
改善
KAIZEN
TEAM

D.E : **Differential equation** (equation containing derivative)

we take O.D.E (Ordinary) with is the same

• Classification :-

1) Order = the highest derivative.

linear × linear

examples :-

1- $y''' + 2xy' = e^y$ order = 3

2- $(y'')^2 + y''' = 1$ order = 3 $(y'')^2 < y'''$

3- $x\frac{dy}{dx} + (y+1)\frac{dy}{dx} = 0$ order = 1

$(e^y, \sqrt{y}, \frac{1}{y})$ ✗

2) Linearity :

derivative of y ... ✓

y', y'', y''' ⇒ should be linear

examples :- and not multiplied by each other

1- $y''' + 2xy' = e^x$ 3rd order linear

2- $y'' + 2x\,y' = e^y$ 2nd order, non-linear

3- $y''' + yy' = 1$ 3rd order, non-linear

Question :- Is $y(t) = 2e^{3t} - e^{-3t}$ solution of $y'' - 9y = 0$

$y' = 6e^{3t} + 3e^{-3t} = 0$

$y'' = 18e^{3t} - 9e^{-3t} = 0$

$18e^{3t} - 9e^{-3t} - 9(2e^{3t} - e^{-3t}) = 0$

$18e^{3t} - 9e^{-3t} = -18e^{3t} + 9e^{-3t} = 0$ yes

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3) separable

ex: $f_x\,dx = g(y)\,dy$

1) $xy\,dx = (y+1)\,dy$

$x\,dx = \frac{y+1}{y}\,dy$

2) $(xy+x)\,dx = 2y\,dy$

$x(y+1)\,dx = 2y\,dy$

$x\,dx = \frac{2y}{y+1}\,dy$

3) $(x+y)\,dx = y\,dy$ non-sep

4) $\frac{dy}{dx} = x+1+xy^2+y^2$

$dy = \frac{(x+1)+(x+1)y^2}{[x+1+xy^2+y^2]}\,dx$

$dy = (x+1)(1+y^2)\,dx$

$\frac{1}{1+y^2} = x+1\,dx$

$\int \frac{1}{1+y^2}\,dy = \int x+1\,dx$

$\tan^{-1} y = \frac{1}{2}x^2 + x + c$

Solve:-

1) $y' = y^2\cos^2 x$

$\frac{dy}{dx} = y^2\cos^2 x \Rightarrow \int y^{-2}\,dy = \int \cos^2 x\,dx$

$-2y^{-1} = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c$ general solution

2) $e^{x+2y}\,dx = e^{3y-x}\,dy$

$e^x e^{2y}\,dx = e^{3y} e^{-x}\,dy$

$\int e^{2y}\,dx = \int e^{3y}\,dy$

$\frac{1}{2}e^{2y} = e^{3y} + c$

$e^{2x} = 2e^{3y} + c$

$C_1 - C_2 = C$

$C\ x-1 = C$

General solution ... ✓

3) $\frac{dy}{dx} = \frac{xy\sin x}{y+1}$ $\quad x=0,\ y=1$

$y(0) = 1$ IVP (condition)

Initial value problem

$\frac{y+1}{y}\,dy = x\sin x\,dx$

$\int 1 + \frac{1}{y}\,dy = \int x\sin x\,dx$

$y + \ln|y| = -x\cos x + \sin x + c$ Particular solution

x	+	$\sin x$
1	−	$-\cos x$
0		$-\sin x$

$c = 1$

$y + \ln|y| = -x\cos x + \sin x + 1$

First Order Linear

قبل ما نبدأ لازم نعرف separable و linear الأول *

- Solve:-

1) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

divide by x

$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}$ ---- ①

y' $P(x)\,y = f(x)$

2) $e^{\int P(x)\,dx} = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = |x^3|$

3) multiply by x^3 → $x^3y' + 3x^2y = \sin x$

هيبقى simplification في

$x^3y' + 3x^2y = \sin x$

(integrating factor . y)' = left side في

$\int (x^3 \cdot y)' = \int \sin x$

$x^3y = -\cos x + c$ ← general solution

2) $y' + \frac{4x}{1+x^2}y = \frac{1}{(1+x^2)^3}$ ---- *

$y' + P(x)\,y = f(x)$

2- $e^{\int P(x)\,dx} = e^{\int \frac{4x}{1+x^2}dx} = e^{2\int \frac{2x}{1+x^2}dx} = e^{2\ln|1+x^2|} = (1+x^2)^2$ = μ

multiply * by $(1+x^2)^2$

$(1+x^2)^2 y' + 4x(1+x)y = \frac{1}{1+x^2}$

$\int ((1+x^2)^2 \cdot y)' = \int \frac{1}{1+x^2}$

$(1+x^2)^2 \cdot y = \tan^{-1}x + c$

$y' + P(x)\,y = f(x)$ ← function y

$\frac{dy}{dx}$

$x' + P(y)\,x = f(y)$ ← function x

$\frac{dx}{dy}$

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3) $\frac{dy}{dx} = \frac{y}{4y^2 - 2x}$ ← ماينفعش اوزع y عشان فتتبلبط

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→ can be Classified as:-

linear in y (مع x عادي)

linear in x

$\frac{dx}{dy} = \frac{4y^2 - 2x}{y}$

$\frac{dx}{dy} = 4y - \frac{2x}{y}$

$x' + \frac{2}{y}\cdot x = 4y$ Linear in x

$x' + P(y)\cdot x = f(y)$

$e^{\int P(y)\,dy} = e^{\int \frac{2}{y}dy} = y^2$

$x' \cdot y^2 + 2y \cdot x = 4y^3$

$\int (y^2 \cdot x)' = \int 4y^3$

$y^2 \cdot x = y^4 + C$

First Order Linear :-

$y' + P(x)y = f(x)$

$y' + P(x)y = f(x) \cdot y^n$ إذا فيه

$n \neq 0, 1$ → not linear

linear ↙ ↘ linear and separable.

$x^2 y' + 2xy = e^x y^{2a-1}$ ← find all values of a that this is linear or Bernoulli

$2a-1 = 0$

$2a-1 = 1$

$a = \frac{1}{2}, 1$

Or $\frac{yy'}{y} + \frac{2xy^2}{y} = \frac{e^x y^{2a-1}}{y}$ ← this linear

$y' + 2xy = e^x y^{2a-2}$ ← $2a-2 = 1$, $2a-2 = 0$ — $a = ??$

Bernoulli D.E

$\frac{1}{\text{دخل}} y' + P(x)y = f(x) \cdot y^n$ ... (*) $n \neq 0, 1$

$\frac{1}{=1}$ دخل

linear غير

$y^n = (y(x))^n$

$\frac{d}{dx}\left[y(x)\right]^n = n\, y^{n-1}(x) \cdot y'(x)$

So $\frac{d}{dx} y^n = n\, y^{n-1} y'$ زي ما عودتنا

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let $u = y^{1-n} = y(x)^{1-n}$

Deff w.r.t. x :- $\frac{}{dx}$

$u' = (1-n)\, y^{-n} \cdot y'$

multiply (*) by $(1-n)\, y^{-n}$

$(1-n)\, y^{-n} y' + (1-n)\, y^{-n} \cdot P(x) y = (1-n) y^{-n} \cdot f(x) y^n$

$(1-n)\, y^{-n} y' + (1-n) P(x) y^{(1-n)} = (1-n) f(x)$

u'

$u' + \underbrace{(1-n) P(x)}_{P(x)} u = \underbrace{(1-n) f(x)}_{f(x)}$ linear in u

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$\frac{1}{\text{دخل}} y' + P(x)y = f(x) \cdot y^n$, $n \neq 0, 1$

Let $u = y^{(1-n)}$ Then

$\frac{1}{\text{دخل}} u' + (1-n) P(x) u = (1-n) \cdot f(x)$

Ex : Solve

1) $x^2 y' + 2xy = y^3$

$y' + \frac{2}{x} y = \frac{1}{x^2} y^3$

$P(x)$ $f(x)$ y^n

$u = y^{(-3)}$ Then $= y^{-2}$

$u' + (1-n) P(x) u = (1-n) f(x)$ linear in u

$u' + \frac{-4}{x} u = \frac{-2}{x^2}$

$u' - \frac{4}{x} u = \frac{-2}{x^2}$ (*)

$e^{\int -\frac{4}{x} dx} = e^{-4 \ln|x|} = (x)^{-4} = \frac{1}{x^4}$

multiply (*) by $\frac{1}{x^4}$

$\frac{1}{x^4} u' - \frac{4}{x^5} u = \frac{-2}{x^6}$

$\left(\frac{1}{x^4} \cdot u\right)' = -2x^{-6}$.

