# Introduction to Operations Research

- Operations research is the area of applying the analytical methods to help in finding an optimal decision (or solution) when solving complex problem. not only a solution (it has to be the best one)

- Operations research problems (models) can be classified into:
  - ✓Deterministic model: All the parameters are known with certainty.
  - ✓Probabilistic model: The occurrence of specific event cannot be perfectly predicted.

we will take this only



# Continue…

- Operations research problems (models) can be classified into:
  - ✓Linear model: All the relationships are linear relationships.

    Example: The relationship between the profit (P) and the number of chairs ($x_1$) and tables ($x_2$) is

$$P = 5 \times x_1 + 8 \times x_2$$

  - ✓Nonlinear model: At least one of the relationships is nonlinear relationship.

    Example: The relationship between the strength of an explosion (E) and the amounts of material A ($x_1$) and material B ($x_2$) is

$$E = 4 \times x_1^3 + 2 \times x_2^2$$

**Graphical method**: the easiest method to solve system contains two variables

## 3.1 PROTOTYPE EXAMPLE

- our target is to find a optimal solution not a solution only

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing
Product 2: A 4 × 6 foot double-hung wood-framed window

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which *mix* of the two products would be *most profitable*. Therefore, an OR team has been formed to study this question.

The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem:

> Determine what the *production rates* should be for the two products in order to *maximize their total profit*, subject to the restrictions imposed by the limited production capacities available in the three plants. (Each product will be produced in batches of 20, so the *production rate* is defined as the number of batches produced per week.) *Any* combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

- the decision variables (course of action) — number of batches
  - 1- determining the production rate for the first product ($x_1$)
  - 2- determining the production rate for the second product ($x_2$)
- the objective:
  - maximize the total profit
  - z
- the constraints (3)
  - limited production capacities available in the three plants — for each plant
- unit rate of production
  - number of batches (each batch has 20 units)

unit rate of production — unit time

- not enogh data
  - i need expert
- extra data
  - i don't know what to do with it

The OR team also identified the data that needed to be gathered:

1. Number of hours of production time available per week in each plant for these new products. (Most of the time in these plants already is committed to current products, so the available capacity for the new products is quite limited.)
2. Number of hours of production time used in each plant for each batch produced of each new product.
3. Profit per batch produced of each new product. (*Profit per batch produced* was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly *constant* regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products, the total profit from each one is approximately this *profit per batch produced* times *the number of batches produced*.)

Obtaining reasonable estimates of these quantities required enlisting the help of key personnel in various units of the company. Staff in the manufacturing division provided the data in the first category above. Developing estimates for the second category of data required some analysis by the manufacturing engineers involved in designing the production processes for the new products. By analyzing cost data from these same engineers and the marketing division, along with a pricing decision from the marketing division, the accounting department developed estimates for the third category.

Table 3.1 summarizes the data gathered.

The OR team immediately recognized that this was a linear programming problem of the classic **product mix** type, and the team next undertook the formulation of the corresponding mathematical model.

**TABLE 3.1** Data for the Wyndor Glass Co. problem

| | Production Time per Batch, Hours (Capacities) | | |
|---|---|---|---|
| | Product | | Production Time Available per Week, Hours (Capacities / Constraints) |
| Plant | 1 | 2 | |
| 1 | 1 hour per one batch | 0 hour | 4 |
| 2 | 0 hour | 2 hour | 12 |
| 3 | 3 hour | 2 hour | 18 |
| Profit per batch | $3,000 | $5,000 | |

- Write the mathematical model

DV:
- $x_1$: number of batches of product 1 that need to be produced per week
- $x_2$: number of batches of product 2 that need to be produced per week

Obj: max profit
- max $z = 3000x_1 + 5000x_2$

Subject to:
- $1x_1 + 0x_2 <= 4$
- $0x_1 + 2x_2 <= 12$
- $3x_1 + 2x_2 <= 18$
- $x_1 >= 0$, $x_2 >= 0$ } non negativity constraints

**Formulation as a Linear Programming Problem**

The definition of the problem given above indicates that the decisions to be made are the

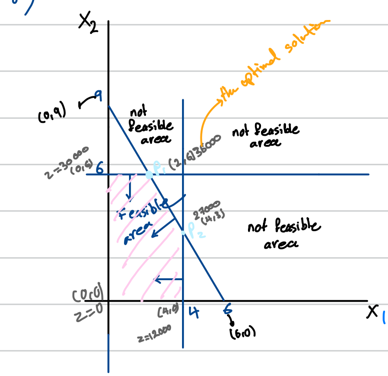
$1x_1 + 0x_2 <= 4 \longrightarrow (x_1 <= 4) \dots x_1 = 4$

$0x_1 + 2x_2 <= 12 \longrightarrow (2x_2 <= 12) \dots 2x_2 = 12$

$3x_1 + 2x_2 <= 18 \longrightarrow (3x_1 + 2x_2 = 18) \longrightarrow$ line $\longrightarrow$
1) slope & intercept
or
2) two points $\longrightarrow$ assume $x_1 = 0 \longrightarrow (0,9)$; $x_2 = 0 \longrightarrow (6,0)$

To know the direction
- بنوخذ point فوق الـ line اذا تحقق يكون لفوق اذا لا يكون لتحت

First quarter only: $x_1 >= 0$, $x_2 >= 0$ } non negativity constraints




complex space → اذا كان في any two points ووصلت بينهم في خط كل الـ points اللي على الخط بيكونوا في الـ space نفسه

convex area → اي تقاطع بوطلع بعض جوا الـ area

يعني بين P2 و بجيب نقطة ثانية بعديها

the optimal solution (the most preferable solution) $\longrightarrow$ one of the corners (because it is convex area)

1) find the corners points
- intersection points: $P_1$: $2x_2 = 12 \rightarrow x_2 = 6$; $3x_1 + 2x_2 = 18 \rightarrow x_1 = 2$ $\longrightarrow (2,6)$
- $P_2$: $x_1 = 4$; $3x_1 + 2x_2 = 18 \rightarrow x_2 = 3$ $\longrightarrow (4,3)$

2) max profit $= z = 3000x_1 + 5000x_2$

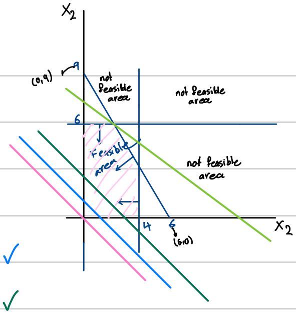
- The number of points in the feasible area is ∞

the Book solution: (it is trail & error)

$Z = 3X_1 + 5X_2$   in thousands

$3X_1 + 5X_2 = 0$

two points  $(0,0), (1, \frac{-1}{5})$




$3X_1 + 5X_2 = 10$  ( within the feasible area) ✓

$3X_1 + 5X_2 = 20$  ( within the feasible area) ✓

$3X_1 + 5X_2 = 36$  ( within the feasible area) ✓ اخر نقطة تقاطع بين ال feasible area والـ objectiv func هي الي بتكون ال optimal point

**3.1-7.** The Whitt Window Company, a company with only three employees, makes two different kinds of hand-crafted windows: a wood-framed and an aluminum-framed window. The company earns \$300 profit for each wood-framed window and \$150 profit for each aluminum-framed window. Doug makes the wood frames and can make 6 per day. Linda makes the aluminum frames and can make 4 per day. Bob forms and cuts the glass and can make 48 square feet of glass per day. Each wood-framed window uses 6 square feet of glass and each aluminum-framed window uses 8 square feet of glass.

The company wishes to determine how many windows of each type to produce per day to maximize total profit.

**(a)** Describe the analogy between this problem and the Wyndor Glass Co. problem discussed in Sec. 3.1. Then construct and fill in a table like Table 3.1 for this problem, identifying both the activities and the resources.

**(b)** Formulate a linear programming model for this problem.

D,I **(c)** Use the graphical method to solve this model.

I **(d)** A new competitor in town has started making wood-framed windows as well. This may force the company to lower the price they charge and so lower the profit made for each wood-framed window. How would the optimal solution change (if at all) if the profit per wood-framed window decreases from \$300 to \$200? From \$300 to 100? (You may find it helpful to use the Graphical Analysis and Sensitivity Analysis procedure in IOR Tutorial.)

I **(e)** Doug is considering lowering his working hours, which would decrease the number of wood frames he makes per day. How would the optimal solution change if he makes only 5 wood frames per day? (You may find it helpful to use the Graphical Analysis and Sensitivity Analysis procedure in IOR Tutorial.)

- decision variables → 1) number of wood windows to be produsèd per day $(X_1)$

2) number of aluminum windows to be produsèd per day $(X_2)$

- the objective → maximize the total profit

$Z = 300X_1 + 150X_2$

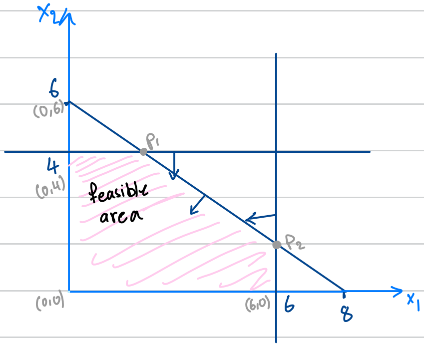
- the constants (subject to) → $X_1 \leq 6$

$X_2 \leq 4$

$6X_1 + 8X_2 \leq 48$ → line → $(0,6)$ $(8,0)$

$X_1 \geq 0$

$X_2 \geq 0$




at $P_1$ : $X_2 = 4$ & $6X_1 + 8X_2 = 48$ → $(2.7, 4)$

at $P_2$ : $X_1 = 6$ & $6X_1 + 8X_2 = 48$ → $(6, 1.5)$

## Design of Radiation Therapy

MARY has just been diagnosed as having a cancer at a fairly advanced stage. Specifically, she has a large malignant tumor in the bladder area (a "whole bladder lesion").

Mary is to receive the most advanced medical care available to give her every possible chance for survival. This care will include extensive *radiation therapy.*

Radiation therapy involves using an external beam treatment machine to pass ionizing radiation through the patient's body, damaging both cancerous and healthy tissues. Normally, several beams are precisely administered from different angles in a two-dimensional plane. Due to attenuation, each beam delivers more radiation to the tissue near the entry point than to the tissue near the exit point. Scatter also causes some delivery of radiation to tissue outside the direct path of the beam. Because tumor cells are typically microscopically interspersed among healthy cells, the radiation dosage throughout the tumor region must be large enough to kill the malignant cells, which are slightly more radiosensitive, yet small enough to spare the healthy cells. At the same time, the aggregate dose to critical tissues must not exceed established tolerance levels, in order to prevent complications that can be more serious than the disease itself. For the same reason, the total dose to the entire healthy anatomy must be minimized.

Because of the need to carefully balance all these factors, the design of radiation therapy is a very delicate process. The goal of the design is to select the combination of beams to be used, and the intensity of each one, to generate the best possible dose distribution. (The dose strength at any point in the body is measured in units called *kilorads.*) Once the treatment design has been developed, it is administered in many installments, spread over several weeks.

In Mary's case, the size and location of her tumor make the design of her treatment an even more delicate process than usual. Figure 3.11 shows a diagram of a cross section of the tumor viewed from above, as well as nearby critical tissues to avoid. These tissues include critical organs (e.g., the rectum) as well as bony structures (e.g., the femurs and pelvis) that will attenuate the radiation. Also shown are the entry point and direction for the only two beams that can be used with any modicum of safety in this case. (Actually, we are simplifying the example at this point, because normally dozens of possible beams must be considered.)

For any proposed beam of given intensity, the analysis of what the resulting radiation absorption by various parts of the body would be requires a complicated process. In brief, based on careful anatomical analysis, the energy distribution within the two-dimensional cross section of the tissue can be plotted on an isodose map, where the contour lines represent the dose strength as a percentage of the dose strength at the entry point. A fine grid then is placed over the isodose map. By summing the radiation absorbed in the squares containing each type of tissue, the average dose that is absorbed by the tumor, healthy anatomy, and critical tissues can be calculated. With more than one beam (administered sequentially), the radiation absorption is additive.

After thorough analysis of this type, the medical team has carefully estimated the data needed to design Mary's treatment, as summarized in Table 3.7. The first column lists the areas of the body that must be considered, and then the next two columns give the fraction of the radiation dose at the entry point for each beam that is absorbed by the respective areas on average. For example, if the dose level at the entry point for beam 1 is 1 kilorad, then an average of 0.4 kilorad will be absorbed by the entire healthy anatomy in the two-dimensional plane, an average of 0.3 kilorad will be absorbed by nearby critical tissues, an average of 0.5 kilorad will be absorbed by the various parts of the tumor, and 0.6 kilorad will be absorbed by the center of the tumor. The last column gives the restrictions on the total dosage from both beams that is absorbed on average by the respective areas of the body. In particular, the average dosage absorption for the healthy anatomy must be *as small as possible,* the critical tissues must *not exceed* 2.7 kilorads, the average over the entire tumor must *equal* 6 kilorads, and the center of the tumor must be *at least* 6 kilorads.

**Formulation as a Linear Programming Problem.** The decisions that need to be made are the dosages of radiation at the two entry points. Therefore, the two decision variables $x_1$ and $x_2$ represent the dose (in kilorads) at the entry point for beam 1 and beam 2, respectively. Because the total dosage reaching the healthy anatomy is to be

**TABLE 3.7** Data for the design of Mary's radiation therapy

| Area | Fraction of Entry Dose Absorbed by Area (Average) | | Restriction on Total Average Dosage, Kilorads |
|---|---|---|---|
| | Beam 1 | Beam 2 | |
| Healthy anatomy | 0.4 | 0.5 | Minimize |
| Critical tissues | 0.3 | 0.1 | ≤ 2.7 |
| Tumor region | 0.5 | 0.5 | = 6 |
| Center of tumor | 0.6 | 0.4 | ≥ 6 |

feasible area ⟶ line here

- decision variables

$x_1$ : dose in Kilorad from the first beam

$x_2$ : dose in Kilorad from the Second beam

- objective

$z = 0.4x_1 + 0.5x_2$

minimize the total absorbed by healthy anatomy

$Z = 0.4x_1 + 0.5x_2$

- Subject to C constrains

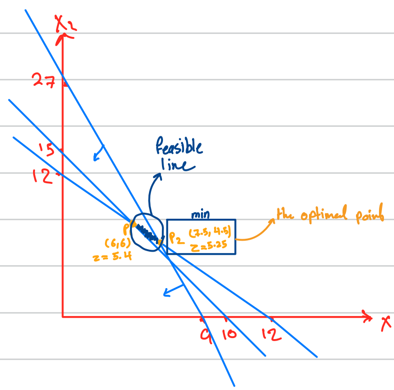
$0.3x_1 + 0.1x_2 \le 2.7 \longrightarrow 0.3x_1 + 0.1x_2 = 2.7 \quad (0,27),(9,0)$

$0.5x_1 + 0.5x_2 = 6 \longrightarrow$ the feasible line (not area)

$0.6x_1 + 0.4x_2 \ge 6 \longrightarrow 0.6x_1 + 0.4x_2 = 6 \quad (0,15),(10,0)$

$x_1 \ge 0$

$x_2 \ge 0$




$P_1 : \left.\begin{array}{l} 0.5x_1 + 0.5x_2 = 6 \\ 0.6x_1 + 0.4x_2 = 6 \end{array}\right\} (6,6)$

$P_2 : \left.\begin{array}{l} 0.5x_1 + 0.5x_2 = 6 \\ 0.3x_1 + 0.1x_2 = 2.7 \end{array}\right\} (7.5, 4.5)$

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OR1 assignment 1

**3.1-9.** The Primo Insurance Company is introducing two new product lines: special risk insurance and mortgages. The expected profit is \$5 per unit on special risk insurance and \$2 per unit on mortgages.

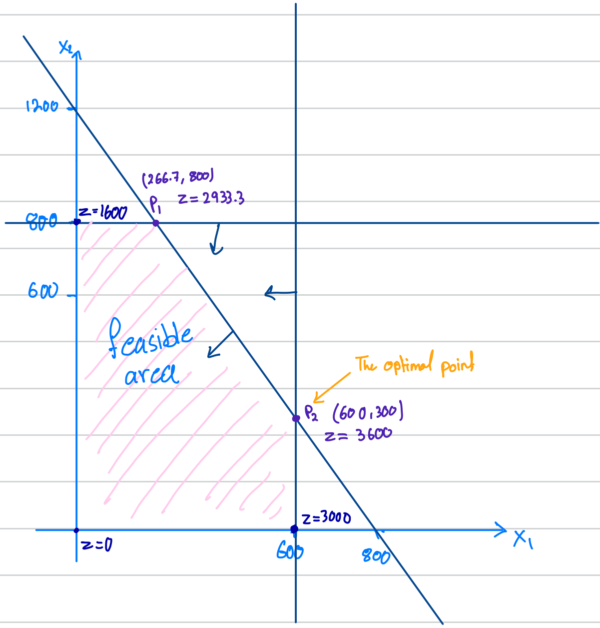
Management wishes to establish sales quotas for the new product lines to maximize total expected profit. The work requirements are as follows:

| Department | Work-Hours per Unit | | Work-Hours Available |
|---|---|---|---|
| | Special Risk | Mortgage | |
| Underwriting | 3 | 2 | 2400 |
| Administration | 0 | 1 | 800 |
| Claims | 2 | 0 | 1200 |

**(a)** Formulate a linear programming model for this problem.

D,I **(b)** Use the graphical method to solve this model.

**(c)** Verify the exact value of your optimal solution from part (*b*) by solving algebraically for the simultaneous solution of the relevant two equations.




