: Computing the steady state transition probabilities: MM R=3*

TORA comparative analysis output for Example 18.6-8 (file *toraEx18.6-8.txt*)

Comparative Analysis

| c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.500 | 5.00 | 4.9980 | 0.0004 | 12.0040 | 11.0044 | 2.4018 | 2.2018 |
| 2 | 0.500 | 5.00 | 8.8161 | 0.0564 | 4.3677 | 2.6045 | 0.4954 | 0.2954 |
| 3 | 0.500 | 5.00 | 9.7670 | 0.1078 | 2.4660 | 0.5128 | 0.2525 | 0.0525 |
| 4 | 0.500 | 5.00 | 9.9500 | 0.1199 | 2.1001 | 0.1102 | 0.2111 | 0.0111 |

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

**Comparative analysis**

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.50000 | 5.00000 | 4.99798 | 0.00040 | 12.00404 | 11.00444 | 2.40178 | 2.20178 |
| 2 | 2 | 0.50000 | 5.00000 | 8.81616 | 0.05638 | 4.36768 | 2.60447 | 0.49542 | 0.29542 |
| 3 | 3 | 0.50000 | 5.00000 | 9.76703 | 0.10779 | 2.46593 | 0.51257 | 0.25247 | 0.05248 |
| 4 | 4 | 0.50000 | 5.00000 | 9.94995 | 0.11993 | 2.10010 | 0.11015 | 0.21107 | 0.01107 |

$$\text{Machinesproductivity} = \frac{\text{Availablemachines- Brokenmachines}}{\text{Availablemachines}} \times 100 = \frac{22 - Ls}{22} \times 100$$

| Repairperson, *R* | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Machines productivity (100%) | 45.44 | 80.15 | 88.79 | 90.45 |
| Marginal increase (100%) | — | 34.71 | 8.64 | 1.66 |

Probabilistic Operations Research

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

For $R = 1, K = 22, n = 0, 1, 2, \ldots\ldots 22, \lambda = 0.5, \mu = 5, \Rightarrow \rho = 0.1$

$$p_0 = \left( \sum_{n=0}^{R} C_n^K \rho^n + \sum_{n=R+1}^{K} C_n^K \frac{n!\rho^n}{R!R^{n-R}} \right)^{-1} = \left( \sum_{n=0}^{1} C_n^{22} \rho^n + \sum_{n=2}^{22} C_n^{22} \frac{n!0.1^n}{1!1^{n-1}} \right)^{-1}$$

$$= \left( \left( C_0^{22}\rho^0 + C_1^{22}\rho^1 \right) + \left( C_0^{22} \frac{0!0.1^0}{1!1^{0-1}} + C_1^{22} \frac{1!0.1^1}{1!1^{1-1}} + C_2^{22} \frac{2!0.1^2}{1!1^{2-1}} + \ldots\ldots + C_{22}^{22} \frac{22!0.1^{22}}{1!1^{22-1}} \right) \right)^{-1}$$

$$= (3.2 + 2471.84)^{-1} = 0.0004$$

$$L_s = \sum_{n=0}^{K} n p_n$$

$$\lambda_{eff} = \lambda(K - L_s)$$



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Industrial Engineering Department

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

For $R = 1, K = 22, n = 0, 1, 2, \ldots\ldots 22, \lambda = 0.5, \mu = 5, \Rightarrow \rho = 0.1$

$p_0 = 0.0004$

$$p_n = \begin{cases} C_n^K \rho^n p_0, & 0 \le n \le R \\ C_n^K \dfrac{n!\rho^n}{R!R^{n-R}} p_0, & R \le n \le K \end{cases} \qquad C_n^K = \frac{K!}{n!(K-n)!}$$

$$p_1 = C_1^{22} 0.1^1 (0.0004) = 0.00004 \qquad C_1^{22} = \frac{22!}{1!(22-1)!} = 22$$

$$p_2 = C_2^{22} \frac{2!0.1^2}{1!1^{2-1}} (0.0004) = 0.0018 \qquad C_2^{22} = \frac{22!}{2!(22-2)!} = 231$$

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$$p_{22} = C_{22}^{22} \frac{22!0.1^{22}}{1!1^{22-1}} (0.0004) = 0.000045 \qquad C_{22}^{22} = \frac{22!}{22!(22-22)!} = 1$$

Probabilistic Operations Research

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K



For $R = 1, K = 22, n = 0, 1, 2, ......22, \lambda = 0.5, \mu = 5, \Rightarrow \rho = 0.1$