2) $y' + x^2 y = \frac{e^{-x^3} \sinh x}{2y^2}$

$y' + x^2 y = (\frac{1}{2} e^{-x^3} \sinh x) \cdot y^{(2)^m}$

let $u = y^3$ $\quad (1-n) = 3$

then

$u' + (1-n) P(x) u = (1-n) P(x)$.

$u' + 3x^2 u = e^{-x^3} \sinh x \quad \dots (*)$ (reduction) reduce طلبه

$e^{\int P(x)\,dx} = e^{\int 3x^2\,dx} = e^{x^3}$

multiply (*) by e^{x^3}

$e^{x^3} \cdot u' + 3x^2 \cdot u = \sinh x$

$\int (e^{x^3} \cdot u)' = \int \sinh x$

$e^{x^3} \cdot u = \cosh x + c$

$e^{x^3} y^3 = \cosh x + c$

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3) $\frac{dy}{dx} = \frac{x}{y - y x^2}$ بقلبي حتى ما يزبط حطلحس

$\frac{dx}{dy} = \frac{y - y x^2}{x}$ linear in x عم بصير خطي

$x' = \frac{y}{x} - yx$ this is not bernoulli

$x' + y \cdot x = y \cdot x^{-1}$ bernoulli in x

let $u = x^{2=(1-n)}$ then.

$u' + (1-n) P(y) u = (1-n) P(y)$

$u' + 2y \cdot u = 2y \quad \dots (*)$

$\int P(y)\,dy = \int 2y = e^{y^2}$

multiply (*) by e^{y^2}

$e^{y^2} \cdot u' + e^{y^2} \cdot 2y \cdot u = 2 e^{y^2} \cdot y$

$u' + (1-n) P(x) u = (1-n) P(x)$

$x' + P(y) x = P(y) x^n$

bernoulli in x

$u = x^{1-n}$

$\int (e^{y^2} \cdot u)' = \int 2y\, e^{y^2}$

$e^{y^2} \cdot u = e^{y^2} + c$

$e^{y^2} \cdot x^2 = e^{y^2} + c$

$x^2 = 1 + c\, e^{-y^2}$

$y e^{y^2}$

طلبا مشتقة $= e^{y^2}$ لأن e^{y^2} موجودة بالحل لأنه كان e^{y^2} موجود.

Or $\frac{dy}{dx} = \frac{x}{y - yx^2}$ solve اول طريقة بال Separable

$y(1-x^2)\,dy = x\,dx$

$\int y\,dy = \int \frac{-2x}{1-x^2} \cdot \frac{-1}{2}\,dx$

$\frac{1}{2} y^2 = -\frac{1}{2} \ln|1-x^2| + c$

$-y^2 = \ln|1-x^2| + c$

choises بال $-3c \quad 2c \quad c$ بنختار

س بوزمر

Homogenous D.E.

$y' = f\left(\frac{y}{x}\right)$

- $f\left(\frac{y}{x}\right)$ variable $\frac{y}{x}$ جميع حدود

→ ex: $f\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 - 2e^{\left(\frac{y}{x}\right)} + 5$

$f = \frac{y-x}{2x+3y}$ اقسم كسر ÷ اقسم كسر

$f = \frac{\left(\frac{y}{x}\right) - 1}{2 + 3\left(\frac{y}{x}\right)}$ ✓

← ؟ و ؟

صحيح يكون كلمة

نفس الـ degree بالبسط و المقام

$x^1 y^1$ $x^2 y^2$

ex

$f = \frac{2x^2 - 3y^2}{xy + x^2}$ ، $\frac{1}{x^2}$ ، $\frac{1}{x^2}$ $= \frac{2 - 3\left(\frac{y}{x}\right)^2}{\frac{y}{x} + 1}$

- wich is homogeneus :-

1) $y' = \frac{2x^2 + 3y^2}{x^{(1)} + 2y^2}$ ∝ not homogeneus

2) $y' = 1 + e^{\frac{x}{y}}$ → $y' = 1 + e^{\left(\frac{y}{x}\right)^{-1}}$ homogeneus.

3) $y' = 2\ln y - \ln x^2 + 1$ → $y' = \ln\frac{y^2}{x^2} + 1$

or $2\left[\ln y - \ln x\right] + 1$ $= \ln\left(\frac{y}{x}\right)^2 + 1$ homo

$\ln\frac{y}{x}$ $= 2\ln\left(\frac{y}{x}\right) + 1$

$y' = n\ln y - \ln x^2 + 5$

what is n that this is homogeneus

- Homogeneous :-

$y' = f\left(\frac{y}{x}\right)$

to solve $u = \frac{y}{x} \Rightarrow y = ux$

(x)لاشتقاق بالنسبة

$y' = xu' + u$

seprable :

- Solve:

1) $\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2}$

$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$ ..... (*)

let $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = xu' + u$

$xu' + u = u + u^2 + 1$

$x\frac{du}{dx} = u^2 + 1$

$\int \frac{du}{u^2+1} = \int \frac{1}{x}\,dx$

$\tan^{-1} u = \ln|x| + c$

$\tan^{-1}\frac{y}{x} = \ln|x| + c$

Or $\frac{y}{x} = \tan\left(\ln|x| + c\right)$

$y = x\tan\left[\ln|x| + c\right]$

$\frac{dy}{dx} = \frac{xy^m + y^2 + x^2 + nx}{x^2}$

$\frac{xy^m}{x^2} + \frac{y^2}{x^2} + 1 + \frac{nx}{x^2}$

$n = ?$ $m = ?$

so it's seprable

$xu' + u = \frac{1-u}{1+u}$

2) $(x+y)\,dy = (x-y)\,dx$

$\frac{dy}{dx} = \frac{x-y}{x+y}$ $\div x^1$ $\div x^1$

$y' = \frac{1 - \left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)}$ ..... (*)

$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = xu' + u$

$x\frac{du}{dx} + u = \frac{1-u}{1+u}$

$x\frac{du}{dx} = \frac{1-u}{1+u} - u$

$\frac{1-u}{1+u} - \frac{u}{1}$

$\frac{1 - u - u - u^2}{1+u}$

$\int \frac{1+u}{1-2u-u^2}\,du = \int \frac{1}{x}\,dx \rightarrow -\frac{1}{2}\int \frac{-2-2u}{1-2u-u^2} = \ln|x| + c$

$-\frac{1}{2}\ln|1-2u-u^2| = \ln|M| + c$

$\ln|1-2u-u^2| = -2\ln x + c$

$\ln\left|1 - 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\right| = -2\ln|x| + c$

