OR2

26 Feb 2025

→ Textbook : OR by Taha , last edition
→ solve suggested prob. , solve many problems

→ marking : mid (30) Final (50) quizes or (20)
in class assign

• Stochastic Processes

Probabilistic
↳ attributes of a system randomly evolve over time. (change)
↳ characteristic aspects → ex: state of a machine
# of customers in a queue or a state
# of students in class from sem to sem.
stock market , data mining [FI]
↳ prediction

Suppose we observe some char. at discreate points
at time (0, 1, 2, ...)
initial

X_t
let x_t denote char. at time t.

discreate → ch9 , ch17
continuous → ch 19

→ ex(1) : state of a machine

$$x_t = \begin{cases} 0 & , \text{poor} \\ 1 & , \text{good} \\ 2 & , \text{excellent} \end{cases} \quad , t = 0, 1, 2 \ldots$$

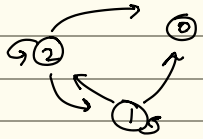
→ x_t is a random variable that means before looking at the system I don't know the state of machine

Set of all possible states = state space.

state كل وحدة في

# of customers in a shop → 5
→ 6
10
} different possible states

State transition diagram :-



1- identify all possible states
2- characterize that covers the transition
→ possible transition
→ Transition Probability

⇒ Our goal is to predict the future behaviour in the system based of the type

0, 1, 2 ⇒ discrete.

$X_t = X_i$

State of power } قيمة
$P(X) = X_i$
Probability of time being at i

→ the system may move from one state to another (transition)

→ The one step transition

Prob. $P(X_1 = X_1 \mid X_0 = X_0)$
↳ conditional probabilities
→ $P(X_{t+1} = x_j \mid X_t = x_i) = P_{ij}$

$j \xleftarrow[\text{step}]{\text{one}} i$ P_{ij}

$i \xleftarrow[\text{two step}]{} i$ P_{ij}^2

:

:

3 March 2025

Stochastic processes.

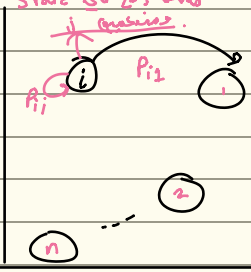
$P(X_t) = X_i$

$P(X_{t+1}=j \mid X_t=i) = P_{ij}$

$\sum_j P_{ij} = 1$, for all i,j

$P_{ij} \geq 0$, " " "

state ... from ... to ... transitions

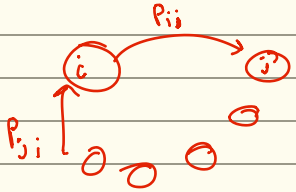


→ $P(X_{t+1}=j \mid X_t=i) = P(X_1=j \mid X_0=i)$

Stationary : time homogeneous (initial time)

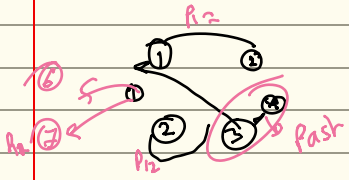
X_0 after one time unit (one step)

I go to j



$P_{t(t+1)} = P_{ij}$ every step

$i \to j \to P_{ij} = 0.25$ after from i to i P_{ii}



→ here the present

① → future

future ... past ... memories

→ Markrove chains:

→ prob, in which how the process evolve in future only depends on the present of the processes

$$P[X_{t+1} = x_{t+1} \mid X_t = x_t, X_{t-1} = x_{t-1}, x_{t-2} = x_{t-2} \dots X_1 = x_1, X_0 = x_0]$$

(Future ← present ← stage ← past)

$$= P[X_{t+1} \mid X_t = x_t]$$

blank or cell memory

بكل ال process انا بطلع بين كل future & present
بين بيعطيني كتير معلومات عن ال future بنقراهم عالسريع

One step transition in Matrix notation :-

- future → columns (j)
- present → rows (i)

P (AUBUC)
↙ ↓ ↘
معطى 2 3

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array} \begin{array}{l} \\ = 1 \\ = 1 \\ = 1 \end{array}$$

P_{11} = transition from station 1 to station 1

P(AUBUC)

A: staying in 1
B: going to 2
C: going to 3

A, B, C are mutually exclusive (if one of them happens the others can't happen)

$$\sum_j P_{ij} = 1 \quad , P_{ij} \geq 0$$

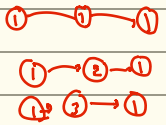
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ex:

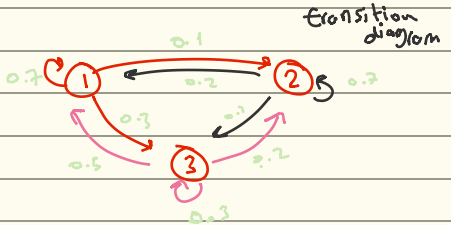
① : Sunny day
② : cloudy
③ : Snow fall

one step transition ↙

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \end{array}$$



P of sunny to sunny in two days



$P[X_{t+2} = ① \mid X_t = ①]$

$0.7 * 0.7 = 0.49$

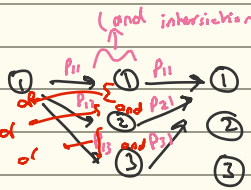
OR

$0.1 * 0.2 = 0.02$

OR

$0.2 * 0.5 = 0.1$

$0.49 + 0.02 + 0.1 = 1.61$



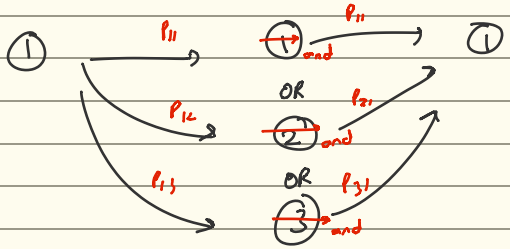
intersect ⟶ $P(A \cap B) = P_A * P_B$ if they are independent

two step

$P_{11} * P_{11} + P_{12} * P_{21} + P_{13} * P_{31} = 0.62$

lack of memory : ما بتذكر شي

independent ← يعني كل خطوة مستقلة عن اللي قبلها



two step transition probability : $P_{ij}^{(2)} = P_{ij}^2$

$P_{11}^{(2)} =$

$P^2 = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} \\ = & \end{bmatrix}$ first row, second column

$P^{(4)} = P * P$

$P^{(2)} = P * P = P^2$

$P^{(3)} = P^{(2)} * P^{(1)}$

$P^{(4)} = P^{(2)} * P^{(2)} = P^{(3)} * P^{(1)}$

first element is multiplying first row with first column

$$P^{(n)} = P^{(n-1)} \cdot P^{(1)}$$

Wed March 5

$$P^{(2)} = P^2$$

$$P^{(3)} = P^2 \cdot P^1$$

n-step Transition Matrix

$$P_{ij}^{(n)} = P\left[X_{t+n}=j \mid X_t = i\right] = \sum_{k=1}^{n} P_{ik}^{n-1} \cdot P_{kj}$$

$$= \sum_{k=1}^{n} P_{ik} \cdot P_{kj}^{n-1}$$

for all states i k j

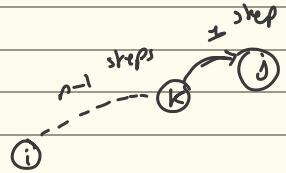
n step

ex: n=10

لـ العاشرة من k ← i

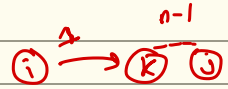
المسافة اللي قبل تنقلت عبين ما وصلت لـ k

بـ steps ثانية



أو يروح لـ k وبعدين انتقل

بين k و j



أو يروح لـ k بـ 5 steps وبعدين

من k لـ j بـ 5 steps

Chapman - Kolmogorov Equation

$$P^{(n)} = P^{n-m} \cdot P^{m}$$

matrix

$$0 < m < n$$

$$P_{ij}^{n} = P_{ik}^{n-m} \cdot P_{kj}^{m}$$

one entry from the matrix

* states i & j

قبل الخوض في الأمثلة يجب أن نعلم أن $m=1$ يجب أن تكون $0 < m < n$ أي أن m بين 0 و n

---

* Absolute probabilities :-

initial state pro. vector

$$a^{(0)} = \{ a_j^{(0)} \quad , \quad j = 1, 2, 3 \dots, m \}$$

$m = 3$

$$a^{(0)} = \{ a_1^0 , a_2^0 , a_3^0 \} = \{ 1 , 0 , 0 \}$$

$a_1^0 = 1$, $a_2^0 = 0$, $a_3^0 = 0$

Prob at the initial to be at station

$1 \to 1$

$2 \to 0$

$3 \to 0$

prob of the next step will be

$$a^1 = \{ a_1^1 , a_2^1 , a_3^1 \} \to m = 3$$

$$= \{ 0.25 \quad 0.5 \quad 0.25 \}$$

Prob after one transition to be at station $1 \to 0.25$

$2 \to 0.5$

$3 \to 0.25$

$$a^{(n)} = \{ a_j^{(n)} \quad , j = 1, 2 \dots, m \}$$

(n) ← # of steps

j ← of station

n = # of steps

m = # of stations

$a^0 = \{1, 0, 0\}$

$a^1 = a^0 \cdot P \rightarrow$ the whole matrix

$a^2 = a^1 \cdot P$

$a^2 = a^0 P^2$ ← two step transition matrix

$a^3 = a^0 \cdot P^3$

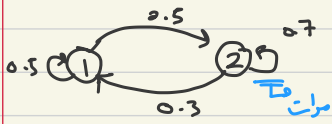
نسبة وين ال system راح يكون

← يتم حساب a^0 كلشي من

$$a^{(n)} = a^0 \cdot P^{(n)} \rightarrow a^{(n)} \text{ absolute probability}$$

= transitions عدد وين ال system راح يكون بعد

exp:-



ممكن مرات

1-0.3=0.7 هاي ما يعطانا اياها هي

1- one step transition matrix

2- $P^{(2)}, P^{(4)}, P^{(5)}, P^{(6)}$

$$P = \begin{matrix} & j: 1 & 2 \\ i: 1 & \\ 2 & \end{matrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$$

احتمالية انه اول ب 1 بعد 2 transition

$$P^{(2)} = P \cdot P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.36 & 0.64 \end{bmatrix}$$

$$P^{(4)} = P^{(2)} \cdot P^{(2)} = \begin{bmatrix} 0.4 & 0.6 \\ 0.36 & 0.64 \end{bmatrix} \cdot \begin{bmatrix} 0.4 & 0.6 \\ 0.36 & 0.64 \end{bmatrix} = \begin{bmatrix} 0.376 & 0.624 \\ 0.374 & 0.626 \end{bmatrix}$$

two transitions

احتمالية اروح من 2 → 1 بعد

$$P^{(8)} = \begin{bmatrix} 0.376 & 0.624 \\ 0.374 & 0.6256 \end{bmatrix} \begin{bmatrix} 0.376 & 0.624 \\ 0.374 & 0.6256 \end{bmatrix} = \begin{bmatrix} 0.375 & 0.625 \\ 0.375 & 0.625 \end{bmatrix}$$

$$P^{(16)} = \begin{bmatrix} 0.375 & 0.625 \\ 0.374 & 0.624 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.36 & 0.64 \end{bmatrix}$$

↳ probability to end up in 1

in one step transition → doesn't depend where I am

→ after more transition the probabilities are steady ⟹ the system is in a steady state.

→ how many steps does it take to get the steady state ⟹ is not our job to know

in a steady state: بنعرف إنه بقدر دائماً من وين بالظبط بس عارفين إنه بقدر يكون في أي مكان

بيت

توزيع احتمالية بعض

in this example $a^{16} = \{0.375 \quad 0.625\}$

probability at 16 transitions to end at 1 / 2

here it's the same.

$a^1 = \{1 \quad 0\}$ ↳ at the initial I'm at state (1)

after 16 transition the probability is ⇒ at

$a^{16} = [1 \quad 0] \begin{bmatrix} \quad \end{bmatrix}^{P^{16}}$

state 0 ex 3 states 5 وعندنا initial

$a^0 = [0 \quad 0 \quad 1 \quad 0 \quad 0]$ → هاي احنا كاتبة بنختبر

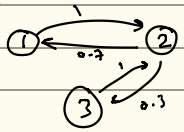
Mon March 10

→ Do we always reach a steady state??

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ex:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



① state transition diagram

② P^2, P^3, P^4, P^5

②

$$P^2 = \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \\ 0.7 & 0 & 0.3 \end{bmatrix} \qquad P^5 = \begin{bmatrix} 0.7 & 0 & 0.3 \\ 0.7 & 0 & 0.3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{22}^{(n)} = \begin{cases} 0, & n \text{ is odd} \\ 1, & n \text{ is even} \end{cases}$$

→ it will keep jumping so we didn't reach steady state.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.9 & 0.1 \end{bmatrix}$$

Find P^8 :

$$P^8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.49 & 0.0167 & 0.165 & 0.326 \\ 0 & 0 & 0.336 & 0.667 \\ 0 & 0 & 0.332 & 0.667 \end{bmatrix}$$

← متساويين تقريباً

→ هنا بردو معايش انا متأكدين بقلب

→ so here I didn't reach steady state.

1 - 6

17-9. A die-rolling game uses a 4-square grid. The squares are designated clockwise as A, B, C, and D with monetary rewards of \$4, – \$2, – \$6, and \$9, respectively. Starting at square A, roll the die to determine the next square to move to in a clockwise direction. For example, if the die shows 2, we move to square C. The game is repeated using the last square as a starting point.

(a) Express the problem as a Markov chain.

(b) Determine the expected gain or loss after the die is rolled 5 times.

4 | -2
A 4 | B 1 5
D 3 | C 2 6
9 | -6

a)

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1/6 & 2/6 & 2/6 & 1/6 \\ 1/6 & 1/6 & 2/6 & 2/6 \\ 2/6 & 1/6 & 1/6 & 2/6 \\ 2/6 & 2/6 & 1/6 & 1/6 \end{bmatrix} \end{array}$$

1 2 3 4 5 6
B C D A D C

OR

$P(1 \cup 5) = P(1) + P(5)$

OR = U = +

and

$P(1 \cap 5)$

and = ∩ = *

1,5 | 2,6

1,5	2,6
A→B	A→C
B→C	B→D
C→D	C→A
D→A	D→B

3	4
A→D	A→A
B→A	B→B
C→B	C→C
D→C	D→D

b)

$$P^5 = \begin{bmatrix} 0.2502 & 0.2497 & 0.2497 & 0.2502 \\ 0.2502 & 0.2502 & 0.2497 & 0.2497 \\ 0.2497 & 0.2502 & 0.2502 & 0.2497 \\ 0.2497 & 0.2497 & 0.2502 & 0.2502 \end{bmatrix}$$

بالاستعانة بالمتجه

$$= 0.2502 \times 4 + 0.2497(-2) + 0.2497(-6) + 0.2502(9)$$

$$\approx 1.25$$

current

17-1. An engineering professor acquires a new computer once every two years. The professor can choose from three models: $M1$, $M2$, and $M3$. If the present model is $M1$, the next computer can be $M2$ with probability .25 or $M3$ with probability .1. If the present model is $M2$, the probabilities of switching to $M1$ and $M3$ are .5 and .15, respectively. And, if the present model is $M3$, then the probabilities of purchasing $M1$ and $M2$ are .7 and .2, respectively. Represent the situation as a Markov chain.