- decision variables → 1) number of units from Special risk ($X_1$)

2) number of units from mortgage ($X_2$)

- the objective → maximize the total profit

$$Z = 5X_1 + 2X_2$$

- the constants (subject to): →

$3X_1 + 2X_2 \leqslant 2400 \longrightarrow 3X_1 + 2X_2 = 2400 \rightarrow (800, 0), (0, 1200)$

$0X_1 + 1X_2 \leqslant 800 \longrightarrow X_2 = 800$

$2X_1 + 0X_2 \leqslant 1200 \longrightarrow X_1 = 600$

$X_1 \geqslant 0$

$X_2 \geqslant 0$

- at $P_1$:

$\left.\begin{aligned} X_2 &= 800 \\ 3X_1 + 2X_2 &= 2400 \end{aligned}\right\} (266.7, 0)$

- at $P_2$:

$\left.\begin{aligned} X_1 &= 600 \\ 3X_1 + 2X_2 &= 2400 \end{aligned}\right\} (600, 300)$

→ The optimal point is (600, 300)

the maximum profit → 3600

**3.1-10.** Weenies and Buns is a food processing plant which manufactures hot dogs and hot dog buns. They grind their own flour for the hot dog buns at a maximum rate of 200 pounds per week. Each hot dog bun requires 0.1 pound of flour. They currently have a contract with Pigland, Inc., which specifies that a delivery of 800 pounds of pork product is delivered every Monday. Each hot dog requires $\frac{1}{4}$ pound of pork product. All the other ingredients in the hot dogs and hot dog buns are in plentiful supply. Finally, the labor force at Weenies and Buns consists of 5 employees working full time (40 hours per week each). Each hot dog requires 3 minutes of labor, and each hot dog bun requires 2 minutes of labor. Each hot dog yields a profit of \$0.88, and each bun yields a profit of \$0.33.

Weenies and Buns would like to know how many hot dogs and how many hot dog buns they should produce each week so as to achieve the highest possible profit.

**(a)** Formulate a linear programming model for this problem.

D,I **(b)** Use the graphical method to solve this model.

- decision variables → 1) number of hot dogs ($x_1$)

2) number of hot dog buns ($x_2$)

- the objective → maximize the total profit

$z = 0.88x_1 + 0.33x_2$

. the constants (subject to): →

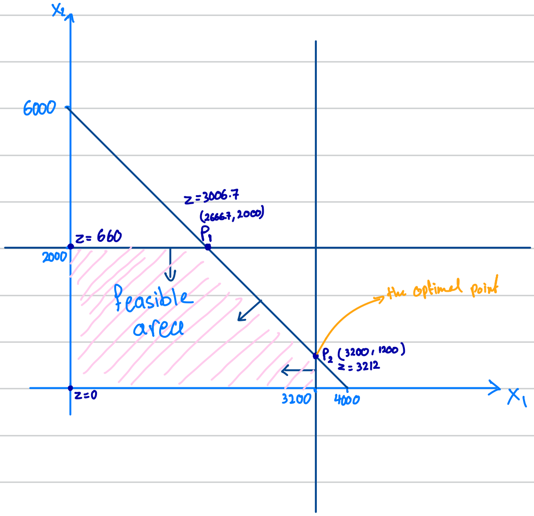
$0.25X_1 \leq 800 \longrightarrow X_1 = 3200$

$0.1X_2 \leq 200 \longrightarrow X_2 = 2000$

$3X_1 + 2X_2 \leq 12000 \longrightarrow 3X_1 + 2X_2 = 12000$ $(0, 6000)$, $(4000, 0)$

$X_1 \geq 0$

$X_2 \geq 0$




- at $P_1$:

$\left.\begin{array}{l} X_2 = 2000 \\ 3X_1 + 2X_2 = 12000 \end{array}\right\} (2666.7, 2000)$

- at $P_2$:

$\left.\begin{array}{l} X_1 = 3200 \\ 3X_1 + 2X_2 = 12000 \end{array}\right\} (3200, 1200)$

→ The optimal point is (3200, 1200)

. the maximum profit → 3212

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**3.2-3.*** This is your lucky day. You have just won a \$20,000 prize. You are setting aside \$8,000 for taxes and partying expenses, but you have decided to invest the other \$12,000. Upon hearing this news, two different friends have offered you an opportunity to become a partner in two different entrepreneurial ventures, one planned by each friend. In both cases, this investment would involve expending some of your time next summer as well as putting up cash. Becoming a *full* partner in the first friend's venture would require an investment of \$10,000 and 400 hours, and your estimated profit (ignoring the value of your time) would be \$9,000. The corresponding figures for the second friend's venture are \$8,000 and 500 hours, with an estimated profit to you of \$9,000. However, both friends are flexible and would allow you to come in at any *fraction* of a full partnership you would like. If you choose a fraction of a full partnership, all the above figures given for a full partnership (money investment, time investment, and your profit) would be multiplied by this same fraction.

Because you were looking for an interesting summer job anyway (maximum of 600 hours), you have decided to participate in one or both friends' ventures in whichever combination would maximize your total estimated profit. You now need to solve the problem of finding the best combination.

**(a)** Describe the analogy between this problem and the Wyndor Glass Co. problem discussed in Sec. 3.1. Then construct and fill in a table like Table 3.1 for this problem, identifying both the activities and the resources.

**(b)** Formulate a linear programming model for this problem.

D,I **(c)** Use the graphical method to solve this model. What is your total estimated profit?

- decision variables → 1) fraction of full partnership in the first venture ($X_1$)

2) fraction of full partnership in the second venture ($X_2$)

- the objective → maximize the total profit

$Z = 9000X_1 + 9000X_2$

- the constants (subject to): →

$10000X_1 + 8000X_2 \leq 12000 \longrightarrow 10000X_1 + 8000X_2 = 12000 \quad (0, 1.5), (1.2, 0)$

$400X_1 + 500X_2 \leq 600 \longrightarrow 400X_1 + 500X_2 = 600 \quad (0, 1.2), (1.5, 0)$

$X_1 \geq 0$

$X_2 \geq 0$

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### Reclaiming Solid Wastes

The SAVE-IT COMPANY operates a reclamation center that collects four types of solid waste materials and treats them so that they can be amalgamated into a salable product. (Treating and amalgamating are separate processes.) Three different grades of this product can be made (see the first column of Table 3.16), depending upon the mix of the materials used. Although there is some flexibility in the mix for each grade, quality standards may specify the minimum or maximum amount allowed for the proportion of a material in the

[7] An equivalent formulation can express each decision variable in natural units for its abatement method; for example, $x_1$ and $x_2$ could represent the number of *feet* that the heights of the smokestacks are increased.

**TABLE 3.16** Product data for Save-It Co.

| Grade | Specification | Amalgamation Cost per Pound ($) | Selling Price per Pound ($) |
|---|---|---|---|
| *A* | Material 1: Not more than 30% of total<br>Material 2: Not less than 40% of total<br>Material 3: Not more than 50% of total<br>Material 4: Exactly 20% of total | 3.00 | 8.50 |
| *B* | Material 1: Not more than 50% of total<br>Material 2: Not less than 10% of total<br>Material 4: Exactly 10% of total | 2.50 | 7.00 |
| *C* | Material 1: Not more than 70% of total | 2.00 | 5.50 |

product grade. (This proportion is the weight of the material expressed as a percentage of the total weight for the product grade.) For each of the two higher grades, a fixed percentage is specified for one of the materials. These specifications are given in Table 3.16 along with the cost of amalgamation and the selling price for each grade.

The reclamation center collects its solid waste materials from regular sources and so is normally able to maintain a steady rate for treating them. Table 3.17 gives the quantities available for collection and treatment each week, as well as the cost of treatment, for each type of material.

The Save-It Co. is solely owned by Green Earth, an organization devoted to dealing with environmental issues, so Save-It's profits are used to help support Green Earth's activities. Green Earth has raised contributions and grants, amounting to $30,000 per week, to be used exclusively to cover the entire treatment cost for the solid waste materials. The board of directors of Green Earth has instructed the management of Save-It to divide this money among the materials in such a way that *at least half* of the amount available of each material is actually collected and treated. These additional restrictions are listed in Table 3.17.

Within the restrictions specified in Tables 3.16 and 3.17, management wants to determine the *amount* of each product grade to produce *and* the exact *mix* of materials to be used for each grade. The objective is to maximize the net weekly profit (total sales income *minus* total amalgamation cost), exclusive of the fixed treatment cost of $30,000 per week that is being covered by gifts and grants.

**Formulation as a Linear Programming Problem.** Before attempting to construct a linear programming model, we must give careful consideration to the proper definition of the decision variables. Although this definition is often obvious, it sometimes becomes the crux of the entire formulation. After clearly identifying what information is really desired and the most convenient form for conveying this information by means of decision variables, we can develop the objective function and the constraints on the values of these decision variables.

**TABLE 3.17** Solid waste materials data for the Save-It Co.

| Material | Pounds per Week Available | Treatment Cost per Pound ($) | Additional Restrictions |
|---|---|---|---|
| 1 | 3,000 | 3.00 | 1. For each material, at least half of the pounds per week available should be collected and treated. |
| 2 | 2,000 | 6.00 | 2. $30,000 per week should be used to treat these materials. |
| 3 | 4,000 | 4.00 | |
| 4 | 1,000 | 5.00 | |

- decision variables → 1) the amount in pounds per week of material 1 that should be used to produce grade A ($x_{A1}$)

$x_{A2}, x_{A3}, x_{A4}$

2) the amount in pounds per week of material 1 that should be used to produce grade B ($x_{B1}$)

$x_{B2}, x_{B3}, x_{B4}$

3) the amount in pounds per week of material 1 that should be used to produce grade C ($x_{C1}$)

$x_{C2}, x_{C3}, x_{C4}$

- the objective → maximize the total profit (selling price − amalgamation cost)

$$Z = (8.5-3)(x_{A1}+x_{A2}+x_{A3}+x_{A4}) + (7-2.5)(x_{B1}+x_{B2}+x_{B3}+x_{B4}) + (5.5-2)(x_{C1}+x_{C2}+x_{C3}+x_{C4})$$

- the constants (subject to): →

$$x_{A1} \le 0.3(x_{A1}+x_{A2}+x_{A3}+x_{A4})$$
$$x_{A2} \ge 0.4(x_{A1}+x_{A2}+x_{A3}+x_{A4})$$
$$x_{A3} \le 0.5(x_{A1}+x_{A2}+x_{A3}+x_{A4})$$
$$x_{A4} = 0.2(x_{A1}+x_{A2}+x_{A3}+x_{A4})$$

$$x_{B1} \le 0.5(x_{B1}+x_{B2}+x_{B3}+x_{B4})$$
$$x_{B2} \ge 0.1(x_{B1}+x_{B2}+x_{B3}+x_{B4})$$
$$x_{B4} = 0.5(x_{B1}+x_{B2}+x_{B3}+x_{B4})$$

$$x_{C1} \le 0.7(x_{C1}+x_{C2}+x_{C3}+x_{C4})$$

table 3.17

$$x_{A1}+x_{B1}+x_{C1} \le 3000$$
$$x_{A1}+x_{B1}+x_{C1} \le 3000$$
$$x_{A1}+x_{B1}+x_{C1} \ge 1500$$
$$x_{A2}+x_{B2}+x_{C2} \le 200$$
$$x_{A2}+x_{B2}+x_{C2} \ge 1000$$
$$x_{A3}+x_{B3}+x_{C3} \le 4000$$
$$x_{A3}+x_{B3}+x_{C3} \ge 2000$$
$$x_{A4}+x_{B4}+x_{C4} \le 1000$$
$$x_{A4}+x_{B4}+x_{C4} \ge 500$$

$$3(x_{A1}+x_{B1}+x_{C1}) + 6(x_{A2}+x_{B2}+x_{C2}) + 4(x_{A3}+x_{B3}+x_{C3}) + 5(x_{A4}+x_{B4}+x_{C4}) = 30000$$

All $x$s $\ge 0$

**3.4-11.*** The Medequip Company produces precision medical diagnostic equipment at two factories. Three medical centers have placed orders for this month's production output. The table below shows what the cost would be for shipping each unit from each factory to each of these customers. Also shown are the number of units that will be produced at each factory and the number of units ordered by each customer.

| From \ To | Unit Shipping Cost for each unit | | | |
|---|---|---|---|---|
| | **Customer 1** | **Customer 2** | **Customer 3** | **Output** |
| Factory 1 | \$600 | \$800 | \$700 | ≤ 400 units |
| Factory 2 | \$400 | \$900 | \$600 | ≤ 500 units |
| Order size | exactly 300 units | exactly 200 units | exactly 400 units | |

A decision now needs to be made about the shipping plan for how many units to ship from each factory to each customer.

**(a)** Formulate a linear programming model for this problem.

C **(b)** Solve this model by the simplex method.

- decision variables → 1) number of units to be shipped from factory 1 to customer 1 ($X_{11}$)

$X_{ij}$ = number of units to be shipped from factory i to customer j

- the objective → minimize the ship

$Z = 600X_{11} + 800X_{12} + 700X_{13} + 400X_{21} + 900X_{22} + 600X_{23}$

the constants (subject to): →

$X_{11} + X_{21} = 300$

$X_{12} + X_{22} = 200$

$X_{13} + X_{23} = 400$

$X_{11} + X_{12} + X_{13} \leq 400$

$X_{21} + X_{22} + X_{23} \leq 500$

all $x_{ij} \geq 0$

**3.4-14*** A cargo plane has three compartments for storing cargo: front, center, and back. These compartments have capacity limits on both *weight* and *space,* as summarized below:

| Compartment | Weight Capacity (Tons) | Space Capacity (Cubic Feet) |
|---|---|---|
| Front | 12 | 7,000 |
| Center | 18 | 9,000 |
| Back | 10 | 5,000 |

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the airplane.

The following four cargoes have been offered for shipment on an upcoming flight as space is available:

| Cargo | Weight (Tons) | Volume (Cubic Feet/Ton) | Profit ($/Ton) |
|---|---|---|---|
| 1 | 20 | 500 | 320 |
| 2 | 16 | 700 | 400 |
| 3 | 25 | 600 | 360 |
| 4 | 13 | 400 | 290 |

it means that you can take any continuous number in between
(not only integers)

Any portion of these cargoes can be accepted. The objective is to determine how much (if any) of each cargo should be accepted and how to distribute each among the compartments to maximize the total profit for the flight.

**(a)** Formulate a linear programming model for this problem.

C **(b)** Solve this model by the simplex method to find one of its multiple optimal solutions.

- decision variables → 1) amount of cargo 1 in tons to shipped and stored in F compartment (X1F)

2) amount of cargo 1 in tons to shipped and stored in C compartment (X1C

3) amount of cargo 1 in tons to shipped and stored in B compartment (X1B

... X2F, X2C, X2B, X3F, X3C, X3B, X4F, X4C, X4B

- the objective → maximize the total profit

Z = 320(X1F + X1C + X1B) + 400(X2F + X2C + X2B) + 360(X3F + X3C + X3B)
+ 290(X4F + X4C + X4B)

the constants (subject to): →

X1F+X2F+X3F+X4F<=12
X1C+X2C+X3C+X4C<=18
X1B+X2B+X3B+X4B<=10

500X1F+700X2F+600X3F+400X4F<=7000
500X1C+700X2C+600X3C+400X4C<=9000
500X1B+700X2B+600X3B+400X4B<=5000

X1F+X1C+X1B<=20
X2F+X2C+X2B<=16
X3F+X3C+X3B<=25
X4F+X4C+X4B<=13

(X1F+X2F+X3F+X4F)/12= (X1C+X2C+X3C+X4C)/18
(X1F+X2F+X3F+X4F)/12= (X1B+X2B+X3B+X4B)/10

ALL X>=0

ch3:

# Assumptions of Linear Programming

2) • Additivity → ex: total profit = profit from chairs + profit from tabels
   - Every function in a linear programming model is the sum of the individual contributions of the activities

3) • Divisibility
   - Decision variables in a linear programming model may have any values (ex: non integer)
     - Including noninteger values
   - Assumes activities can be run at fractional values





# Assumptions of Linear Programming

4) • Certainty relate to Parameters of the model not D.V
   - Value assigned to each parameter of a linear programming model is assumed to be a known constant
   - Seldom satisfied precisely in real applications
     - Sensitivity analysis used

5$ – 10$

- Parameters ⟶ the values that given in the question
- D.V ⟶ the unknowns ($x_1, x_2, \ldots$)

ch4 :

## - Simplex method:

### 3.1 PROTOTYPE EXAMPLE

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing
Product 2: A 4 × 6 foot double-hung wood-framed window

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which *mix* of the two products would be *most profitable.* Therefore, an OR team has been formed to study this question.

product($x_1$)
product($x_2$)

The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem:

target

> Determine what the *production rates* should be for the two products in order to *maximize their total profit,* subject to the restrictions imposed by the limited production capacities available in the three plants. (Each product will be produced in batches of 20, so the *production rate* is defined as the number of batches produced per week.) *Any* combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

The OR team also identified the data that needed to be gathered:

1. Number of hours of production time available per week in each plant for these new products. (Most of the time in these plants already is committed to current products, so the available capacity for the new products is quite limited.)
2. Number of hours of production time used in each plant for each batch produced of each new product.
3. Profit per batch produced of each new product. (*Profit per batch produced* was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly *constant* regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products, the total profit from each one is approximately this *profit per batch produced* times *the number of batches produced.*)

Obtaining reasonable estimates of these quantities required enlisting the help of key personnel in various units of the company. Staff in the manufacturing division provided the data in the first category above. Developing estimates for the second category of data required some analysis by the manufacturing engineers involved in designing the production processes for the new products. By analyzing cost data from these same engineers and the marketing division, along with a pricing decision from the marketing division, the accounting department developed estimates for the third category.

Table 3.1 summarizes the data gathered.

The OR team immediately recognized that this was a linear programming problem of the classic **product mix** type, and the team next undertook the formulation of the corresponding mathematical model.