$\ln|\cdots| + 2\ln(\cdots) = c$

$\ln|\cdots| + \ln(\cdots)^2 = c$

$\ln\left(1-2u-u^2\right)x = c$

$\ln\left(x - 2ux - u^2x\right) = c$ $e^c = c$

$x - 2y - \frac{y^2}{x}$ or $x^2(1-u-u^2) = c$

3) $\frac{dy}{dx} = \frac{y}{x}(\ln y - \ln x + 1)$

$f(\frac{y}{x})$

$y' = \{\frac{y}{x}(\ln(\frac{y}{x}) + 1)\}$ Homogeneous

$xu' + u = u(\ln u + 1)$

n?? → homo

$\frac{dy}{dx} = \frac{y}{x}(2\ln y - \ln x^n + 1)$

$\ln \frac{y^2}{x^n}$ $n=2$

$\frac{dy}{dx} = \frac{y}{x}(2\ln y - \ln x + 1)$ not Homo

$(\ln \frac{y^2}{x} + 1$

* Solve :-

1) $\frac{dy}{dx} = (x+y+2)^3$ ، $f(x+y+2)$

الطريقة الأولى

$u = x+y+2$

$u' = 1+y' \Rightarrow y' = u' - 1$

$u' - 1 = u^3$ sep

$\frac{du}{dx} = u^3$

$\int \frac{1}{u^3} du = \int 1\, dx$

$-\frac{1}{2}u^{-2} = x + c$

$u^{-2} = -2x + c$

$(x+y+2)^{-2} = -2x+c$

$f(x+y) = (x+y+2)^3 - 1$

$u = x+y$

$u' = 1+y' \Rightarrow y' = u' - 1$

$u' - 1 = (u+2)^3 - 1$

$\int \frac{1}{(u+2)^3} du = \int 1\, dx$

$-\frac{1}{2}\int (u+2)^{-2} = x + c$

$\frac{1}{(u+2)^2} = -2x + c$

$\frac{1}{(x+y+2)^2} = -2x + c$

اذا تقدر اخذ المتغير و constant

HOW TO SOLVE

• $y' = f(ax+by+c)$

ex:

$f(2x+y-3) = (2x+y-3)^3 - 2(2x+y-3) + 5$

وحدة واحدة

نفرض u = function وتشتق بالنسبة لـ (x)

معظم الأسئلة $c = 0$

ex: e^{2x+y}

$f(2x+y) = \sqrt{2x+y} + e^{2x}e^{y} + c$

$f(ax+by+c)$

$y' = f(ax+by+c)$

$\frac{d}{dx}(u = ax+by+c)$

$u' = a + by'$

Seprable

$y' = x^2+2xy+y^2-5$

$y' = (x+y)^2 = 5$

اذا ما وصلنا لشي معب

$f = \frac{2x-y+5}{2y-4x+10}$ $f = \frac{2x+y+1}{x+3y+5}$

$2x = b$

$f = \frac{(2x-y)+5}{-2(2x-y)+10}$

اقرب ① تعامل x في بعض وتعامل الباقي

وتقرب ② تعامل الباقي وتعامل بعضها

اذا ① = ② ← معادلة تربيعية

2) $y' = \sqrt{x+y} - 1$

شغل امكانك

$u = x+y$

$u' = 1+y' \Rightarrow y' = u' - 1$

$u' - 1 = \sqrt{u} - 1$

$\int u^{-\frac{1}{2}} du = \int 1\, dx$

$2u^{\frac{1}{2}} = x + c$

$2\sqrt{x+y} = x + c$

$f(x) = 2x^3 + 5x + 3$

$f'(x) = 6x^2 + 5$

$u(x,y) = x^2 + y^2 + xy^2$

$\frac{\partial u}{\partial x} = 2x + y^2$

$\frac{\partial u}{\partial y} = 2y + 2xy$

$u(x,y) = \sin(xy) + y^2$

$u_y = x\cos(xy) + 2y$

$u_x = y\sin(xy)$

$\int [2xy + \cos(xy)]\, dy = xy^2 + \frac{\sin(xy)}{x} + f(x)$

$y \to$ valuable, $x \to$ constant — not c — constant

partial integral.

$\int [e^{2xy} + y^2]\, dx = \frac{1}{2y} e^{2xy} + y^2 x + f(y)$

constant

exact D.E :-

$M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Exact

the solution :

$u(x,y) = c$

has two properties

$\int \frac{\partial u}{\partial x} dx = \int M\, dx$; $\int \frac{\partial u}{\partial y} dy = \int N\, dy$

$u = \int M\, dx + y(y)$ $\quad$ $u = \int N\, dy + f(x)$

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example:

solve:-

IF $\int f'(y) = f$, $f(y) = 0$

1) $(e^x + y)\,dx + (x - \sin y)\,dy = 0$

$M(x,y)$ $\quad$ $N(x,y)$

$\int M\, dx = e^x + xy$

$\int N\, dy = xy + \cos(y)$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$ - Exact

the solution is $u(x,y) = c$

$u(x,y) = e^x + xy + \cos(y)$

$\frac{du}{dx} = M$

$\int \frac{du}{dx} dx = \int e^x + y\, dx$

$u(x,y) = e^x + yx + f(y)$

Diff the equ. with respect to y

$\frac{\partial u}{\partial y} = x + f'(y)$

$\cos(y) + c_1$

$\frac{\partial u}{\partial y} = N$

$x - \sin(y) = x + f'(y)$

$\int f'(y) = \int -\sin(y)\, dy$

IF $f'(y) = \sin y + x$ then x is constant

$\frac{\partial u}{\partial y} = x - \sin(y)$

$f(y) = \cos(y) + c_1$

$u(x,y) = e^x + xy + \cos(y) + c_1$ already

2) example : solve:-

$(2x\cos(y) + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy$

$M(x,y)$ $\qquad$ $N(x,y)$

$\frac{\partial M}{\partial y} = -2x\sin y + 3x^2$

$\frac{\partial N}{\partial x} = 3x^2 - 2x\sin(y)$

$\int M\,dx = x^2\cos(y) + x^3y$

$\int N\,dy = x^3y + x^2\cos(y) - \frac{y^2}{2}$

$x^2\cos(y) + x^3y - \frac{y^2}{2} = c$

If $(2x\cos y + 3x^n y)\,dx + (x^3 - x^2\sin y - y)\,dy$ is exact find n

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$x^n = x^2$

$n = 2$

$(2x\cos y + 3x^2y)\,dy + (f(x) - x^2\sin y - y)\,dx = 0$ is exact find $f(x)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$-2x\sin y + 3x^2 = f'(x) - 2x\sin y$

$\int f'(x)\,dx = \int 3x^2\,dx$

$f(x) = x^3 + c$

بما ان $f(1) = 2$ اذا

$f(1) = 2$

$1 + c = 2$

$c = 1$

$f(x) = x^3 + 1$

Intergrating factor

$M(x,y)\,dx + N(x,y)\,dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ not exact

- if μ is a function such that $\mu M\,dx + \mu N\,dy = 0$ is exact, then μ is called integrating factor

$\mu(x,y)$ or $\mu(x)$ or $\mu(y)$

-

19/11 ↓ med ~~level~~ 4:30 → 5:30 student ch1+2

$\mu(x)$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = R(x)$ = constant or function of x

$\mu(x) = e^{\int R(x)\,dx}$

$\mu(y)$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = R(y)$ ↓ constant or function of y

$\mu(y) = e^{\int R(y)\,dy}$

EXAMPLE: Solve:-

1) $\underbrace{[y^2 - 3xy - 2x^2]}_{M}\,dx + \underbrace{[xy - x^2]}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 2y - 3x$

not exact.

$\frac{\partial N}{\partial x} = y - 2x$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y - 3x - y + 2x = y - x$

$\frac{y-x}{N} = \frac{y-x}{x(y-x)} = \frac{1}{x} = R(x)$

صار يكون لدي خيار، صير خطوة.