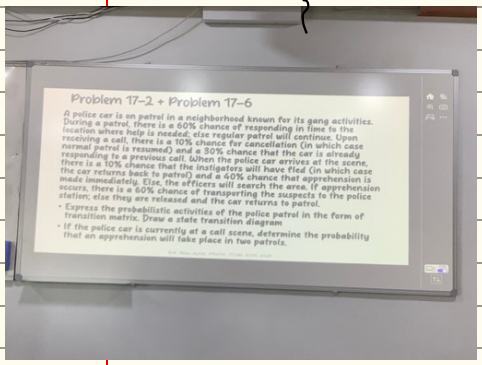
2. determine the prob. that the professor will purchase the current model in 4 years

17-5. Consider Problem 17-1. Determine the probability that the professor will purchase the current model in 4 years.

$$P = \begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \end{array} \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 0.65 & 0.25 & 0.1 \\ 0.5 & 0.35 & 0.15 \\ 0.7 & 0.2 & 0.1 \end{bmatrix} \end{array}$$

one step = 2 years

P^2



Problem 17-2 + Problem 17-6

A police car is on patrol in a neighborhood known for its gang activities. During a patrol, there is a 60% chance of responding in time to the location where help is needed; else regular patrol will continue. Upon receiving a call, there is a 10% chance for cancellation (in which case normal patrol is resumed) and a 30% chance that the car is already responding to a previous call. When the police car arrives at the scene, there is a 10% chance that the instigators will have fled (in which case the car returns back to patrol) and a 40% chance that apprehension is made immediately. Else, the officers will search the area. If apprehension occurs, there is a 60% chance of transporting the suspects to the police station; else they are released and the car returns to patrol.

- Express the probabilistic activities of the police patrol in the form of transition matrix. Draw a state transition diagram
- If the police car is currently at a call scene, determine the probability that an apprehension will take place in two patrols.

Part (cancel)

respond (previous)

At scene

apprehend

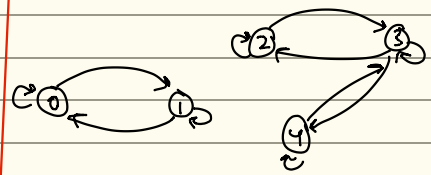
transport

	Part	respond	At scene	apprehend	transport
		0.6			

wednesday, mar 12

* Section 17.3 :-

ex 1 :



① Reachable (Accessible) :-

→ A state j is reachable from state i if there is a path from i to j

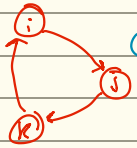
↳ a sequence of transitions with positive possibility

$P_{ij}^{(n)} > 0$ for some $n \geq 1$

→ no arrow → no possibility

→ arrow → positive possibility.

here ④ is reachable from ②



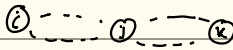
② Communicate :-

→ two states communicate, if both are accessible from eachother.

→ it's a two way path (communication is giving & taking)

the communicating can be with more than one step

ex :-



i & j communicate , j & k communicate

→ so i & K communicate.

self transition :



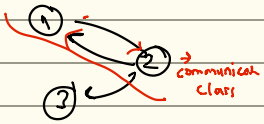
if the only P_{ii} is $P_{ii} = 1$ absorbing state

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③ Closed Set :-

if there is a closed set يعني لو في closed set خلص ما بقدر اطلع Closed set من

→ a set of states is said to be closed if no state outside the set is reachable from any state from inside the set (the closed set should communicate)



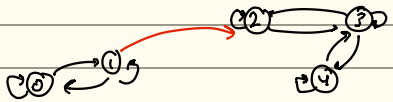
→ in ex 1: 2,3,4 → closed set

1,0 → closed set

if I have more than two states communicate it's called communication class communicating

---

ex2 :



2,3,4 → is still closed.

1,0 → isn't closed set

because you can reach outside them

اذا رجعت ما بقدر اطلع منه

not every communicating class is closed set

every closed set is communicating class

④ **Irreducible system (Markov chain.)** :-

→ if all states communicate (two way between every state), all states consitute a closed set with no subset of the states is closed — one closed set not two

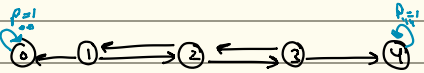
→ in ex1: it's reducible system (you can divide the system into more than one closed set)

---

⑤ **Absorbing state** :-

→ it's a special case of a closed state, it is a singal state with transition prob $P_{ii} = 1$

ex 3 :



$P_{ii} = 1$

if you go from 1→0 or 3→4 you are stuck there

---

⑥ **Transient state** :

→ state i is a transient state if there is a transition out of state i to some other state & from which state i can never be reached again.

→ in ex1: 1, 2, 3 → are transient state

---

⑦ **Re current state** :-

→ any state that is not transient (there is a prob to be back at this state)

⇨ for any state i let f_i : prob that starting from i the process will ever re enter state i.

→ for a recurrent state $f_i = 1$

→ for a transient state $f_i < 1$

I'm not sure that I'll ever come to it again because of an absorbing state.

absorbing state is a recurrent

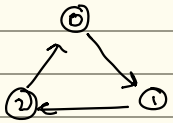
(8) **Periodic state :-**

→ state i is periodic with period $t > 1$ if a return is possible in $\{t, 2t, 3t, \ldots\}$ steps

multiple of t

→ t is the shortest path

ex 4 :-



P_{00} → 3 step

P_{11} → 3 step

P_{22} → 3 step

→ in a set if one state is periodic, all the others are also periodic, and the period for all of them = t (same fixed #)

(9) **aperiodic state :**

one word.

the number of steps is not fixed. (periodic else)

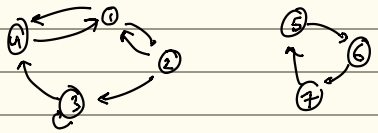
Monday Mar 17

→ An irreducible finite M.c. in which all states are aperiodic is called ERGODIC M.C

ERGODIC = irreducible + aperiodic

all states are one closed set

ex:



1,2,3,4 → ERGODIC

1,2,3,4,5,6,7 → not ERGODIC because it's not irreducible

note:-

if I have a self transition (P_{ii}) the system is aperiodic

* If the system is ERGODIC then it will get to a steady state.

→ Sufficient condition for convergence to steady state probabilities are that the entire state space be finite, irreducible, aperiodic & Ergodic.

---

* $P^2 = P^{(1)} \cdot P^{(1)}$

$P^{10} = P^9 \cdot P^{(1)}$

$P^{(n+1)} = P^n \cdot P^1$

$P^{(n)} = P^{(n)} \cdot P^{(1)}$

for n sufficiently large → $P^{(n+1)} = P^{(n)}$

لما n تكبر كتير

steady state.

because it's already at the steady state.

↳ we need to solve this equation for $P^{(n)}$

---

ex: $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$

in steady state.

$P_{11}^n = P_{21}^n$

$P_{12}^n = P_{22}^n$

steady state.

$$P^{(n)} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} P_{11}^n & P_{12}^n \\ P_{21}^n & P_{22}^n \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} \cdot \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

now we solve for π_1, π_2

ex: $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$

$$\begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

redundant equation

$\pi_1 = 0.6\pi_1 + 0.2\pi_2$

$\pi_2 = 0.4\pi_1 + 0.8\pi_2$

نشيل ↓

$\pi_1 = 0.6\pi_1 + 0.2\pi_2$

$\pi_2 = 0.4\pi_1 + 0.8\pi_2$

$\pi_1 + \pi_2 = 1$

$0.2\pi_2 = 0.4\pi_1$ (لو 1 فوق 1 بيساوي 1)

$\pi_2 = 2\pi_1$

$\pi_1 + 2\pi_1 = 1$

$\pi_1 = \frac{1}{3}$

$\pi_2 = 1 - \frac{1}{3} = \frac{2}{3}$

$\pi_1 + \pi_2 = 1$ ← redundant. لازم نجيب واحدة

# of states = m

# of steps = n

$$\begin{cases} \pi_j = \sum_{i=1}^{m} \pi_j \, p_{ij}, \quad \text{for } j = 1, \dots, m \\ \sum_{i=1}^{m} \pi_{ij} = 1 \end{cases}$$

---

ex: Find the steady state probabilities

$$P = \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.4 & 0.55 \end{bmatrix}$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.4 & 0.55 \end{bmatrix}$$

$$\pi_1 = 0.3\pi_1 + 0.1\pi_2 + 0.05\pi_3$$

$$\pi_2 = 0.6\pi_1 + 0.6\pi_2 + 0.4\pi_3$$

$$\pi_3 = 0.1\pi_1 + 0.3\pi_2 + 0.55\pi_3$$

$$0.4\pi_2 = 0.6\pi_1 + 0.4\pi_3$$

$$\pi_2 = 1.5\pi_1 + \pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = 0.1017$$

$$\pi_2 = 0.5254$$

$$\pi_3 = 0.3728$$

30% state (1) | 50% state (2) | 10% of time the system is on state (1)

$$P = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

$$\pi_1 = 0.333$$

$$\pi_2 = 0.333$$

$$\pi_3 = 0.333$$

Σ row = 1 / Σ column = 1

↳ doubly stochastic

$\pi_s = \frac{1}{\#\text{ of states}}$ ⇐ هيك في

Wed, Mar 19th

π_j = Prob. that if we inspect the system at any instance, we'll find the system in state j.

Or: the percentage of time the system spends in state j in the long run.

ex: $\pi_1 = 0.25$, $\pi_2 = 0.5$, $\pi_3 = 0.25$ → this is the expectation of where we'll find the system

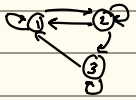
prob. + percentage of time

→ Mean first Return time (mean recurrence time) :

(it's a random variable.)

$\mu = \frac{1}{\pi_j}$, $j = 1, 2, \ldots, n$

μ = #steps to go from $j \to j$



aperiodic

expected number of transitions before the syst. returns back to state j for the first time

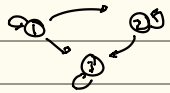
→ if it was periodic, (atleast 1 can predict. but here I can't

---

ex: $P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$ classify the states!

3: rec, absorbing & closed set

1, 2: transient



---

a) $\begin{matrix} & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{bmatrix}$:



periodic

t = 3

b) $\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0 & 1 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

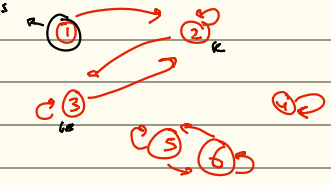
not periodic

but state 4 is absorbing.

c)

	1	2	3	4	5	6
1	0	1	0	0	0	0
2	0	0.5	0.5	0	0	0
3	0	0.2	0.5	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	0.4	0.6
6	0	0	0	0	0.2	0.8

trans



5,6 closed set

4 absorbing set & closed set

1,2,3 closed set

→ it's not ergodic but

I can divide it to 5,6 → system

2,3 → system

1 → (2,3) one big absorbing state

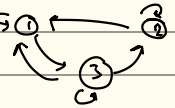
1 → transient → long term prob = 0

steady state

zero = Π

d)

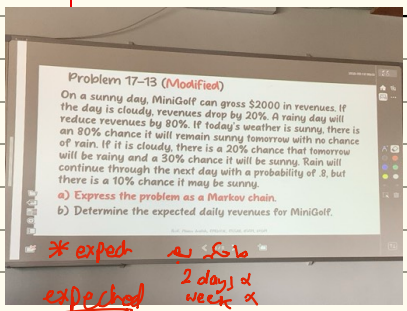
	1	2	3	
1	0.1	0	0.9	= 1
2	0.7	0.3	0	
3	0.2	0.7	0.1	
	= 1			



Irreducible system

doubly stochastic

$\Pi_1 = \Pi_2 = \Pi_3 = 0.33$



* expected

2 days of week

expected

drop

1 = cloudy

2 = rainy

3 = sunny

revenue = 2000

↓ 0.2 (cloudy) (1)

↓ 0.8 rainy (2)

2000 × 0.8 = 1600

→ expected daily

sunny → 2000

rainy → 400

cloudy → 600

weather

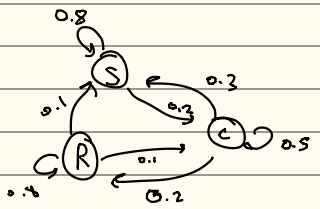
	1	2	3
1	0.5	0.2	0.3
2	0.1	0.8	0.1
3	0.2	0	0.8

prob of the

$\Pi_S = 0.5$

$\Pi_R = 0.25$

$\Pi_C = 0.25$

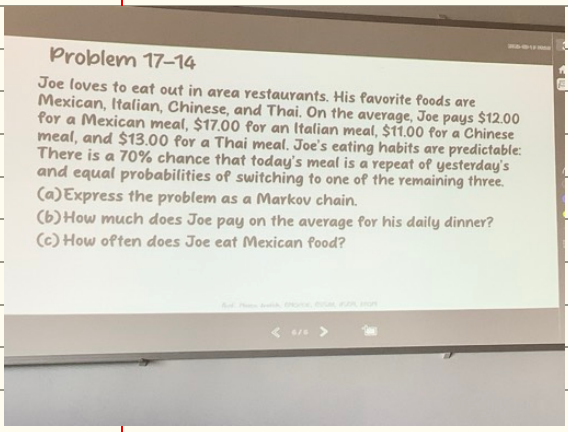


revenue

$r_S = 0.5 \times 2000$

$r_C = 0.25 \times 1600$

$r_R = 0.25 \times 400$



Problem 17–14

Joe loves to eat out in area restaurants. His favorite foods are Mexican, Italian, Chinese, and Thai. On the average, Joe pays $12.00 for a Mexican meal, $17.00 for an Italian meal, $11.00 for a Chinese meal, and $13.00 for a Thai meal. Joe's eating habits are predictable: There is a 70% chance that today's meal is a repeat of yesterday's and equal probabilities of switching to one of the remaining three.

(a) Express the problem as a Markov chain.

(b) How much does Joe pay on the average for his daily dinner?

(c) How often does Joe eat Mexican food?