**TABLE 3.1** Data for the Wyndor Glass Co. problem

| | Production Time per Batch, Hours (Capacities) | | |
|---|---|---|---|
| | Product | | |
| Plant | 1 | 2 | Production Time Available per Week, Hours (Capacities) |
| 1 | 1 hour per one batch | 0 hour | 4 |
| 2 | 0 hour | 2 hour | 12 |
| 3 | 3 hour | 2 hour | 18 |
| Profit per batch | $3,000 | $5,000 | |

### Formulation as a Linear Programming Problem

The definition of the problem given above indicates that the decisions to be made are the

$Z \longrightarrow$ is always has under it an identity

- Always the number of basic variables equal to the number of constraints

$\max\ z = 3x_1 + 5x_2$

$1x_1 + 0x_2 <= 4$

$0x_1 + 2x_2 <= 12$

$3x_1 + 2x_2 <= 18$

$x_1 >= 0$

$x_2 >= 0$

1) first we have to check that this mathematical method is written in the standard form

شروط

- the objective has to be maximization
- All the constraints should be $\leq$ with positive right hand side
- All the decision variables should be $\geq 0$

2) Identifying an initial solution

X1<=4.........x1+x3=4

2x2<=12........2x2+x4=12

3x1+2x2<=18.......3x1+2x2+x5=18

X1,x2>=0

X3,x4,x5 slack variables >=0

slack variables

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -5 | 0 | 0 | 0 | 0 | ← obj row |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | |
| $R_2$ | 0 | 0 | 2 | 0 | 1 leader | 0 | 12 | corresponding |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 | |

3) Initialization

- Basic variables $\longrightarrow$ variables that have values
- Nonbasic variables $\longrightarrow$ variables that have zero values

- Variables that have Identity under them $\longrightarrow$ Basic variables

$\hookrightarrow$ here : $x_3, x_4, x_5$

$x_3 = 4 \longrightarrow$ (the Right hand side of the row of the leader at $x_3$)

$x_4 = 12 \longrightarrow$ (the Right hand side of the row of the leader at $x_3$)

$x_5 = 18 \longrightarrow$ (the Right hand side of the row of the leader at $x_3$)

- Nonbasic variables $\longrightarrow$ $x_1 = 0$ , $x_2 = 0$

## 4) Optimality:

ابلش اتحرك باتجاه معين يزيدلي ال objective function

(Is this solution Optimal?)

→ To know if the solution is optimal or not

→ we have to look at the objective row

→ if you have an negative coefficient in the objective row then this solution is not optimal.

- select the most negative variable → which is $X_2$ here (-5)

∴ $X_2$ is entering variable ( $X_2$ will become a basic variable)

entering var

obj row → $R_0$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -5 | 0 | 0 | 0 | 0 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 |

## 5) Feasibility

→ How much can i add to the entering variable ($X_2$) while continuing to stay in the feasible rejion

take the right hand side for the constraints ($R_1$, $R_2$, $R_3$) and then divide them by the coefficient of $X_2$ (RHS / coef entering var)

| Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|
| 1 | -3 | -5 | 0 | 0 | 0 | 0 | |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | $4/0 = \infty$ |
| 0 | 0 | 2 | 0 | 1 | 0 | 12 | $12/2 = 6$ minimum |
| 0 | 3 | 2 | 0 | 0 | 1 | 18 | $18/2 = 9$ |

pivot element (must make it a leader)

in order to stay in the feasible area we have to select the minimum

- the Leaving Variable is the variable that its leader in the this row → $X_4$

∴ $X_4$ → leaving var

| | | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|---|
| obj row → | $R_0$ | 1 | -3 | 0 | 0 | 2.5 | 0 | 30 | |
| | $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $4/0 = \infty$ |
| | $R_2$ | 0 | 0 | 1 | 0 | 1/2 | 0 | 6 | $12/2 = 6$ |
| | $R_3$ | 0 | 3 | 0 | 0 | -1 | 1 | 6 | $18/2 = 9$ |

1) $R_2 \to R_2/2$
2) $R_0 \to 5R_2 + R_0$
3) $R_3 \to 2R_2 + R_3$

6) Repeat step 4 & 5 (Optimality & Feasibility) until you reach the optimal solution

- Basic variables → $x_3 = 4$, $x_2 = 6$, $x_5 = 6$
- Nonbasic variables → $x_1 = 0$, $x_4 = 0$

| | | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|---|
| obj row → | $R_0$ | 1 | -3 | 0 | 0 | 2.5 | 0 | 30 | |
| | $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $4/1 = 4$ |
| | $R_2$ | 0 | 0 | 1 | 0 | 1/2 | 0 | 6 | $6/0 = \infty$ |
| | $R_3$ | 0 | 3 | 0 | 0 | -1 | 1 | 6 | $6/3 = 2$ minimum |

3 → is the pivot element

$x_1$ → is the entering variable
$x_5$ → is the leaving variable

| | | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|---|
| obj row → | $R_0$ | 1 | 0 | 0 | 0 | 1.5 | 1 | 36 | |
| | $R_1$ | 0 | 0 | 0 | 1 | 1/3 | -1/3 | 2 | $4/1 = 4$ |
| | $R_2$ | 0 | 0 | 1 | 0 | 1/2 | 0 | 6 | $6/0 = \infty$ |
| | $R_3$ | 0 | 1 | 0 | 0 | -1/3 | 1/3 | 2 | $6/3 = 2$ |

1) $R_3 \to R_3/3$
2) $R_1 \to -1R_3 + R_1$
3) $R_0 \to 3R_3 + R_0$

- continue step 6:

  - Basic variables $\longrightarrow$ $x_3 = 2$ , $x_2 = 6$ , $x_1 = 2$
  - Nonbasic Variables $\longrightarrow$ $x_4 = 0$ , $x_5 = 0$

- All the coefficient are positive
  - $\hookrightarrow$ optimal solution

    $\boxed{z = 36}$

- the slack variables $\longrightarrow$ بمثلو الفائض ( unutilized capacity )

$x_1 \leq 4$ $\quad$ $x_3 = 2$ $\longrightarrow$ $x_1 + x_3 = 4$

$2x_2 \leq 12$ $\quad$ $x_4 = 0$ الفائض صفر $\longrightarrow$ $2x_2 + x_4 = 12$

$3x_1 + 2x_2 \leq 18$ $\quad$ $x_5 = 0$ $\longrightarrow$ $3x_1 + 2x_2 + x_5 = 18$

slack $\text{var}_s$

$x_1 \geq 0$

$x_2 \geq 0$

what if analysis:

- I want to increase the number of hours from 4 to 5 in the first constraint will this effect my solution?
  - $\hookrightarrow$ No, because i have excess (فائض) ($x_3 = 2$)

- I want to increase the right hand side for the third constraint by one unit how that will effect my solution?
  - $\hookrightarrow$ will increase the objective value (total profit)
    - $\hookrightarrow$ How do we know the amount of increase?
      - $\hookrightarrow$ shadow price $\longrightarrow$ ( Coefficient of the slack variables in the optimal simplex table ) which is

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | 0 | 0 | 1.5 | 1 | 36 | |
| $R_1$ | 0 | 0 | 0 | 1 | 1/3 | -1/3 | 2 | 4/1 = 4 |
| $R_2$ | 0 | 0 | 1 | 0 | 1/2 | 0 | 6 | 6/0 = ∞ |
| $R_3$ | 0 | 1 | 0 | 0 | -1/3 | 1/3 | 2 | 6/3 = 2 |

- if we do increase the RHS for the first constraint by one unit
  - $\hookrightarrow$ the optimal value (36) will increase by zero unit
- if we do increase the RHS for the second constraint by one unit
  - $\hookrightarrow$ the optimal value (36) will increase by 1.5 unit
- if we do increase the RHS for the third constraint by one unit
  - $\hookrightarrow$ the optimal value (36) will increase by one unit

the values of $x_1$ & $x_2$ will remain the same

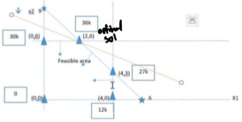
$x_1, x_2 \geq 0$ $\quad$ they are the same

- Post optimality analysis : in this case if we do change the cф

1) Slight changes in the RHS

2) Slight changes in the coefficients of the variables in the objective function

3)




Z=3x1+5x2 in thousands

3x1+5x2=36 (0,4) (20/3, 0)

point remain optimal → the last intersection between the objective & the feasible area

→ we can change the slope of the objective function in certain amount but it remains trapped between two slopes (the two slopes are the slopes of the constraints that their intersection point gave me the optimal solution.

Optimal solution is the intersection of

$2x_2=12$

$3x_1+2x_2=18$

(2,6) optimal point remains the same ……….. the slope of the objective function should be bounded by the slopes of these two lines ($2x_2=12$; $3x_1+2x_2=18$)

lec9

D,I **4.4-7.** Work through the simplex method step by step (in tabular form) to solve the following problem.

Maximize $\quad Z = 2x_1 - x_2 + x_3,$

subject to

$$\begin{aligned} 3x_1 + x_2 + x_3 &\le 6 \\ x_1 - x_2 + 2x_3 &\le 1 \\ x_1 + x_2 - x_3 &\le 2 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

1) it is in the standard form

2) $Z - 2X_1 - 1X_2 + X_3 = 0$

$3X_1 + X_2 + X_3 + X_4 = 6$

$X_1 - X_2 + 2X_3 + X_5 = 1$

$X_1 + X_2 - X_3 + X_6 = 2$

$X_1 \geq 0$

$X_2 \geq 0$

$X_4, X_5, X_6 \longrightarrow$ slacking variables

3) - Basic variables $\longrightarrow$ $X_3 = 1$, $X_4 = 20$, $X_5 = 10$, $X_6 = 5$

- Nonbasic Variables $\longrightarrow$ $X_1 = 0$, $X_2 = 0$

4)

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | RH |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -2 | 1 | -1 | 0 | 0 | 0 | 0 |
| $R_1$ | 0 | 3 | 1 | 1 | 1 | 0 | 0 | 6 |
| $R_2$ | 0 | 1 | -1 | 2 | 0 | 1 | 0 | 1 |
| $R_3$ | 0 | 1 | 1 | -1 | 0 | 0 | 1 | 2 |

5)

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | RH | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -2 | 1 | -1 | 0 | 0 | 0 | 0 | |
| $R_1$ | 0 | 3 | 1 | 1 | 1 | 0 | 0 | 6 | 6/3 = 2 |
| $R_2$ | 0 | 1 | -1 | 2 | 0 | 1 | 0 | 1 | 1/1 = 1 min |
| $R_3$ | 0 | 1 | 1 | -1 | 0 | 0 | 1 | 2 | 2/1 = 2 |

1 is the pivot element

Povit element $\longrightarrow$ بدنا نخلي كل الي فوقها وتحتها صفر وهو يكون واحد

$X_1 \longrightarrow$ is the entering variable

$X_5 \longrightarrow$ is the leaving variable

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | RH |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | -1 | 3 | 0 | 2 | 0 | 2 |
| $R_1$ | 0 | 0 | 4 | -5 | 1 | -3 | 0 | 3 |
| $R_2$ | 0 | 1 | -1 | 2 | 0 | 1 | 0 | 1 |
| $R_3$ | 0 | 0 | 2 | -3 | 0 | -1 | 1 | 1 |

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | RH | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | -1 | 3 | 0 | 2 | 0 | 2 | |
| $R_1$ | 0 | 0 | 4 | -5 | 1 | -3 | 0 | 3 | 3/4 |
| $R_2$ | 0 | 1 | -1 | 2 | 0 | 1 | 0 | 1 | 1/-1 |
| $R_3$ | 0 | 0 | 2 | -3 | 0 | -1 | 1 | 1 | 1/2 |

دايمًا اذا كان (RHS/coef ent var) سالب

$\longleftarrow$ بهمله

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | RH |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 3/2 | 0 | 3/2 | 1/2 | 5/2 |
| $R_1$ | 0 | 0 | 0 | 1 | 1 | -1 | -2 | 1 |
| $R_2$ | 0 | 1 | 0 | 1/2 | 0 | 1/2 | 1/2 | 3/2 |
| $R_3$ | 0 | 0 | 1 | -3/2 | 0 | -1/2 | 1/2 | 1/2 |

$\longrightarrow$ The Optimal solution

$\boxed{Z = 5/2}$

* Example

max:
$\hookrightarrow z = 3x_1 + 3x_2 + 2x_3$

Sub to:
$\hookrightarrow x_1 - x_2 + 4x_3 \leq 7$
$2x_1 + 3x_2 + x_3 \leq 8$
$x_1, x_2, x_3 \geq$ zero

it is in the standard form

$Z - 3X_1 - 2X_2 = 0$

$X_1 - X_2 + 4X_3 + X_4 = 7$

$2X_1 + 3X_2 + X_3 \leq 8$

$X_1, X_2, X_3 \geq 0$

$X_4, X_5 \longrightarrow$ slacking variables

- Basic variables $\longrightarrow$ $X_3 = 4$ , $X_4 = 6$ , $X_5 = 18$
- Nonbasic Variables $\longrightarrow$ $X_1 = 0$ , $X_2 = 0$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | $\boxed{-3}$ | $\boxed{-3}$ | -2 | 0 | 0 | 0 |
| $R_1$ | 0 | 1 | -1 | 4 | 1 | 0 | 7 |
| $R_2$ | 0 | 2 | 3 | 1 | 0 | 1 | 8 |

(Tai for entering variable)
$\hookrightarrow$ we have two numbers that are most negative
$\hookrightarrow$ choose any one of these two

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -3 | -5 | 0 | 0 | 0 | |
| $R_1$ | 0 | 1 | -1 | 4 | 1 | 0 | 8 | $8/4 = \boxed{2}$ |
| $R_2$ | 0 | 2 | 3 | 1 | 0 | 1 | 2 | $2/1 = \boxed{2}$ |

(Tai for Leaving variable)
$\hookrightarrow$ we have two numbers that are minimum
$\hookrightarrow$ choose any one of these two

OR1 assignment 2

**4.1-2.** Consider the following problem.

Maximize $\quad Z = 3x_1 + 2x_2,$

subject to

$$2x_1 + x_2 \leq 6$$
$$x_1 + 2x_2 \leq 6$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

1) it is in the standard form

2) $Z - 3X_1 - 2X_2 = 0$

$2X_1 + X_2 + X_3 = 6$

$X_1 + 2X_2 + X_4 = 6$

$X_1 \geq 0$

$X_2 \geq 0$

$X_3, X_4 \longrightarrow$ slacking variables

3) - Basic variables $\longrightarrow$ $X_3 = 6$, $X_4 = 6$

- Nonbasic Variables $\longrightarrow$ $X_1 = 0$, $X_2 = 0$

4)

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | RHS |
|---|---|---|---|---|---|---|
| $R_0$ | 1 | −3 | −2 | 0 | 0 | 0 |
| $R_1$ | 0 | 2 | 1 | 1 | 0 | 6 |
| $R_2$ | 0 | 1 | 2 | 0 | 1 | 6 |

5)

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | −3 | −2 | 0 | 0 | 0 | |
| $R_1$ | 0 | 2 | 1 | 1 | 0 | 6 | 6/2 = 3 minimum |
| $R_2$ | 0 | 1 | 2 | 0 | 1 | 6 | 6/1 = 6 |

2 is the pivot element

$X_1 \longrightarrow$ is the entering variable

$X_3 \longrightarrow$ is the leaving variable

$R_1 \longrightarrow R_1/2$

$R_0 \longrightarrow 3R_1 + R_0$

$R_2 \longrightarrow -R_1 + R_2$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | RHS |
|---|---|---|---|---|---|---|
| $R_0$ | 0 | 0 | −1/2 | 3/2 | 0 | 9 |
| $R_1$ | 0 | 1 | 1/2 | 1/2 | 0 | 3 |
| $R_2$ | 0 | 0 | 3/2 | −1/2 | 1 | 3 |

6) - Basic variables $\longrightarrow$ $X_3 = 6$, $X_4 = 6$

- Nonbasic Variables $\longrightarrow$ $X_1 = 0$, $X_2 = 0$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | −1/2 | 3/2 | 0 | 9 | |
| $R_1$ | 0 | 1 | 1/2 | 1/2 | 0 | 3 | $3/\frac{1}{2} = 6$ |
| $R_2$ | 0 | 0 | 3/2 | −1/2 | 1 | 3 | $3/\frac{3}{2} = 2$ min |

3/2 is the pivot element

$X_2 \longrightarrow$ is the entering variable

$X_4 \longrightarrow$ is the leaving variable

$R_2 \longrightarrow \frac{2}{3} R_2$

$R_1 \longrightarrow -\frac{1}{2} R_2 + R_1$

$R_0 \longrightarrow \frac{1}{2} R_2 + R_0$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | RHS |
|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | 0 | 4/3 | 1/3 | 10 |
| $R_1$ | 0 | 1 | 0 | 2/3 | −1/3 | 2 |
| $R_2$ | 0 | 0 | 1 | −1/3 | 2/3 | 2 |

The optimal solution

$\boxed{Z = 10}$

- Find the optimal solution in simplex method & graphical method:

max
$\rightarrow z = 3x_1 + 2x_2$

Subject to
$\rightarrow x_1 \leq 4$

$2x_2 \leq 6$

$3x_1 + 2x_2 \leq 18$

Me

$x_1 , x_2 \geq 0$

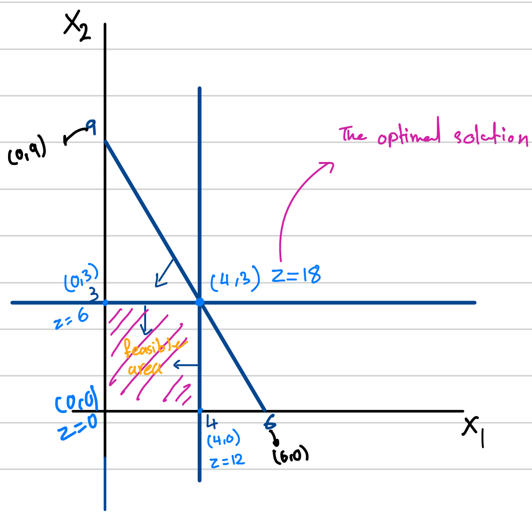
## - In graphical method:

$x_1 = 4$

$x_2 = 3$

$3x_1 + 2x_2 = 18 \longrightarrow$ when $x_1 = 0 \longrightarrow (0,9)$

when $x_2 = 0 \longrightarrow (6,0)$




$\rightarrow$ The optimal point is $(4,3)$

$\boxed{z = 18}$

## In Simplex method:

1) it is in the standard form

2) $Z - 3X_1 - 2X_2 = 0$

$X_1 + X_3 = 4$

$2X_2 + X_4 = 6$

$3X_1 + 2X_2 + X_5 = 18$

$X_1 \geq 0$

$X_2 \geq 0$

$X_3, X_4, X_5 \longrightarrow$ slacking variables

3) - Basic variables $\longrightarrow$ $X_3 = 4$ , $X_4 = 6$ , $X_5 = 18$

- Nonbasic variables $\longrightarrow$ $X_1 = 0$ , $X_2 = 0$

4)