$\mu(x) = e^{\int \frac{1}{x}\,dx} = x$

$[xy^2 - 3x^2y - 2x^3]\,dx + [x^3y - x^3]\,dy = 0$

$\frac{1}{2}x^2y^2 - x^3y - \frac{1}{2}x^4$

$\frac{1}{2}xy^2 - x^3y$

$\frac{1}{2}x^2y^2 - x^3y - \frac{1}{2}x^4 - \frac{1}{2}x^4$

2) $\underbrace{[3x^2y + y^2]}_{M}\,dx + \underbrace{[2x^2 + 3xy]}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 2x^2 + 2y$

$\frac{\partial N}{\partial x} = 4x + 3y$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{-(3x^2 + y)}{-(3x^2y + y^2)} = \frac{1}{y}$

$\mu = e^{\int \frac{1}{y}\,dy} = y$

multiply ⊗ by y :-

$[3x^2y^2 + y^3]\,dx + [2x^2y + 3xy^2]\,dy = 0$

$x^3y^2 + y^3x$

$x^3y^2 + xy^3$

$u(x,y) = x^3y^2 + xy^3 = c$

$$y'' + P(x)\,y' + q(x)\,y = f(x)$$

(iii) If $f(x) = 0$ → then the D.E is called homogenious, otherwise it's called non-homogenious

ex :-

$y'' - 2y' + y = 0$ ✓ $\quad y'' + y' - 2 = 0$ ×

$x^2 y'' + e^x y = 0$ ✓ $\quad y'' + y' = +2$

Consider :-

$$y'' + P(x)y' + q(x)y = 0 \quad \cdots (*)$$

(iv) let y_1, y_2 be two solutions of (*).

Then all $\begin{cases} c_1 y_1 \\ c_2 y_2 \\ y_1 \pm y_2 \\ c_1 y_1 \pm c_2 y_2 \end{cases}$ are solution of star

second order :-
two diffrent solution (independant)
if they were not diffrent they are me solution

Def. let y_1, y_2 be two solution of (*), Then y_1, y_2 are dependent iff $y_1 = c y_2$ $\left(\frac{y_1}{y_2} = c\right)$
Otherwise, they are called independent.

---

ex:-

(v) $y_1 = e^{2x}$, $y_2 = e^{3+2x}$ } one solution.

$$\frac{y_1}{y_2} = \frac{e^{2x}}{e^{3+2x}} = \frac{e^{2x}}{e^3 \cdot e^{2x}} = \frac{1}{e^3}$$ } constant → y_1, y_2 dependent

→ $W(e^{2x}, e^{3+2x}) = e^{2x} \cdot 2e^{3+2x} - e^{3+2x} \cdot 2e^{2x} = 0$

(vi) $y_1 = 2$, $y_2 = \ln x$ } independant.

$$\frac{y_2}{y_1} = \frac{\ln x}{2}$$ } not constant

$W(2, \ln x) = 2\frac{1}{x} - \ln x \cdot 0 = \frac{2}{x} \neq 0$

Wronskian :-

let y_1, y_2 be two solution of (*)

Then the wronskian is defined by :

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

example y_1, y_2, y_3

$$\begin{matrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ \vdots & & \end{matrix}$$

• Theorem :-

let y_1, y_2 be two solutions of (*),

$W(y_1, y_2) \neq 0 \Rightarrow y_1, y_2 \rightarrow$ are independent

$W(y_1, y_2) = 0 \rightarrow y_1, y_2 \rightarrow$ are dependent.

---

ex:- ① If $W(f, g) = 3e^{4t}$, $f(t) = e^{2t}$, $g(0) = 3$

$f' = 2e^{2t}$

find $g(t)$:-

$$W(f, g) = 3e^{4t}$$

$$fg' - gf' = 3e^{4t}$$

$$e^{2t} g' - g 2e^{2t} = \frac{3e^{4t}}{e^{2t}}$$

(*) ... $g' - 2g = 3e^{2t}$ ⇒ g is the solution of this first order

$$e^{\int -2\,dt} = e^{-2t}$$

multiply by e^{-2t} (*)

$$\int (e^{-2t} \cdot g)' \, dt = \int 3\,dt$$

$$e^{-2t} \cdot g(t) = 3t + c$$

$$g(t) = e^{2t}[3t + c] \rightarrow g(0) = 3$$

$$c = 3$$

$$g(t) = e^{2t}(3t + 3)$$

② If $W(f,g) = 2$, find $W(f, f+2g)$.

$$W(f, f+2g) = f(f+2g)' - f'(f+2g)$$
$$f(f'+2g') = f(f+2g)'$$
$$ff' + 2fg' = ff' + f'2g$$
$$2fg' = 2f'g$$
$$2[fg' - f'g]$$
$$2\,W(f,g) = 4$$

notes :-

1) $W(f,f) = 0$

2) $W(f,g) = -W(g,f)$

3) $W(af, bg) = ab\,W(f,g)$

4) $W(f+2g, f-g) =$
$W(f,f) - W(f,g) + 2W(g,f) - 2W(g,g) - 3W(f,g)$

---

* **Abel's Theorem:-**

Let y_1, y_2 be two solutions of (*) on $I = (a,b)$ (open interval)

(*) $y'' + P(x)y' + q(x)y = 0$

then $W(y_1, y_2)(x) = c\, e^{-\int P(x)\,dx}$

If $c = 0$ ... $W = 0$ ...

interval ... $W(y_1, y_2)(x) = c\, e^{-\int P(x)dx}$

- $W(y_1, y_2)(x) \neq 0$
  Or
- $W(y_1, y_2)(x) = 0$ on the all interval because $c = 0$

Ex :- If y_1, y_2 are two solutions of :-

$t y'' + 2y' + te^t y = 0$

$W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(5)$

$y'' + \frac{2}{t} y' + e^t y = 0$

$$e^{-\int P(t)\,dt} = e^{-\int \frac{2}{t} dt} = e^{-2\ln t} = \frac{1}{t^2}$$

$$W(y_1, y_2)(t) = \frac{c}{t^2}$$

$$W(y_1, y_2)(2) = 3 = \frac{c}{4}$$
$$c = 12$$

$$W(t) = \frac{12}{t^2} \rightarrow W(5) = \frac{12}{25}$$

$y'' = f(x, y')$ — y missed

reduction from second order to first order

$u = y'$, $u' = y''$ then

$$u' = f(x,y) \Rightarrow \frac{du}{dx} = f(x,y)$$

---

example:- solve:

① $y'' + \frac{2}{x} y' = \frac{1}{x^2}$ — y - missed

a ary-missing :. الاختصار لما بتيجي تسوا السين 2 p. f

$u = y'$ $u' = y''$ then

$$u' + \frac{2}{x} u = \frac{1}{x^2}$$ First Order linear

$$e^{\int \frac{2}{x} dx} = x^2$$

$$\int (u \cdot x^2)' dx = \int 1 \, dx$$

$$u \cdot x^2 = x + c_1$$

$$u = \frac{x + c_1}{x^2}$$

$$y' = \frac{x + c_1}{x^2}$$ sep

Agian D.E بنكاملها

$$\frac{dy}{dx} = \frac{x + c_1}{x^2}$$

$$\int dy = \int \frac{x + c_1}{x^2} dx$$

$$y = \ln x + \frac{c_1}{x} + C$$

$-c_1 = +c_1$

$\int \frac{1}{x^2} = -\frac{1}{x}$

- Note : y/x - missing ممكن تجيب خطر مكانك بيرمتش 2 في مسائل

$y'' = f(y, y')$ — x- missing.