M , I , C , T

↓ ↓ ↓ ↓

12 17 11 13

$$\text{avg} = \frac{30 + 23}{4} = \frac{53}{4} =$$

	M	I	C	T
M	0.7	0.1	0.1	0.1
I	0.1	0.7	0.1	0.1
C	0.1	0.1	0.7	0.1
T	0.1	0.1	0.1	0.7

↓

How → M

$\pi_M \quad \pi_I \quad \pi_C \quad \pi_T$

$$\pi_M = 0.7\pi_M + 0.1\pi_I + 0.1\pi_C + 0.1\pi_T$$

$$\pi_I = 0.1\pi_M + 0.7\pi_I + 0.1\pi_C + 0.1\pi_T$$

$$\pi_C = 0.1\pi_M + 0.1\pi_I + 0.7\pi_C + 0.1\pi_T$$

$$\pi_M + \pi_I + \pi_C + \pi_T = 1$$

$\pi_M = 0.25$

$\pi_I = 0.25$

$\pi_C = 0.25$

$\pi_T = 0.25$

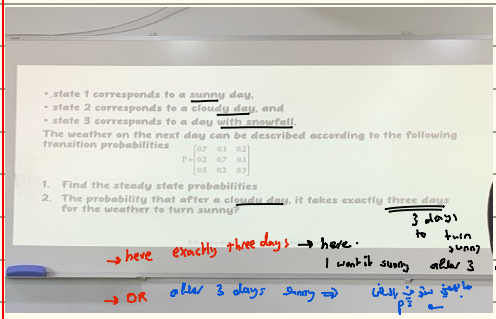
0.25 x price

↓ probability

how often = 4 days

lec 8 :

Mon , March 24



2-

1 = sunny 2 = cloudy 3 = snow

1- $P = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = 0.513 \qquad \pi_2 = 0.297 \qquad \pi_3 = 0.189.$$

→ I want it to go from cloudy to sunny in 3 days
but I just want sunny at the end not in the middle transitions

* this is called * First Passage Prob. :

→ the prob that the system leaves state i &
enters state j for the first time after n steps

$i \leftarrow j$ in 3 steps

$$f_{ij}^{(n)} = P(X_n = j, X_{n-1} \neq j, X_{n-2} \neq j, \dots, X_0 = i)$$

ex : $f_{ij}^{3} = P^{3} -$ ____

3 steps like $c \to s \to \dots \to s$
$c \to snow \to s \to s$
...

→ the first passage Prob $f_{ij}^{(n)}$ is computed as $p_{ij}^{(n)}$ minus the probabilities of the n-step path in wich state j occure before the nth step

$$f_{ij}^{(1)} = p_{ij}^{(1)}$$

$$f_{ij}^{(2)} = p_{ij}^{(2)} - f_{ij}^{(1)} \cdot p_{jj}^{(1)}$$

$$f_{ij}^{(3)} = p_{ij}^{(3)} - f_{ij}^{(2)} p_{jj}^{(1)} - f_{ij}^{(1)} p_{jj}^{(2)}$$

صيغة وقت ما نأخذ
بعد خطوتين من نفس مرتين

نظريات من نعد بد يوصل
بعد من نظريات نفس ... وصلت د ز يوصل

$$f_{ij}^{(n)} = p_{ij}^{(n)} - f_{ij}^{(n-1)} p_{jj}^{(1)} - f_{ij}^{(n-2)} p_{jj}^{2} \dots\dots - f_{ij}^{(1)} p_{jj}^{n-1}$$

$$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} p_{ij}^{(k)} p_{jj}^{(n-k)}$$

$$P = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

find :-

a) $F^2 = \begin{bmatrix} f_{11}^2 & f_{12}^2 & f_{13}^2 \\ f_{21}^2 & \vdots & \vdots \\ \vdots & \vdots & f_{33}^2 \end{bmatrix}$, b) $F^3 = \begin{bmatrix} f_{11}^3 & \cdots & \cdots \\ \vdots & f_{22}^3 & \vdots \\ \cdots & \cdots & f_{33}^3 \end{bmatrix}$

$$f_{23}^{(1)} = p_{23}^{(1)} = 0.1$$

$$p^2 = \begin{bmatrix} 0.61 & 0.18 & 0.21 \\ 0.23 & 0.63 & 0.14 \\ 0.54 & 0.25 & 0.21 \end{bmatrix}$$

$$f_{23}^{(2)} = p_{23}^{(2)} - f_{23}^{(1)} \cdot p_{33}^{(1)}$$

$p_{23}^2 = 0.14$, $p_{33}^{(1)} = 0.1$

$$0.14 - p_{23}^{(1)} \cdot 0.1$$

$$0.14 - (0.1 \cdot 0.1)$$

$$= 0.14 - 0.03$$

$$= 0.11$$

$f_{11}^{(2)} = 0.12$

$$f_{31}^{2} = p_{31}^{2} - f_{31}^{(1)} \cdot p_{11}^{(1)}$$

$$= 0.54 - (0.5 \cdot 0.7) = 0.19$$

ex : p_{12}^3 , f_{12}^3 ← F , P بين ← الفرق

p_{12}^3 : 3 steps بعد 2 من الحالة

f_{12}^3 : 3 steps بعد 2 من الحالة for the first time ever

$$f_{11}^{(3)} = p_{11}^{(3)} - f_{11}^{(2)} \cdot p_{11}^{(1)} - f_{11}^{(1)} \cdot p_{11}^{(2)}$$

$$f_{21}^{(3)} = p_{21}^{(3)} - f_{21}^{(2)} p_{11}^{(1)} - f_{21}^{(1)} \cdot p_{11}^{(2)}$$

f_{ij} : Prob. starting from i that we ever reach j

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

$f_{ij} < 1$	$P_{ii} < 1$
$f_{ij} = 1$	$P_{ii} = 1$

in ERGODIC I'm sure to go back

---

lec 9

first Passage probability f_{ij}

first Passage time : of changing from state i to state j is be number of transition need by Markov chain m.c process as it goes from state i to j for the first time

$T_{ij} = n$ # of transition / f_{ij} given

T_{ij} random variable

قبل ما يغير الحيز حيرين

الوقت اللي بيحتاجه عشان يطلع من i لـ j

أول مرة

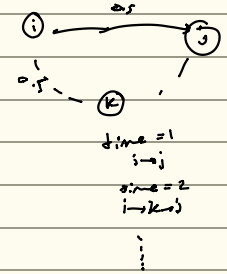
→ we need the expected value $E(T_{ij})$

$$E(T_{ij}) = M_{ij}$$

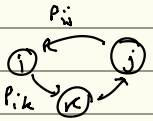
expected value for T_{ij} :

$$M_{ij} = \sum_{n=1}^{\infty} n f_{ij}^{(n)}$$

$$E(x) = \sum x P_{xy}$$



(mean)) expected value. متوقع يعني أوقف ال random variable



$i \to j$ / $i \to k \to j$

we start from (i) for each $k \neq j$

* the 1st passage there will be 2 + expected 1st passage time.

from k to j

$$M_{ij} = 1 + M_{kj}$$

one step ← كم خطوة من i لـ j

expected ← متوقع

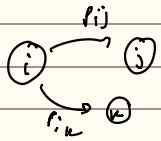
$k \neq j$

$$M_{ij} = P_{ij}(1) + \sum_{k \neq j} P_{ik}(1 + M_{kj})$$

اللي بيطلع من i لـ j =

A OR B → union → Plus

$P(A \cup B) = P(A) + P(B)$



$M_{ij} = P_{ij} + \sum_{k \neq j} P_{ik} + \sum_{k \neq j} P_{ik} M_{kj}$ → =1 ⇒ sum of probability from i

$= 1 + \sum_{k \neq j} P_{ik} M_{kj}$

$M_{ij} = 1 + \sum_{k \neq j} P_{ik} M_{kj}$ (1)

M_{ij} = # steps from k → j + step from i → k

$1 + \sum_{k \neq j} P_{ik} M_{kj}$

$$P = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

how many days do we need to go from (3) to (1)

$M_{31} = 1 + \begin{cases} k=2 \rightarrow P_{32} M_{21} \\ k=3 \rightarrow P_{33} M_{31} \end{cases}$ → $k \neq j+1 \rightarrow k \neq 2$

$M_{21} = 1 + \begin{cases} k=2 \rightarrow P_{22} M_{21} \\ k=3 \rightarrow P_{23} M_{31} \end{cases}$

M_{31} , M_{21}

$1 + P_{32} M_{21} + P_{33} M_{31}$

one way

$$\begin{cases} M_{31} = 1 + P_{32} M_{21} + P_{33} M_{31} \\ M_{21} = 1 + P_{22} M_{21} + P_{23} M_{31} \\ 0.2\, M_2 + 0.3\, M_{31} = 1 \\ 0.7 M_2 + 0.1 M_{31} = 1 \end{cases}$$

($P_{32} = 0.2$, $P_{33} = 0.3$, $P_{22} = 0.7$, $P_{23} = 0.1$)

$M_{21} = 4.2105$ step

$M_{31} = 2.6316$ step

M_{ij} is defined a term & the M_{kj} P_{ik} are constant ⇒ $N_m - 1$

expected value ← discrete ← متغير عشوائي

one way لحساب expected value لـ M_{21}

* another way :-

$$[M_{ij}] = (I - N_j)^{-1} \cdot 1 \quad , j \neq i$$

I = $(n-1)$ identity matrix

N_j = transition matrix (P) minus j^{th} row & j^{th} column P target state j

1 = column vector with all elements = 1

$N_1 = N_j = \begin{bmatrix} 0.7 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}$ / $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$$

$$\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.7 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = \begin{bmatrix} 0.3 & -0.1 \\ -0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = \text{inverse} * \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

M_{21} =

M_{31} =

lec 10

Mon, Apr 7th

ex: first passage time. needs practice.

$$\mathbf{P} = \begin{pmatrix} .30 & .60 & .10 \\ .10 & .60 & .30 \\ .05 & .40 & .55 \end{pmatrix}$$

find Pass from 2,3 $\Rightarrow$ 1 M_{21}, M_{31}

" " " 1,3 $\rightarrow$ 2 M_{12}, M_{32}

" " " 1,2 $\rightarrow$ 3 M_{23}, M_{13}

$$\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = \left[I - N_1\right]^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.55 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

↳ number to go to 1

$$\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = \begin{pmatrix} 0.4 & -0.3 \\ -0.4 & 0.45 \end{pmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 173.33 & -6.66 \\ -0.3 & -0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = \begin{bmatrix} 12.5 \\ 13.33 \end{bmatrix}$$

this is the expected # of transitions to go from state 2 $\rightarrow$ 1

$0.19 - 0.12 = 0.06$

17.6 : Analysis of absorbing states → here we're looking at non-ergodic

we'll take. NON-ERGODIC M.C :-

→ steady state conditions are different for N.E.M.C (non ergodic) than for E.M.C (ergodic)

in the exam we look ergodic or non-ergodic

→ in N.E.M.C we have transient & absorbing states :-

for transient	for absorbing
π_i = zero for transient state	one absorbing state → $\pi_i = 1$
	more than one absorbing state → does not exist.

→ So we are interested in :- → weather in E.M.C or N.E.M.C

1) prob of reaching absorption (π)

2) expected # of transitions to reach absorption (M)

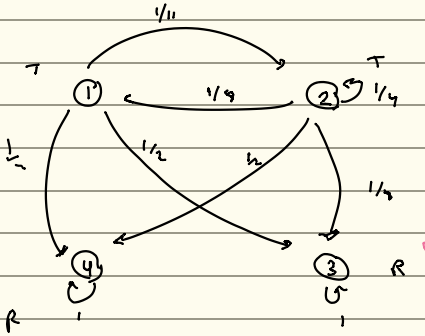
IMPORTANT :-

π = prob of reaching the state.

M = # of steps to reach

---

ex :-



$\pi_1 = 0$ $\pi_2 = 0$

$\{\pi_3, \pi_4\}$ → do not exist. or undefined.

* the probability to end up in (4) is different if you start in (1) OR (2) (steady state dependence)

So that's why we don't have steady state probability

if state k is an absorbing state.

then $a_{ik} = \sum_{j=1}^{m} P_{ij} a_{jk}$, for $i = 1, \dots, m$

i: from state i — j: to state j

absorption prob.

Analysis of N.E.M.C (with absorbing states)

can be carried out more conveniently using Matricies.

$P =$ $m \times m$

1) rearrange P to P^* — t transient states in P

rows + columns: نقل الحالات الانتقالية transient states

$P^* =$ t rows $\begin{bmatrix} N & R \\ \text{circle} & I \end{bmatrix}$ ($m-t$ rows; t columns; $m-t$ columns)

abs ← absorbing state

N: square matrix $t \times t$

R: // // $t \times m-t$

zeros matrix O — // // $m-t, t$

---

2) $F = (I - N)^{-1} \rightarrow t \times t$

↳ the fundamental matrix

I here isn't the same as I size = N size.

→ the elements = the expected # of times the system will be in it's varieth transient state before absorption occurs

from transient to transient

$N_{t \times t}$

t

$F = t \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 + 0 \end{bmatrix}$

starting from state ① how many times I'm gonna go to ② before ending up in absorption

how many times I go to transient before ending in absorption.

elements of $F \rightarrow f_{ij}$

$$T_{ij} = \sum_{j=1}^{t} P_{ij}$$ ← rows مجموع

unit vector × الصف = مجموع

$$= F \cdot 1$$ ← unit vector

$$[T] = [I - N]^{-1} \cdot 1$$

unit vector

transient states عدد الحالات

T :- the number of transitions I take in the transient states

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_t \end{pmatrix}$$

---

example: $P \rightarrow P^*$ $n=4$, $t=2$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R \quad N \quad 0 \quad I$$

$$P^* = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 2 \\ 3 \\ 1 \\ 4 \end{matrix}$$

$$N = \begin{bmatrix} 0 & 0.3 \\ 0 & 0 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 0.7 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

For every row

Σ row = 1 in P^* also because I only rearranged.

$$F = [I - N]^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.3 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 1 \end{bmatrix}$$

Lec 11
Wed, Apr 9th

Analysis N.E.M.C :-

$$P \rightarrow P^* = \begin{array}{c} \\ t \\ m-t \end{array} \begin{array}{c} \begin{array}{cc} t & m-t \end{array} \\ \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} \end{array}$$

1 ... m states

t → transient

m-t → Ab

→ square, i.e. same size

$R, 0$

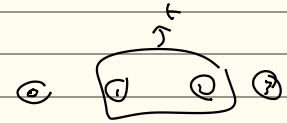
→ N, I → definitely square

→ $F = \left[\frac{I}{t} - N\right]^{-1}$

same size as N

↳ f_{ij}

ex:



$$F = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \end{array} \quad t$$

expected # of transitions from $i \to j$ before ending up in absorption

Row = # transitions I take from the row (state) before absorption

prob → π

transition → $M_{ij} = \frac{1}{\pi_i}$ } → in ergodic

Non-ergodic

→ prob of absorption (A)

→ # transition in transient

↳ this is (T)

$$P^* = \begin{bmatrix} N & R \\ 0 & I \end{bmatrix}$$

→ So prob of absorption (A) is the ergodic term (...) π

→ # transitions in transient (T) is the ergodic term (...) M

$$A = F \cdot R$$

$[t \cdot m-t]$

$$A = \begin{array}{c} \\ t \end{array} \begin{array}{c} \overset{m-t}{\overbrace{\phantom{[a_{ik}]}}} \\ \begin{bmatrix} a_{ik} \end{bmatrix} \end{array}$$

$i \to t$, $k \to m$

So A is the probability in the long run to end up in the absorbing states from the transient

example: 0, 3 → m-t

1, 2 → t

$$A = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 3 \end{array} \\ \begin{bmatrix} a_{10} & a_{13} \\ a_{20} & a_{23} \end{bmatrix} \end{array}$$

→ I can calculate the elements of A (a_{ik}) from the formula we took OR $A = F.R$ (matrices)

in our example:-

$$A = F.R = \begin{bmatrix} 1 & 0.7 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.15 \\ 0.5 & 0.5 \end{bmatrix}$$

⇒ the row summation in A should equal = 1 because in the long run we always end in absorption (Prob. of end up in absorbing = 1)

→ we took → p^*, T_i A, F

N, R

Example 17.6 (modified)

A product is processed on two sequential machines, I and II. Inspection takes place after a product unit is completed on either machine. There is a 5% chance that the unit will be junked before inspection. After inspection, there is a 3% chance the unit will be junked, and a 7% chance of being returned to the same machine for reworking. Else, a unit passing inspection on both machines is good.