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -2 | 0 | 0 | 0 | 0 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 6 |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 |

5)

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -2 | 0 | 0 | 0 | 0 | |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $4/1 = 4$ min |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 6 | $6/0 = \infty$ |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 | $18/3 = 6$ |

is the pivot element

$X_1 \longrightarrow$ is the entering variable

$X_3 \longrightarrow$ is the leaving variable

$R_0 = 3R_1 + R_0$ $\qquad$ $R_3 = -3R_1 + R_3$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | -2 | 3 | 0 | 0 | 12 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 6 |
| $R_3$ | 0 | 0 | 2 | -3 | 0 | 1 | 6 |

6) - Basic variables $\longrightarrow$ $X_1 = 4$ , $X_4 = 6$ , $X_5 = 6$

- Nonbasic variables $\longrightarrow$ $X_2 = 0$ , $X_3 = 0$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | -2 | 3 | 0 | 0 | 12 | |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $4/0 = \infty$ |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 6 | $6/2 = 3$ |
| $R_3$ | 0 | 0 | 2 | -3 | 0 | 1 | 6 | $6/2 = 3$ |

Tie for leaving variable (I can choose any one of these two rows)

is the pivot element

$R_2 \longrightarrow \frac{1}{2} R_2$ $\qquad$ $R_3 \longrightarrow -2R_2 + R_3$

$R_0 \longrightarrow 2R_2 + R_0$

| | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | 0 | 3 | 1 | 0 | 18 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 1 | 0 | 1/2 | 0 | 3 |
| $R_3$ | 0 | 0 | 0 | -3 | -1 | 1 | 0 |

$\longrightarrow$ The optimal solution

$\boxed{Z = 18}$

$X_1 = 4$

$X_2 = 3$

- Find the optimal solution in simple x method & graphical method:

max
$\hookrightarrow z = 3x_1 + 2x_2$

Subject to
$\hookrightarrow x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

Me

$x_1, x_2 \geq 0$

## In Simple x method:

1) it is in the standard form

2) $Z - 3X_1 - 2X_2 = 0$

$X_1 + X_3 = 4$

$2X_2 + X_4 = 12$

$3X_1 + 2X_2 + X_5 = 18$

$X_1 \geq 0$

$X_2 \geq 0$

$X_3, X_4, X_5 \longrightarrow$ slacking variables

3) - Basic variables $\rightarrow$ $X_3 = 4$, $X_4 = 6$, $X_5 = 18$

- Non basic variables $\rightarrow$ $X_1 = 0$, $X_2 = 0$

4)

| | z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -2 | 0 | 0 | 0 | 0 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 |

5)

| | z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -3 | -2 | 0 | 0 | 0 | 0 | |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | 4/1 = 4 min |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 12 | 12/0 = ∞ |
| $R_3$ | 0 | 3 | 2 | 0 | 0 | 1 | 18 | 18/3 = 6 |

$\rightarrow$ is the pivot element

$X_1 \longrightarrow$ is the entering variable

$X_3 \longrightarrow$ is the leaving variable

$R_0 = 3R_1 + R_0$ $\quad$ $R_3 = -3R_1 + R_3$

| | z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | -2 | 3 | 0 | 0 | 12 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| $R_3$ | 0 | 0 | 2 | -3 | 0 | 1 | 3 |

6) - Basic variables $\rightarrow$ $X_1 = 4$, $X_4 = 6$, $X_5 = 6$

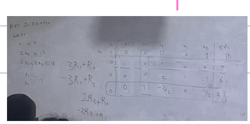
- Non basic variables $\rightarrow$ $X_2 = 0$, $X_3 = 0$

| | z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS | RHS/coef ent var |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | -2 | 3 | 0 | 0 | 12 | |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | 4/0 = ∞ |
| $R_2$ | 0 | 0 | 2 | 0 | 1 | 0 | 6 | 6/2 = 3 |
| $R_3$ | 0 | 0 | 2 | -3 | 0 | 1 | 6 | 6/2 = 3 |

Tie for leaving variable (I can choose any one of these two rows)

$\rightarrow$ is the pivot element

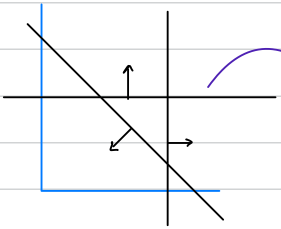
| | z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | 0 | 0 | 0 | 1 | 18 |
| $R_1$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $R_2$ | 0 | 0 | 0 | 3 | 1 | -1 | 6 |
| $R_3$ | 0 | 0 | 2 | -3/2 | 0 | 1/2 | 3 |



31/oct

4:35min

## * Infeasible :



no feasible area
(Infeasible area)

mistake on the mathematical model

∴ so i can't use the simple x method

use big M method

**3.4-10.** Larry Edison is the director of the Computer Center for Buckly College. He now needs to schedule the staffing of the center. It is open from 8 A.M. until midnight. Larry has monitored the usage of the center at various times of the day, and determined that the following number of computer consultants are required:

| Time of Day | Minimum Number of Consultants Required to Be on Duty |
|---|---|
| 8 A.M.–noon | 4 |
| Noon–4 P.M. | 8 |
| 4 P.M.–8 P.M. | 10 |
| 8 P.M.–midnight | 6 |

Two types of computer consultants can be hired: full-time and part-time. The full-time consultants work for 8 consecutive hours in any of the following shifts: morning (8 A.M.–4 P.M.), afternoon (noon–8 P.M.), and evening (4 P.M.–midnight). Full-time consultants are paid $40 per hour.

Part-time consultants can be hired to work any of the four shifts listed in the above table. Part-time consultants are paid $30 per hour.

An additional requirement is that during every time period, there must be at least 2 full-time consultants on duty for every part-time consultant on duty.

Larry would like to determine how many full-time and how many part-time workers should work each shift to meet the above requirements at the minimum possible cost.

**(a)** Formulate a linear programming model for this problem.

C **(b)** Solve this model by the simplex method.

Solve it using simplex method

D.V : $x_{fm}$ = number of full time workers that should work the morning shift ($x_{fa}$, $x_{fe}$)

$x_{p_1}$ = number of part time workers that should work the $1^{st}$ shift ($x_{p_2}$, $x_{p_3}$, $x_{p_4}$)

Objective → minimize the cost

→ $Z = 40(8)(x_{fm} + x_{fa} + x_{fe}) + (30)(4)(x_{p_1} + x_{p_2} + x_{p_3} + x_{p_4})$

sub to : Xfm+xp1>=4

Xfm+xfa+xp2>=8

Xfa+xfe+xp3>=10

Xfe+xp4>=6

Xfm>=2xp1

Xfm+xfa>=2xp2

Xfa+xfe>=2xp3

Xf2>=2xp4

All x>=0

**Design of Radiation Therapy**

MARY has just been diagnosed as having a cancer at a fairly advanced stage. Specifically, she has a large malignant tumor in the bladder area (a "whole bladder lesion").

Mary is to receive the most advanced medical care available to give her every possible chance for survival. This care will include extensive *radiation therapy.*

Radiation therapy involves using an external beam treatment machine to pass ionizing radiation through the patient's body, damaging both cancerous and healthy tissues. Normally, several beams are precisely administered from different angles in a two-dimensional plane. Due to attenuation, each beam delivers more radiation to the tissue near the entry point than to the tissue near the exit point. Scatter also causes some delivery of radiation to tissue outside the direct path of the beam. Because tumor cells are typically microscopically interspersed among healthy cells, the radiation dosage throughout the tumor region must be large enough to kill the malignant cells, which are slightly more radiosensitive, yet small enough to spare the healthy cells. At the same time, the aggregate dose to critical tissues must not exceed established tolerance levels, in order to prevent complications that can be more serious than the disease itself. For the same reason, the total dose to the entire healthy anatomy must be minimized.

Because of the need to carefully balance all these factors, the design of radiation therapy is a very delicate process. The goal of the design is to select the combination of beams to be used, and the intensity of each one, to generate the best possible dose distribution. (The dose strength at any point in the body is measured in units called *kilorads.*) Once the treatment design has been developed, it is administered in many installments, spread over several weeks.

In Mary's case, the size and location of her tumor make the design of her treatment an even more delicate process than usual. Figure 3.11 shows a diagram of a cross section of the tumor viewed from above, as well as nearby critical tissues to avoid. These tissues include critical organs (e.g., the rectum) as well as bony structures (e.g., the femurs and pelvis) that will attenuate the radiation. Also shown are the entry point and direction for the only two beams that can be used with any modicum of safety in this case. (Actually, we are simplifying the example at this point, because normally dozens of possible beams must be considered.)

For any proposed beam of given intensity, the analysis of what the resulting radiation absorption by various parts of the body would be requires a complicated process. In brief, based on careful anatomical analysis, the energy distribution within the two-dimensional cross section of the tissue can be plotted on an isodose map, where the contour lines represent the dose strength as a percentage of the dose strength at the entry point. A fine grid then is placed over the isodose map. By summing the radiation absorbed in the squares containing each type of tissue, the average dose that is absorbed by the tumor, healthy anatomy, and critical tissues can be calculated. With more than one beam (administered sequentially), the radiation absorption is additive.

After thorough analysis of this type, the medical team has carefully estimated the data needed to design Mary's treatment, as summarized in Table 3.7. The first column lists the areas of the body that must be considered, and then the next two columns give the fraction of the radiation dose at the entry point for each beam that is absorbed by the respective areas on average. For example, if the dose level at the entry point for beam 1 is 1 kilorad, then an average of 0.4 kilorad will be absorbed by the entire healthy anatomy in the two-dimensional plane, an average of 0.3 kilorad will be absorbed by nearby critical tissues, an average of 0.5 kilorad will be absorbed by the various parts of the tumor, and 0.6 kilorad will be absorbed by the center of the tumor. The last column gives the restrictions on the total dosage from both beams that is absorbed on average by the respective areas of the body. In particular, the average dosage absorption for the healthy anatomy must be *as small as possible,* the critical tissues must *not exceed* 2.7 kilorads, the average over the entire tumor must *equal* 6 kilorads, and the center of the tumor must be *at least* 6 kilorads.

**Formulation as a Linear Programming Problem.** The decisions that need to be made are the dosages of radiation at the two entry points. Therefore, the two decision variables $x_1$ and $x_2$ represent the dose (in kilorads) at the entry point for beam 1 and beam 2, respectively. Because the total dosage reaching the healthy anatomy is to be

■ **TABLE 3.7** Data for the design of Mary's radiation therapy

| Area | Fraction of Entry Dose Absorbed by Area (Average): Beam 1 | Beam 2 | Restriction on Total Average Dosage, Kilorads |
|---|---|---|---|
| Healthy anatomy | 0.4 | 0.5 | Minimize |
| Critical tissues | 0.3 | 0.1 | $\le$ 2.7 |
| Tumor region | 0.5 | 0.5 | $=$ 6 |
| Center of tumor | 0.6 | 0.4 | $\ge$ 6 |

Big M method

Solve it using simplex method

→ we can not use the simplex method → then we will use

because this model is not written in the standard form

1) minimization → maximization

2) add artificial variables to the constraints that have (=)

3) add artificial variables & Surplus to the constraints that have ($\ge$) → negative slack variable

- decision variables

$x_1$ : dose in Kilorad from the first beam

$x_2$ : dose in Kilorad from the Second beam

- objective

minimize the total absorbed by healthy anatomy

$M \longrightarrow$ a very big number

$Z = 0.4x_1 + 0.5x_2 \xrightarrow{1)} \max \rightarrow -Z = -0.4x_1 - 0.5x_2 - Mx_{a_1} - Mx_{a_2}$

→ To force the objective make $Mx_{a_1} = \text{zero}$, $Mx_{a_2} = \text{zero}$

$-Z + 0.4x_1 + 0.5x_2 + Mx_{a_1} + Mx_{a_2} = 0$

Subject to constrains

$0.3x_1 + 0.1x_2 \le 2.7 \longrightarrow 0.3x_1 + 0.1x_2 + x_3 = 2.7$

artificial variable

$0.5x_1 + 0.5x_2 = 6 \longrightarrow 0.5x_1 + 0.5x_2 + x_{a_1} = 6$

$0.6x_1 + 0.4x_2 \ge 6 \longrightarrow 0.6x_1 + 0.4x_2 - x_4 + x_{a_2} = 6$

Surplus

$x_1 \ge 0$

$x_2 \ge 0$

$x_3 \ge 0$ slack var

$x_4 \ge 0$ surplus var

$x_{a_1}, x_{a_2} \ge 0$ Artificial var

not feasible solution

~~Restoring the proper GF~~

$R_0 \to -MR_2 + R_0$

$R_0 \to -MR_3 + R_0$

| | Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a_1}$ | $x_{a_2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | -1 | 0.4 | 0.5 | 0 | 0 | M | M | 0 |
| $R_1$ | 0 | 0.3 | 0.1 | 1 | 0 | 0 | 0 | 2.7 |
| $R_2$ | 0 | 0.5 | 0.5 | 0 | 0 | 1 | 0 | 6 |
| $R_3$ | 0 | 0.6 | 0.4 | 0 | -1 | 0 | 1 | 6 |

the negative here it does not mean anything

- Basic variables → $x_3 = 2.7$, $x_{a_1} = 6$, $x_{a_2} = 6$

- Nonbasic variables → $x_1 = 0$, $x_2 = 0$, $x_4 = 0$

| | Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a_1}$ | $x_{a_2}$ | RHS | RHS/coef of ent var |
|---|---|---|---|---|---|---|---|---|---|
| $R_0$ | -1 | 0.4-1.1M | 0.5-0.9M | 0 | M | 0 | 0 | -12M | |
| $R_1$ | 0 | 0.3 | 0.1 | 1 | 0 | 0 | 0 | 2.7 | 2.7/0.3 = 9 |
| $R_2$ | 0 | 0.5 | 0.5 | 0 | 0 | 1 | 0 | 6 | 6/0.5 = 12 |
| $R_3$ | 0 | 0.6 | 0.4 | 0 | -1 | 0 | 1 | 6 | 6/0.6 = 10 |

pivot element

- As long as the artificial variables are taking values other than zero → This tableau is not feasible.

note

infeasible + optimal $\Longrightarrow$ the mathematical model is infeasible

$R_1 \longrightarrow R_1 / 0.3$ $\qquad$ $R_2 \longrightarrow -0.5R_1 + R_2$

$R_0 \longrightarrow ((-0.4+1.1M)R_1 + R_0$ $\qquad$ $R_3 \longrightarrow -0.6R_1 + R_3$

دليل انو بتحرك باتجاه
الـ feasible rejoin
لانو بتقل

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a_1}$ | $x_{a_2}$ | RHS | RHS/coef of ent var |
|---|---|---|---|---|---|---|---|---|---|
| $R_0$ | −1 | 0 | 0.366 − 0.53M | −1.3 + 3.67M | M | 0 | 0 | −3.6 − 2.1M | |
| $R_1$ | 0 | 1/3 | 1/3 | 10/3 | 0 | 0 | 0 | 9 | 9 /(1/3) = 27 |
| $R_2$ | 0 | 1/3 | 1/3 | −5/3 | 0 | 1 | 0 | 1.5 | 1.5 /(1/3) = 4.5 |
| $R_3$ | 0 | 0.2 | 0.2 | −2 | −1 | 0 | 1 | 0.6 | 0.6 / 0.2 = 3 |

pivot element

$R_3 \longrightarrow R_3 / 0.2$ $\qquad$ $R_1 \longrightarrow -\frac{1}{3} R_3 + R_1$

$R_0 \longrightarrow -(0.366 - 0.53M)R_3 + R_0$ $\qquad$ $R_2 \longrightarrow -\frac{1}{3} R_3 + R_2$

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a_1}$ | $x_{a_2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | −1 | 0 | 0 | 2.33 − 1.66M | 1.83 − 1.66M | 0 | −1.83 + 2.66M | −4.7 − 0.5M |
| $R_1$ | 0 | 1 | 0 | 6.67 | 1.67 | 0 | −1.67 | 8 |
| $R_2$ | 0 | 0 | 0 | 1.67 | 1.67 | 1 | −1.67 | 0.5 |
| $R_3$ | 0 | 0 | 1 | −10 | −5 | 0 | 5 | 3 |

$R_2 \longrightarrow R_2 / 1.67$ $\qquad$ $R_1 \longrightarrow -1.67 R_2 + R_1$

$R_0 \longrightarrow -(1.83 - 1.66M)R_2 + R_0$ $\qquad$ $R_3 \longrightarrow 5R_2 + R_3$

| | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a_1}$ | $x_{a_2}$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | −1 | 0 | 0 | 0.5 | 0 | (−1.1 + 1M) | M | −5.25 |
| $R_1$ | 0 | 1 | 0 | 5 | 0 | −1 | 0 | 7.5 |
| $R_2$ | 0 | 0 | 0 | 1 | 1 | 0.6 | −1 | 0.3 |
| $R_3$ | 0 | 0 | 1 | −5 | 0 | 3 | 0 | 4.5 |

optimal & feasible ✓

(artificial variables)

its feasible because $\longrightarrow$ $x_{a_1}, x_{a_2}$ = zero

Basic variables $\longrightarrow$ $x_1 = 7.5$, $x_2 = 4.5$, $x_4 = 0.3$, $z = 5.25$

$z = 5.25 \longrightarrow$ negative cancel negative

Big M method

D,I **3.4-5.** Use the ~~graphical method~~ to solve this problem:

Minimize $Z = 3x_1 + 2x_2$, $\longrightarrow$ Maximize $-z = -3x_1 - 2x_2 - Mx_{a_1} - Mx_{a_2}$

$\hookrightarrow -z + 3x_1 + 2x_2 + Mx_{a_1} + Mx_{a_2} = 0$

subject to

$$x_1 + 2x_2 \leq 12 \longrightarrow x_1 + 2x_2 + x_3 = 12$$
$$2x_1 + 3x_2 = 12 \longrightarrow 2x_2 + 3x_2 + x_{a_1} = 12$$
$$2x_1 + x_2 \geq 8 \longrightarrow 2x_1 + x_2 + x_4 + x_{a_2} = 8$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

| | z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|---|
| R0 | -1 | 3 | 2 | 0 | 0 | M | M | 0 |
| R1 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 12 |
| R2 | 0 | 2 | 3 | 0 | 0 | 1 | 0 | 12 |
| R3 | 0 | 2 | 1 | 0 | -1 | 0 | 1 | 8 |