① $u = y' \rightarrow u' = y''$

اشتق بالنسبة لـ x

$u' = f(y, u)$

$\frac{du}{dx} = f(y,u)$

$u \frac{du}{dy}$

$y' = u$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

②

$$\frac{du}{dx} = u \frac{du}{dy}$$

- Solve :-

$y\,y'' + (y')^2 = 0$ ← معادلة لا يوجد بها x-missing

$u = y' \rightarrow u' = y''$

$u' = \frac{du}{dx} = u\frac{du}{dy}$

$y\,u' + u^2 = 0$

$y\,u\frac{du}{dy} + u^2 = 0$

- divide by u

$y\frac{du}{dy} = -u$

$\int \frac{du}{u} = \int \frac{dy}{y}$

$\ln u = -\ln y + c_1$

$\ln u + \ln y = c_1$

$\ln uy = c_1$

$uy = c_1 \rightarrow u = \frac{c_1}{y}$

$y' = \frac{c_1}{y}$

$\int y\,dy = \int c_1\,dx$

$\frac{1}{2}y^2 = c_1 x + c_2$ → FINAL ANSWER

مثلة من هيك النوع

Consider $y'' + 2y(y')^3 = 0$ ← x missing

If $u = y'$ then the D.E can be written as :-

$u' + 2yu^3 = 0$

$u\frac{dy}{dy} + 2yu^3 = 0$

Separable

$\frac{du}{u^2} = -2y\,dy$

A) $\frac{dy}{u^2} = -2y^2\,dy$

B) $\frac{du}{u^2} = -2y\,dx$

C) $\frac{du}{dx} + 2y$

1 $y'' + p(x)\,y' + q(x)\,y = 0$

Given y_1 is solution, we can find another solution (independent) as:

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

Ex:- Consider $xy'' + 2y' + xy = 0$ Second Order linear

Given $y_1 = \frac{\sin x}{x}$, find y_2:

$y_1 = c_1 y_1$

$y'' + \frac{2}{x} y' + y = 0$

$e^{-\int p(x)\,dx} = e^{-\int \frac{2}{x}\,dx} = x^{-2}$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2} = y_1 \int \frac{x^{-2}}{\frac{\sin^2 x}{x^2}}\,dx = \frac{\sin x}{x} \int \frac{1}{\sin^2 x}\,dx$$

$$= \frac{\sin x}{x} \int \csc^2 x\,dx$$

$$= \frac{\sin x}{x} \cdot -\cot x = \frac{\sin x}{x} \cdot -\frac{\cos x}{\sin x} = \frac{-\cos x}{x} =$$

$$\frac{\cos x}{x}$$

The general solution is-

$$y = c_1 y_1 + c_2 y_2$$

* $y'' + ay' + by = 0$

let $y = e^{rx}$ be a solution

$r^2 e^{rx} + are^{rx} + be^{rx} = 0$

$r^2 + a + b = 0$

Charactistic equation

(r) يسمى

remark: $y' + ay = 0$

$$\frac{dy}{dx} = -ay$$

$$\int \frac{dy}{y} = \int -a\,dx$$

$$\ln|y| = -ax + c \rightarrow y = e^{-ax} \cdot e^{c}$$

$$\rightarrow y = e^{-ax}\, c = e^{-ax}$$

e^{rx}

---

example:- Solve.

$2y'' - 2y' - 4y = 0$

$2r^2 - 2r - 4 = 0$

$r^2 - r - 2 = 0$

Case (1) $\Rightarrow \Delta = +$

$(r-2)(r+1) = 0$

$r = 2$, $r = -1$

e^{rx} يسمى

$y_1 = e^{2x}$, $y_2 = e^{-x}$

general solution:

$y = c_1 e^{2x} + c_2 e^{-x}$

(///) $\{y_1, y_2\}$ is called Fundemental set solution or Basis of solutions

---

example:- Solve

$2y'' - 3y' + y = 0$ homogenious + constant coefficients + linear

$2r^2 - 3r + 1 = 0$

$(2r-1)(r-1) = 0$

$r = \frac{1}{2}$, $r = 1$

$y_1 = e^{\frac{1}{2}x}$, $y_2 = e^{x}$

general solution

$y = c_1 e^{\frac{x}{2}} + c_2 e^{x}$

- there is two conditions to find c_1, c_2 in the general solution we'll take them later. ☺

example :- find a 2nd- order linear D.E when solution is $y(x) = c_1 e^{2x} + c_2 e^{5x}$ ($r_1 = 2$, $r_2 = 5$)

$(r-2)(r-5) = 0$

$r^2 - 7r + 10 = 0$

$y'' - 7y' + 10y = 0$

(1) $\Rightarrow y$

$r \Rightarrow y'$

$r'' = y''$

we have r_1 r_2 :-

The characteristic equation is

$r^2 - (r_1 + r_2) r + r_1 r_2$

---

example :- Solve

$y'' - 2y' - y = 0$

$r^2 - 2r - 1 = 0$

$a=1 \quad b=-2 \quad c=-1$

$\Delta = b^2 - 4ac = 8$ complex → جذور مركبة

$r = \frac{-b \pm \sqrt{\Delta}}{2}$

$r = \frac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

$r_1 = 1+\sqrt{2} \qquad r_2 = 1-\sqrt{2}$

$y_1 = e^{(1+\sqrt{2})x}, \quad y_2 = e^{(1-\sqrt{2})x}$

$c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$

---

example :- find a 2nd-order linear D.E whose Basis of solution is $\{ e^{(2-\sqrt{2})x}, e^{(2+\sqrt{2})x} \}$ (r_1, r_2)

Case (2) → $\Delta = 0$

$r^2 - (r_1 + r_2) r + r_1 \cdot r_2$

$r^2 - 4r + 2 = 0$

$y'' - 4y' + 2y = 0$

---

example :- Solve

$y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$ ← roots

$(r-3)(r-3) = 0$

$r = 3$

$y_1 = e^{3x}$

2nd order اذا بالمعادلة من

يد حلين

JJ + constant coefficient + homogeneous هي من

مستقرة

ف

$y_2 = x y_1$

$y = x e^{3x}$

general solution :-

$c_1 e^{3x} + c_2 x e^{3x}$

$y_2 = y_1 \int \frac{e^{-\int p x \, dx}}{(y_1)^2} \, dx$

example :- find a 2^{nd}-order linear O.E when Solution is given by : $y(x) = (ax+b)e^{4x}$

$r_1 = r_2$

$be^{4x} + axe^{4x}$

$y_2 = x y_1$ $\quad r_1 = 4$, $r_2 = 4$

$r^2 - 8r + 16 = 0$

$y'' - 8y' + 16y = 0$

Remark :-

$\sqrt{4} = 2$

$\sqrt{-1} = i$

$\boxed{i = -1}$

$\sqrt{-5} = \sqrt{5}\, i$

$\sqrt{-9} = \sqrt{9}\,\sqrt{-1} = 3i$

$\sqrt{-16} = 4i$

ex

$a+bi$

$2+3i$ → complete complex number

$5i$ → imaginary complex number

7 → complex real

not real

---

example :- Solve

$r^2 - 2r + 2 = 0$

$a = 1 \quad b = -2 \quad c = 2$

$\Delta = 4 - 4(1)(2) = -4$ → negative

$r = \dfrac{2 \pm \sqrt{-4}}{2}$

$r = \dfrac{2 \pm 2i}{2}$

$r = 1 \pm i$

real part λ $\quad \mu$ imaginary Part

Case (3) → $\Delta = -1$

(iii) $\lambda \pm \mu i$

$\lambda + \mu i$ $\quad$ $\lambda - \mu i$

$y_1 = e^{(\lambda+\mu i)x}$

$y_2 = e^{(\lambda-\mu i)x}$

→ we want solutions without i (all reals)

$y_1 = e^{\lambda x}(\cos(\mu x))$

$y_2 = e^{\lambda x}\sin(\mu x)$

example :- Solve.