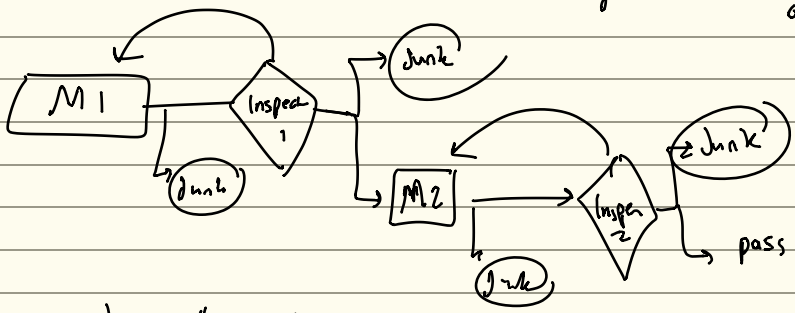
a) Express the problem as a Markov chain.

b) For a part starting at machine I, determine the average number of visits to each state.

c) If a batch of 1000 units is started on machine I, determine the average number of completed good units. A → absorption

Junked → Junk

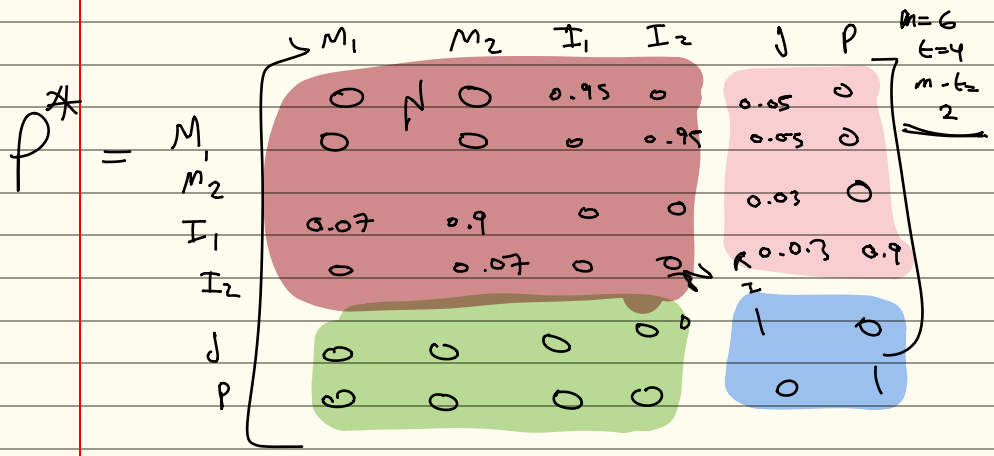
I	
II	
Junk	0.05
good.	0.03 / 0.07



$P =$

	I	II	Junk	Insp 1	Insp 2	pass
I	0	0	0.05	0.95	0	0
II	0	0	0.05	0	0.95	0
Junk	0	0	1	0	0	0
Insp 1	0.07	0.9	0.03	0	0	0
Insp 2	0	0.07	0.03	0	0	0.9
pass	0	0	0	0	0	1

N_1


P*
=
M1
M2
I1
I2
J
P
0.95
0.05
0.95
0.05
0.07
0.9
0.03
0.07
0.03
0.9
1
1
m=6
t=4
m-t=2

Mon, April 21th

Continuous M.C :-

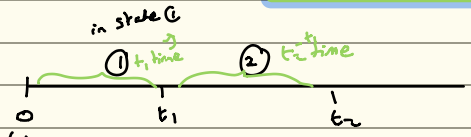
in previous sections, we assumed that the time Parameter (t) was descreate ($t=0, 1, 2, \dots$)

→ Starting at time 0 & letting the time parameter(t) run cont for $t \geq 0$

let $X(t)$ be the state of the system at time t.

→ $X(t)$ will take on one of its possible m values over some interval $0 \leq t \leq t_1$

the system ← $X(t)$ ⇐ (t) بالوقت ... يعني
↳ will be in one of m values
$X(t)$: possible states → (m)



→ time is possible to be not an integer.

this time is random

absolute ... ⑥ ... time C (h) → then it will go to any other m state.

so it's possible that it is $0, 1, 2, \dots$

the time between each step is ... it's not fixed anymore.
random variable.

→ then it will jump to another state over the next interval $t_1 \leq t \leq t_2$ & it will keep jumping.

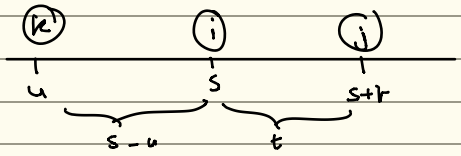
A random process $[x(t), t \geq 0]$ is a continuous time M.C if for all $s, t \geq 0$ & non-negative integers i, j, k

$$P[x(t+s)=j \mid x(s)=i, x(u)=k, 0 \leq u < s]$$

$$= P(x(t+s)=j \mid x(s)=i]$$

↳ because the past is irrelevant.

I want where are you now?



all are random variables

- u → Past
- s → Present
- t → Future

متغیر این تصمیم

→ in addition if

$$P[x(s+t)=j \mid x(s)=i]$$ is independ of s

then the c.m.c is said to have Stationary or time homogeneous transitions probabilities. (they are independent of where I'm now)

→ to simplify the notation

$$P_{ij}(t) = P[x(s+t)=j \mid (s)=i]$$

$$P_j(t) = P[x(t)=j]$$

Prob. that the system at time=t is at $\frac{j}{T_i}$ state

$\Rightarrow P_{ij}(t)$ are transition Prob. that satisfy the following :-

$$0 \leq P_{ij}(t) \leq 1$$

$$\sum_{j}^{m} P_{ij}(t) = 1 \quad , \forall i \rightarrow$$ i من الحالة إلى كل الحالات المتوقعة = 1

$$\sum_{j}^{m} P_j(t) = 1 \rightarrow$$ at Time t احتمال أن الـ system يكون بإحدى الحالات المتوقعة

Chapman – Kolomogorov equ for C.M.C

$$P_{ij}(s+t) = \sum_{K} P_{ik}(t)\, P_{kj}(s)$$

Holding time :-

→ whenever a C.M.C enters a state i it spends an amount of time is called dwell time or holding time.

→ once this time ends the system will be at a different state because holding time ends when its out of state i

state الحالة i هي الحالة holding time الوقت الذي يقضيه في الحالة P_{ii} يساوي ←

→ the holding time in state i is exponentially distributed with mean = $1/v_i$ → rate

at the expiration of the holding time → the process makes a transition to another state j with Prob P_{ij}

$t \sim Exp(1/v_i)$

When the holding time ends

→ P_{ij} must satisfy $\sum_{j} P_{ij} = 1$ → transition to any state time الانتقال إلى أي حالة عند انتهاء الـ holding time هو ←

& $P_{ii} = 0$

A C.t.M.C is a Stochastic Process that moves from state to state in accordance with a D.t.M.C But is such that the amount of time it spends in each state before proceeding to the next state. is exponentially distr., in addition the amount of time the process spends in state i & the next state visited must be Independent R.V

Random variable.

Wed, Apr 23

Cont. time M.C.

→ holding time is exponential

$E(T_i) = 1/\nu_i$

ν_i :- transitions rate out of state i

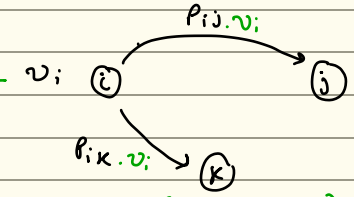
expected # of times that the process leaves state i per unit time spent in state i

ex: $\nu_i = 3$

$(T_i) = 1/3$ ← بالساعة طلع ٣ مرات

state i بـ $(\frac{20}{mins})$ متصت بداية

وخلص كنت طالعة مرة!



عدد المرات اللي طلع state i من ← ν_i

لوين ← طلع

p_{ik}: احتمالية اروح لـ k

$p_{ik}.\nu_i$: Rate when in state i that the process makes a transition to state k, (عدد مرات اني اروح لـ k)

we consider the transition equ. for state(i) for the small time interval Δt

---

⇒ Prob. that when the process is in state i a transition to state $j \neq i$ will take place in the next small time interval, Δt is approximately $p_{ij}.\nu_i.\Delta t$

احتمالية اني اروح من i لـ j بالـ next interval

ex: next minute, ½ minute.

كل ما كان Δt اقل كان approximation احسن

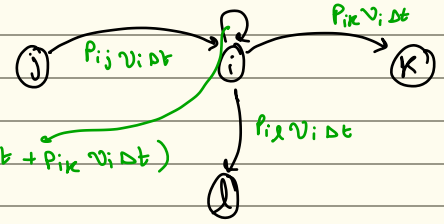
⇒ Prob. it leaves state i in the next time interval $(\Delta t) = \sum_{j \neq i} P_{ij} v_i \Delta t$

لـ بين احتمال انها تطلع طابعتي لوين

⇒ Prob. that no transition out of i will take place in $\Delta t = 1 - \sum_{j \neq i} P_{ij} v_i \Delta t$

---

ex:-



$1 - (P_{il} v_i \Delta t + P_{ij} v_i \Delta t + P_{ik} v_i \Delta t)$

احتمالية انه كمان Δt الـ system is in (i) ← $P_i(t + \Delta t)$

$$= P_i(t)\left[1 - \sum_{j \neq i} P_{ij} v_i \Delta t\right] + \sum_{j \neq i} P_j(t) \cdot P_{ji} v_j \Delta t$$

[i في . (i كنت بـ) وطلعت من] + time(t) عند كنت . وخلال Δt اجيت عند state (i)

state(j) بـ

process was in (i) and didn't move + entering (i) from (j)

j is any other state.

P_i: تابع قديم : prob that the system is in state i

---

steady state: ⇒ In steady state $P_j(t) \to P_j$ لـ بدل بيصير عـ t

يعني الوقت اللي بيقضي بحيث بيتغير نفس الـ prob.

balance at steady state.

$$v_i \pi_i = \sum_{j \neq i} \pi_j P_{ji} v_i$$

balance equation

rate out of state i of transition = rate in state i of transition

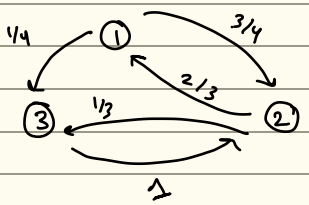
$\sum \pi_i = 1$

because ← one of the equations will be redunt.

جاي بالامتحان **

Question 8-

Solve for steady state probabilities (π_1, π_2, π_3)



$v_1 = 4$

$v_2 = 3$

$v_3 = 4$

$$v_i \pi_i = \sum_{j \neq i} \pi_j v_j P_{ji}$$

$$v_1 \pi_1 = v_2 \pi_2 P_{21} + v_3 \pi_3 P_{31} \Rightarrow 4\pi_1 = \left(3\pi_2 \left(\frac{2}{3}\right) + 4\pi_3(0)\right)$$

$$v_2 \pi_2 = v_1 \pi_1 P_{12} + v_3 \pi_3 P_{32} \Rightarrow 3\pi_2 = \left(4\pi_1 \left(\frac{3}{4}\right) + 4\pi_3(1)\right)$$

redundant $\quad v_3 \pi_3 = v_1 \pi_1 P_{13} + v_2 \pi_2 P_{23} \Rightarrow 4\pi_3 = \left(4\pi_1 (1/4) + 3\pi_2 (1/3)\right)$

$\pi_1 + \pi_2 + \pi_3 = 1$

$\pi_2 = 2\pi_1 \leftarrow \quad 4\pi_1 = 2\pi_2 \quad (1)$

$3\pi_2 = 3\pi_1 + 4\pi_3 \quad (2)$

$\pi_1 + \pi_2 + \pi_3 \quad (3)$

$3(2\pi_1) = 3\pi_1 + 4\pi_3 \rightarrow$ in (3)

$3\pi_1 = 4\pi_3 \rightarrow \pi_3 = \frac{3}{4}\pi_1 \rightarrow$ in (3)

$$\pi_1 + 2\pi_1 + \frac{3}{4}\pi_1 = 1$$

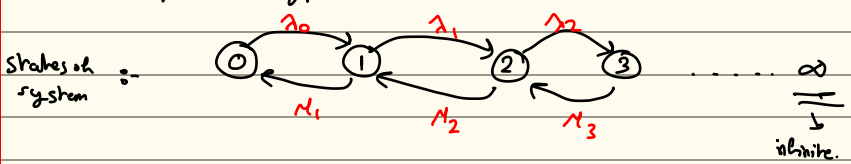
$$3.75\,\pi_1 = 1$$

$$\pi_1 = \frac{1}{3.75} = 0.266 \rightarrow \pi_2 = 2 \times 0.266 = 0.532$$

$$\pi_3 = \frac{3}{4} \times 0.266 = 0.1995$$

Birth & Death processes :-

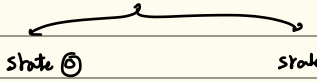
→ special type of C.T.M.C



ex:→ these states are # costumers in Que

So state ⓪ → no costumers

state ① → زبون جديد (birth)



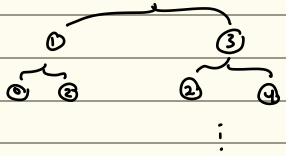
state في كل ← يا بجوج

death birth

only one person can add

because Δt is so small

only one person can come or go



⇒ consider C.T.M.C with state 0,1,2,3 ... ∞

if $P_{ij} = 0$ when $j \neq i+1$ or $j \neq i-1$

then the M.C is called A birth & death process .