Restoring $\longrightarrow R_0 \to -MR_2 + R_0$ , $R_0 \to -MR_3 + R_0$

| | z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS | |
|---|---|---|---|---|---|---|---|---|---|
| R0 | -1 | 3-4M | 2-4M | 0 | M | 0 | 0 | (-20M) | |
| R1 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 12 | 6 |
| R2 | 0 | 2 | 3 | 0 | 0 | 1 | 0 | 12 | 4 |
| R3 | 0 | 2 | 1 | 0 | -1 | 0 | 1 | 8 | 8 |

$R_2 \to R_2/3$ , $R_3 \to -1R_2 + R_3$ , $R_1 \to -2R_2 + R_1$ , $R_0 \to (-2+4M)R_2 + R_0$

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS | | |
|---|---|---|---|---|---|---|---|---|---|
| -1 | 1.66-1.33M | 0 | 0 | M | (-0.66+1.33M) | 0 | (-8-4M) | | |
| 0 | -0.333333333 | 0 | 1 | 0 | -0.666666667 | 0 | 4 | -12 | ignore |
| 0 | 0.666666667 | 1 | 0 | 0 | 0.333333333 | 0 | 4 | 6 | |
| 0 | 1.333333333 | 0 | 0 | -1 | -0.333333333 | 1 | 4 | 3 | |
| | x1 is ent | | xa2 is leav | | | | | | |

$R_3 \to R_3/1.3$ $R_2 \to -0.66R_3 + R_2$ , $R_1 \to 0.33R_3 + R_1$ , $R_0 \to (-1.66+1.33M)R_3 + R_0$

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 1.25 | (-0.25+M) | (-1.25+M) | -13 |
| 0 | 0 | 0 | 1 | -0.25 | -0.75 | 0.25 | 5 |
| 0 | 0 | 1 | 0 | 0.5 | 0.5 | -0.5 | 2 |
| 0 | 1 | 0 | 0 | -0.75 | -0.25 | 0.75 | 3 |

its optimal & feasible ✓

**4.6-10.** Follow the instructions of Prob. 4.6-9 for the following problem.

Minimize $Z = 3x_1 + 2x_2 + 7x_3,$ $\longrightarrow$ $-Z = -3X_1 - 2X_2 - 7X_3 - MX_{a_1} - MX_{a_2}$

subject to

$$-x_1 + x_2 \qquad = 10$$
$$2x_1 - x_2 + x_3 \geq 10$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

$X_1 + X_2 + X_{a_1} = 10$

$2X_1 - X_2 + X_3 + X_{a_2} - X_4 = 10$

$\left.\begin{array}{l} X_1 \\ X_2 \\ X_3 \end{array}\right\} \geq 0$

$X_4 \geq 0$

$X_{a_1}, X_{a_2} \geq 0$

$-Z + 3X_1 + 2X_2 + 7X_3 + MX_{a_2} + MX_{a_2} = 0$

Restoring the proper GF

$R_0 \rightarrow -MR_2 + R_0$

$R_0 \rightarrow -MR_3 + R_0$

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 3 | 2 | 7 | 0 | M | M | 0 |
| 0 | -1 | 1 | 0 | 0 | 1 | 0 | 10 |
| 0 | 2 | -1 | 1 | -1 | 0 | 1 | 10 |

Restoring GF

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 3-M | 2 | (7-M) | M | 0 | 0 | (-20M) |
| 0 | -1 | 1 | 0 | 0 | 1 | 0 | 10 |
| 0 | 2 | -1 | 1 | -1 | 0 | 1 | 10 |

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 3.5-.5M | 5.5-.5M | 1.5+.5M | 0 | (-1.5+.5M) | (-15-15M) |
| 0 | 0 | 0.5 | 0.5 | -0.5 | 1 | 0.5 | 15 |
| 0 | 1 | -0.5 | 0.5 | -0.5 | 0 | 0.5 | 5 |

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 2 | 5 | (-7+M) | (-5+M) | -120 |
| 0 | 0 | 1 | 1 | -1 | 2 | 1 | 30 |
| 0 | 1 | 0 | 1 | -1 | 1 | 1 | 20 |

(-3.5+.5M)R1+R0
-120

Two phase method

### Design of Radiation Therapy

MARY has just been diagnosed as having a cancer at a fairly advanced stage. Specifically, she has a large malignant tumor in the bladder area (a "whole bladder lesion").

Mary is to receive the most advanced medical care available to give her every possible chance for survival. This care will include extensive *radiation therapy.*

Radiation therapy involves using an external beam treatment machine to pass ionizing radiation through the patient's body, damaging both cancerous and healthy tissues. Normally, several beams are precisely administered from different angles in a two-dimensional plane. Due to attenuation, each beam delivers more radiation to the tissue near the entry point than to the tissue near the exit point. Scatter also causes some delivery of radiation to tissue outside the direct path of the beam. Because tumor cells are typically microscopically interspersed among healthy cells, the radiation dosage throughout the tumor region must be large enough to kill the malignant cells, which are slightly more radiosensitive, yet small enough to spare the healthy cells. At the same time, the aggregate dose to critical tissues must not exceed established tolerance levels, in order to prevent complications that can be more serious than the disease itself. For the same reason, the total dose to the entire healthy anatomy must be minimized.

Because of the need to carefully balance all these factors, the design of radiation therapy is a very delicate process. The goal of the design is to select the combination of beams to be used, and the intensity of each one, to generate the best possible dose distribution. (The dose strength at any point in the body is measured in units called *kilorads.*) Once the treatment design has been developed, it is administered in many installments, spread over several weeks.

In Mary's case, the size and location of her tumor make the design of her treatment an even more delicate process than usual. Figure 3.11 shows a diagram of a cross section of the tumor viewed from above, as well as nearby critical tissues to avoid. These tissues include critical organs (e.g., the rectum) as well as bony structures (e.g., the femurs and pelvis) that will attenuate the radiation. Also shown are the entry point and direction for the only two beams that can be used with any modicum of safety in this case. (Actually, we are simplifying the example at this point, because normally dozens of possible beams must be considered.)

For any proposed beam of given intensity, the analysis of what the resulting radiation absorption by various parts of the body would be requires a complicated process. In brief, based on careful anatomical analysis, the energy distribution within the two-dimensional cross section of the tissue can be plotted on an isodose map, where the contour lines represent the dose strength as a percentage of the dose strength at the entry point. A fine grid then is placed over the isodose map. By summing the radiation absorbed in the squares containing each type of tissue, the average dose that is absorbed by the tumor, healthy anatomy, and critical tissues can be calculated. With more than one beam (administered sequentially), the radiation absorption is additive.

After thorough analysis of this type, the medical team has carefully estimated the data needed to design Mary's treatment, as summarized in Table 3.7. The first column lists the areas of the body that must be considered, and then the next two columns give the fraction of the radiation dose at the entry point for each beam that is absorbed by the respective areas on average. For example, if the dose level at the entry point for beam 1 is 1 kilorad, then an average of 0.4 kilorad will be absorbed by the entire healthy anatomy in the two-dimensional plane, an average of 0.3 kilorad will be absorbed by nearby critical tissues, an average of 0.5 kilorad will be absorbed by the various parts of the tumor, and 0.6 kilorad will be absorbed by the center of the tumor. The last column gives the restrictions on the total dosage from both beams that is absorbed on average by the respective areas of the body. In particular, the average dosage absorption for the healthy anatomy must be *as small as possible,* the critical tissues must *not exceed* 2.7 kilorads, the average over the entire tumor must *equal* 6 kilorads, and the center of the tumor must be *at least* 6 kilorads.

**Formulation as a Linear Programming Problem.** The decisions that need to be made are the dosages of radiation at the two entry points. Therefore, the two decision variables $x_1$ and $x_2$ represent the dose (in kilorads) at the entry point for beam 1 and beam 2, respectively. Because the total dosage reaching the healthy anatomy is to be

**TABLE 3.7** Data for the design of Mary's radiation therapy

| Area | Fraction of Entry Dose Absorbed by Area (Average): Beam 1 | Beam 2 | Restriction on Total Average Dosage, Kilorads |
|---|---|---|---|
| Healthy anatomy | 0.4 | 0.5 | Minimize |
| Critical tissues | 0.3 | 0.1 | $\leq$ 2.7 |
| Tumor region | 0.5 | 0.5 | $=$ 6 |
| Center of tumor | 0.6 | 0.4 | $\geq$ 6 |

Solve it using simple x method

= we can not use the simple x method here — then we will use

because this model is not written in the standard form

1) minimization $\longrightarrow$ maximization
2) add artificial variables to the constraints that have $(=)$
3) add artificial variables & surplus to the constraints that have $(\geq)$

surplus $\rightarrow$ negative slack variable

- decision variables

$x_1$ : dose in Kilorad from the first beam

$x_2$ : dose in Kilorad from the Second beam

- objective

minimize the total absorbed by healthy anatomy

$Z = 0.4x_1 + 0.5x_2 \xrightarrow{1)} \max \rightarrow -Z = -0.4x_1 - 0.5x_2 - Mx_{a_1} - Mx_{a_2}$

$M \longrightarrow$ a very big number

To force the objective make $Mx_{a_1} = \text{zero}$, $Mx_{a_2} = \text{zero}$

Subject to ( constrains

$0.3x_1 + 0.1x_2 \leq 2.7 \longrightarrow 0.3x_1 + 0.1x_2 + x_3 = 2.7$

$0.5x_1 + 0.5x_2 = 6 \longrightarrow 0.5x_1 + 0.5x_2 + x_{a_1} = 6$

$0.6x_1 + 0.4x_2 \geq 6 \longrightarrow 0.6x_1 + 0.4x_2 - x_4 + x_{a_2} = 6$

$x_1 \geq 0$

$x_2 \geq 0$

$x_3 \geq 0$ slack var

$x_4 \geq 0$ surplus var

$x_{a_1}, x_{a_2} \geq 0$ AV

not feasible Solution

the last table is feasible

In two phase method

we will identify two objectives

the first phase objective → to minimize the artificial variable → phase 1: $\text{Min } z = x_{a_1} + x_{a_2} \longrightarrow \text{Max } -Z + x_{a_1} + x_{a_2} = 0$

just reaching a feasible solution

the second phase objective → the original variable that we do have in the original problem → phase 2: $\text{Min } z = 0.4x_1 + 0.5x_2$, $\text{Max } -z = -0.4x_1 - 0.5x_2$

## first phase

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0.3 | 0.1 | 1 | 0 | 0 | 0 | 2.7 |
| 0 | 0.5 | 0.5 | 0 | 0 | 1 | 0 | 6 |
| 0 | 0.6 | 0.4 | 0 | -1 | 0 | 1 | 6 |

Restoring Gaussian method

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS | |
|---|---|---|---|---|---|---|---|---|
| -1 | -1.1 | -0.9 | 0 | 1 | 0 | 0 | -12 | |
| 0 | 0.3 | 0.1 | 1 | 0 | 0 | 0 | 2.7 | 9 |
| 0 | 0.5 | 0.5 | 0 | 0 | 1 | 0 | 6 | 12 |
| 0 | 0.6 | 0.4 | 0 | -1 | 0 | 1 | 6 | 10 |
| | x1 is ent | | x3 is leav | | | | | |

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS | |
|---|---|---|---|---|---|---|---|---|
| -1 | 0 | -0.53333 | 3.666667 | 1 | 0 | 0 | -2.1 | |
| 0 | 1 | 0.333333 | 3.333333 | 0 | 0 | 0 | 9 | 27 |
| 0 | 0 | 0.333333 | -1.66667 | 0 | 1 | 0 | 1.5 | 4.5 |
| 0 | 0 | 0.2 | -2 | -1 | 0 | 1 | 0.6 | 3 |
| | x2 is ent | | xa2 is leav | | | | | |

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS | | |
|---|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | -1.66666667 | -1.66667 | 0 | 2.666667 | -0.5 | | |
| 0 | 1 | 0 | 6.66666667 | 1.666667 | 0 | -1.66667 | 8 | 1.2 | |
| 0 | 0 | 0 | 1.66666667 | 1.666667 | 1 | -1.66667 | 0.5 | 0.3 | |
| 0 | 0 | 1 | -10 | -5 | 0 | 5 | 3 | -0.3 | ignore |

we can take either $X_3$ or $X_4$ as the most negative

if we choose $X_4$

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 5 | 0 | -1 | 0 | 7.5 |
| 0 | 0 | 0 | 1 | 1 | 0.6 | -1 | 0.3 |
| 0 | 0 | 1 | -5 | 0 | 3 | 0 | 4.5 |

| z | x1 | x2 | x3 | x4 | RHS | | Restoring GF |
|---|---|---|---|---|---|---|---|
| -1 | 0.4 | 0.5 | 0 | 0 | 0 | | |
| 0 | 1 | 0 | 5 | 0 | 7.5 | | |
| 0 | 0 | 0 | 1 | 1 | 0.3 | | |
| 0 | 0 | 1 | -5 | 0 | 4.5 | | |

| z | x1 | x2 | x3 | x4 | RHS |
|---|---|---|---|---|---|
| -1 | 0 | 0 | 0.5 | 0 | -5.25 |
| 0 | 1 | 0 | 5 | 0 | 7.5 |
| 0 | 0 | 0 | 1 | 1 | 0.3 |
| 0 | 0 | 1 | -5 | 0 | 4.5 |

| z | x1 | x2 | x3 | x4 | xa1 | xa2 | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | -5 | -4 | 5 | 6 |
| 0 | 0 | 0 | 1 | 1 | 0.6 | -1 | 0.3 |
| 0 | 0 | 1 | 0 | 5 | 6 | -5 | 6 |

optimal & feasible ✓

second phase:

1) delete the artificial variables (after make it feasible & optimal in the first phase)

2) delete the objective row & replace it with
$\text{Max } -z = -0.4x_1 - 0.5x_2$

## Second phase

| z | x1 | x2 | x3 | x4 | RHS |
|---|---|---|---|---|---|
| -1 | 0.4 | 0.5 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | -5 | 6 |
| 0 | 0 | 0 | 1 | 1 | 0.3 |
| 0 | 0 | 1 | 0 | 5 | 6 |

we can not judge if it is feasible or optimal until Restoring

Restoring Gaussian method

| z | x1 | x2 | x3 | x4 | RHS |
|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | -0.5 | -5.4 |
| 0 | 1 | 0 | 0 | -5 | 6 |
| 0 | 0 | 0 | 1 | 1 | 0.3 |
| 0 | 0 | 1 | 0 | 5 | 6 |

feasible but not optimal

| z | x1 | x2 | x3 | x4 | RHS |
|---|---|---|---|---|---|
| -1 | 0 | 0 | 0.5 | 0 | -5.25 |
| 0 | 1 | 0 | 5 | 0 | 7.5 |
| 0 | 0 | 0 | 1 | 1 | 0.3 |
| 0 | 0 | 1 | -5 | 0 | 4.5 |

optimal & feasible ✓

**4.1-7.** Describe graphically what the simplex method does step by step to solve the following problem.

Minimize $\quad Z = 5x_1 + 7x_2 + MX_{a_1} + MX_{a_2} + MX_{a_3}$

subject to

$2x_1 + 3x_2 \geq 42 \longrightarrow 2X_1 + 3X_2 - X_3 + X_{a_1} = 42$

$3x_1 + 4x_2 \geq 60 \longrightarrow 3X_1 + 4X_2 - X_4 + X_{a_2} = 60$

$x_1 + x_2 \geq 18 \longrightarrow X_1 + X_2 - X_5 + X_{a_3} = 18$

and

$x_1 \geq 0, \qquad x_2 \geq 0.$

$\boxed{\text{Phase 1}} \longrightarrow$ Min $z = X_{a_1} + X_{a_2} + X_{a_3} \longrightarrow$ Max $-z + X_{a_1} + X_{a_2} + X_{a_3} = 0$

$\boxed{\text{Phase 2}} \longrightarrow$ Min $z = 5X_1 + 7X_2 \longrightarrow$ Max $-z + 5X_1 + 7X_2 = 0$

Note:

RHS for the objective row $= -\sum$ RHS for the rows that have the Basic artificial variables

$\boxed{\text{Phase 1}}$:

| z | x1 | x2 | x3 | x4 | x5 | xa1 | xa2 | xa3 | RHS |
|---|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 2 | 3 | -1 | 0 | 0 | 1 | 0 | 0 | 42 |
| 0 | 3 | 4 | 0 | -1 | 0 | 0 | 1 | 0 | 60 |
| 0 | 1 | 1 | 0 | 0 | -1 | 0 | 0 | 1 | 18 |

Restoring Gaussian method

| z | x1 | x2 | x3 | x4 | x5 | xa1 | xa2 | xa3 | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| -1 | -6 | -8 | 1 | 1 | 1 | 0 | 0 | 0 | -120 | |
| 0 | 2 | 3 | -1 | 0 | 0 | 1 | 0 | 0 | 42 | 14 |
| 0 | 3 | 4 | 0 | -1 | 0 | 0 | 1 | 0 | 60 | 15 |
| 0 | 1 | 1 | 0 | 0 | -1 | 0 | 0 | 1 | 18 | 18 |

x2 is ent var. xa1 is leaving

| z | x1 | x2 | x3 | x4 | x5 | xa1 | xa2 | xa3 | RHS | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| -1 | -0.66667 | 0 | -1.66667 | 1 | 1 | 2.666667 | 0 | 0 | -8 | | |
| 0 | 0.666667 | 1 | -0.33333 | 0 | 0 | 0.333333 | 0 | 0 | 14 | -42 | ignore it |
| 0 | 0.333333 | 0 | 1.333333 | -1 | 0 | -1.33333 | 1 | 0 | 4 | 3 | |
| 0 | 0.333333 | 0 | 0.333333 | 0 | -1 | -0.33333 | 0 | 1 | 4 | 12 | |

| z | x1 | x2 | x3 | x4 | x5 | xa1 | xa2 | xa3 | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| -1 | -0.25 | 0 | 0 | -0.25 | 1 | 1 | 1.25 | 0 | -3 | |
| 0 | 0.75 | 1 | 0 | -0.25 | 0 | 0 | 0.25 | 0 | 15 | 20 |
| 0 | 0.25 | 0 | 1 | -0.75 | 0 | -1 | 0.75 | 0 | 3 | 12 |
| 0 | 0.25 | 0 | 0 | 0.25 | -1 | 0 | -0.25 | 1 | 3 | 12 |

we can select any one of them as an entering variable (Tie for entering var)

we can select any one of them as an leaving variable (Tie for leaving var)

| z | x1 | x2 | x3 | x4 | x5 | xa1 | xa2 | xa3 | RHS |
|---|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | -1 | 3 | 0 | 1 | -3 | 6 |
| 0 | 0 | 0 | 1 | -1 | 1 | -1 | 1 | -1 | 0 |
| 0 | 1 | 0 | 0 | 1 | -4 | 0 | -1 | 4 | 12 |

feasible & optimal ✓

## Phase 2 :

| z | x1 | x2 | x3 | x4 | x5 | RHS |
|---|---|---|---|---|---|---|
| -1 | 5 | 7 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | -1 | 3 | 6 |
| 0 | 0 | 0 | 1 | -1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | -4 | 12 |