$$y'' - 2y' + 4y = 0$$

$$r^2 - 2r + 4 = 0$$

$a=1 \quad b=-2 \quad c=4$

$$\Delta = -12$$

$$r = \frac{2 \pm \sqrt{-12}}{2}$$

12
3 6
4 3
$2\sqrt{3}$

$$r = \frac{2 \pm 2\sqrt{3}i}{2}$$

$r_{1,2} = 1 \pm \sqrt{3}i$ → λ , μ

$y_1 = e^{\lambda x} \cos(\mu x)$

$y_2 = e^{\lambda x} \sin \mu(x)$

$$y_1 = e^x \cos(\sqrt{3}x)$$

$$y_2 = e^x \sin(\sqrt{3}x)$$

ناخد λ بإشارتها

أما μ نأخذه موجب

(+) بغير إشارة

---

example :- (الحالة العكسية) find a 2nd order D.E When solution is :

$$y(x) = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

$c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$

$\lambda = +2$

$\mu = \pm 3$

$2 \pm 3i$

$2+3i$ (r_2) $\quad 2-3i$ (r_1)

$(r+3i)(r-3i)$

$r^2 - 4r + 4 - 3^2(i)^2$

$r^2 - 4r + 4 - 9(-1)$

$r^2 - 4r + 13 = 0$

$$y'' - 4y' + 13y = 0$$

$r = \lambda \pm \mu i$ → complex

$(\lambda + \mu i)x$

$y_1 = e$

$y_2 = e^{(\lambda - \mu i)x}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

Real

$y_1 = e^{\lambda x}\cos(\mu x)$

$y_2 = e^{\lambda x}\sin(\mu x)$

$y_1 = e^{\lambda x} e^{(\mu x)i} = e^{\lambda x}(\cos(\mu x) + i\sin(\mu x))$

$y_2 = e^{\lambda x} e^{-(\mu x)i} = e^{\lambda x}(\cos(\mu x) - i\sin(\mu x))$

$y_1 + y_2 = 2e^{\lambda x}\cos(\mu x)$

---

example :- Solve. $y'' - 4y' + 13y = 0$

homo with constant coefficients

x-missing

$r^2 - 4r + 13 = 0$

$a=1 \quad b=-4 \quad c=13$

$\Delta = 16 - 52 = -36$ → complex roots

$r = \frac{4 \pm 6i}{2} = 2 \pm 3i$

$\lambda \quad \mu$

$y_1 = e^{2x}\cos(3x)$ $\quad y_2 = e^{2x}\sin(3x)$

general solution $= c_1 e^{2x}\cos(\beta x) + c_2 e^{2x}\sin(\beta x)$

---

example :- find a 2nd-order linear D.E whose solution is given by

$y(x) = e^{3x}[c_1\cos(4x) + c_2\sin(4x)]$

$\lambda \quad \mu$

$r = 3 \pm 4i$

$r^2 - (r_1 + r_2)r + r_1 r_2$

$r^2 - 6r + 9 - 16(i)^2$

$r^2 - 6r + 9 + 16$

$i^2 = -1$

$(3+4i)(3-4i)$

$(3)^2 - (4i)^2$

مزدوج

$\lambda^2 + \mu^2$

$r^2 - 6r + 25$

$y'' - 6y + 25y = 0$

example : If $\{ e^{(-1+i)x}, e^{(-1-i)x} \}$ is the basis of $y'' + ay' + by = 0$ find a, b

$\lambda = -1$, $\mu = 1$

$r = -1 \pm i$ $\quad (-1+i, -1-i)$

$r^2 - (r_1 + r_2) r + r_1 r_2$

$r^2 + 2r + 2 = 0$

$y'' + 2y' + 2y = 0$ $\quad a = 2$ $\quad b = 2$

Euler - Cauchy Equations

$a x^2 y'' + bx y' + cy = 0$

where a, b, c are constants

$a\, r(r-1) + br + c = 0$

$y'' + ay' + by = 0$

→ charastic equation

$y = e^{rx}$

① $r_1 \neq r_2$

$e^{r_1 x}$, $e^{r_2 x}$

② $r_1 = r_2 = r$

e^{rx} , $x e^{rx}$

③ $r = \lambda \pm \mu i$

$e^{\lambda x} \cos(\mu x)$

$e^{\lambda x} \sin(\mu x)$

$x^2 y'' + axy' + by = 0$, $x > 0$

$y = x^r$

① $r_1 \neq r_2$

x^{r_1} , x^{r_2}

② $r_1 = r_2 = r$

x^r , $x^r \ln x$

③ $r = \lambda \pm \mu i$

$x^\lambda \cos(\mu \ln x)$

$x^\lambda \sin(\mu \ln x)$

example : Solve $2x^2y'' + 3xy' - y = 0$ ✓

$2r(r-1) + 3r - 1$

$2r^2 - 2r + 3r - 1$

$2r^2 + r - 1 = 0$

$(2r - 1)(r + 1)$

$r = \frac{1}{2}$ $r_2 = -1$

$y_1 = x^{\frac{1}{2}}$ $y_2 = x^{-1}$

$y(x) = c_1\sqrt{x} + \frac{c_2}{x}$

---

example :- Solve $x^2y'' - 5xy' + 9y = 0$

$r(r-1) - 5r + 9 = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r = 3$

$y_1 = x^3$ $y_2 = x^3 \ln x$

$y(x) = c_1 x^3 + c_2 x^3 \ln x$

---

example :- Solve $x^2y'' - 5xy' + 13y = 0$

$r(r-1) - 5r + 13 = 0$

$r^2 - 6r + 13 = 0$

$a = 1$ $b = -6$ $c = 13$

$\Delta = 36 - 52 = -16$

$r = \frac{6 \pm 4i}{2} = 3 \pm 2i$

$y_1 = x^3 \cos(2\ln x)$

$y_2 = x^3 \sin(2\ln x)$

$y(x) = c_1 x^3 \cos(2\ln x) + c_2 x^3 \sin(2\ln x)$

---

example :- solve $(xy'' + 4y' = 0)\ x$

$x^2y'' + 4xy' = 0$

$r^2 + 3r = 0$

$r(r+3) = 0$

$r = 0$, $r = -3$

$y_1 = x^0 = 1$

$y_2 = x^{-3}$

$y(x) = c_1 + \frac{c_2}{x^3}$

$x^2y'' + 4y' = 0$

y missing

$x^2u' + 4u = 0$

⋮ sep

$\{x^{r_1}, x^{r_2}\}$ r_1 r_2

$\{x^2, x^3\}$

$r_1 = 2$

$r_2 = 3$

$r^2 - 3r + 6 = 0$

$r(r-1) - 2r + 6 = 0$

$x^2y'' - 2xy' + 6y = 0$

x^2y'' $-2xy'$

$[x^{r_1}, x^{r_2}]$ Coucher ou...

↳ Chara...stic equ.

$r(r-1) \rightarrow$ ...

$r^2 + ar = r(r-1)$

$ar = -r + b$

$y'' + ay' + by = f(x)$

y_L

y_p → three types of undetermined coefficients: poly, expo, sin/cos

y_p → Variation of parameters

example :- Solve

$y'' - 4y = 4x + 1$ non-homo

$r^2 - 4 = 0$

$r = \pm 2$

$y_1 = e^{2x}$ $y_2 = e^{-2x}$

$* y_L = c_1 e^{2x} + c_2 e^{-2x}$

$4x+1$ ← لأن عندنا

let $y_{p(x)} = ax + b$ ← لأن عندنا x^1

$y_{(p(x))} = ax^2 + bx + c$ ← لو عندنا x^2

$y_p' = a$

$y_p'' = 0$

$0 - 4(Ax+b) = 4x+1$

$-4ax - 4b = 4x + 1$

$-4ax = 4x$

$a = -1$

$-4b = 1$

$b = \frac{-1}{4}$

$y_p = -x - \frac{1}{4}$

The general solution: → undetermined coefficients

$y = y_L + y_p$

$c_1 e^{2x} + c_2 e^{-2x} + (-x - \frac{1}{4})$

c_1 / c_2 ← كوامل غير محروفة

homogenious مثل

$y'' + ay' + by = f(x)$

* The undetermined coefficients accepts three types of $f(x)$ صانة الشكل (بس)