اي Prob غير اللي قبل وبعد بالبنط = zero

⇒ for birth & death process we define transition rate from state i:

$\lambda_i = v_i \, P_{i(i+1)}$

$\mu_i = v_i \, P_{i(i-1)}$ by default $\mu_0 = zero$

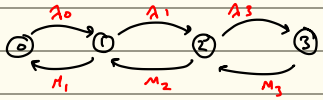
$P_{i(i+1)} = \frac{\lambda_i}{\lambda_i + \mu_i}$

$P_{i(i-1)} = \frac{\mu_i}{\lambda_i + \mu_i}$

$P_{i(i+1)} + P_{i(i-1)} = 1$

Wed Apr 30th

Birth & death processes e-



i is where I'm starting

→ $\lambda_i = \nu_i \; P_{i(i+1)}$

→ $\mu_i = \nu_i \; P_{i(i-1)}$

$P_{i(i+1)} = \frac{\lambda_i}{\lambda_i + \mu_i}$

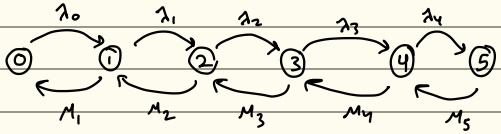
$P_{i(i-1)} = \frac{\mu_i}{\lambda_i + \mu_i}$

$\nu_i = \lambda_i + \mu_i$

transition from i → i+1 ↑ , ↗ transition from i → i-1

↙ # transition out of i

{i-1, i+1} → only options to go



balance equations-

in = out

$\nu_i = \lambda_i + \mu_i$

$P_n = \left(\frac{\lambda_0 \dots \lambda_{n-1}}{\mu_1 \quad \mu_n} \right) \cdot P_0$

$\sum_{i=0}^{\infty} P_i = 1$

for getting π :

balance equs-

ex: $\pi_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} \cdot \pi_0$

In = out

state: 0 $\quad \mu_1 \pi_1 = \pi_0 \lambda_0 \Rightarrow \pi_1 = \frac{\lambda_0}{\mu_1} \cdot \pi_0$

state 1 $\quad \lambda_0 \pi_0 + \mu_2 \pi_2 = \lambda_1 \pi_1 + \mu_1 \pi_1 \Rightarrow \pi_2 = \frac{\lambda_1}{\mu_2} \pi_1 = \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} \cdot \pi_0$

state 2 $\quad \lambda_1 \pi_1 + \mu_3 \pi_3 = \lambda_2 \pi_2 + \mu_2 \pi_2 \Rightarrow \pi_3 = \frac{\lambda_2}{\mu_3} \cdot \pi_2 = \frac{\lambda_2}{\mu_3} \cdot \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} \cdot \pi_0$

" 3 $\quad \lambda_2 \pi_2 + \mu_4 \pi_4 = \lambda_3 \pi_3 + \mu_3 \pi_3 \Rightarrow \pi_4 = \lambda_4 / \mu_4 \cdot \pi_4$

" 4 $\quad \lambda_3 \pi_3 + \mu_5 \pi_5 = \lambda_4 \pi_4 + \mu_4 \pi_4 \Rightarrow \pi_5 = \frac{\lambda_5}{\mu_5} \cdot \pi_4$

" 5 $\quad \lambda_4 \pi_4 = \lambda_5 \pi_5 \Rightarrow$

$$\pi_1 = \frac{\lambda_0}{M_1}\,\pi_0$$

$$\pi_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}\cdot \pi_0$$

$$\pi_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3}\cdot \pi_0$$

$\vdots$

one is gonna be redundant

$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

$\sum \pi_i = 1$

$\Rightarrow$ if λ_i & M_i are independent of time

---

one of the models we'll study in chapter 18 $\Leftarrow$ ATM example :-

Average time between arrivals = 30 sec

Average service time = 24 sec.

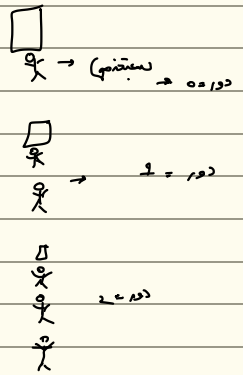
over 1 equal

$$\pi_1 = \frac{\lambda_0}{\mu_1}\cdot \pi_0$$

λ = 2 cost/min

μ = 2.5 costumer/min

arm



الشرط < الشرط

اذا كان المقام يقل عن البسط فهذا يعني

steady state $\Rightarrow$ the cashier will not finish working.

time between inputs = $\frac{1}{\lambda}$

λ : rate per time of inputs = $\frac{1}{\text{time input}}$

0 1 2 3 4 5

$\lambda = 2$ l/min

$\mu = 2.5$ l/min

$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 \cdots$

$\mu_0 = \mu_1 = \mu_2$

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \ldots \mu_n} \cdot \pi_0$$

$$\pi_1 = \frac{\lambda_0}{\mu_1} \pi_0 = \frac{2}{2.5} \pi_0$$

$$\pi_2 = \frac{2 \times 2}{2.5 \times 2.5} \pi_0$$

$$\pi_3 = \frac{2 \times 2 \times 2}{2.5 \times 2.5 \times 2.5} \cdot \pi_0$$

$$\pi_4 = \frac{2 \times 2 \times 2 \times 2}{2.5 \times 2.5 \times 2.5 \times 2.5} \cdot \pi_0$$

$$\pi_5 = \frac{2 \times 2 \times 2 \times 2 \times 2}{2.5 \times 2.5 \times 2.5 \times 2.5 \times 2.5} \cdot \pi_0$$

$\{ \pi_i = 1$

$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

$\frac{4}{5}\pi_0 + \frac{16}{25}\pi_0 + \frac{64}{125}\pi_0 + \frac{256}{625}\pi_0 + \frac{1024}{3125}\pi_0 = 1$

$\Rightarrow 2.69\,\pi_0 = 1$

$\pi_0 = 0.27106$

π_1

$\lambda/\mu = 0.8$

steady state

$\pi_0 = 0.27 \Rightarrow$ prob ... (text illegible)

(text illegible) prob that he finds no one $= \pi_0$

prob to find que $= 1 - \pi_0$ ... (text illegible)

Transeint sol. :

$a_i(t)$: Prob that the system is in state i at time t

$a(t)$: state vector $\left(a_0(t) \quad a_1(t) \quad a_2(t) \quad \dots \quad a_m(t)\right)$

(1) Δt : small time interval

n : # steps required to represent t.

$$\rightarrow \quad n = \frac{t}{\Delta t(1)} = \frac{t}{\Delta}$$

$t = 1$ min , $\Delta = 0.05$ min

$$n = \frac{1}{0.05} = 20$$

in this one minute I have 20 parts I want the system after 20 steps → so it becomes discrete.

$a(1\text{min}) = a^{(n=20)} = a^0 P^{20}$

at time = 0 (initial) there was no

$a^0 = \left(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right)$ يعني احتمال انه يكون

at time = zero $= a(t=0) = a^{(0)}$

يعني في البداية

system فيه بال

$$\frac{20 \text{ steps in this example}}{\text{دقيقة}} = \frac{1}{\Delta t}$$

$$a(t=1) = a^0 P^{20}$$

↳ I need $P^{(1)}$

→ How to find $P^{(1)}$

↳ one-step

$\lambda_0 = \lambda_1 = \cdots \lambda$

$\mu_1 = \mu_2 \cdots \mu$

$$R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & \lambda & 0 & 0 & 0 & 0 \\ \mu & 0 & \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu & 0 & \lambda \\ 0 & 0 & 0 & 0 & \mu & 0 \end{bmatrix} \end{array}$$

transition rate matrix

from trans → absorbing

↳ for birth & death

→ R ساده

R ⇒ P

μ تولد و λ مرگ

$$P_{i \to j} = P_{ij} \cdot \nu_i \cdot \Delta t \qquad i \neq i$$

$$= \begin{cases} \lambda_i \Delta t, & j = i+1 \\ \mu_i \Delta t, & j = i-1 \end{cases}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1-(\lambda.\Delta t) & \lambda.\Delta t & 0 & 0 & 0 & 0 \\ \mu.\Delta t & \boxed{1} & \lambda.\Delta t & 0 & 0 & 0 \\ 0 & \mu.\Delta t & \boxed{1} & \lambda\Delta t & 0 & 0 \\ 0 & 0 & \mu.\Delta t & \boxed{1} & \lambda.\Delta t & 0 \\ 0 & 0 & 0 & \mu.\Delta t & \boxed{1} & \lambda.\Delta t \\ 0 & 0 & 0 & 0 & \mu.\Delta t & 1-\mu\Delta t \end{bmatrix} \end{array}$$

1 → 4

two terms + diagonal

0, 5

one term, diagonal

$$\boxed{1} = 1 - (\lambda.\Delta t + \mu.\Delta t)$$

$\lambda = 2$

$\mu = 2.5$ → $\frac{2\text{ day}}{P}$

$\frac{\text{day}}{\Delta t}$

Mon, May 5th

lec 15

Ch18:- Queuing Theory

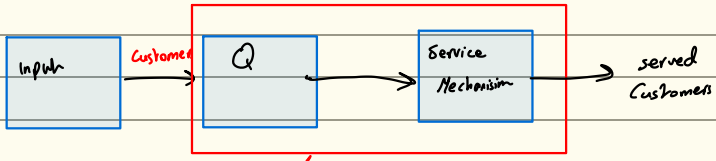
→ elements of queuing model :

1- customers

2- enter the queuing system (Q syst.) & join the Q if service is not available.

3- member (customers) are selected for service by some Q rule discipline

4- required service is performed by service mechanism



Queuing system: Q + service mechanism

---

① → Source of Input:-

- calling pop
  - a. size → Infinite / finite
  - b. arrived rate (λ)
    - ↳ Poisson Distr.
      (arrival of customers, events per time)
    - → Inter arrival time = $\frac{1}{\lambda}$ → expo dist.
      - ↳ time between arrivals

Infinite: ex: # people going to subway

finite :- ex: # machines waiting to be fixed in a factory.

↳ the que can be virtual, ex:- airplanes in sky waiting to land.

---

note① the service mechanism could be :

services, airports, maintinance.

the Q discipline :- is a rule about who is getting to get the service first.

Customer → finite : machines, جزء من شيء

Customer → infinite : I have a shop, people are infinite.

② → Que : where customer wait for service.

a. size → finite. (like who there is 5 in Que & you can't come)
↳ infinite. you can come with as u come

b. channel → Single
↳ Multipale

→ multipale you have different ques & the service
↳ more than 1

like the cashier, But the single one big line.

---

③ → Queue Disciplines-

↳ it represents order in which customers are called for service.

1- FIFO first in first out (like banks, cashiers)
2- LIFO last " " " (like a fridge the one you take is the last)
3- SIRO service in random order
4- Some order of priority

---

in the que →

Customer Behavior :-

Jockey — que بطل يغير محله بال
Balk — que بيخاف انه محلهش حد بال
Renege. — بيقف بال que بيخاف حد انه يروح

we won't study this, we assume: that when a cust comes they get in the que unless it's finite and full.

---

④ → Service Mechanisim :-

a- service channel

#services → Single
↳ Multipale — in the service not que has multipale places

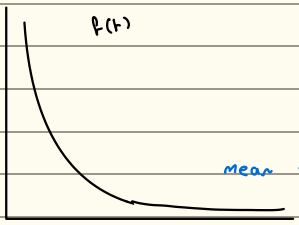
multipale → Parallel
↳ Series (more complex)

b- service rate (μ)

② ③ time → service time = 1/μ

* Note: customers in system = # in que + # in service.

→ 18.3 B- Role of expo dist.



$$f(t) = \begin{cases} \lambda e^{-\lambda t} & , t \geq 0 \\ 0 & , \text{else} \end{cases}$$

mean = expected value $E(T) = 1/\lambda$

variance $V(T) = 1/\lambda^2$

→ λ is rate of customers / hour ex: $\lambda = 3$ per hour

$1/\lambda = 0.333$ hour

$= 20$ min.

$$P(t < T) = \int_0^T \lambda e^{-\lambda t} = 1 - e^{-\lambda t}$$

$$P(t > T) = e^{-\lambda t}$$

Poisson:

$$P(X = x) = \frac{(\lambda t)^x}{x!} \cdot e^{-\lambda t} \quad , x \geq 0$$

$E(t) = \lambda t$

$V(t) = \lambda t$

Question B- at JD.edu by a Poisson Dist. at an avg of

λ ← 3 visitor/hr, find :- the prob that the next visitor is :

a- within 10 min

b- after 30 min passes

c- in exactly 15 min

d- within the first min •

e- " " " two min •

f- " two min •

↑ within the second min

$\lambda = 3$ v/hr

customer = ⊗ → in Poisson

time = h → expo (inter arrival time)

if the customers arrive in Poisson then

the time is expo

a- $P(t<10_{min}) = 1 - e^{-\lambda t}$

$= 1 - e^{-3 \cdot (\frac{10}{60})}$ hr

$= 1 - e^{-0.5}$

لازم نحول الـ t لـ hr

لأن λ بـ hr

3/hr و 10 min

$P(t>10) = P(X=0)\big|_{t=10}$

الوقت بين two customers is >10 so in 10 min customers

$P(t<10) = 1 - P(X=0)\big|_{t=10}$

---

lec 16

wed May 7

b- $P(t>30) = e^{-3 \cdot \frac{30}{60}}$

c- $P(t=15) = 0$ because exactly → zero

صايرة من متغير متصل لمتقطع

d- $P(t<1)$

e- within the first two minutes → $P(t<2)$

$= 1 - e^{-3 \cdot \frac{2}{60}}$

$= 0.095$

$P(t<2) - P(t<1)$

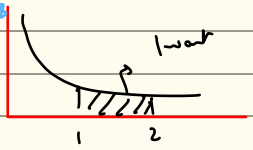
f- within the second minute. $P(1<t<2)$

لحد أكتر من 1 لأقل من 2

$= (1 - e^{-3\frac{2}{60}}) - (1 - e^{-3\frac{1}{60}})$

$0.095 - 0.048$

$= 0.047$



ما بجل لـ 1 سؤال بوخذ اربع

$P(t<2) - P(t<1)$

بدها conditional probability

$P(t<4)$

g- before 4 min, given that more than 2 min already passed without a visitor

$P(t<4 \mid t>2)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ التقاطع اللي بينهم هو

$= \frac{P(A<t<B)}{P(B)} = \frac{0.047}{0.095}$

$= 0.49$

$$P(t<2) = 0.095$$

$$P(1<t<2) = P(t<2) - P(t<1) =$$

$$0.095 - 0.048 = 0.047$$

$$1 - e^{-3.4/2.0}$$

$$\frac{P(2<t<4)}{P(t>2)} = \frac{P(t<4) - P(t<2)}{0.905}$$

$$= \frac{0.080106}{0.905} = 0.095$$

same thing because it has lack of memory

example: 18.7.4.1

18.4 : pure birth & death models

→ birth: arrivals of customers into the Qing syst.