Restoring Gaussian method

| z | x1 | x2 | x3 | x4 | x5 | RHS | | |
|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 2 | -1 | -102 | | |
| 0 | 0 | 1 | 0 | -1 | 3 | 6 | 2 | |
| 0 | 0 | 0 | 1 | -1 | 1 | 0 | 0 | |
| 0 | 1 | 0 | 0 | 1 | -4 | 12 | -3 | ignore |

| z | x1 | x2 | x3 | x4 | x5 | RHS | | |
|---|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 2 | -1 | -102 | | |
| 0 | 0 | 1 | 0 | -1 | 3 | 6 | 2 | |
| 0 | 0 | 0 | 1 | -1 | 1 | 0 | 0 | |
| 0 | 1 | 0 | 0 | 1 | -4 | 12 | -3 | ignore |

| z | x1 | x2 | x3 | x4 | x5 | RHS |
|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 1 | 1 | 0 | -102 |
| 0 | 0 | 1 | -3 | 2 | 0 | 6 |
| 0 | 0 | 0 | 1 | -1 | 1 | 0 |
| 0 | 1 | 0 | 4 | -3 | 0 | 12 |

optimal & feasible ✓

x1=12
x2=6
z=102

this case called Degeneration

the RHS that i chose was zero → getting the same objective

x1=12
x2=6
z=102

* Notes:

# 4.5 Tie Breaking in the Simplex Method

Case 1
- Tie for the entering basic variable
  - Decision may be made arbitrarily

Case 2
- Tie for the leaving basic variable
  - Matters theoretically but rarely in practice
  - Choose arbitrarily

Case 3
- Condition of no leaving basic variable
  - Z is unbounded
  - Indicates a mistake has been made

ex:

| z | x1 | x2 | x3 | x4 | RHS | |
|---|---|---|---|---|---|---|
| 1 | -5 | -2 | 0 | 0 | 0 | |
| 0 | -1 | 2 | 1 | 0 | 2 | ignore |
| 0 | 0 | 1 | 0 | 1 | 4 | ignore |

Max z=5x1+2x2

Sub to

-x1+2x2<=2

X2<=4

X1,x2>=0

Unbounded obj

if we increases the coefficient to ∞ all constrains will be satisfied

∴ then

## Tie Breaking in the Simplex Method

- Multiple optimal solutions
  - Simplex method stops after one optimal BF solution is found
  - Often other optimal solutions exist and would be meaningful choices
  - Method exists to detect and find other optimal BF solutions

when one of the determinant constraints is parallel to the objective function → not always is a multiple optimal solutions

ex:

| z | x1 | x2 | x3 | x4 | x5 | RHS | |
|---|---|---|---|---|---|---|---|
| 1 | -3 | -2 | 0 | 0 | 0 | 0 | |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | 4 |
| 0 | 0 | 2 | 0 | 1 | 0 | 12 | |
| 0 | 3 | 2 | 0 | 0 | 1 | 18 | 6 |

| z | x1 | x2 | x3 | x4 | x5 | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | -2 | 3 | 0 | 0 | 12 |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| 0 | 0 | 2 | -3 | 0 | 1 | 6 |

| z | x1 | x2 | x3 | x4 | x5 | RHS |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 18 |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 0 | 0 | 0 | 3 | 1 | -1 | 6 |
| 0 | 0 | 1 | -1.5 | 0 | 0.5 | 3 |

optimal ✓

- the value of $X_3$ in the objective row is zero, which means:
  if i take it as an entering variable → the value of the objective will remain the same

| z | x1 | x2 | x3 | x4 | x5 | RHS | Optimal | z=18 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 18 | x1=4 | |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | x2=3 | |
| 0 | 0 | 0 | 3 | 1 | -1 | 6 | x4=6 | |
| 0 | 0 | 1 | -1.5 | 0 | 0.5 | 3 | | |

| z | x1 | x2 | x3 | x4 | x5 | RHS | | Optimal | z=18 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 18 | | x1=2 | |
| 0 | 1 | 0 | 0 | -0.33333 | 0.333333 | 2 | | x2=6 | |
| 0 | 0 | 0 | 1 | 0.333333 | -0.33333 | 2 | | x3=2 | |
| 0 | 0 | 1 | 0 | 0.5 | 0 | 6 | | | |

Two points gave us the same optimal solution

linear mathematical model convex space

all the points that on that line (4,3) to (2,6) they are optimal solutions as well

$\rightarrow$ How we can find this points?

$\rightarrow$ using interpolation

$\rightarrow$ $P_1 = (4,3)$

$P_2 = (2,6)$

$P_n = V P_1 + (1-V) P_2$

$V = [0,1]$

when $V = 0.3 \longrightarrow P_n = 0.3(4,3) + (1-0.3)(2,6)$

$P_n = (2.6, 5.1)$

must if we put $P_n$ in the objective give us 18

$Z = 3X_1 + 2X_2 = 3(2.6) + 2(5.1) = 18$

$\rightarrow$ the number this of points is $\infty$

* important notes :

$\boxed{\text{Case 1}}$ :

Max z=3x1+5x2 $\longrightarrow z=-3X_p+5X_2$

Subject to :

X1<=4 $\longrightarrow -X_p \leq 4$

X1+X2<=6 $\longrightarrow -X_p+X_2 \leq 6$

X1+3x2<=18 $\longrightarrow -X_p+3X_2 \leq 18$

X1<=0 $\quad X_p=-X_1 \longrightarrow X_p \geq 0 \longrightarrow X_1=-X_p$

X2>=0

$\hookrightarrow$ using simple x method

$\boxed{\text{case 2}}$ :

Max z=3x1+5x2 $\longrightarrow z=3(X_p-50)+5X_2$

Sub to

X1<=5 $\longrightarrow (X_p-50) \leq 5$

2x1+4x2<=10 $\longrightarrow 2(X_p-50)+4X_2 \leq 10$

X1>=-50 $\longrightarrow X_p=X_1+50 \geq 0 \longrightarrow X_1=X_p-50$

X2>=0

$\boxed{\text{Case 3}}$ :

- unconstrained $\longrightarrow (-\infty,\infty)$

Max z=3x1+4x2 $\longrightarrow z=3(X_{1p}-X_{1n})+4X_2$

Sub to

3x1+2x2<=10 $\longrightarrow 3(X_{1p}-X_{1n})+2X_2 \leq 10$

3x1+x2>=5 $\longrightarrow 3(X_{1p}-X_{1n})+X_2 \geq 5$

X1 is unconstrained

X2>=0

$X_1=\underbrace{X_{1p}}_{\text{positive}}-\underbrace{X_{1n}}_{\text{negative}}$ ex: if $X_1=10 \longrightarrow X_{1p}=10$, $X_{1n}=0$

if $X_1=-20 \longrightarrow X_{1p}=0$, $X_{1n}=20$

**4.6-2.** Consider the following problem.

Maximize $\quad Z = 4x_1 + 2x_2 + 3x_3 + 5x_4 - Mx_{a1} - Mx_{a2}$

subject to

$2x_1 + 3x_2 + 4x_3 + 2x_4 = 300 \longrightarrow 2x_1 + 2x_2 + 4x_3 + 2x_4 + x_{a1} = 300$

$8x_1 + \phantom{3}x_2 + \phantom{4}x_3 + 5x_4 = 300 \longrightarrow 8x_1 + x_2 + x_3 + 5x_4 + x_{a2} = 300$

and

$x_j \geq 0, \qquad$ for $j = 1, 2, 3, 4$.

$\boxed{\text{phase 1}}$

$\boxed{\text{phase 1}} \longrightarrow \min\ z = x_{a1} + x_{a2} \longrightarrow \max\ -z = -x_{a1} - x_{a2}$

$\boxed{\text{phase 2}} \longrightarrow \max\ z - 4x_1 - 2x_2 - 3x_3 - 5x_4 = 0$

**(a)** Using the Big $M$ method, construct the complete first simplex tableau for the simplex method and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

I **(b)** Work through the simplex method step by step to solve the problem.

**(c)** Using the two-phase method, construct the complete first simplex tableau for phase 1 and identify the corresponding initial (artificial) BF solution. Also identify the initial entering basic variable and the leaving basic variable.

I **(d)** Work through phase 1 step by step.

**(e)** Construct the complete first simplex tableau for phase 2.

I **(f)** Work through phase 2 step by step to solve the problem.

**(g)** Compare the sequence of BF solutions obtained in part (*b*) with that in parts (*d*) and (*f*). Which of these solutions are feasible only for the artificial problem obtained by introducing artificial variables and which are actually feasible for the real problem?

C **(h)** Use a software package based on the simplex method to solve the problem.

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 2 | 3 | 4 | 2 | 1 | 0 | 300 |
| 0 | 8 | 1 | 1 | 5 | 0 | 1 | 300 |

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | -10 | -4 | -5 | -7 | 0 | 0 | -600 |
| 0 | 2 | 3 | 4 | 2 | 1 | 0 | 300 |
| 0 | 1 | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{5}{8}$ | 0 | $\frac{1}{8}$ | $\frac{300}{8}$ |

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | $-\frac{11}{4}$ | $-\frac{15}{4}$ | $-\frac{3}{4}$ | 0 | $\frac{10}{8}$ | -225 |
| 0 | 0 | $\frac{11}{4}$ | $\frac{15}{4}$ | $\frac{3}{4}$ | 1 | $-\frac{2}{8}$ | 225 |
| 0 | 1 | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{5}{8}$ | 0 | $\frac{1}{8}$ | $\frac{300}{8}$ |

| Z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | $x_{a2}$ | RHS |
|---|---|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | $\frac{11}{15}$ | 1 | $\frac{1}{5}$ | $\frac{4}{15}$ | $-\frac{1}{15}$ | 60 |
| 0 | 1 | $\frac{1}{30}$ | 0 | $\frac{3}{5}$ | $-\frac{1}{30}$ | $\frac{2}{15}$ | 30 |

# Phase 2:

first phase

lead at $x_{1,3}$

Basic

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| 1 | -4 | -2 | -3 | -5 | 0 |
| 0 | 0 | $\frac{11}{15}$ | 1 | $\frac{1}{5}$ | 60 |
| 0 | 1 | $\frac{1}{30}$ | 0 | $\frac{3}{5}$ | 30 |

Restoring

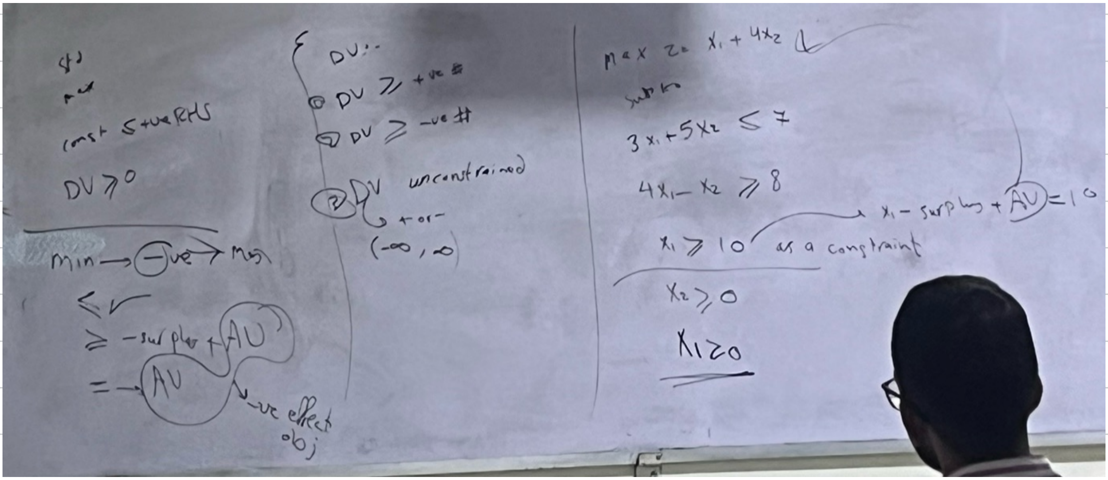
| | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|---|
| | 1 | 0 | $\frac{1}{3}$ | 0 | -2 | 300 |
| 3 | 0 | 0 | $\frac{11}{15}$ | 1 | $\frac{1}{5}$ | 60 |
| 4 | 0 | 1 | $\frac{1}{30}$ | 0 | $\frac{3}{5}$ | 30 |

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | RHS |
|---|---|---|---|---|---|
| 1 | $\frac{10}{3}$ | $\frac{4}{9}$ | 0 | 0 | 400 |
| 0 | $-\frac{1}{3}$ | $\frac{13}{18}$ | 1 | 0 | 50 |
| 0 | $\frac{5}{3}$ | $\frac{1}{18}$ | 0 | 1 | 50 |

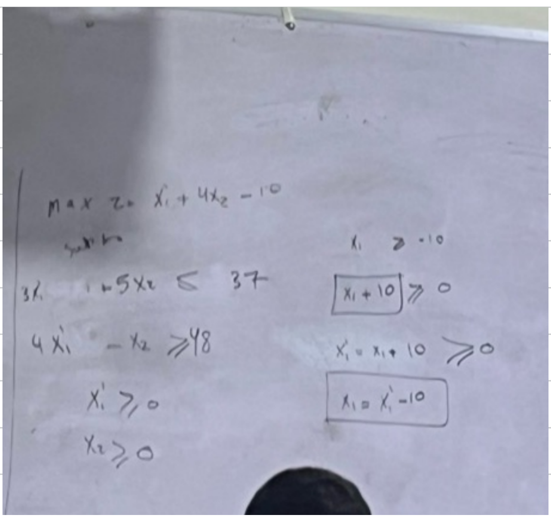
end

* Notes:

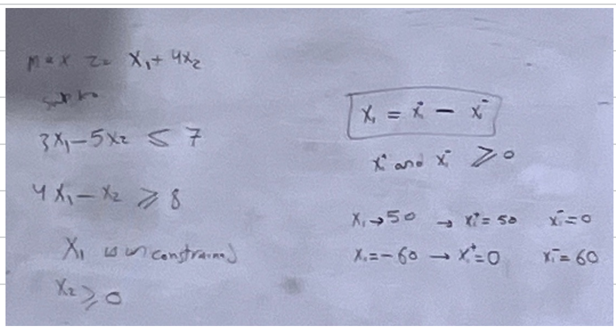
case 1




case 2:




case 3:



assignment in two phase method

$$\max z = x_1^+ - x_1^- + 4x_2$$

sub to

$$3x_1^+ - 3x_1^- - 5x_2 \leq 7$$

$$4x_1^+ - 4x_1^- - x_2 \geq 8$$

$$x_1^+ \geq 0$$

$$x_1^- \geq 0$$

$$x_2 \geq 0$$

$$\boxed{x_1 = x_1^+ - x_1^-}$$

$x_1^+$ and $x_1^-$ $\geq 0$

$x_1 \rightarrow 50 \rightarrow x_1^+ = 50 \quad x_1^- = 0$

$x_1 = -60 \rightarrow x_1^+ = 0 \quad x_1^- = 60$

Max z=x1+4x2
Sub to
3x1-5x2<=7
4x1-x2>=8
x1 unconstrained
x2>=0
Using two phase method

Me

# OR1 assignment 4

$x_1 = x_1^+ - x_1^-$

$z = x_1^+ - x_1^- + 4x_2 + Mx_{a1}$

$3x_1^+ - 3x_1^- - 5x_2 + x_3 = 7$

$4x_1^+ - 4x_1^- - x_2 - x_4 + x_{a1} = 8$

Phase 1 $\longrightarrow$ Min $z = x_{a1} \longrightarrow$ Max $-z + x_{a1} = 0$

Phase 2 $\longrightarrow$ Max $z = x_1^+ - x_1^- + 4x_2 \longrightarrow z - x_1^+ + x_1^- - 4x_2 = 0$

$\boxed{\text{Phase 1}}$ :

| | z | $x_1^+$ | $x_1^-$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | RHR |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $R_1$ | 0 | 3 | -3 | -5 | 1 | 0 | 0 | 7 |
| $R_2$ | 0 | 4 | -4 | -1 | 0 | -1 | 1 | 8 |

Restoring $\longrightarrow R_0 \rightarrow -R_2 + R_0$

| | z | $x_1^+$ | $x_1^-$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | RHR | |
|---|---|---|---|---|---|---|---|---|---|
| $R_0$ | -1 | -4 | 4 | 1 | 0 | 1 | 0 | -8 | |
| $R_1$ | 0 | 3 | -3 | -5 | 1 | 0 | 0 | 7 | $7/3 = 2.33$ |
| $R_2$ | 0 | 4 | -4 | -1 | 0 | -1 | 1 | 8 | $8/4 = 2$ |

$R_2 \longrightarrow R_2/4$ , $R_0 \longrightarrow 4R_2 + R_0$ , $R_1 \longrightarrow -3R_2 + R_1$

| | z | $x_1^+$ | $x_1^-$ | $x_2$ | $x_3$ | $x_4$ | $x_{a1}$ | RHR |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $R_1$ | 0 | 0 | 0 | -17/4 | 1 | 3/4 | -3/4 | 1 |
| $R_2$ | 0 | 1 | -1 | -1/4 | 0 | -1/4 | 1/4 | 2 |

optimal & feasible

$\boxed{\text{phase 2}}$ :

| | z | $x_1^+$ | $x_1^-$ | $x_2$ | $x_3$ | $x_4$ | RHR |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | -1 | 1 | 4 | 0 | 0 | 0 |
| $R_1$ | 0 | 0 | 0 | -17/4 | 1 | 3/4 | 1 |
| $R_2$ | 0 | 1 | -1 | -1/4 | 0 | -1/4 | 2 |