① poly

② expo

③ cos / sin (or)

poly سواء كان ١ او ٥ حوامل

كلية زي ماهو الا ال constant : poly degree = 0

example : Solve

$$y'' - 2y' = 6e^{-2x}$$

expo

$y'' - 2y' = 0$ homogenious

$r^2 - 2r = 0$

$r(r-2) = 0$

$r = 0 \quad , \quad r = 2$

$y_1 = 1 \quad , \quad y_2 = e^{2x}$

$$\boxed{y_L = c_1 + c_2 e^{2x}}$$

$y_{p(x)} = a e^{-2x}$ نفس صاحبو

$y'_p = -2a e^{-2x}$

$y''_p = 4a e^{-2x}$

$4a e^{-2x} - 2(-2a e^{-2x}) = 6e^{-2x}$

$4a e^{-2x} + 4a e^{-2x} = 6 e^{-2x}$

$8a e^{-2x} = 6e^{-2x} \Rightarrow 8a = 6 \Rightarrow \boxed{a = \frac{3}{4}}$

لو $f(x) = 6 \times e^{-2x}$

$y_{p(x)} = (ax+b) e^{-2x}$

$y_p = \frac{3}{4} e^{-2x}$

general solution: $y = y_L + y_p = \underbrace{c_1 + c_2 e^{2x}}_{\text{from the homogenious}} + \frac{3}{4} e^{-2x}$

---

example: Solve

$y'' + 2y' = 4 \sin x$

$y_{p(x)} = a \sin x + b \cos x$

$y'_p = a \cos x - b \sin x$

$y''_p = -a \sin x - b \cos x$

$y'' + 2y' = 0$ homo

$r^2 + 2r = 0$

$r = 0 \quad , \quad r = -2$

$y_1 = 1 \quad , \quad y_2 = e^{-2x}$

$y = c_1 + c_2 e^{-2x}$

$-a \sin x - b \cos x + 2a \sin x - 2b \cos x = 4 \sin x$

$(-a - 2b) \sin x + (2A - B) \cos x = 4 \sin x$

$-a - 2b = 4$

$2A - B = 0 \Rightarrow \boxed{B = 2A}$

$-A - 4A = 4$

$A = \frac{-4}{5} \qquad B = \frac{-8}{5}$

general solution $= c_1 + c_2 e^{-2x} + \frac{-4}{5} \sin x - \frac{8}{5} \cos x$

لو $f(x) = \sin x$ بيصير y_p بيصير $\sin$ و $\cos$ لانو مشتقات $\sin$ هي $\cos$ و $\sin$

example $f(x) = 2x \cos(2x)$

$y_p = (ax+b)(\cos(2x) + \sin(2x))$

* Solve :

1) $y'' - y = 4e^x$

homo

$y'' - y = 0$

$r^2 - r = 0$

$r = 0, r = 1$

$y_1 = 1 \;,\; y_2 = e^x$

$y = c_1 + c_2 e^x$

$y_{p(x)} = Ae^x$

$Ae^x - Ae^x = 4e^x$

$0 = 4e^x$

$\exp 0 \neq 0$ خطأ

$y_{p(x)} = Axe^x$

$y_p' = Axe^x + Ae^x$

$y_p'' = Axe^x + Ae^x + Ae^x$

$\cancel{Axe^x} + 2Ae^x - \cancel{Axe^x} = 4e^x$

$2Ae^x = 4e^x$

$A = 2$

$y_p = 2xe^x$

Always Check for similarity with homo

general solution:

$y = y_h + y_p = c_1 + c_2 e^x + 2xe^x$

---

* write a suitable form for $y_{p(x)}$:-

① $y'' - 2y' = 2x + 4e^{-2x}$

homo

$y'' - 2y' = 0$

$r^2 - 2r = 0$

$r_1 = 0, r_2 = 2$

$y_1 = 1 \;,\; y_2 = e^{2x}$

constant

$y_p = x(Ax+b) + Ce^{-2x}$

$y_{p(x)} = x(Ax+b) + Ce^{-2x}$ → suitable

Solve

A/b/C

---

② $y'' + 4y = 2x\cos x + 1$

$r^2 + 4 = 0$

$r = \sqrt{-4}$

$r = \pm 2i$

$y_1 = \cos 2x \quad y_2 = \sin 2x$

$\lambda = 0$

$y_p = (ax+b)\cos x + (cx+d)\sin x + e$

$y_p = (Ax+b)\cos + (Cx+d)\sin x + e$

③ $y''+2y'+2y = 2xe^{-x}\sin x + 4x^2 - 1$

$r^2+2r+2=0$

$\Delta = -4$

$r = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = (-1 \pm i)$

$\lambda \quad \mu$

$y_1 = e^{-x}\cos x \qquad y_2 = e^{-x}\sin x$

$y_{p(x)} = (Ax+B)e^{-x}\sin x + (Cx+D)e^{-x}\cos x$

$xe^{-x}\sin x \qquad e^{-x}\sin x$

$+\left(A_1x^2 + A_2x + A_0\right)$

$y'' + 4y = \sin^2 x - 2x$

$= -\frac{1}{2}\cos 2x + \frac{1}{2} - 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$y_p = A\cos x + B\sin x + (Cx+D)$

---

④ $y'' + 4y' = 1 - e^{2x} + x\sin x\cos x$

$x\sin x\cos x = \frac{1}{2} x \sin 2x$

$\sin x\cos x = \frac{1}{2}\sin 2x$

$y'' + 4y' = 0$

$r^2 + 4r = 0$

$r = 0, \; r = -4$

$y_1 = 1, \; y_2 = e^{-4x}$

$y_{p(x)} = Ax + Be^{2x} + (Cx+D)\sin 2x + (Ex+F)\cos 2x.$

$A_1 + A_2 x \ldots$

Variation of parameters

$y'' + ay' + by = r(x)$ ... non-homogenious. $r(x) \neq 0$

undetermined.

① y_h (make it homo and solve) y_1, y_2

② y_p →

$y = y_h + y_p$

expo, poly, sin/cos, constant — $r(x)$

y'' معامل 1 *

بدون $r(x)$

$$y_{p(x)} = -y_1 \int \frac{y_2}{W} r(x) + y_2 \int \frac{y_1}{W} r(x)$$

$W(y_1, y_2)$

y_1, y_2 من y_h

---

example : Solve

① $x^2y'' - 2xy' - 4y = 12x^{-3}$ ← do not use undetermined : ① coefficient not constant ② r(x) not poly

$y = y_h + y_p = \frac{c_1}{x} + c_2x^4 + \frac{6}{7}x^{-3}$

$x^2y'' - 2xy' - 4y = 0$ couch-E

$r(r-1) - 2r - 4 = 0$

$r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$r = 4, -1$

$y_1 = x^{-1}, y = x^4$

$y_h = \frac{c_1}{x} + c_2x^4$

③ $W(\frac{1}{x}, x^4) = \frac{4x^3}{x} - \frac{-1}{x^2} \cdot x^4$

$= 4x^2 + x^2 = 5x^2$

$y_{p(x)} = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$

$= \frac{-1}{x} \int \frac{x^4}{5x^2} 12x^{-5}\,dx + x^4 \int \frac{x^{-1}}{5x^2} 12x^{-5}$

$= \frac{-12}{5x} \int x^{-3}\,dx + \frac{12}{5}x^4 \int x^{-8}\,dx$ ← suitable form

$= \frac{-12}{5x} \frac{x^{-2}}{-2} + \frac{12}{5}x^4 \frac{x^{-7}}{-7}$

$\frac{12}{10}x^{-3} - \frac{12}{35}x^{-3} = (\frac{12}{10} - \frac{12}{35})x^{-3} = \frac{6}{7}x^{-3}$