$p_0 = 0.0004, p_1 = 0.00004 \quad ..... \quad p22_0 = 0.000045$

$$L_s = \sum_{n=0}^{K} np_n = \sum_{n=0}^{22} 0p_0 + ....22p_{22} = 12.9984$$

$$\lambda_{eff} = \lambda(K - L_s) = 0.5(22 - 12.9984) = 4.9980$$

$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{12.9984}{4.9980} = 2.40$$

$$W_q = W_s - \frac{1}{\mu} = 2.40 - \frac{1}{5} = 2.2$$

$$L_q = W_q \lambda_{eff} = 2.2 \times 4.9980 = 11.0$$

# Machine Servicing Model—(M/M/R):(GD/K/K), R < K

Toolco operates a machine shop with 22 machines. On the average, a machine breaks down every 2 hours. It takes an average of 12 minutes to complete a repair. When a machine breaks down, one of four repairpersons is called upon to do the repair. Both the time between breakdowns and the repair time are exponential. Toolco is interested in analyzing this situation and knowing the number of repairpersons needed to keep the shop running "smoothly."

| # | c | λ | μ | n | K | $C_n^K$ | ρ | $\rho^n$ | R | $R^{n-R}$ | $C_n^K \rho^n$ | $C_n^K(n!\rho^n/R!R^{n-R})$ | p0 | $p_n$ | $L_s$ | λeff | Ws | Wq | Lq |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0.5 | 5 | 0 | 22 | 1 | 0.10 | 1.0000 | 1.00 | 1.00 | 1.00 | | 0.0004 | 0.0004040 | 12.0032 | 4.9984 | 2.4014 | 2.2014 | 11.0035 |
| | | | | 1 | | 22 | | 0.1000 | | 1.00 | 2.20 | | | 0.0000404 | | | | | |
| | | | | 2 | | 231 | | 0.0100 | | 1.00 | | 4.62000 | | 0.0018666 | | | | | |
| | | | | 3 | | 1540 | | 0.0010 | | 1.00 | | 9.24000 | | 0.0037333 | | | | | |
| | | | | 4 | | 7315 | | 0.0001 | | 1.00 | | 17.55600 | | 0.0070932 | | | | | |
| | | | | 5 | | 26334 | | 0.0000 | | 1.00 | | 31.60080 | | 0.0127678 | | | | | |
| | | | | 6 | | 74613 | | 0.0000 | | 1.00 | | 53.72136 | | 0.0217052 | | | | | |
| | | | | 7 | | 170544 | | 0.0000 | | 1.00 | | 85.95418 | | 0.0347283 | | | | | |
| | | | | 8 | | 319770 | | 0.0000 | | 1.00 | | 128.93126 | | 0.0520925 | | | | | |
| | | | | 9 | | 497420 | | 0.0000 | | 1.00 | | 180.50377 | | 0.0729295 | | | | | |
| | | | | 10 | | 646646 | | 0.0000 | | 1.00 | | 234.65490 | | 0.0948084 | | | | | |
| | | | | 11 | | 705432 | | 0.0000 | | 1.00 | | 281.58588 | | 0.1137700 | | | | | |
| | | | | 12 | | 646646 | | 0.0000 | | 1.00 | | 309.74447 | | 0.1251470 | | | | | |
| | | | | 13 | | 497420 | | 0.0000 | | 1.00 | | 309.74447 | | 0.1251470 | | | | | |
| | | | | 14 | | 319770 | | 0.0000 | | 1.00 | | 278.77002 | | 0.1126323 | | | | | |
| | | | | 15 | | 170544 | | 0.0000 | | 1.00 | | 223.01602 | | 0.0901059 | | | | | |
| | | | | 16 | | 74613 | | 0.0000 | | 1.00 | | 156.11121 | | 0.0630741 | | | | | |
| | | | | 17 | | 26334 | | 0.0000 | | 1.00 | | 93.66673 | | 0.0378445 | | | | | |
| | | | | 18 | | 7315 | | 0.0000 | | 1.00 | | 46.83336 | | 0.0189222 | | | | | |
| | | | | 19 | | 1540 | | 0.0000 | | 1.00 | | 18.73335 | | 0.0075689 | | | | | |
| | | | | 20 | | 231 | | 0.0000 | | 1.00 | | 5.62000 | | 0.0022707 | | | | | |
| | | | | 21 | | 22 | | 0.0000 | | 1.00 | | 1.12400 | | 0.0004541 | | | | | |
| | | | | 22 | | 1 | | 0.0000 | | 1.00 | | 0.11240 | | 0.0000454 | | | | | |
| Sum | | | | | | | | | | | 3.20 | 2471.84 | | | | | | | |

When changing the value of R in cell O5, it is necessary to adjust the calculations for column R, column S, and column T

*Excel file: Queuing Questions: M/M/R: GD/K/K*

Probabilistic Operations Research