Restoring $\longrightarrow$ $R_0 \rightarrow R_0 + R_2$

| | z | $x_1^+$ | $x_1^-$ | $x_2$ | $x_3$ | $x_4$ | RHR |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 0 | 0 | -17/4 | 0 | -1/4 | 2 |
| $R_1$ | 0 | 0 | 0 | -17/4 | 1 | 3/4 | 1 |
| $R_2$ | 0 | 1 | -1 | 1/4 | 0 | -1/4 | 2 |

| | z | $x_1^+$ | $x_1^-$ | $x_2$ | $x_3$ | $x_4$ | RHR |
|---|---|---|---|---|---|---|---|
| $R_0$ | 1 | 17 | -17 | 0 | 0 | -9/2 | 36 |
| $R_1$ | 0 | 17 | -17 | 0 | 1 | -7/2 | 35 |
| $R_2$ | 0 | -4 | -4 | 1 | 0 | -1 | 8 |

$\boxed{\text{unbounded}}$

# Matrices

ch 5

## Ch5

Max z= 3x1+5x2

Subject to

X1<=4 $\longrightarrow$ $X_1 + X_3 = 4$

2x2<=12 $\longrightarrow$ $2X_2 + X_4 = 12$

3x1+2x2<=18 $\longrightarrow$ $3X_1 + 2X_2 + X_5 = 18$

X1,x2>=0

## The Simplex Method in Matrix Form

- Standard form for the linear programming model
  - In matrix form

Maximize $Z = \mathbf{cx}$,

subject to

$\mathbf{Ax} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$,

where **c** is the row vector

$$\mathbf{c} = [c_1, c_2, \ldots, c_n],$$

**x**, **b**, and **0** are the column vectors such that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and **A** is the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$





Parameters of z are 3 & 5

(original variables)

$C = [3 \quad 5]$ : coefficients of the decision variables in the objective

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}$ : coefficients of the decision variables in the constraints

$b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$ : RHS of the constraints

$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

## 5.3 A Fundamental Insight

- Coefficients of the slack variables
  - Reveal how the entire rows of the current simplex tableau were obtained from the rows in the initial tableau

standard form

■ **TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Z | Coefficient of: Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-\mathbf{c}$ | $\mathbf{0}$ | 0 |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{A}$ | $\mathbf{I}$ | $\mathbf{b}$ |
| Any | $Z$ | (0) | 1 | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

obj row

$[\quad]_{1 \times n} * [\quad]_{n \times 1}$

Max $Z = CX$ (row matrix * column matrix) = one value (Scaler) $_{(1\times1)}$ (the value of the objective)

Subject to:

$AX = b$

$X \geq 0$

$[A \mid I] * \begin{bmatrix} X \\ X_S \end{bmatrix} = b$

from the slack variables

$X_S \longrightarrow$ slack variables

coefficient of the original variables in the objective

the RHS of any simplex tableau

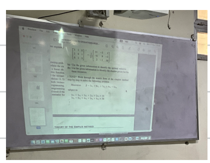
B : coefficients of the basic variables in the constraints $\rightarrow \boxed{m \times m}$

$C_B$ : coefficients of the basic variables in the objective

$\boxed{\text{\# of Basic variables = \# of constraints}}$

| z | x1 | x2 | x3 | x4 | x5 | RHS |
|---|---|---|---|---|---|---|
| 1 | -3 | -5 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| 0 | 3 | 2 | 0 | 0 | 1 | 18 |

Simplex tableau



$$\begin{array}{cccc} & x_3 & x_4 & x_5 \\ B = & \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \end{array}$$

- There are 3 types of matrices that are fixed: (Along the solution)
  - → $c$ & $A$ & $b$

$C_B = [0 \ \ 0 \ \ 0]$

- To know if the tableau is optimal or not
  - → I have to know the coefficient of the original variables & the slack variables in the objective.

$C_B * B^{-1} * A - C = -C = [-3 \ \ -5]$

$C_B * B^{-1} = \text{zero matrix} = [0 \ \ 0 \ \ 0]$

zero matrix

Identity

$C_B * B^{-1} = \text{zero matrix}$

$\text{zero matrix} * A = \text{zero matrix}$

$\text{zero matrix} - C = -C$

$B = B^{-1}$ here because it is an identity here

→ not optimal because there are negative coefficient
  - → the most negative is $-5$ which is related to $X_2$
    - → then $X_2$ is entering variable

- To know the leaving variable
  - → I have to find the coefficient of the most negative ($X_2$) here

$$B^{-1} * A = \begin{array}{cc} x_1 & x_2 \\ \left[\begin{matrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{matrix}\right] \end{array}$$

$$B^{-1} * b = \left[\begin{matrix} 4 \\ 12 \\ 18 \end{matrix}\right]$$

$$(\text{RHS} / \text{coef of ent var}) = \left[\begin{matrix} 4/0 \\ 12/2 \\ 18/2 \end{matrix}\right] \begin{matrix} \text{ignore} \\ \text{min} \\ \ \end{matrix}$$

select the minimum which is the second row here

→ then the leaving variable is $X_4$

the same column of $X_2$ from the original matrix

then $B$ & $C_B$ will be $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ , $C_B = [0 \quad 5 \quad 0]$

- we will Repeat all the steps

c & A & b fixed

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$C_B = [0 \quad 5 \quad 0]$$

$$C_B B^{-1} * A - C = \begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ [-3 \quad 0] \end{array}$$

$$C_B B^{-1} = \begin{array}{c} \begin{matrix} x_3 & x_4 & x_5 \end{matrix} \\ [0 \quad 2.5 \quad 0] \end{array}$$

not optimal

entering variable $\longrightarrow X_1$

$$B^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B^{-1} * b = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} \qquad \text{RHS/coff of ent var} = \begin{bmatrix} 4/1 \\ 6/0 \\ 6/3 \end{bmatrix} \text{min}$$

$\therefore X_5$ is leaving variable

$$C_B * B^{-1} * b = [30]$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad C_B = [0 \quad 5 \quad 3]$$

coefficient of the original var $= C_B * B^{-1} * A - C = [0 \quad 0]$

" " " slack " $= C_B B^{-1} = [0 \quad 1.5 \quad 1]$

$\Big\}$ optimal ✓

RHS : $B^{-1} * b = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$

objective value : $C_B * B^{-1} * b = [36]$

| Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $c_B B^{-1}$ | | | |
| $Z$ | (0) | 1 | | $c_B B^{-1} A - c$ | | 1 | 1 | 0 | $c_B B^{-1} b$ |
| $x_2$ | (1) | 0 | | | | 1 | 3 | 0 | |
| $x_6$ | (2) | 0 | | $B^{-1} A$ | | 0 | 1 | 1 | $B^{-1} b$ |
| $x_3$ | (3) | 0 | | | | 1 | 2 | 0 | |

$B^{-1}$

**5.1-4.** Consider the following problem.

Maximize $\quad Z = 2x_1 - x_2 + x_3,$

subject to

$$3x_1 + x_2 + x_3 \le 60$$
$$x_1 - x_2 + 2x_3 \le 10$$
$$x_1 + x_2 - x_3 \le 20$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

After slack variables are introduced and then one complete iteration of the simplex method is performed, the following simplex tableau is obtained.

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Z$ | (0) | 1 | 0 | −1 | 3 | 0 | 2 | 0 | 20 |
| | $x_4$ | (1) | 0 | 0 | 4 | −5 | 1 | −3 | 0 | 30 |
| | $x_1$ | (2) | 0 | 1 | −1 | 2 | 0 | 1 | 0 | 10 |
| | $x_6$ | (3) | 0 | 0 | 2 | −3 | 0 | −1 | 1 | 10 |

**(a)** Identify the CPF solution obtained at iteration 1.
**(b)** Identify the constraint boundary equations that define this CPF solution.

$c = [2 \ -1 \ 1]$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ B = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$c_B = [0 \ 0 \ 0]$

$$\left.\begin{aligned} c_B * B^{-1} * A - C &= [-2 \ \ 1 \ -1] \\ c_B * B^{-1} &= [0 \ 0 \ 0] \end{aligned}\right\}$$

not optimal

$x_1$ is an entering variable

$$B^{-1} * A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$x_5$ is leaving variable

$$B^{-1} b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_B = [0 \;\; 2 \;\; 0]$$

$$\left.\begin{array}{l} C_B * B^{-1} * A - C = [0 \;\; -1 \;\; 3] \\ C_B * B^{-1} = [0 \;\; 2 \;\; 0] \end{array}\right\}$$

not optimal

$x_2$ is entering variable

$$B^{-1} * A = \begin{bmatrix} 0 & 4 & -5 \\ 1 & -1 & 2 \\ 0 & 2 & -5 \end{bmatrix}$$

$$B^{-1} * b = \begin{bmatrix} 30 \\ 10 \\ 10 \end{bmatrix} \rightarrow \text{RHS / coef of int var} = \left.\begin{array}{lll} 30/4 & = 7.5 & \\ 10/-1 & = -10 & \text{ignore} \\ 10/2 & = 5 & \text{min} \end{array}\right\}$$

$x_6$ is leaving variable

$$C_B * B^{-1} * b = [20]$$

$$B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_B = [0 \;\; 2 \;\; -1]$$

$$\left.\begin{array}{l} C_B * B^{-1} * A - C = [0 \;\; 0 \;\; 1.5] \\ C_B * B^{-1} = [0 \;\; 1.5 \;\; 0.5] \end{array}\right\}$$

optimal ✓

$$B^{-1} * b = \begin{array}{c} \begin{bmatrix} 10 \\ 15 \\ 5 \end{bmatrix} \end{array} \begin{array}{c} x_4 \\ x_1 \\ x_2 \end{array}$$

} value of the variables

$$C_B * B^{-1} * b = [25]$$

} value of the optimal value

$x_3$
$x_1$
$x_2$

a) Identify the corner points feasible (CPF)

write the values of the original variables?

at this iteration: $x_1 = 10$, $x_2 = 0$, $x_3 = 0$

b) Identify the constraint boundary eq that define this CPF solution?

which of the constraints that the slack variables have zero value

$x_5$ that related to the second constraint

**5.1-15.** Consider the following problem.

$$\text{Maximize} \quad Z = 3x_1 + 4x_2 + 2x_3,$$

subject to

$$x_1 + x_2 + x_3 \le 20$$
$$x_1 + 2x_2 + x_3 \le 30$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let $x_4$ and $x_5$ be the slack variables for the respective functional constraints. Starting with these two variables as the basic variables for the initial BF solution, you now are given the information that the simplex method proceeds as follows to obtain the optimal solution in two iterations: (1) In iteration 1, the entering basic variable is $x_2$ and the leaving basic variable is $x_5$; (2) in iteration 2, the entering basic variable is $x_1$ and the leaving basic variable is $x_4$.

Follow the instructions of Prob. 5.1-14 for this situation.

$$c = [3 \quad 4 \quad 2]$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

$$B = \begin{array}{cc} x_4 & x_5 \\ \hline \end{array}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_B = \begin{array}{c} x_4 \quad x_5 \\ [0 \quad 0] \end{array}$$

In the first iteration $\longrightarrow$ $x_2$ is entering var, $x_5$ is leaving var $\Big\}$ that means $\longrightarrow$ $B = \begin{array}{c} x_4 \quad x_2 \\ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \end{array}$, $C_B = [0 \quad 4]$

In the second iteration $\longrightarrow$ $x_1$ is entering var, $x_4$ is leaving var $\Big\}$ that means $\longrightarrow$ $B = \begin{array}{c} x_1 \quad x_2 \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$, $C_B = [3 \quad 4]$

to find the variables value $\longrightarrow$ $B^{-1} * b = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \longrightarrow x_1 = 10, \; x_2 = 10$

to find the objective value $\longrightarrow$ $C_B * B^{-1} * b = [70]$

if the question doesn't tell me the this is optimal $\longrightarrow$ i have to check it

$$C_B * B^{-1} * A - C = [0 \quad 0 \quad 1]$$
$$C_B * B^{-1} = [2 \quad 1]$$

optimal ✓

**5.1-14.** Consider the following problem.

$$\text{Maximize} \quad Z = 2x_1 + 2x_2 + 3x_3,$$

subject to

$$2x_1 + x_2 + 2x_3 \le 4$$
$$x_1 + x_2 + x_3 \le 3$$

and

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

Let $x_4$ and $x_5$ be the slack variables for the respective functional constraints. Starting with these two variables as the basic variables for the initial BF solution, you now are given the information that the simplex method proceeds as follows to obtain the optimal solution in two iterations: (1) In iteration 1, the entering basic variable is $x_3$ and the leaving basic variable is $x_4$; (2) in iteration 2, the entering basic variable is $x_2$ and the leaving basic variable is $x_5$.

**(a)** Develop a three-dimensional drawing of the feasible region for this problem, and show the path followed by the simplex method.

**(b)** Give a geometric interpretation of why the simplex method followed this path.

**(c)** For each of the two edges of the feasible region traversed by the simplex method, give the equation of each of the two constraint boundaries on which it lies, and then give the equation of the additional constraint boundary at each endpoint.

**(d)** Identify the set of defining equations for each of the three CPF solutions (including the initial one) obtained by the simplex method. Use the defining equations to solve for these solutions.

**(e)** For each CPF solution obtained in part ($d$), give the corresponding BF solution and its set of nonbasic variables. Explain how these nonbasic variables identify the defining equations

$c = [2 \ 2 \ 3]$

$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C_B = [0 \ 0]$

Identify the constraint boundary eq that define this CPF solution ?

which of the constraints that the slack variables have zero value

first constraint $\xrightarrow{\text{bec}} x_4 = 0$

sec " $\xrightarrow{\text{bec}} x_5 = 0$

$x_1 \geq 0 \xrightarrow{\text{bec}} x_1 = 0$

First iteration $\longrightarrow$ $x_3$ then $x_4$

Second iteration $\longrightarrow$ $x_2$ then $x_5$

$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ , $C_B = [3 \ 2]$

$B^{-1} * b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow x_3 = 1, \ x_2 = 2$

$C_B * B^{-1} * b = [7]$

the coefficient of the slack variables in this tableau represent $\longrightarrow$ The shadow price

To double check if it is optimal or not $\longrightarrow$ $C_B * B^{-1} - C = [1 \ 0 \ 0]$

$C_B * B^{-1} = [1 \ 1]$

optimal

D **5.3-1.*** Consider the following problem.

Maximize $\quad Z = x_1 - x_2 + 2x_3,$

subject to

$$\begin{aligned} 2x_1 - 2x_2 + 3x_3 &\le 5 \\ x_1 + x_2 - x_3 &\le 3 \\ x_1 - x_2 + x_3 &\le 2 \end{aligned}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 \ge 0.$$

Let $x_4$, $x_5$, and $x_6$ denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

| Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| | | | Coefficient of: | | | | | | |
| $Z$ | (0) | 1 | $C_B B^{-1} A - C$ | | | 1 | 1 | 0 | $C_B B^{-1} b$ |
| $x_2$ | (1) | 0 | $B^{-1}A$ | | | 1 | 3 | 0 | $B^{-1} b$ |
| $x_6$ | (2) | 0 | | | | 0 | 1 | 1 | |
| $x_3$ | (3) | 0 | | | | 1 | 2 | 0 | |

$c_B B^{-1}$

$B^{-1} \rightarrow BV$

**(a)** Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations.

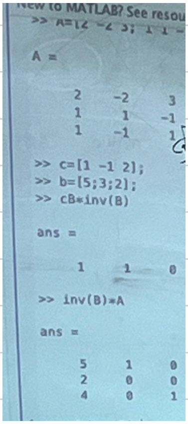
**(b)** Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

$c = (1, -1, 2)$

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \qquad X_b = \begin{bmatrix} x_2 \\ x_6 \\ x_3 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \qquad cB = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}$$



D **5.3-3.** Consider the following problem.