→ death: departures of served customer

pure birth : normal poisson dist.

pure death : it like ex: inventory I have it a level (full) and keep taking to a some level then reorder & get it to the same first full level.

Pure death : Only departures take place, system with N customers

with no arrivals allowed, departures take place at rate of μ cust/unit time

when it was pure birth

poisson & expo same thing

$$P(X=x) = \frac{(\lambda t)^x}{x!} \cdot e^{-\lambda t} \qquad x \neq 0$$

poisson dist.

$P(X=1)$ → ... zero customers or ...

---

in pure death: we can use poisson.

Δt is very small & even we take only one.

Max # cust $\frac{N}{2}$ and we take from that.

$N=5$ iphone 11 ← iphone shapes

← يعني عندي 5 أنواع وبيها وحدات
وبين وحدة وبيعين مثلاً وصلة ...
(error)

Stock	departures
5	0
4	1
3	2
2	3
1	4
0	5

$N=2$

the state is basically (0) ... customers

N = current + departure.
present
ex 3 + 2
5 = 3 + 2

this is the prob. that at time t we have n products in system

$$P_n(t) = \frac{(\lambda t)^{N-n}}{(N-n)!} \cdot e^{-\lambda t}, \quad n = 1,2,3,\ldots N$$

↳ # departures $= N - n$ ← عدد المغادرين من النظام

$$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t)$$

typo في الكتاب $n-1$ → $n=1$

---

example : 18.4.2

The florist section in a grocery store stocks 18 dozen roses at the beginning of each week. On the average, the florist sells 3 dozens a day (one dozen at a time), but the actual demand follows a Poisson distribution. Whenever the stock level reaches 5 dozens, a new order of 18 new dozens is placed for delivery at the beginning of the following week. Because of the nature of the item, all roses left at the end of the week are disposed of. Determine the following:

(a) The probability of placing an order in any one day of the week.

(b) The average number of dozen roses discarded at the end of the week.

Because purchases occur at the rate of $\mu = 3$ dozens per day, the probability of placing an order by the end of day t is

poisson

N = n at departure + orders

$N = 18$ dozen

$\mu = 3$ dozen / day

prob of placing an order $= P_n \leq 5$

↳ placing an order is when $P_n \leq 5$

Prob on any day of the week

$P(1st\ day \cup 2nd \cup 3rd \cup \ldots\ldots)$

$P(1st) + P(2nd) + P(3rd) + P(4th) + P(5th) + P(6th) + P(7th)$

day 1 → $t=1$

day 2 → $t=2$

$$P_n(t) = \frac{\mu t^{(N-n)}}{(N-n)!} \cdot e^{-\mu t}$$

stock

18

17

16.

5

4

3

2

1

0 $\rightarrow 1 - \sum_{n=1}^{18} P_n(t)$

---

ex8-

$M = 3$ $\quad P_{1,2,3,4,5,0}$

stock	departure
5	0
4	1
3	2
2	3
1	4
0	5

$$P_n(t) = \frac{(\lambda t)^{N-n}}{(N-n)!} e^{-\mu t}$$

$$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t)$$

$P_0 = 0.1847$

$P_1 = 0.163$

$P_2 = 0.224$

$P_3 = 0.224$

$P_4 = 0.149$

$P_5 = 0.0497$

Mon, May 12th

lec 17:-

18.5 General Poisson Queing models:-

↳ section 3, [illegible]

→ if system is dependent on time it is said to be Transient. steady state → independent of time

↳ transient: [illegible]

→ State of the system n :-

# of customers in the queing system

↳ Que + in-service.

→ $n(t)$: # customers in the Qing system at time t, $t \geq 0$

→ $P_n(t)$: Prob. of n customers in the system at time t.

[illegible]

independent of time [illegible]

→ In STEADY STATE conditions: the # customers in the Syst. is independent of the initial state & the elapsed time.

P_n

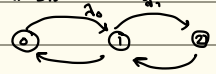
G.P.Q.M:-

① combines both arrivals (birth) and departures (death)

② {Inter arrival times & Service times} → they follow the exponential because they are time b/w poisson.

③ based on steady state conditions

④ the model assumes that both arrival & departure rates (λ, μ) are state dependents

λ, μ can differ from state to state: ex: 

μ_n departures rate [illegible]

λ_n arrival rate given in the Qing system. $\lambda_0 \neq \lambda_1$ → it can be.

if we open another counter ⇒ μ will be2x

⑤ $\sum_{n=0}^{\infty} P_n$ $\qquad P_n = \frac{(\lambda_0 \cdots \lambda_{n-1})}{(\mu_1 \cdots \mu_n)} P_0$

these prob. are used to determine the system measures of performance:-

1- the Average que length (variability) ← [illegible]

Exp # in the Que

2- waiting time (Avg or expected)

3- the Avg utilization of the facility

ex: I have 2 cashiers but I use only 1 ← [illegible]

utilization = $\frac{1}{2}$ time per customer $= \frac{1}{\mu}$ ← μ → customers per time

---

Example 6: Example 18.5-1+problem 18.39

B&K Groceries operates with three checkout counters. The manager uses the following schedule to determine the number of counters in operation, depending on the number of customers in line:

Number of customers in store	Number of counters in operation
1 to 3	1
4 to 6	2
More than 6	3

Customers arrive in the counters area according to a Poisson distribution with a mean rate of 10 customers per hour.

The average checkout time per customer is exponential with mean 12 minutes.

1. Determine the steady-state probability p_n of n customers in the checkout area.
2. The probability that only one counter will be open
3. Expected number of idle counters
4. The probability distribution of the number of open counters.
5. The average number of busy counters

$\lambda_n = \lambda = 10$

$\mu = \frac{1}{12} \frac{\text{Customer}}{\text{min}} \times \frac{60 \text{ min}}{\text{hour}}$

$= 5$ cust / hour

μ : cust per hour

↑ servers ↳ μ ↑

n

counter = 1 (1-3) → $\mu = 5$ cust/min

counter = 2 (4→6) → $\mu = 10$ cust/min

counter = 3 → $\mu = 15$ cust/min

$$\mu_n = \begin{cases} 5, & n = 0, 1, 2, 3 \\ 10, & n = 4, 5, 6 \\ 15, & n = 7, 8, \cdots \infty \end{cases} \quad (7 \le n < \infty)$$

$n \ge 6$

$n \ge 7$

---

① P_n in steady state.

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{10}{5} P_0 = 2P_0$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 = \frac{100}{25} P_0 = 4P_0$$

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} = \left(\frac{10}{5}\right)^3 P_0 = 8P_0$$

$$P_4 = \frac{\lambda_1 \lambda_2 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3 \mu_4} P_0 = \frac{(10)^4}{(5)^3 \times 10^1} P_0 = \frac{10^3}{5^3} P_0 = 8 P_0$$

$P_5 = \frac{\lambda_0 \cdots \lambda_4}{\mu_1 \cdots \mu_5} P_0 = 8P_0$

① counter open

$P_6 = 8P_0$

$\omega = P_1 + P_2 + P_3$

$P_7 = \frac{\lambda_0 \cdots \lambda_6}{\mu_1 \cdots \mu_7} \cdot P_0 = \frac{10^7 \cdot 4}{(5)^3 (10)^4 (15)} = \frac{16}{3} P_0$

$P_8 = \frac{32}{9} P_0$

$P_9 = \frac{64}{27} P_0$

$\sum_{n=0}^{\infty} P_n = 1$

$P_0 + 2P_0 + 4P_0 + 8P_0 + 8P_0 + 8P_0 + \frac{16}{3} P_0 + \frac{32}{9} P_0 + \frac{64}{27} P_0 = 1$

$P_0 + P_0 \left(2+2+4+8+8+8+\frac{16}{3}+\frac{32}{9}+\frac{64}{27}\right) = 1$

$P_0 \left(1+2+4+8+8\left(\frac{2}{3}\right)+8\left(\frac{2}{3}\right)^2+8\left(\frac{2}{3}\right)^3+8\left(\frac{2}{3}\right)^4+\cdots\right) = 1$

$P_0 \left(1+30+8\left[1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots\right]\right) = 1$

$\left(\frac{2}{3}\right)^0$, $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

$P_0 \left(31+ \quad // \quad // \quad //\right) = 1$

$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, $|x| < 1$, $\sum x^i$

هر عدد به توان صفر

$x^0 = 1$

$P_{\text{more than 7}} = 1 - \sum_{n=0}^{7} P_n$

$P_0 \left(31 + \frac{1}{1-2/3}\right) = 1$

$55 P_0 = 1 \rightarrow P_0 = \frac{1}{55}$

b- only one counter open

یکی باز است:

اگر احتمال P_0 تا P_3 =

$P_0 + P_1 + P_2 + P_3$

(c) → expected # of idle counters :-

↳

# of customers	Idle	Busy	Probability
n ≥ 7	0	3	$1 - \sum_{n=0}^{6} P_n$
4, 5, 6	1	2	$P_4 + P_5 + P_6$
1, 2, 3	2	1	$P_1 + P_2 + P_3$
0	3	0	$P_0 = \frac{1}{55}$

expected # of idle counter =

$E(x) = \sum x f(x)$

↳ $x \times p(x)$

$x f(x)$

number of

$E(x) = 3 \cdot P_0 + 2(P_1 + P_2 + P_3) + 1(P_4 + P_5 + P_6) + 0\left(1 - \sum_{n=0}^{6} P_n\right)$

possible values of Idle:	Probability of each value.
0	P(0) =
1	P(1) =
2	P(2) =
3	P(3) =

expected number of idle

↳ idle

Probability dist of # of customers

* probability dist :-

↳ possi. value, what corresponding possible probability of value

probability dist

$E(\text{idle}) + E(\text{busy}) = 3$

number # number

Wed, May 14th

19.6 : specified Poisson Queue (Model)

→ Kendall lee notation (a/b/c) (d/e/f)

a: Arrivals dist
b: departures "
c: # of parallel servers

d: Que discipline
e: max # in the syst (include the being served)
f: size of calling population.

(a b c) (M/M/1) (d e f) (FIFO/20/∞)

M_i : Markovian (M/M/1...
D_i : constant (deterministic)
E_k : Erlang or gamma dist
G: general (generate) with.

A, b → markovian

b can be ∞

d: FIFO
LIFO
SIRO
G.D (general discipline)

f: population → ∞ / finite.

STEDY STATE Measures of performance:

1- L_s :- exp. # of customers in the Qing syst.

$L_n = L_s$

$$E(n) = L_s = \sum_{n=0}^{\infty} n P_n$$

2- L_q : Exp. # of customers in the Queue (Queue length)

L_s # in syst. → random, n≥c
L_q # in Que → random

n = # cust in syst
c: # parallel servers

$$L_q = \sum_{n=c+1}^{\infty} (n-c) P_n$$

n ≤ c → no queue.

3- $\bar{c}$: exp. # of busy servers

$\bar{c} = L_s - L_q$ ($L_s = L_q + \bar{c}$)

↳ L_q (not Queue)

exp # of idle servers = $c - \bar{c}$ (c → max # of servers)

4- W_s : exp waiting time in system $= L_s/\lambda$

$W_s = W_q$ + service time

5- W_q = exp waiting time in Que $= L_q/\lambda$

λ → معدل الي جايين عندي

Que length بس ممكن يكون limited. يعني فعليا مش كل ما يوصل

relationship (L_s , W_s)

little law :

$L_s = \lambda_{effect} \cdot W_s$ (rate)

$L_q = \lambda_{eff} \cdot W_q$

cust = cust/time × time

λ_{eff} → الي هو الي وصلوا

$\lambda_{eff} \leq \lambda$

$\lambda = \lambda_{eff}$ بنفترض

G ← عدد الي جايين

بـ limit → ∞

$\rho = L_s - L_q$

$\rightarrow W_s = W_q + 1/\mu$

↳ service time.

Facility utilization $= \frac{\bar{c}}{c}$ busy servers / total

$\bar{c} = L_s - L_q = \frac{\lambda_{effect}}{\mu}$

* $\rho = \frac{\lambda}{\mu}$ arrival rate / service rate — عدد الي جايين / عدد الي بيخلص

↳ لقو بيشتغل ؟!

$= L_s - L_q$

$(1-\rho)$

$\rho < 1$ لازم $\rho \geq 1$

There is one place where $\rho > 1$ ← queue will explode

C = 2

Q:- for a 2 server Que ($\lambda = 9$ cust/hour), we found that

$P_0 = 0.4$, $P_1 = 0.3$, $P_2 = 0.2$, $P_3 = 0.1$

find

1- L_s | 3- ρ | 5- W_q

2- L_q | 4- W_s | 6- service time.

1- $L_s = \sum_{n=0}^{\infty} n P_n = 0 \times 0.4 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.1$

$= 0.3 + 0.4 + 0.3 = 1$

2- $L_q = \sum_{n=c+1}^{\infty} (n-c) P_n$

$L_s = 1 \times P_3 = 1 \times 0.1 = 0.1$

3- $\rho = \frac{\lambda}{\mu} = L_s - L_q = 1 - 0.1 = 0.9$

4- $W_s = \frac{L_s}{\lambda} = 1/8 = 0.125$ h

5- $W_q = \frac{L_q}{\lambda} = 0.1/8 = 0.0125$ h

6- service time $= \frac{1}{\mu}$ $\rho = \frac{\lambda}{\mu} \rightarrow \mu = \frac{\lambda}{\rho} \rightarrow 1/\mu = \rho/\lambda$

$\therefore W_s - W_q = 0.125 - 0.0125 = 0.1125$

0.9

Example 18.6-1 Problem 18-48

Visitors' parking at Ozark College is limited to five spaces only. Cars making use of this space arrive according to a Poisson distribution at the rate of six cars per hour. Parking time is exponentially distributed with a mean of 30 minutes. Visitors who cannot find an empty space on arrival may temporarily wait inside the lot until a parked car leaves. That temporary space can hold only three cars. Other cars that cannot park or find a temporary waiting space must go elsewhere. Determine the following:

1. The probability, p_n, of n cars in the system.
2. The effective arrival rate for cars that actually use the lot.
3. The average number of cars in the lot. L_s
4. The average time a car waits for a parking space inside the lot.
5. The average number of *occupied* parking spaces.
6. The average utilization of the parking lot.
7. Compute the average number of cars that will not [illegible] parking lot during an 8-hr period

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λ = 6 cars/hour

service time = $\frac{1}{\mu}$ = 30 min

$\mu = \frac{1}{30} = \dots$

μ = 2 cars/hour

λ_{lost} in

we have 5 servises

leaving

μ = cars/hour

$\mu_n = 2n$

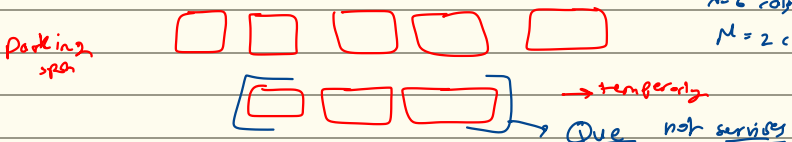
limited.