$r(x) = 12x^{-3} \leftarrow 12x^{-5}$

y'' معامل x^2 ← يجب أن يكون 1

← Find a suitable form for y_p لازم نعرف شكل الـ y_p ،

$$y_p = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$$

suitable form

بدنا نطلع الـ homogenious للتعويض فيه والباقي

ـ بنجيبلي اياه بعدين منكمل مع الباقي

Suitable form for y_p الـ

undiltermined. — variation

A, B, C — صفر ما بتجي من تكاملات

A_0, A_1, A_2 or

$0 \int + \int$

---

* لازم نعرف $r(x)$ عشان منه نطلع y'' عشان نعوض

---

suitable form

$$x^2y'' - 2xy' - 4y = g(x)$$

$$r(x) = \frac{g(x)}{x^2}$$

هيك

$$x^2y'' - 2xy' - h(x)y = g(x)$$

y_1, y_2 بس معطينا

$$\int \frac{(x-3)+3}{x-3}$$

$$\int 1 + \frac{3}{x-3}$$

$$x + 3\ln|x-3| = x = x + \ln(x-3)^3 = e^x + (x-3)^3$$

CH3:- Higher ODE

Solves-

① $y''' - 2y'' - 5y' + 6y = 0$ (homo with constant)

$r^3 - 2r^2 - 5r + 6 = 0$

$r=1 \rightarrow (r-1)$ is a factor

	1	-2	-5	6 ← coeffs
		+ 1	-1	-6
x	↓			
	1	-1	-6	0 ← r.s = 0

$r^2 - r - 6 = 0$ ← factor

$(r-1)(r-3)(r+2) = 0$

$r=1, r=3, r=-2$

$y_1 = e^x, y_2 = e^{3x}, y_3 = e^{-2x}$

$y = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$

② $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$ homo with constant

$r^5 - 3r^4 + 3r^3 - r^2 = 0$

$r^2(r^3 - 3r^2 + 3r - 1) = 0$

$(r^3 - 1) - 3r(r-1)$

$(r-1)(r^2 + r + 1) - 3r(r-1)$

$(r-1)(r^2 - 2r + 1)$

$r^2 (r-1)(r-1)$

$r\ r$

So 0 0 1 1 1

$y_1 = 1 \quad y_2 = x \quad y_3 = e^x$

$y_4 = xe^x \quad y_5 = x^2 e^x$

$y = c_1 + c_2 x + c_3 e^x + c_4 xe^x + c_5 x^2 e^x$

③ $y^{(4)} + 2y'' + y = 0$ homo with constant

$r^4 + 2r^2 + 1 = 0$

$(r^2)^2 + 2(r^2) + 1 = 0$ تربيعية معادلة

$(r^2+1)(r^2+1) = 0$

$r^2 + 1 = 0$ or $r^2 + 1 = 0$

$r = \pm i \qquad r = \pm i$

$y_1 = \cos x \qquad y_3 = x\cos x$

$y_2 = \sin x \qquad y_4 = x\sin x$

$y = c_1 \cos x + c_2 \sin x + c_3 x\cos x + c_4 \sin x$

④ $x^3 y''' - 3x^2 y'' + 6xy' - 6y$ (r, $r(r-1)$, $r(r-1)(r-2)$, 1)

* Couchy Euler (3rd order)

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1)$ $\quad r^2 - 2r - 3$

$(r-1)(r(r-2) - 3r + 6) = 0$

$(r-1)(r^2 - 5r + 6) = 0$

$(r-1)(r-3)(r-2) = 0$

$r = 1, 3, 2$

$y_1 = x \quad y_2 = x^3 \quad y_3 = x^2$

$y = c_1 x + c_2 x^3 + c_3 x^2$

$x^2\left(y'' = \frac{y'}{x}\right)$ y-missing

or جزء

$x^2 y'' - xy' = 0$

Couch - Euler

Lms
LR

(4) $y^{iv} - y''' - 1 = 0$ → non homogeneous term

$y = 1$ — constant ??

$y^{(4)} - y''' = 0$

$y_{p(x)} = Ax^3$

$y'_p = 3Ax^2$ $\quad y'''_p = 6Ax$

$y''''_p = 6A$ $\quad y^{(5)}_p = 0$

undetermined

$r^4 - r^3 = 0$

$r^3(r-1) = 0$

$r = 0, r = 0, r = 0, r = 1$

$y = c_1 + c_2x + c_3x^2 + c_4e^x$

$0 - 6A = 1$

$A = \frac{-1}{6}$

$y_p = \frac{-1}{6}x^3$

$y = c_1 + c_2x + c_3x^2 + c_4e^x - \frac{x^3}{6}$

Higher - Order Variation of parameters → third order في

$$y''' + p(x)\,y'' + q(x)\,y' + g(x)\,y = r(x) \quad \text{(x) ≠ 0}$$

① $y_h = C_1 y_1 + C_2 y_2 + C_3 y_3$

② $y_p = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$

ناخذ د معامل y'''

- example: $W(x, x^2, x^3) = \begin{vmatrix} + x & - x^2 & + x^3 \\ - 1 & + 2x & - 3x^2 \\ + 0 & - 2 & + 6x \end{vmatrix}$

رتبة صف ثانية

عامود ثانية

even + 4 = 2+2

$= x\begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - (1)\begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix} + (0)\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$

منختار عامود اوصفر فيه اجهاد اكثر

$= x(12x^2 - 6x^2) - (6x^3 - 2x^3)$

$= 6x^4 - 4x^3 = \boxed{2x^3}$ W

أول عامود

① $W_1(x, x^2, x^3) = \begin{vmatrix} + 0 & x^2 & x^3 \\ - 0 & 2x & 3x^2 \\ + 1 & 2 & 6x \end{vmatrix} = 1\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$

العامود الأول ← صفر

---

② $W_2(x, x^2, x^3) = \begin{vmatrix} + x & - x^3 \\ - 1 & + 3x^2 \end{vmatrix} = -\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = -(3x^3 - x^3) = -2x^3$

عامود ثانية صف ٢

3+2=5 odd (-)

لا يستخدم العامود الثانية

---

③ $W_3(x, x^2, x^3) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$2x^2 - x^2 = x^2$

٣+٢=٥ فردي

لا يستخدم عامود

example:- Solve:

suitable form

بالامكان نبقيها

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 24x^4$$

coefficients is not constant so we can't use undetermined

$r(x) = 24x$

$r(r-1)(r-2) - 3r(r-1) + 6r - 6$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1)$

$(r-1)(r^2 - 2r - 3r + 6) = 0$

$(r-1)(r^2 - 5r + 6) = 0$

$(r-1)(r-2)(r-3) = 0$

$y_1 = x \quad y_2 = x^2 \quad y_3 = x^3$

$y_h = c_1x + c_2x^2 + c_3x^3$

$y = y_h + y_p$

$= c_1x + c_2x^2 + c_3x^2 + 4x^4$

حسب الـ y_1, y_2, y_3 بالصفحة اللي قبل

$$y_p = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$$

$$x \int \frac{x^4}{2x^3} 24x + x^2 \int \frac{-2x^3}{2x^3} 24x + x^3 \int \frac{x^2}{2x^3} 24x$$

$$12x \int x^2 dx - 24x^2 \int x\,dx + 12x^3 \int dx$$ → suitable form

$$12x\left(\frac{1}{3}x^3\right) - 24x^2\left(\frac{1}{2}x^2\right) + 12x^3(x)$$

$$4x^4 - 12x^4 + 12x^4 = 4x^4$$

CH4:-

system of Differintal equation

$$y_1' = a y_1 + b y_2$$

$$y_2' = c y_1 + d y_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a y_1 + b y_2 \\ c y_1 + d y_2 \end{bmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Question writtin in this form write it in this form

$$\underline{y}' = A \underline{y}$$

$\underline{y}$ → y vector