Maximize $\quad Z = 6x_1 + x_2 + 2x_3,$

subject to

$$2x_1 + 2x_2 + \frac{1}{2}x_3 \leq 2$$

$$-4x_1 - 2x_2 - \frac{3}{2}x_3 \leq 3$$

$$x_1 + 2x_2 + \frac{1}{2}x_3 \leq 1$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Let $x_4$, $x_5$, and $x_6$ denote the slack variables for the respective constraints. After you apply the simplex method, a portion of the final simplex tableau is as follows:

| Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | 0 | | 0 | 2 | 0 | 2 | |
| $x_5$ | (1) | 0 | 0 | | 0 | 1 | 1 | 2 | |
| $x_3$ | (2) | 0 | 0 | | 1 | −2 | 0 | 4 | |
| $x_1$ | (3) | 0 | 1 | | 0 | 1 | 0 | −1 | |

$c_B B^{-1} A - c$

$B^{-1} A$

$c_B B^{-1} b$

$B^{-1} b$

$c = [6 \ \ 1 \ \ 2]$

$A = \begin{bmatrix} 2 & 2 & 1/2 \\ -4 & -2 & -3/2 \\ 1 & 2 & 1/2 \end{bmatrix}$

$c_B = [0 \ \ 2 \ \ 6]$

$b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$c_B B^{-1} A - c$

$B^{-1}A = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$

$B^{-1}b = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = [0 \ 2 \ 6] \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$

$C = [6 \ \ 1 \ \ 2]$

$A = \begin{bmatrix} 2 & 2 & 1/2 \\ -4 & -2 & -3/2 \\ 1 & 2 & 1/2 \end{bmatrix}$

$b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$B = \begin{array}{c} \begin{array}{ccc} x_5 & x_3 & x_1 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 2 \\ 1 & -3/2 & -4 \\ 0 & 1/2 & 1 \end{bmatrix} \end{array}$

$c_B = [0 \ \ 2 \ \ 6]$

$c_B B^{-1} A - C = [0 \ \ 7 \ \ 0]$

$B^{-1} A = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$c_B B^{-1} b = [6]$

$B^{-1} b = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$

# chapter

# 6

ch6

# 6.1 The Essence of Duality Theory

- Every linear programming problem has an associated problem called the dual
  - Original problem is known as the primal
  - These relationships prove useful in a variety of ways
- Consider a maximization primal problem in standard form
  - Dual is a minimization problem
  - Dual uses same parameters in different locations

Original problem → Primely problem
Second problem → Dual Problem
→ equivalent problems
- both of them will lead to the same optimed solution
- there is some relationships between the dual & the primal

(Primely)
Original problem → maximization
Dual problem → Minimization

- maximization → Minimization
- the same parameters but in different locations

parameters → values in the second stage from the development of the mathematical model (collecting the data)

Decision variables → we don't know their values (we have to find them by solving the mathematical model)

## 3.1 PROTOTYPE EXAMPLE

The WYNDOR GLASS CO. produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing
Product 2: A 4 × 6 foot double-hung wood-framed window

Product 1 requires some of the production capacity in Plants 1 and 3, but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which *mix* of the two products would be *most profitable.* Therefore, an OR team has been formed to study this question.

product($X_1$)
product($X_2$)

The OR team began by having discussions with upper management to identify management's objectives for the study. These discussions led to developing the following definition of the problem:

> target
>
> Determine what the *production rates* should be for the two products in order to *maximize their total profit,* subject to the restrictions imposed by the limited production capacities available in the three plants. (Each product will be produced in batches of 20, so the *production rate* is defined as the number of batches produced per week.) *Any* combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

unit time

The OR team also identified the data that needed to be gathered:

1. Number of hours of production time available per week in each plant for these new products. (Most of the time in these plants already is committed to current products, so the available capacity for the new products is quite limited.)
2. Number of hours of production time used in each plant for each batch produced of each new product.
3. Profit per batch produced of each new product. (*Profit per batch produced* was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly *constant* regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products, the total profit from each one is approximately this *profit per batch produced* times *the number of batches produced.*)

Obtaining reasonable estimates of these quantities required enlisting the help of key personnel in various units of the company. Staff in the manufacturing division provided the data in the first category above. Developing estimates for the second category of data required some analysis by the manufacturing engineers involved in designing the production processes for the new products. By analyzing cost data from these same engineers and the marketing division, along with a pricing decision from the marketing division, the accounting department developed estimates for the third category.

Table 3.1 summarizes the data gathered.

The OR team immediately recognized that this was a linear programming problem of the classic **product mix** type, and the team next undertook the formulation of the corresponding mathematical model.

**TABLE 3.1** Data for the Wyndor Glass Co. problem

| | Production Time per Batch, Hours (Capacities) | | |
|---|---|---|---|
| | Product | | |
| Plant | 1 | 2 | Production Time Available per Week, Hours (Capacities) |
| 1 | 1 hour per one batch | 0 hour | 4 |
| 2 | 0 hour | 2 hour | 12 |
| 3 | 3 hour | 2 hour | 18 |
| Profit per batch | $3,000 | $5,000 | |

### Formulation as a Linear Programming Problem

The definition of the problem given above indicates that the decisions to be made are the

**TABLE 6.1** Primal and dual problems for the Wyndor Glass Co. example

*Primal Problem in Algebraic Form*

Maximize $Z = 3x_1 + 5x_2$,
subject to
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
and $x_1 \ge 0, \quad x_2 \ge 0.$

*Dual Problem in Algebraic Form*

Minimize $W = 4y_1 + 12y_2 + 18y_3$,
subject to
$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$
and
$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

*Primal Problem in Matrix Form*

Maximize $Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,
subject to
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$
and
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

*Dual Problem in Matrix Form*

Minimize $W = [y_1, y_2, y_3]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$
subject to
$$[y_1, y_2, y_3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \ge [3, 5]$$
and
$$[y_1, y_2, y_3] \ge [0, 0, 0].$$

# Dual model

1) Primal model

max $z = 3X_1 + 5X_2$ — dual decision variables

$1X_1 + 0X_2 <= 4 \longrightarrow y_1$

$0X_1 + 2X_2 <= 12 \longrightarrow y_2$

$3X_1 + 2X_2 <= 18 \longrightarrow y_3$

$X_1 >= 0$

$X_2 >= 0$

Dual problem
identify new decision variables

the number of constaints in the primal problem = the number of decision variable in the dual problem

each constraint in the primal problem is corresponding to a decision variable in the dual problem

- each constraint in the dual problem is related to a decision variable in the primal problem

- if the primal problem was maximization the dual problem will be minimization
- if the primal problem was minimization the dual problem will be maximization

عكس بعض دايمًا

- the objective in dual $\longrightarrow$ (RHS for each constraint from the primal * Decision variable of the Dual)

2) Dual model

- Any model have Dual model

min $w = 4y_1 + 12y_2 + 18y_3$

sub to: standard form

$y_1 + 0y_2 + 3y_3 \geq 3$

$0y_1 + 2y_2 + 2y_3 \geq 5$

$y_1, y_2, y_3 \geq 0$

- (Coefficients of $X_1$ in the three Constraints * corresponding dual decision variable)
- the RHS $\longrightarrow$ the coefficient of $X_1$ in the objective in the primal problem

## The Essence of Duality Theory

| *Primal Problem* | *Dual Problem* |
|---|---|
| Maximize $Z = \sum_{j=1}^{n} c_j x_j,$ | Minimize $W = \sum_{i=1}^{m} b_i y_i,$ |
| subject to | subject to |
| $\sum_{j=1}^{n} a_{ij} x_j \leq b_i,$ for $i = 1, 2, \ldots, m$ | $\sum_{i=1}^{m} a_{ij} y_i \geq c_j,$ for $j = 1, 2, \ldots, n$ |
| and | and |
| $x_j \geq 0,$ for $j = 1, 2, \ldots, n.$ | $y_i \geq 0,$ for $i = 1, 2, \ldots, m.$ |

## The Essence of Duality Theory

- Coefficients in the objective function of the primal problem:
  - Are right-hand sides of the functional constraints in the dual problem
- Right-hand sides of the functional constraints in the primal problem:
  - Are the coefficients in the objective function of the dual problem

## The Essence of Duality Theory

- Coefficients of a variable in the functional constraints of the primal problem:
  - Are the coefficients in a functional constraint of the dual problem

ex:

Primal model:

max $z = 4x_1$

Sub to:

$x_1 + 2x_2 \leq 5 \longrightarrow y_1$

$x_2 \leq 3 \longrightarrow y_2$

$x_1, x_2 \geq 0$

Me

Dual model:

min $w = 5y_1 + 3y_2$

Sub to:

$y_1 + 0y_2 \geq 4$

$2y_1 + 1y_2 \geq 0$

$y_1, y_2 \geq 0$

---

ex:

Primal model:

max $z = 4x_1 + x_2$

Sub to:

$3x_1 + x_3 \leq 5 \longrightarrow y_1$

$2x_2 + 3x_3 \leq 7 \longrightarrow y_2$

$x_1, x_2, x_3 \geq 0$

Me

Dual model:

min $w = 5y_1 + 7y_2$

Sub to:

$3y_1 \geq 4$

$2y_2 \geq 1$

$y_1 + 3y_2 \geq 0$

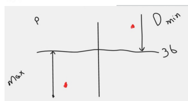
$y_1, y_2 \geq 0$

# The Essence of Duality Theory

- Origin of the dual problem
  - Duality theory based on the fundamental insight presented in Chapter 5
- Summary of primary-dual relationships
  - Weak duality property
  - Strong duality property
  - Complementary solutions property
  - Complementary optimal solutions property
  - Symmetry property

# The Essence of Duality Theory

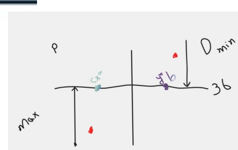
- Summary of primary-dual relationships (cont'd.)
  - Duality theorem




- Weak duality property
  - If **x** is a feasible solution for the primal problem and **y** is a feasible solution for the dual problem, then **cx ≤ yb**.

# The Essence of Duality Theory

- Strong duality property
  - If **x*** is an optimal solution for the primal problem and **y*** is an optimal solution for the dual problem, then **cx* = y*b**.




- Complementary solutions property
  - At each iteration, the simplex method simultaneously identifies a CPF solution **x** for the primal problem and a complementary solution **y** for the dual problem
    - Where **cx = yb**

feasible in primal
infeasible in dual


the values of $y_{1,2,3\ldots}$ → is the coefficients of the slack variables in the objective

& represent the shadow price also

| | z | x1 | x2 | x3 | x4 | x5 | RHS |
|---|---|---|---|---|---|---|---|
| R0 | 1 | -3 | -5 | 0 | 0 | 0 | 0 |
| R1 | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| R2 | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| R3 | 0 | 3 | 2 | 0 | 0 | 1 | 18 |

## The Essence of Duality Theory

- Complementary solutions property (cont'd.)
  - If **x** is not optimal for the primal problem, then **y** is not feasible for the dual problem
- Complementary optimal solutions property
  - The simplex method identifies (at its final iteration) an optimal solution **x*** for the primal problem and a complementary optimal solution **y*** for the dual problem
    - Where **cx*** = **y*b**

## The Essence of Duality Theory

- Symmetry property
  - For any primal problem and its dual problem
    - All relationships between them must be symmetric

feasible in primal ⟷ infeasible in dual

infeasible in primal ⟷ feasible in dual

- Duality theorem
  - Identifies the only possible relationships between the primal and dual problems
  - If one problem has feasible solutions and a bounded objective function, then so does the other problem
    - Both weak and strong duality properties apply

if Primal was bounded & feasible (more than one sol) → the dual will be bounded & Feasible

## The Essence of Duality Theory

- Duality theorem (cont'd.)
  - If one problem has feasible solutions and an unbounded objective function, then the other problem has no feasible solutions
  - If one problem has no feasible solutions, then the other problem either has no feasible solutions or an unbounded objective function

primal has a feasible solution & unbounded → the dual will be infeasible

feasible region → the objective is sub optimality

-above the optimal solution → super optimal (but infeasible)

## The Essence of Duality Theory

- Applications
  - Dual problem can be solved directly by the simplex method to identify an optimal solution for the primal problem
    - Can be useful if one of the problems has fewer functional constraints
  - Evaluation of a proposed solution for the primal problem
  - Economic interpretation of the dual problem
    - Insights for the primal problem

# Primal-Dual Relationships

**TABLE 6.10** Classification of basic solutions

| | | Satisfies Condition for Optimality? | |
|---|---|---|---|
| | | **Yes** | **No** |
| **Feasible?** | **Yes** | Optimal | Suboptimal |
| | **No** | Superoptimal | Neither feasible nor superoptimal |

**TABLE 6.11** Relationships between complementary basic solutions

| Primal Basic Solution | Complementary Dual Basic Solution | Both Basic Solutions: Primal Feasible? | Both Basic Solutions: Dual Feasible? |
|---|---|---|---|
| Suboptimal | Superoptimal | Yes | No |
| Optimal | Optimal | Yes | Yes |
| Superoptimal | Suboptimal | No | Yes |
| Neither feasible nor superoptimal | Neither feasible nor superoptimal | No | No |

primal has a feasible solution & unbounded → the dual will be infeasible

feasible region → the objective is sub optimality

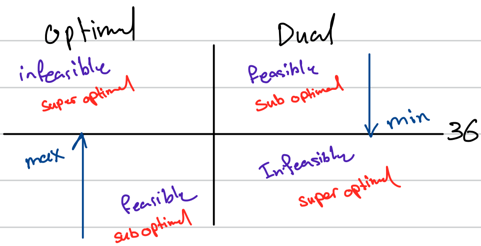
-above the optimal solution → super optimal (but infeasible)

feasible & optimal → optimal sol

feasible & not optimal → sub optimal

infeasible & optimal → super optimal

infeasible & not optimal → Neither feasible nor super optimal



- If the mathematical model is not written in the standard form we have to use:

## Adapting to Other Primal Forms

- Sensible-odd-bizarre (SOB) method for determining the form of constraints in the dual
  - Formulate the primal problem in either maximization or minimization form
    - Dual problem will be in other form
  - Label the different forms of the functional and variable constraints as being sensible, odd, or bizarre
    - See Table 6.14 for guidance

# *SOB Method :

**max**

const
- $\leq$ (S) sensible
- $=$ (O) ood
- $\geq$ (B) Biazrre

- if the constsraint Sensible its dual decision variable will be Sensible

⋮

DV
- $\geq 0$ (S) sensible
- $\leq 0$ (B) Biazrre
- unconstraint (O) ood

**min**

const
- $\leq$ (B) Biazrre
- $=$ (O) ood
- $\geq$ (S) sensible

DV
- $\geq 0$ (S) sensible
- $\leq 0$ (B) Biazrre
- unconstraint (O) ood

ex:

Max z=3x1+4x2-x3

Subject to:

X1+2x2+x3>=10 $\rightarrow y_1 \rightarrow$ Bizarre (B) $\xrightarrow{\text{when}}$ $y_1 \leq 0$

X1+x2+x3<=20 $\rightarrow y_2 \rightarrow$ Sensible (S) $\xrightarrow{\text{when}}$ $y_2 \geq 0$

X2+3x3=15 $\rightarrow y_3 \rightarrow$ odd (O) $\rightarrow$ $y_3$ is unconstrained

X1>=0 S

X2 <=0 B

X3 is unconstrained O

M

Dual model :

min $w = 10y_1 + 20y_2 + 15y_3$

Sub to :

related to $x_1$: $y_1 + y_2 \geq 3$

related to $x_2$: $2y_1 + y_2 + y_3 \leq 4$

related to $x_3$: $y_1 + y_2 + 3y_3 = -1$

$y_1 \leq 0$

$y_2 \geq 0$

$y_3$ is unconstrained

ex:

Min z=3x1+4x2-x3

Subject to:

X1+2x2+x3>=10 $\longrightarrow y_1 \longrightarrow$ S when $y_1 \geq 0$

X1+x2+x3<=20 $\longrightarrow y_2 \longrightarrow$ B when $y_2 \leq 0$

X2+3x3=15 $\longrightarrow y_3 \longrightarrow$ O when $y_3$ is unconstrained

X1>=0 S

X2 <=0 B

X3 is unconstrained O

Me

Dual model

max $w = 10y_1 + 20y_2 + 15y_3$

sub to:

related to $x_1$ $y_1 + y_2 \leq 3$

related to $x_2$ $2y_1 + y_2 + y_3 \geq 4$

related to $x_3$ $y_1 + y_2 + 3y_3 = -1$

$y_1 \geq 0$

$y_2 \leq 0$

$y_3$ is unconstrained

Question 2: For the following linear programming problem, use the SOB method to construct its dual problem.

$$\begin{array}{ll} \textit{Maximize} & Z = 3x_1 + 4x_2 \\ \textit{Subject to} & \end{array}$$

$$\begin{array}{llll}
4x_1 + 3x_2 \geq 4 & \longrightarrow y_1 & B & \xrightarrow{\text{when}} y_1 \leq 0 \\
2x_1 + 3x_2 = 6 & \longrightarrow y_2 & O & \longrightarrow y_2 \text{ is unconstrained} \\
4x_1 + x_2 \leq 4 & \longrightarrow y_3 & S & \longrightarrow y_3 \geq 0 \\
\underset{S}{x_1 \geq 0}, \quad \underset{B}{x_2 \leq 0} & & &
\end{array}$$

Dual :

Min $w = 4y_1 + 6y_2 + 4y_3$

Sub to:

$4y_1 + 2y_2 + 4y_3 \geq 3$

$3y_1 + 3y_2 + y_3 \leq 4$

$y_1 \leq 0$

$y_2$ is unconstrained

$y_3 \geq 0$

Question 2: For the following linear programming problem, use the SOB method to construct its dual problem.

$$\begin{array}{ll} \textit{Maximize} & Z = 3x_1 + 4x_2 \\ \textit{Subject to} & \\ & 4x_1 + 3x_2 \geq 4 \\ & 2x_1 + 3x_2 = 6 \\ & 4x_1 + x_2 \leq 4 \\ & x_1 \geq 1, \quad x_2 \leq 0 \end{array}$$

Dual

Min w=4y1+6y2+4y3+y4

Subject to
4y1+2y2+4y3+1y4>=3
3y1+3y2+y3+0y4=4
Y1<=0
Y2 unconstrained
Y3 >=0
Y4<=0

Max $z=1x_1+0x_2+3x_3$

Subject to

$X_1+x_2+x_3>=10$

$1x_1+0x_2+1x_3<=12$

$X_1>=0$

$X_2$ unconstrained

$X_3>=0$

Dual Min $w=10y_1+12y_2$

Subject to:

$Y_1+y_2>=1$ → Redundant (we can skip it)

$Y_1+2y_2=0$

$Y_1+y_2>=3$

$Y_1<=0$

$Y_2>=0$

redundant → it means that this constraint won't effect your solution (its already satisfied by the solution you obtained)

Max z=3x1+5x2

Subject to

X1<=4

2x2<=12

3x1+2x2<=18

X1,x2>=0

Optimal point (2,6)

Optimal value=36

Add a constraint

or

change C & A & RHS at the same time

or

Add a new decision variable

1) If i add a new constraint what will happen?

$\hookrightarrow x_1 + x_2 \leq 10$

you have to check if the new constraint is redundant or not

$\hookrightarrow$ if the new constraint is satisfied with the optimal point

$\hookrightarrow$ it is redundant

redundant $\rightarrow$ it means that this constraint won't effect your solution (its already satisfied by the solution you obtained)

2) If i add a new decision variable what will happen?

$\hookrightarrow x_3 \rightarrow$

Max z=3x1+5x2+1x3

Subject to

X1+x3<=4

2x2<=12

3x1+2x2+x3<=18

X1,x2>=0

$\rightarrow$ the dual will be

Dual

Min w=4y1+12y2+18y3

Subject to

Y1+3y3>=3

2y2+2y3>=5

Y1+y3>=1

shadow price

$\hookrightarrow y_1 = 0$

$y_2 = 1.5$

$y_3 = 1$

$\rightarrow$ To know if it is redundant

$\hookrightarrow$ if the new constraint is satisfied with $y_1 = 0$, $y_3 = 1$

$\hookrightarrow$ it is redundant