1/service time

$n = 1, 2, 3, 4, 5$ ← Cargo.

λ = 6 cars/hr

μ = 2 cars/hour



1- $\lambda_{eff} = \lambda - \lambda P_8 = 6 - 0.1345 = 5.8737$

$\lambda = \lambda_{eff} + \lambda_{lost}$

$\lambda_{loss} = \lambda P_8$

$\mu_n = \begin{cases} 2n, & n = 1,2,3,4,5 \\ 10, & n = 6,7,8 \end{cases}$ ← waiting in Q

1- $P_n = \frac{\lambda_0 \dots \lambda_{n-1}}{\mu_n \dots \mu_1} \cdot P_0$

$P_1 = \frac{\lambda_0}{\mu_1} P_0 = \frac{6}{2} P_0 = 3P_0$

$P_2 = \frac{\lambda_0 \cdot \lambda_1}{\mu_1 \mu_2} P_0 = \frac{(6)^2}{2 \cdot 4} P_0$

$= \frac{36}{8} P_0$

$\lambda - (P_8 \lambda)$

λ_{eff}

$P_3 = \frac{\lambda_0 \cdot \lambda_1 \cdot \lambda_2}{\mu_1 \mu_2 \mu_3} P_0 = \frac{\overset{18}{\cancel{36}} \times 6}{\underset{1}{\cancel{2}} \times 4 \times \underset{1}{\cancel{6}}} P_0 = \frac{18}{4} P_0$

make ~ 9

$P_4 = \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3 \mu_4} P - \dots$

$P_0 + P_0(3 + 4.5 + 4.5 + 3.375 + 2.025 + 1.215 + 0.729 + 0.4374) = 1$

$P_0 = 0.04812$

$P_8 = 0.0210$

2% من الوقت يكون ممتلئ

$W_s = W_q + 1/\mu$

system

1

3- $L_s = \sum_{n=0}^{8} n p_n = 1p_1 + 2p_2 + \dots = 3.12$

avg = in any time I come

empty space at any time = $8 - L_s$ $\quad L_q = \lambda_{eff} \cdot W_q$

L_q / λ_{eff} ← 4- $W_q = W_s + 1/\mu$ ← service time

$= \frac{L_s}{\lambda_{eff}} + \frac{1}{\mu}$ $\quad \frac{L_q}{\lambda_{eff}}$ $\quad L_q = \sum_{n=6}^{8} (n-c)\, p_n$

$= \frac{3.12}{5.87} + 0.5 = 0.0326$ $\quad$ c = 5

$=$

$= p_6 + 2p_7 + 3p_8$

5- $\bar{c} = L_s - L_q = 2.9368$ packing spaces $=$

6- Utilization $= \frac{\bar{c}}{c} = \frac{2.9368}{5} = 0.58736$

7- work number ⇒ I have rate (λ_{loss})

in an hour

$8 \times \lambda_{loss} =$ → rate I lose in 8 hours

number $= \frac{1}{8\lambda_{loss}}$

18-6.2 Single server

(M/M/1) : (GD/∞/∞)

↳ one service ↳ General discipline

$\lambda < \mu$, $\rho < 1$ (M/M/1) (G.D/∞/∞)

$P_1 = \frac{\lambda_0}{\mu_1} P_0$

$P_2 = \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} P_0$

$P_n = \frac{\lambda_0 \dots \lambda_{n-1}}{\mu_1 \dots \mu_n} P_0$

because of, as:

$\lambda_0 = \lambda_1 = \lambda_n = \lambda$

$\mu_1 = \mu_2 = \mu_n = \mu$ } single server

$\rho = \frac{\lambda}{\mu}$ rate

$P_1 = \rho . P_0$ $P_2 = \rho^2 . P_0$

↳ specialized.

$P_n = \rho^3 . P_0$

$P_0 + P_1 + P_2 \dots = 1$

$P_0 + \rho P_0 + \rho^2 P_2 + \rho^3 P_0 \dots = 1$

$P_0 [1 + \rho + \rho^2 + \rho^3 + \dots] = 1$

$P_0 \left[\frac{1}{1-\rho}\right] = 1$

$P_0 = 1 - \rho$

one server → there is no λ_{left} and λ_{bad}.

$C = 1$ → in this

$\bar{c} \rightarrow (0 \rightarrow 1)$ in this situation

$L_s = \frac{\rho}{1-\rho}$, $L_q = L_s - \bar{c}$ $= \rho^2/(1-\rho)$ $= L_s - \rho$

$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - \lambda}$ $W_q = \frac{1}{\mu} \cdot \frac{\rho}{1-\rho}$

ملاحظة: يكون أكبر من 1 عشان أقدر أحط one server في

$\bar{c} = \rho$

$C=1$

Q: $\mu = 10$ cus/hr

$\lambda = (1,5,9)$ for each λ:

1) ρ 2) L_q 3) w_q
2) L_s 3) w_s

$\lambda = 5$ hr

1) $\rho = \frac{\lambda}{\mu} = \frac{5}{10} = 0.5$ $\quad \bar{c} = \rho$

2) $L_s = \frac{\rho}{1-\rho} = 1$

3) $L_q = L_s - \bar{c} = 1 - 0.5 = 0.5$

4) $W_q = \frac{1}{10} \times 1 = 0.1$

5) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{10-5} = \frac{1}{5} = 0.2$

$\lambda = 1$

1) $\rho = \frac{1}{10} = 0.1 = \bar{c}$

2) $L_s = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = 0.11$

3) $L_q = 0.111 - 01 = 0.0111$

4) $W_s = 0.11$

5) $W_q = 0.011$

$\lambda = 9$

1) $\rho = \frac{9}{10} = 0.9 = \bar{c}$

2) $L_s = \frac{0.9}{0.1} = 9$

3) $L_q = 9 - 0.9 = 8.1$

4) $W_s = 1$

5) $W_q = 0.9$

$\lambda = 9.9$

$\rho = \frac{9.9}{10} = 0.99$

$L_s = \frac{0.99}{0.01} = 0.99$

$L_q = 98.01$

$W_s = 10$

$W_q = 9.9$ hr

$\rho = 1 \rightarrow \lambda = \mu \rightarrow L_s = \frac{1}{0} \rightarrow \infty$ it will explode

(M/M/1) (GD/∞/∞)

Example: Example 18.6-2 (Modified)

Automatic car wash is a one-bay facility. Cars arrive according to a Poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot or on the street bordering the wash facility if the bay is busy. The time for washing and cleaning a car is exponential with a mean of 10 minutes. This means that, for all practical purposes, there is no limit on the size of the system.

Calculate

1. The utilization 2. Ls 3. Lq 4. Ws 5. Wq
2. If we have 2 parking spaces, what is the probability that an arriving car will enter the facility?
3. If we have 4 parking spaces, what is the probability that an arriving car will not enter the facility?
4. The manager of the facility wants to determine the size of the parking lot.

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$\lambda = 4$

$\frac{1}{\mu} = 10$ min

$\mu = 0.1$ a minute $= 6$ car/hr

1 — 1) Utilization $= \frac{\bar{c}}{c} = \bar{c} = \rho = \frac{\lambda}{\mu} = \frac{4}{6} = 0.66$

2) $L_s = \frac{0.66}{1-0.66} = 2$

3) $L_q = 2 - 0.666 = \frac{4}{3} = 1.33$ cars

$W_q = \frac{1}{\mu}$

$= L_q$

2 — 2 p

↳ f

$P_0 = (1-\rho) = 0.$

$+(0.66^2 \times 0.3)$

bay.

the facil

space

3 — if

arrive → parking.

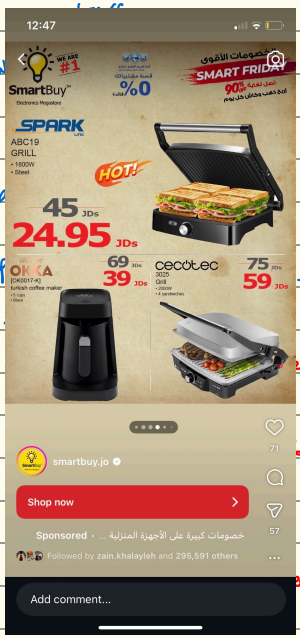
For 4 spaces → $P_0 + P_1 + P_2 + P_3 = 86\%$

sorry

5 spaces → 91.2 %

5 ⇐ 90% ... high

5 places



$P_0 + P_1 + P_2 + \dots + P_s \geq$ level of service

ex: 90% → 0.9

$(1-\rho) + \rho(1-\rho) + \rho^2(1-\rho) \dots + \rho^s(1-\rho) \geq 0.9$

الشكال غير

$(1-\rho)\left[1 + \rho + \rho^2 + \rho 3 \dots \rho^s\right] \geq 0.9$

↳ not to ∞ so ...

$\frac{(1-\rho)}{(1-\rho)} \quad \frac{1-\rho^{s+1}}{1-\rho} \geq 0.9$

$1 - \rho^{s+1} \geq 0.9$

$\rho^{s+1} \leq 0.1$

$s \geq 4.52$

$s \geq 5$

$\rho^{s+1} = 0.9$

$0.5^{x+1} = 0.9$

S = parking space.

lec 20:-

21/5/2025

Wednesday

Example (M/M/1) (GD/∞/∞)

Sports fans arrive at a ticket counter at the Amman Sports City Arena by a Poisson process with rate 105 customers per hour. Customers are served by a single cashier. The service time has an exponential distribution with mean 30 seconds.

Calculate:

1. The service rate μ
2. The utilization $\rho = \lambda/\mu = \frac{102}{120}$
3. $L_s = \rho/(1-\rho) = 7$
4. L_q = 6 min
5. W_s = 4 min
6. W_q = 3.5 min

$\lambda = 105$ cust/hr

$\mu = 120$ cust/hr

$\mu = \frac{1}{30 \text{ second}}$

يعني في 1 ثانية 2 customer

في دقيقة 120 customer

---

another special ← (2) (M/M/1) : (GD/N/∞)

servers = 1

Max # of customers in the sys = N

→ new arrivals are not allowed when the # of cust in the system reaches N

1- $$\lambda_n = \begin{cases} \lambda, & n = 0, 1, \ldots, N-1 \\ 0, & n = N \end{cases}$$

2- $\mu_n = \mu, \quad n \neq 0$

3- $\rho = \frac{\lambda}{\mu}$ → ask Shaimaa

4- $$P_n = \begin{cases} \rho^n P_0, & n \leq N \\ 0, & n > N \end{cases}$$

5- $\sum_{n=0}^{N} P_n = 1$

6- $$P_0 = \begin{cases} \frac{1-\rho}{1-\rho^{N+1}}, & \rho \neq 1 \quad \rho < \rho > \\ \frac{1}{N+1}, & \rho = 1 \end{cases}$$

بالنسبة لـ N ما بتأثر على ρ إذا $\rho < 1$

7- $\lambda = \lambda_{eff} + \lambda_{lost}$

$\lambda_{lost} = \lambda P_N$

8- $L_s = \begin{cases} \dfrac{\rho\left(1-(N+1)\rho^N + N\rho^{N+1}\right)}{(1-\rho)(1+\rho^{N+1})}, & \rho \neq 1 \\ \dfrac{N}{2}, & \rho = 1 \end{cases}$

L_s → قانونية

$W_s \rightarrow L_s$

$W_q \rightarrow W_s$

$V_q \rightarrow W_q$

9- $w_s, w_q \longrightarrow L_s = \lambda_{eff} * w_s$

$w_s = w_q + \frac{1}{\mu}$

10- $L_q = \lambda_{eff} * w_q$

Example: Example 18.6-4 + Problem 18-59

Consider the car wash facility of Example 18.6-2. Suppose that the facility has a total of four parking spaces. If the parking lot is full, newly arriving cars balk to other facilities.

The owner wishes to determine the impact of the limited parking space on losing customers to the competition.

a) Probability that an arriving car will go into the wash bay immediately on arrival.

b) Expected waiting time until a service starts.

c) Expected number of empty parking spaces.

d) Probability that all parking spaces are occupied.

e) Percent reduction in average service time that will limit the average time in the system to about 10 minutes.

$W_s = 22.4 \rightarrow W_s = 10$

$N = 5$ → $N = 4 + 1$

because we have 1 bay

$\lambda_{lost} = \lambda P_N$

$P_N = P_5$

$P_0 = 0.365048$

$\rightarrow P_5 = 0.04892$

$\lambda_{lost} = 4 \times 0.04892$ خسارة بالساعة

$= 0.19229$ car/hr

≈ 4.6 car/day $\times 24$

الخسارة باليوم

اذا بدو كم بخسر مصاري

معادلة الساعة × index

parking spaces are the que

λ not changing

$\mu \uparrow \rightarrow W_s = 10$

a) P_0

b) W_q

c) $4 - L_q$ ← max # of parking spaces

max q ← expected # of cars in Que

d) P_5: # of cars in sys are 5 → 4 in Que, 1 in bay (service)

e) 22.4 min → 10 min

e) → avg time in syst (W_s)

$W_s = 22.4$

6 percent reduction to 10

W_s old = 10 min

new = 6 min reduced = 10 − 6 = 4

بدو نقلل الوقت في النظام

W_s = 10 min عشان نحقق الهدف

$[\lambda, M, W_s]$

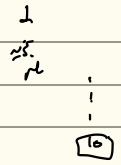
λ ثابت

M نغير

$W_s \rightarrow 22.4$

trial and error

عشان نحقق الهدف



Wed, May 25th
lec 19

Queing networks.

Jackson network → (M, M)

↳ (M/M/c) : (G.D/∞/∞) → for it to be Jackson network

c=1 , c>1

Example

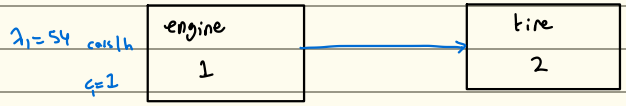
The last two things that are done to a car before its manufacture is complete are installing the engine and putting on the tires. An average of 54 cars per hour arrive requiring these two tasks. One worker is available to install the engine and can service an average of 60 cars per hour. After the engine is installed, the car goes to the tire station and waits for its tires to be attached. Three workers serve at the tire station. Each works on one car at a time and can put tires on a car in an average of 3 minutes. Both interarrival times and service times are exponential.

1. Determine the mean queue length at each work station. L_q
2. Determine the total expected time that a car spends waiting for service.

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two stations → Que before station1 + Que before station2

- they have the same λ (because he said all the cars going through 1 go through 2



$\lambda_1 = 54$ cars/h, $c_1 = 1$, $\mu_1 = 60$ cars/h

$c_2 = 3$, $1/\mu_2 = 3$ min, $\frac{1}{3} = 0.33 \times 60$, $\mu_2 = 20$ cars/hour

(Queuing)2 ... ρ<1 ... not explode

L_{q_1}, W_{q_1} ... $\frac{\rho}{c} < 1$ ... ρ ...

$$\rho = \frac{\lambda}{\mu} = \frac{54}{60} = 0.9$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{0.9}{0.1} = 8.1$$

$$W_q = 0.15 \text{ hr} = 9 \text{ min}$$

$$\rho = \frac{54}{20} = 2.7$$

$$\frac{\rho}{c} = \frac{2.7}{3} = 0.9 < 1$$

↳ will not explode

$L_q = 7.35$ cars

$W_q = 0.136$ hr = 8.17 min

b) 17.17 min

$W_{q_1} + W_{q_2}$ ←

Jackson network

Example

All batch jobs submitted to a computer center must first pass through an input processor before moving on to the central processor station where the bulk of the work is performed.

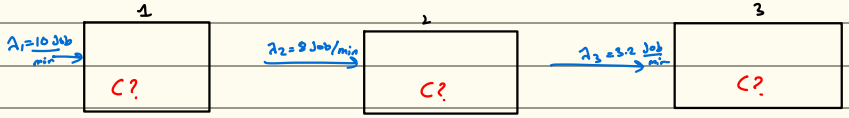
Because of errors, only 80% of the jobs go through the central processor; the remaining 20% are rejected. Of the jobs that pass through the central processor successfully, 40% are routed to a printer station where a hard copy is produced.

Jobs arrive randomly at the computer center at an average rate of 10 per minute. To handle the load, each station may have several processors operating in parallel. The times for the three steps have exponential distributions with means as follows: 10 seconds for an input processor, 3 seconds for a central processor, and 10 seconds for a printer. When all the processors at a station are in use, an arriving job must wait in a queue. All queues are assumed to have unlimited capacity.

- Our goal is to find the minimum number of processors of each type
- and compute the average time required for a job to pass through the system.
- Where is the bottleneck?

$Ws_1 + Ws_2 + Ws_3$

* Important :-
not all the jobs which go throug 1 go to 2



$1/\mu_1 = 10$ seconds, $\mu_1 = 6$ Job/min

$1/\mu_2 = 3$ seconds, $\mu_2 = 20$ Job/min

$1/\mu_3 = 70$ seconds

$\mu_3 = \frac{1}{70} \times 60 = 0.857$

$\rho_1 = \frac{10}{6} = 1.67 \rightarrow \frac{\rho_1}{c_1} < 1 \rightarrow \rho_1 < c_1$

$1.67 < c_1$, $c_1 = 2$ — it's the # of servers so you always round up

$\rho_2 = \frac{8}{20} \Rightarrow \frac{2}{20} < c_2 \rightarrow c_2 = 1$

$\rho_3 = \frac{3.2}{0.857} = 3.73 \rightarrow 3.73 < c_3 \rightarrow c_3 = 4$

$\frac{\rho}{c} < 1$ لازم يكون اقل من c بحيث

$L_{q_1} = 8.7886$ Jobs

$L_{s_1} = L_{q_1} + \rho = 5.455$ Jobs

$W_{s_1} = L_{s_1}/\lambda = 0.5455$ min

$W_{q_1} = L_{q_1}/\lambda = 0.37886$ min

$W_{service} = L_s - L_q = 0.16654$ min

$W_{s_2} = \frac{1}{\mu - \lambda} = \frac{1}{20 - 8} = 0.08333$ min

$L_{q_2} = 0.2667$ Jobs

$L_{q_2} + \rho \leftarrow L_{s_2} = 0.667$ Jobs

$W_{q_2} = 0.0333$ min

$W_{service} = 0.05$ min

$L_{q_3} = 11.99$ Jobs

$L_{s_3} = 15.72$ Jobs

$W_{q_3} = 3.7469$ min

$W_{s_3} = 4.9$ min

$W_{service_3} = (W_s - W_q)$ min

$= 1.1531$ min

$W_{Total} = 0.0833 + 0.5455 + 4.9$

the bottleneck → is the taking the longest time

here the station 3

# of servers ... ليش اخذنا هذا العدد

Section in the book 8- Self service

$(M/M/\infty) : (GD/\infty/\infty)$

self service

Atm isn't self service because the Atm is the server

so no human server $\neq$ self service.

in self service $\rightarrow$ we do the Job.

$\lambda_n = \lambda$, for $n \geq 0$

$\mu_n = n \cdot \mu$, $n \geq 0$

$P_n = \frac{P_0 \rho^n}{n!}$, $n \geq 0$

$\rho^0 = e^{-\rho}$

$L_s = \rho$, L_q = zero

W_q = zero

$W_s = 1/\mu$ time in the system = $1/\mu$ وقت في النظام

$W_{service} = W_s$

---

test $\rightarrow$ the one doing labs your ' they didn't say that

Problem 18-92.

New drivers are required to pass written tests before they are given road driving test. These tests are usually administered in the city hall. Records at the City of Springdale show that the average number of written tests is 100 per 8-hr day. The average time needed to complete the test is about 30 minutes. However, the actual arrival of test takers and the time each spends on the test are totally random.

Determine the following:

a) The average number of seats the test hall should provide.

b) The probability that the number of test takers will exceed the average number of seats provided in the test hall.

c) The probability that no tests will be administered in any one day.

λ = 100/8hr = 12.5 test/h

$1/\mu$ = 30 min

$\frac{1}{30} \times 60$ μ = 2 test/h

a) $L_s = \rho = \frac{\lambda}{\mu} = \frac{12.5}{2} = 6.25$

$\approx$ 7 seats

b) $P_{n>8} = 1 - \sum_{n=0}^{7} P_n$

c) $P_0 = e^{-\rho} = e^{-6.25}$

Mon 26/5

$(M/M/C) : (GD/\infty/\infty)$

- C: # of servers → multiple servers (c)
- GD → $\lambda_{eff} = \lambda$ (كل الداخلين بيخدموا ولا بيخلوا)

- C: identical parallel servers
- λ: arrival rate
- μ: service rate /server

if $n \leq c$ → all n customers are in service

$n > c$ → $n-c$ waiting line

all c are busy

Effect of using C identical servers :- is a proportional increase in the facility service rate. μ

$\lambda_n = \lambda \quad , n \geq 0$

↳ λ if there is n in the system

↳ this means it doesn't depend on the customer number

$$\mu_n = \begin{cases} n\mu & , n < c \\ c\mu & , n \geq c \end{cases}$$

- $n\mu$: busy servers = n
- $c\mu$: busy servers = c max
- service rate depends on # customers in system

service rate: كمية الشغل اللي بيخدموا في الوحدة الزمنية

n_{busy} = ACC

$$\mu_n = \min(n, c) \times \mu$$

↳ service rate /server

$\rho = \lambda/\mu$ ← rate per server

$$\frac{\rho}{c} < 1$$

in single server :- $\rho < 1$ → to not explode

A) $$P_n = \begin{cases} \frac{\rho^n}{n!} P_0, & n < c \\ \frac{\rho^n}{c!\,(c)^{n-c}} P_0, & n \geq c \end{cases}$$

B) P_0 is determined from $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}\left(\frac{1}{1-\rho/c}\right)\right]}$$

C) $$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \cdot P_0$$

$\lambda_{eff} = \lambda$ → little's law.

D) $L_s = L_q + \rho$

E) $W_q = L_q/\lambda$

F) $W_s = W_q + 1/\mu$

note :-

ρ/c → Utilization

Traffic Intensity

Efficiency

reduction in means

Example

Sports fans arrive at a ticket counter at the Amman Sports City Arena by a Poisson process with rate 105 customers per hour. Customers are served by a single cashier. The service time has an exponential distribution with mean 30 seconds. → c = 1

The service rate is 1/(30 seconds), or 120 customers per hour. Calculate:

c = 2

1. The utilization r is λ/μ = 105/120 = 0.875.
2. Ls = 0. 875 / (1 - .875) = 7 customers → 1. 09 cust
3. Lq = 0.875^2 /(1- .875) = 6.125 customers → 0.206 cust
4. Ws = (1/105)(7) = 0.0667 hours = 4 minutes 0.62 min
5. Wq =(1/105)(6.125) =0.0583 hours = 3.5 minutes

$L_q \cdot C \rightarrow 0.206$

1) measure P_0 to get everything $\frac{\rho}{c} < 1$

$\frac{0.871}{2} < 1$ ✓

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}\left[\frac{1}{1-\frac{\rho}{c}}\right]}$$

$$= \frac{1}{\frac{\rho^0}{0!} + \frac{\rho^1}{1!} + \frac{\rho^2}{2!}\left[\frac{1}{1-\frac{\rho}{2}}\right]}$$

$$= \frac{1}{1 + 0.875 + \frac{0.875^2}{2}\left[\frac{1}{1-\frac{0.875}{2}}\right]} = 0.391$$

$$L_q = \frac{\rho^{c+1}}{(c-1)!(c-\rho)^2} \cdot P_0$$

$$= \frac{(0.875)^3}{1!\,(2-0.875)^2} \times 0.391 = 0.206 \text{ cust.}$$

$L_s = L_q + \rho$

$= 0.206 + 0.875 = 1.08$ cust

$W_q = L_q/\lambda = 0.206/105 =$

$=$ Cust / 105 Cust/hr

≈ 0.00196 hr

$W_s = L_s/\lambda = 0.0103$ hr

$= 0.62$ min

$= 6.12$ min

$(M|M|c):(GD|N|\infty)$, $c \leq N$

λ_{eff} ↵ # customers $\leq$ allowed in pool $\hookrightarrow N \to \infty$

$\lambda = \lambda_{eff} + \lambda_{lost}$ ↳ $= \lambda P_N$

$\rho/c < 1$ ↵ شرط لازم برای پایداری صف

λ_n = $\begin{cases} \lambda & , 0 \leq n \leq N-1 \\ 0 & , n > N \end{cases}$ per building ↵

μ_n = $\begin{cases} n\mu & , 0 < n < c \\ c\mu & , c \leq n \leq N \end{cases}$ ↵ per server

---

A) $P_n = \begin{cases} \frac{\rho^n}{n!} \cdot P_0 & , 0 < n < c \\ \frac{\rho^n}{c!\, c^{n-c}} \cdot P_0 & , c \leq n \leq N \end{cases}$

B) $P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c \left[1 - \left(\frac{\rho}{c}\right)^{N-c+1}\right]}{\left[c! \left(1 - \frac{\rho}{c}\right)\right]}}$, $\rho/c < 1$

$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!}(N-c+1)}$, $\rho/c = 1$

C) $L_q = \sum_{n=c}^{N} (n-c) P_n$

no que ↵ c ، بیشتر ↵

$L_q = \frac{\rho^{c+1}}{(c-1)!\,(c-\rho)^2} \left[1 - \left(\frac{\rho}{c}\right)^{N-c+1} - (N-c+1)\left(1-\frac{\rho}{c}\right)\left(\frac{\rho}{c}\right)^{N-c}\right] P_0$

↳ if $\rho/c \neq 1$

→ if $\rho/c = 1$:

$$L_q = \frac{\rho^c (N-c)(N-c+1)}{2\,c!} \cdot p_0 \quad , \frac{\rho}{c} = 1$$

$\lambda = \lambda_{eff} + \lambda_{lost}$

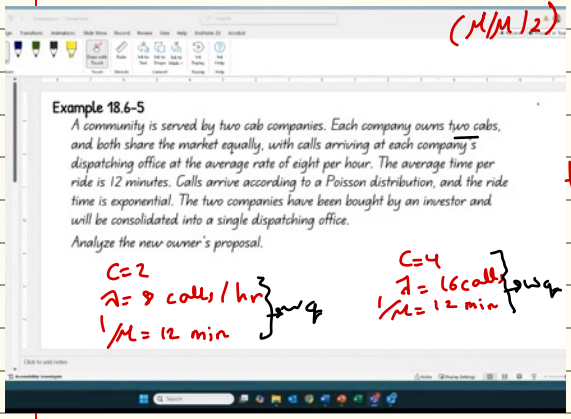
$\lambda_{lost} = \lambda p_N$

W_q , W_s , L_s

↓

little's law

$(M/M/2)$:( )



Example 18.6-5

A community is served by two cab companies. Each company owns two cabs, and both share the market equally, with calls arriving at each company's dispatching office at the average rate of eight per hour. The average time per ride is 12 minutes. Calls arrive according to a Poisson distribution, and the ride time is exponential. The two companies have been bought by an investor and will be consolidated into a single dispatching office.

Analyze the new owner's proposal.

$C=2$, $\lambda = 8$ calls/hr, $1/\mu = 12$ min } for W_q

$C=4$, $\lambda = 16$ calls, $1/\mu = 12$ min } for W_q

two dispatching office

8 cars

↓

one office

↓

$\lambda = 16$

analyze new proposal

performance measure مقياس الأداء

$L_s \rightarrow W_q + 1/\mu$

L_q W_q

لا يهمنا بالوقت

يستخدم

two cases لا يستخدم

$\frac{1}{\mu} = 12 \rightarrow \mu = \frac{1}{12}$ min

$\mu = 60\frac{min}{hr} \times \frac{1\ ride}{12\ min} = 5$

$W_q = L_q / \lambda$, $\rho = \frac{\lambda}{\mu} = \frac{8}{5}$

$= 1.6$

↳ $\rho = 1.6$

$\rho/c < 1$

$L_q = 2.8416$

$W_q = 0.35552\ hr = 21.33\ min$

Case 2:

$\rho = 16/5 = 3.2$ $\quad \frac{\rho}{c} = \frac{3.2}{4} = 0.8 < 1$

$L_q = 2.385$ cash

$W_q = 0.149\ hr = 8.94\ min$

the case #2 is better single dispatch office.